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Bailey pairs for the q-hypergeometric integral pentagon identity

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Abstract In this work, we construct new Bailey pairs for the integral pentagon identity in terms of q-hypergeometric functions. The pentagon identity considered here represents the equality of the partition functions of certain three-dimensional supersymmetric dual theories. It can be also interpreted as the star-triangle relation for the Ising-type integrable lattice model.

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1 Introduction

Bailey's lemma [1,2] is a powerful tool to derive hypergeometric identities (ordinary, trigonometric, and elliptic type). In this work, we construct new integral Bailey pairs for the pentagon identity in terms of q-hypergeometric functions. The pentagon identity can be interpreted as a Pachner's 3-2 move for triangulated three-dimensional manifolds. Such identities also play a role in the study of supersymmetric gauge theories, integrable models, knot theory, etc.[1]

Let $q, z \in \mathbb{Z}$ with $|q| < 1$. We define the infinite q-product

$$(z;q)_\infty := \prod_{k=0}^{\infty}(1 - zq^k). \tag{1.1}$$

We also adopt the following convention

$$(a, b; q)_\infty := (a; q)_\infty (b; q)_\infty. \tag{1.2}$$

Theorem 1.1 *Let $a_1, a_2, a_3, b_1, b_2, b_3, q \in \mathbb{C}$ and integers $m_i, n_i \in \mathbb{Z}$. Then*

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \frac{dz}{2\pi i z} (-q^{\frac{1}{2}})^{\sum_{i=1}^3 \frac{|m_i+m|}{2}+\frac{|n_i-m|}{2}} z^{-\sum_{i=1}^3 (\frac{|m_i+m|}{2}-\frac{|n_i-m|}{2})} \\
&\quad \times \prod_{i=1}^3 a_i^{-\frac{|m_i+m|}{2}} b_i^{-\frac{|n_i-m|}{2}} \frac{(q^{1+\frac{|m_i+m|}{2}}\frac{1}{a_i z}, q^{1+\frac{|n_i-m|}{2}}\frac{z}{b_i}; q)_\infty}{(q^{\frac{|m_i+m|}{2}} a_i z, q^{\frac{|n_i-m|}{2}}\frac{b_i}{z}; q)_\infty} \\
&= (-q^{\frac{1}{2}})^{\sum_{i,j=1}^3 \frac{|m_i+n_j|}{2}} \prod_{i,j=1}^3 (a_i b_j)^{-\frac{|m_i+n_j|}{2}} \\
&\quad \times \frac{(q^{1+\frac{|m_i+n_j|}{2}}\frac{1}{a_i b_j}; q)_\infty}{(q^{\frac{|m_i+n_j|}{2}} a_i b_j; q)_\infty},
\end{aligned} \tag{1.3}$$

where the balancing conditions are

$$\prod_{i=1}^3 a_i b_i = q, \tag{1.4}$$

$$\sum_{i=1}^3 m_i + n_i = 0, \tag{1.5}$$

and $\mathbb{T}$ represents the positively oriented unit circle.

We would like to mention that the integral identity represents the supersymmetric duality for three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories with the flavor symmetry[2] $SU(3) \times SU(3) \times U(1)$. This identity can also be writ-

[1] See some recent works [3–11].

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[2] In this case parameters a_i and b_i stand for the flavor symmetry and z is the fugacity for the $U(1)$ gauge group.

ten as the star-triangle relation[3] for some integrable model of statistical mechanics.

The proof of the form above is given in [8] for the balancing conditions[4]

$$\prod_{i=0}^{3} a_i = \prod_{i=0}^{3} b_i = q^{\frac{1}{2}}, \quad \sum_{i=0}^{3} m_i = \sum_{i=0}^{3} n_i = 0. \tag{1.6}$$

The absolute values can be eliminated by the identity [12]

$$\frac{(q^{1+\frac{|m|}{2}}/z;q)_\infty}{(q^{\frac{|m|}{2}}z;q)_\infty} = (-q^{-\frac{1}{2}}z)^{\frac{|m|-m}{2}} \frac{(q^{1+\frac{m}{2}}/z;q)_\infty}{(q^{\frac{m}{2}}z;q)_\infty}, \tag{1.7}$$

and one ends up with the following q-hypergeometric sum/integral identity [6–8]

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \prod_{i=1}^{3} \frac{(q^{1+\frac{m+m_i}{2}}\frac{1}{a_i z}, q^{1+\frac{n_i-m}{2}}\frac{z}{b_i};q)_\infty}{(q^{\frac{m+m_i}{2}}a_i z, q^{\frac{n_i-m}{2}}\frac{b_i}{z};q)_\infty} \frac{1}{z^{3m}} \frac{dz}{2\pi i z} \\ = \frac{1}{\prod_{i=1}^{3} a_i^{m_i} b_i^{n_i}} \prod_{i,j=1}^{3} \frac{(q^{1+\frac{m_i+n_j}{2}}\frac{1}{a_i b_j};q)_\infty}{(q^{\frac{m_i+n_j}{2}}a_i b_j;q)_\infty}.\end{aligned} \tag{1.8}$$

2 Integral pentagon identity

In [6–8] it was shown that the identity (1.3) can be written as an integral pentagon identity

$$\begin{aligned}\sum_{m\in\mathbb{Z}} \int_{\mathbb{T}} \frac{dz}{2\pi i z} \prod_{i=1}^{3} \mathcal{B}[a_i, n_i+m; b_i z^{-1}, m_i - m] \\ = \mathcal{B}[a_1 b_2, n_1+m_2; a_3 b_1; n_3+m_1] \\ \times \mathcal{B}[a_2 b_1, n_2+m_1; a_3 b_2, n_3+m_2],\end{aligned} \tag{2.1}$$

where we define the following function as

$$\begin{aligned}\mathcal{B}_m[a,n;b,m] = (-q^{\frac{1}{2}})^{\frac{|n|}{2}+\frac{|m|}{2}+\frac{|n+m|}{2}} a^{-\frac{|n|}{2}} b^{-\frac{|m|}{2}} (ab)^{\frac{|n+m|}{2}} \\ \times \frac{(q^{1+\frac{|n|}{2}}a^{-1}, q^{1+\frac{|m|}{2}}b^{-1}, q^{\frac{|n+m|}{2}}ab;q)_\infty}{(q^{\frac{|n|}{2}}a, q^{\frac{|m|}{2}}b, q^{1+\frac{|n+m|}{2}}(ab)^{-1};q)_\infty}.\end{aligned} \tag{2.2}$$

In a general sense, any algebraic relation for operators $\mathcal{B}$

$$\mathcal{B}\mathcal{B}\mathcal{B} = \mathcal{B}\mathcal{B} \tag{2.3}$$

which can be interpreted as a 2–3 Pachner move of a triangulated three-dimensional manifold is called a pentagon relation [4,5]. Note that the integral pentagon identity (2.1) for the $\mathcal{N}=2$ supersymmetric $S^2\times S^1$ partition functions is supposed to be related to some topological invariant of corresponding 3-manifold via $3d$–$3d$ correspondence [12,13] that connects three-dimensional $\mathcal{N}=2$ supersymmetric theories and triangulated 3-manifolds. There are several examples of pentagon identities arising from supersymmetric gauge theory computations, see, e.g. [6–15].

3 Bailey pairs

Rogers–Ramanujan type identities are being continuously used in the solution of the integrable models, namely to derive the Yang–Baxter and the pentagon identities. In fact, a well-known example of this usage is conducted during the investigations of the hard hexagon model by Baxter. It turns out that Bailey discovered a systematic way to derive these types of identities [1,2,16,17]. As generalized by Andrews [18,19], there exists an iterative scheme to derive infinitely many of these identities if one pair, called a Bailey pair is known. This forms the so-called Bailey chain. The induction step of generating the particular Bailey pairs is referred to as the Bailey lemma for the chain we consider.

A generalization of the Bailey pairs approach to the integral identities is firstly done by Spiridonov in [20,21]. The construction of integral Bailey pairs yields new powerful verifications of various supersymmetric dualities [22,23], generating solutions to the Yang-Baxter equation [24–27], etc.

Accordingly, the generalized version of the Bailey chain is a couple of infinite sequences of holomorphic functions $\{\alpha_n^{(i)}\}_{n\geq 0}$ and $\{\beta_n^{(i)}\}_{n\geq 0}$ such that there exists an identity independent of i which connect $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ as

$$\beta_n^{(i)} = F_n(\alpha_0^{(i)}, \alpha_1^{(i)}, \ldots, \alpha_n^{(i)}), \tag{3.1}$$

where F can be an operator which may now include sum or integrals. Here, $\alpha_n^{(i)}$ and $\beta_n^{(i)}$ are constructed according to

$$\alpha_n^{(i)} = G(\alpha_0^{(i)}, \alpha_1^{(i)}, \ldots, \alpha_{n-1}^{(i)}), \tag{3.2}$$

$$\beta_n^{(i)} = H(\beta_0^{(i)}, \beta_1^{(i)}, \ldots, \beta_{n-1}^{(i)}), \tag{3.3}$$

where G and H represent integral-sum operators.

Definition 3.1 Let $\{\alpha_m(z;t)\}_{m\in\mathbb{Z}}$ and $\{\beta_m(z;t)\}_{m\in\mathbb{Z}}$ be two sequences of functions. They are said to form a Bailey pair with respect to the parameter t iff

$$\begin{aligned}\beta_m(w;t) = \sum_{n\in\mathbb{Z}} \int dz \mathcal{B}[twz^{-1}, m-n \\ + n_t, tw^{-1}z, -m+n+n_t]\, \alpha_n(z;t).\end{aligned} \tag{3.4}$$

[3] In this case parameters a_i, b_i, and z stand for the continuous spin variables.

[4] Yet, as $SU(3)\times SU(3)\times U(1)$ has five independent parameters, the above form must be correct even for the more general balancing conditions in (1.4, 1.5).

Lemma 3.1 *If $\{\alpha_m(z;t)\}_{m\in\mathbb{Z}}$ and $\{\beta_m(z;t)\}_{m\in\mathbb{Z}}$ form a Bailey pair with respect to t, then the following sequences*

$$\alpha'_n(w;st) = \mathcal{B}[tuw, n+n_u+n_t, s^2, 2n_s]\alpha_n(w;t) \tag{3.5}$$

$$\begin{aligned}\beta'_n(w;st) = \sum_{m\in\mathbb{Z}}\int \frac{dx}{2\pi i x}\mathcal{B}[swx^{-1}, -m+n \\ + n_s; ux, n_u+m]\mathcal{B}[st^2uw, n+2n_t \\ + n_u+n_s, sw^{-1}x, -n+m+n_s]\beta_m(x;t)\end{aligned} \tag{3.6}$$

form a Bailey pair with respect to the parameter st.

Proof We have to show that

$$\begin{aligned}\beta'_n(w,st) = \sum_{p\in\mathbb{Z}}\int \mathcal{B}[stwy^{-1}, n-p+n_s \\ + n_t, sty^{-1}x, -n+p+n_s+n_t]\alpha'_p(y,st)dy.\end{aligned} \tag{3.7}$$

Inserting (3.4) in (3.6), we first calculate the left-hand side of the equality (3.7)

$$\begin{aligned}\beta'_n(w;st) &= \sum_{m\in\mathbb{Z}}\int_{\mathbb{T}}\frac{dx}{2\pi i x}\mathcal{B}[swx^{-1}, n+n_s-m, ux, m+n_u] \\ &\quad\times\mathcal{B}[st^2uw, n+n_u+2n_t+n_s, sw^{-1}x, m-n+n_s] \\ &\quad\times\sum_{p\in\mathbb{Z}}\int_{\mathbb{T}}\mathcal{B}[txy^{-1}, m-p \\ &\quad + n_t, tx^{-1}y, -m+p+n_t]\alpha_p(y,t)dy \\ &= \sum_{m\in\mathbb{Z}}\sum_{p\in\mathbb{Z}}\int\int\mathcal{B}[swx^{-1}, -m+n+n_s, ux, m+n_u] \\ &\quad\times\mathcal{B}[st^2uw, n+n_u+2n_t+n_s, sw^{-1}x, -n+m+n_s] \\ &\quad\times\mathcal{B}[txy^{-1}, m-p+n_t, tx^{-1}y, -m+p+n_t] \\ &\quad\times\alpha_p(y,t)dy\frac{dx}{2\pi i x}\end{aligned} \tag{3.8}$$

Hence, by regrouping the terms accordingly, we obtain[5]

$$\begin{aligned}&\sum_{p\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int(-q^{\frac{1}{2}})^{\frac{|m+n_u|}{2}+\frac{|m-n+n_s|}{2}+\frac{|m-p+n_t|}{2}+\frac{|n-m+n_s|}{2}+\frac{|m-n_u|}{2}+\frac{|p-m+n_t|}{2}} \\ &\times(ux)^{-\frac{|m+n_u|}{2}}(sw^{-1}x)^{-\frac{|m-n+n_s|}{2}}(ty^{-1}x)^{-\frac{|m-p+n_t|}{2}}(swx^{-1})^{-\frac{|n-m+n_s|}{2}}(s^2t^2q^{-1}ux)^{\frac{|m-n_u|}{2}}(tyx^{-1})^{-\frac{|p-m+n_t|}{2}} \\ &\times\frac{(q^{1+\frac{|n-m+n_s|}{2}}(swx^{-1})^{-1})_\infty}{(q^{\frac{|n-m+n_s|}{2}}swx^{-1})_\infty}\frac{(q^{1+\frac{|m+n_u|}{2}}(ux)^{-1})_\infty}{(q^{\frac{|m+n_u|}{2}}ux,)_\infty}\frac{(q^{1+\frac{|m-n+n_s|}{2}}(sw^{-1}x)^{-1})_\infty}{(q^{\frac{|m-n+n_s|}{2}}sw^{-1}x)_\infty}\frac{(q^{1+\frac{|m-n_u|}{2}}s^2t^2q^{-1}ux)_\infty}{(q^{\frac{|m-n_u|}{2}}s^{-2}t^{-2}qu^{-1}x^{-1})_\infty} \\ &\times\frac{(q^{1+\frac{|m-p+n_t|}{2}}(ty^{-1}x)^{-1})_\infty}{(q^{\frac{|m-p+n_t|}{2}}ty^{-1}x)_\infty}\frac{(q^{1+\frac{|p-m+n_t|}{2}}(tyx^{-1})^{-1})_\infty}{(q^{\frac{|p-m+n_t|}{2}}tyx^{-1})_\infty}(-q^{\frac{1}{2}})^{-\frac{|n-n_t|}{2}+|n_t|+\frac{|n+n_t|}{2}}(swu)^{\frac{|n-n_t|}{2}}(st^2uw)^{-\frac{|n+n_t|}{2}}(q^{-1}t^2)^{|n_t|} \\ &\times\frac{(q^{1+\frac{|n-n_t|}{2}}swq^{-1}u)_\infty}{(q^{\frac{|n-n_t|}{2}}(swq^{-1}u)^{-1})_\infty}\frac{(q^{1+\frac{|n+n_t|}{2}}(st^2uw)^{-1})}{(q^{\frac{|n+n_t|}{2}}st^2uw)_\infty}\frac{(q^{1+|n_t|}q^{-1}t^2)_\infty}{(q^{|n_t|}qt^{-2})_\infty}\alpha_p(y,t)dy\frac{dx}{2\pi i x}\end{aligned} \tag{3.9}$$

where we required the sum of the powers of x to vanish, namely

$$n_u+n_s+n_t = 0 \tag{3.10}$$

Upon renaming the variables as

$$a_1 = u \to m_1 = n_u \quad b_1 = sw \to n_1 = n+n_s \tag{3.11}$$

$$a_2 = sw^{-1} \to m_2 = -n+n_s \quad b_2 = qs^{-2}t^{-2}u^{-1} \to n_2 = n_u \tag{3.12}$$

$$a_3 = ty^{-1} \to m_3 = -p+n_t \quad b_3 = tx \to n_3 = p+n_t \tag{3.13}$$

we identify the integral relation (1.3). Also, observe that the constraint (3.10) resulted in the balancing condition (1.5). We hence get upon simplification and regrouping of the terms

$$\begin{aligned}&\sum_{p\in\mathbb{Z}}\int\alpha_p(y,t)dy(-q^{\frac{1}{2}})^{\frac{|n-p-n_u|}{2}+\frac{|-n+p-n_u|}{2}-|n_u|} \\ &\times(stwy^{-1})^{-\frac{|n-p-n_u|}{2}}(stw^{-1}y)^{-\frac{|p-n-n_u|}{2}}(s^2t^2)^{|n_u|} \\ &\times\frac{(q^{1+\frac{|n-p-n_u|}{2}}(stwy^{-1})^{-1})_\infty}{(q^{\frac{|n-p-n_u|}{2}}stwy^{-1})_\infty}\frac{(q^{1+\frac{|p-n-n_u|}{2}}(stw^{-1}y)^{-1})_\infty}{(q^{|p-n-n_u|}stw^{-1}y)_\infty} \\ &\times\frac{(q^{\frac{|n_u|}{2}}s^2t^2)_\infty}{(q^{1+\frac{|n_u|}{2}}s^{-2}t^{-2})_\infty} \\ &\times(-q^{\frac{1}{2}})^{\frac{|p-n_s|}{2}+|n_s|-\frac{|n_s+p|}{2}}(tuy)^{-\frac{|p-n_s|}{2}}(s^2)^{-|n_s|}(s^2tuy)^{\frac{|p+n_s|}{2}} \\ &\times\frac{(q^{1+\frac{|p-n_s|}{2}}(tuy)^{-1})_\infty}{(q^{\frac{|p-n_s|}{2}}tuy)_\infty}\frac{(q^{1+|n_s|}s^{-2})_\infty}{(q^{|n_s|}s^2)_\infty}\frac{(q^{1+\frac{|p+n_s|}{2}}s^2tuy)_\infty}{(q^{\frac{|p+n_s|}{2}}(s^2tuy)^{-1})_\infty},\end{aligned} \tag{3.14}$$

[5] For convenience q of the q-product is omitted.

which is the desired operator equality

$$\sum_{p\in\mathbb{Z}}\int dy\mathcal{B}[stwy^{-1}, n_s+n_t+n-p, stw^{-1}y, n_s + n_t - n + p)\mathcal{B}[tyu, n_t+p+n_u, s^2, 2n_s] = \sum_{p\in\mathbb{Z}}\int dy\mathcal{B}[stwy^{-1}, n_s+n_t + n-p, stw^{-1}y, n_s+n_t-n+p). \quad (3.15)$$

□

4 Conclusions

In this work, we have constructed a new integral Bailey pair for the pentagon identity in the form of q-hypergeometric functions. One can use this Bailey construction to obtain new supersymmetric dualities for linear quiver theories. Namely, any relation between Bailey pairs $\alpha^{(n)}$ and $\beta^{(n)}$ gives integral identities corresponding to the equality of partition functions of certain dual linear quivers, see e.g. [22,23].

We would like to mention that the pentagon identity presented here can also be written as the star-triangle relation for some integrable lattice model of statistical mechanics. It would be interesting to construct the Bailey pairs corresponding to the star-triangle form of the same integral identity.

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Data availability This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical study and no experimental data has been listed.]

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