BLACK HOLES AND COSMIC CENSORSHIP

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ABSTRACT

BLACK HOLES AND COSMIC CENSORSHIP

This thesis starts with a brief review of causality conditions, gravitational collapse and singularity theorems. Cosmic censorship conjecture is introduced and the need for such a conjecture to preserve the deterministic nature of physics is explained. Examples of thought experiments which are designed to test the validity of cosmic censorship conjecture, are given. These examples involving cases of both particles and fields, are considered to have conceptual importance. Spinor algebra and analysis is reviewed. Newman-Penrose formalism is introduced. The derivations of Teukolsky's solutions for test fields in Kerr space-time are given. Using these solutions an original thought experiment involving an extremal Kerr black hole and a test electromagnetic is constructed. It is shown that cosmic censorship conjecture is not violated in this case. The examples of thought experiments with particles that start with nearly extremal black holes instead of extremal ones are given. Another original thought experiment is constructed using integer-spin test fields and a nearly extremal Kerr black hole. It is found that cosmic censorship conjecture is violated in this case as long as radiative and self-force effects are ignored. Finally the case of neutrino fields are considered as an example of half integer-spin. It is shown that neutrino fields lead to a generic violation of the conjecture which is unique among all the attempts since the conjecture was introduced.

ÖZET

KARA DELİKLER VE KOZMİK SANSÜR

Bu tezde öncelikle nedensellik şartları, çekimsel çöküş ve tekillik kuramları gözden geçirildi. Kozmik sansür hipotezi tanıtıldı ve fiziğin deterministik doğasını korumak için böyle bir hipotezin gerekliliği açıklandı. Kozmik sansür hipotezinin geçerliliğini sınamak amacıyla tasarlanan düşünce deneylerinden örnekler verildi. Parçacık ve alan durumlarını kapsayan bu örneklerin kavramsal önem taşıdığı düşünülmektedir. Spinör cebiri ve analizi gözden geçirildi. Newman-Penrose formalizmi tanıtıldı. Kerr uzay-zamanınında test alanları için Teukolsky çözümleri verildi. Bu çözümler kullanılarak uçdeğerdeki bir Kerr kara deliği ile test elektromanyetik alanının etkileşimini inceleyen orjinal bir düşünce deneyi tasarlandı. Bu durumda kozmik sansür hipotezinin ihlal edilmediği gösterildi. Uçdeğerdeki kara delikler yerine uçdeğere yakın kara delikleri kullanan düşünce deneylerinden örnekler verildi. Tam sayı spinli test alanları ve uçdeğere yakın bir Kerr kara deliği kullanılarak başka bir orjinal düşünce deneyi tasarlandı. Genelçekimsel ışıma ile enerji kaybı ve öz-kuvvet etkileri ihmal edildiği sürece, bu deneyde kozmik sansür hipotezinin ihlal edildiği gösterildi. Son olarak buçuklu spin durumuna örnek teşkil eden nötrino alanları ele alındı. Nötrino alanlarının hipotezin genel bir ihlaline yol açtığı gösterildi, ki bu durum hipotez ortaya atıldığından beri yapılan tüm denemeler arasında tektir.

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LIST OF SYMBOLS

a	Angular momentum per unit mass
A_μ	Electromagnetic vector potential
$\mathcal{B}$	Black hole region
$\mathcal{C}$	Complex space
D	Domain of dependence
E	Energy
$F_{\mu\nu}$	Electromagnetic field tensor
g_{ab}	Metric tensor
H	Cauchy horizon
$\mathcal{H}$	Event horizon
i^a	Spin base
I^-, I^+	Chronological past, chronological future
$\mathcal{I}^+$	External infinity
J^-, J^+	Causal past, causal future
J_e	Electric four current
J_m	Magnetic four current
l^a, m^a, n^a	Newman Penrose tetrad vectors
$\mathcal{L}_x$	Lie derivative
M	Manifold
o^a	Spin base
p	Point
Q	Charge
r_+	Spatial coordinate of the event horizon
R_{ab}	Ricci tensor
R_{abcd}	Riemann tensor
S	Surface
$\mathcal{T}$	Trapped surface
T_{ab}	Energy momentum tensor

T_p	Tangent space
$\alpha, \beta, \gamma, \epsilon, \kappa, \lambda$	Newman Penrose spin coefficients
δ, Δ	Newman Penrose derivative operators
ϵ_{AB}	Spinor symplectic structure
Θ	Expansion scalar
ϕ_0, ϕ_1, ϕ_2	Newman Penrose electromagnetic scalars
ϕ_{AB}	Spinor equivalent of Maxwell tensor

LIST OF ACRONYMS/ABBREVIATIONS

CCC	Cosmic censorship conjecture
JS	Jacobson Sotiriou
NP	Newman Penrose
WCCC	Weak cosmic censorship conjecture

1. INTRODUCTION

In this chapter we review causality conditions, gravitational collapse and singularity theorems. We introduce the cosmic censorship conjecture and explain the need for such a conjecture to preserve the deterministic nature of physics.

1.1. Causal Structure

In pre-relativity physics two events p and q may be causally related if they lie on the past or the future of each other. There exists a three dimensional surface of simultaneity. It is impossible for an observer to be present at two events p and q that lie on this surface, i.e. only such events can be causally unrelated. In special relativity, however the existence of a maximum speed, namely the speed of light leads to the so called light cone at an event which forms a boundary between the set of events that are causally related to p and those that are not. The light cone is is the set of points that can only be reached by a light signal and forms a three dimensional set.

Events that lie outside the light cone are space-like related to p which means there exists a frame in which they are simultaneous. The set of events which are not causally related to p is a 4 dimensional set, in contrast to pre-relativity physics.

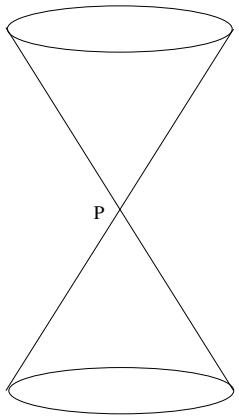


Figure 1.1. Light cone.

In a general space-time manifold (M, g_{ab}), for every $p \in M$ the tangent space T_pM

is isomorphic to Minkowski space-time. The light cone of p is the light cone passing through the origin of the tangent space. (M, g_{ab}) is time orientable if a continuous designation of past and future can be made, i.e. a continuous time-like vector field can be chosen. On a time-like curve λ the tangent vector is time-like for each $p \in \lambda$.

The chronological future of p, denoted by $I^+(p)$, is defined as the set of points that can be reached by a future directed time-like curve starting from p:

$$I^+(p) = \{q \in M; \exists \lambda(t), \lambda(0) = p, \lambda(1) = q\}$$

where $\lambda(t)$ is a future directed time-like curve. $I^+(p)$ is an open subset of M. Note that $p \notin I^+(p)$ unless there exists a closed time-like curve beginning and ending at p.

The causal future $J^+(p)$ is defined in the same way as the set of points that can be reached by a future directed, causal curve, i.e. a time-like or null curve, starting from p. In Minkowski space-time $I^+(p)$ consists of all points reached by time-like geodesics starting from p and the boundary of $I^+(p)$ –denoted by $\dot{I}(p)$– is generated by null geodesics starting from p. In general space-time this is locally true if we restrict ourselves to a neighbourhood U of p and geodesics γ starting from p and staying entirely within U. Note that $I^+(S) = \mathrm{int}[\mathrm{J}^+(\mathrm{S})]$; since the interior of a set is the difference of the set from its boundary, the boundaries of the chronological and the causal futures are always equal, $\dot{I}(S) = \dot{J}(S)$.

A surface S is achronal if there do not exist $p, q \in S$ such that $q \in I^+(p)$, i.e. $I^+(S) \cap S = \emptyset$. The boundary of the chronological future of a set always forms a well behaved, three dimensional, achronal surface. Moreover, it forms an embedded submanifold of M.

We have defined the set of events $I^+(S)$ and $J^+(S)$ that could be influenced by a set of events, S. Let us consider the set of events which are entirely determined by

S. The future domain of dependence of S, $D^+(S)$ is defined by:

$$D^+(S) = \{p \in M \,|\text{Every past inextendible curve through } p \text{ intersects } S\}$$

A curve is called inextendible if it does not have an end point. *Nothing can travel faster than light* so any signal sent to $p \in D^+(S)$ must have registered in S. Given initial conditions on S we should be able to predict what happens at p. We also expect to be able to retrodict the conditions at all $q \in D^-(S)$ with the same "initial" conditions. The full domain of dependence of S is defined by:

$$D(S) = D^+(S) \cup D^-(S)$$

Thus, $D(S)$ represents the complete set of events for which all conditions should be determined by a knowledge of the conditions on S. A closed achronal set Σ for which $D(\Sigma) = M$, the manifold itself, is called a *Cauchy surface*. Σ is an embedded, 3 dimensional, C^0 submanifold of M. Since Σ is achronal we may think of it as the representation of an "instant" of time.

The *future Cauchy horizon* of an achronal set S is defined by

$$H^+(S) = \overline{D^+(S)} - I^-[D^+(S)]$$

where $\overline{D^+(S)}$ denotes the closure of $D^+(S)$, i.e. the union of $D^+(S)$ with its boundary. $H^+(S)$ and the analogously defined $H^-(S)$ measure the failure of S to be a Cauchy surface. For that purpose we define the full Cauchy horizon.

$$H(\Sigma) = H^+(\Sigma) \cup H^-(\Sigma)$$

S is a Cauchy surface if and only if $H(S) = \emptyset$ A manifold (M, g_{ab}) which possesses a Cauchy surface Σ is called globally hyperbolic. It can be proven that $H(S)$ is the

boundary of $D(S)$ by using $\mathrm{int[D(S)]} = \mathrm{I^-[D^+(S)] \cap I^+(D^-(S)]}$. The boundary of a set is the difference of the closure from the interior:

$$\dot{D}(S) = \overline{D(S)} - \mathrm{int[D(S)]} = \overline{\mathrm{D(S)}} - (\mathrm{I^-[D^+(S)] \cap I^+(D^-(S)]})$$

and

$$\overline{D(S)} = \overline{D^+(S)} \cup \overline{D^-(S)}$$

$$\Rightarrow \dot{D}(S) = (\overline{D^+(S)} \cup \overline{D^-(S)}) - (I^-[D^+(S)] \cap I^+(D^-(S)])$$

$$\Rightarrow \dot{D}(S) = (\overline{D^+(S)} \cup \overline{D^-(S)} - I^-[D^+(S)]) \cup (\overline{D^+(S)} \cup \overline{D^-(S)} - I^+[D^-(S)])$$

Since $\overline{D^-(S)} \subset I^-(S) \subset I^-[D^+(S)]$, the difference of $\overline{D^-(S)}$ from $I^-[D^+(S)]$ is the empty set. Similarly $\overline{D^+(S)} \subset I^+(S) \subset I^+[D^-(S)]$, so $\overline{D^+(S)} - I^+[D^-(S)] = \emptyset$.

$$\Rightarrow \dot{D}(S) = (\overline{D^+(S)} - I^-[D^+(S)]) \cup (\overline{D^-(S)} - I^+[D^-(S)]) = H^+(S) \cup H^-(S) = H(S)$$

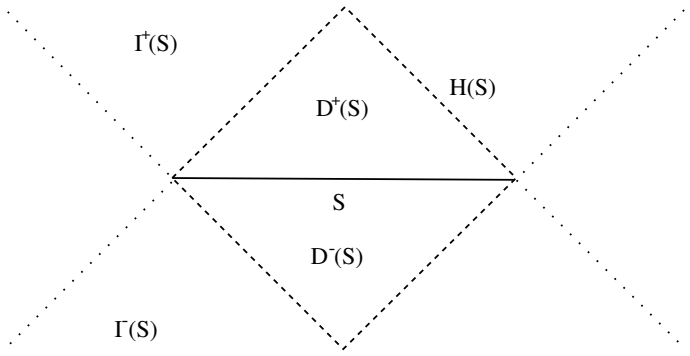


Figure 1.2. Domain of dependence, and Cauchy horizon of a space-like surface.

The first attempt of a precise formulation of causal behaviour that we shall discuss

is the strong causality condition. A space-time (M, g_{ab}) is said to be strongly causal if in every neighbourhood U of $p \in M$ there exists a neighbourhood V of p such that no causal curve intersects V more than once. This prevents the causal curves from coming arbitrary closed to intersecting themselves near p so the formation of closed causal curves is prohibited even by a small modification of g_{ab}. If a space-time (M, g_{ab}) is globally hyperbolic, then it is strongly causal.

However, a modification of g_{ab} in an arbitrary small neighbourhood of two or more points could still produce closed causal curves. The condition of avoiding to approach the verge of violating causality is expressed more satisfactorally by the notion of stable causality defined as follows: Let t^a be a time-like vector at point $p \in M$. We define $\breve{g}_{ab} = g_{ab} - t_a t_b$ where g_{ab} is the space time metric. The light cone of $\breve{g}_{ab}$ is larger than that of g_{ab} so every time-like and null vector of g_{ab} is a time-like vector of $\breve{g}_{ab}$. A space-time (M, g_{ab}) is said to be stably causal if there exists a continuous non-vanishing time-like vector field t^a such that the space-time $(M, \breve{g}_{ab})$ possesses no closed time-like curves. Stable causality is equivalent to the existence of a global time function f on the space-time such that $\nabla^a f$ is a past directed time-like vector field. Along every future directed time-like curve with tangent v^a, we have $g_{ab} v^a \nabla^b f > 0$, thus $v(f) > 0$. Hence, f strictly increases along every future directed time-like curve in M, so there can be no closed time-like curves in M since f can not return to its original value. Also one can show that $\breve{g}^{ab} \nabla_b f$ is a time-like vector in the metric $\breve{g}_{ab}$. By the same argument it follows that no closed time-like curves exist in the space-time $(M, \breve{g}_{ab})$. Thus, (M, g_{ab}) is stably causal. We also note that stable causality implies strong causality.

1.2. Gravitational Collapse and Singularity Theorems

Gravitation is essentially unstable. This instability becomes relevant when large enough concentrations of mass are present, though it may not be manifest under normal conditions where the pull of gravity is counteracted by other forces. In particular, a star which has exhausted its thermal and nuclear energy, can not sustain itself against its own gravitational pull, so a gravitational collapse ensues. The collapsing star may end up as a white dwarf supported by electron degeneracy pressure if its mass is less

than the Chandrasekar limit $M_c \simeq 1.4M_\odot$. For $M > M_c$ the radius will continue to decrease unless neutron Fermi pressure stops collapse, forming a neutron star. This is possible for a maximum mass of $M_{\max} \sim 3M_\odot$; a more massive star should continue to collapse to an asymptotically infinite density. This state of infinite space-time curvature is called a *singularity*.

A space-time possesses a singularity if it fails to satisfy causal geodesic completeness which requires that every time-like and null geodesic can be extended to arbitrarily large affine parameter value both into the future and into the past. In the case of time-like geodesics we can use proper time as such a parameter. In essence, causal geodesic completeness means that photons or freely moving particles can not appear or disappear off the edge of the universe.

According to singularity theorems of Hawking and Penrose [1], a space-time M can not satisfy causal geodesic completeness if, together with Einstein's equations, the following four conditions hold:

- M contains no closed time-like curves
- The energy condition is satisfied at every point.

$$R_{ab}t^a t^b \leq 0 \tag{1.1}$$

  where R_{ab} is the Ricci tensor and t^a is time-like. Note we use signature $(+,-,-,-)$. With Einstein's equations this becomes

$$T_{ab}t^a t^b \geq (1/2)T^c_c \tag{1.2}$$

  The energy condition (1.1) ensures that gravitation is always attractive, in the sense that neighbouring geodesics near any point accelerate towards each other.
- The generality condition is satisfied, i.e. every causal geodesic γ contains a point

for which

$$k_{[a}R_{b]cd[e}k_{f]}k^c k^d \neq 0 \tag{1.3}$$

where k^a is tangent to γ. For time-like γ this becomes

$$R_{abcd}k^b k^c \neq 0 \tag{1.4}$$

In any physically realistic model, (1.3) is expected to hold for each γ

- M contains a *trapped surface*.

If every null congruence orthogonal to a space-like two surface has negative *expansion*, then such a surface is trapped. The expansion scalar Θ is defined as

$$\Theta = B^{ab}h_{ab} \tag{1.5}$$

where $B_{ab} = \nabla_b \xi_a$, $h_{ab} = g_{ab} + \xi_a \xi_b$, and ξ^a is the vector field induced by the congruence. Along any geodesic in the congruence Θ indeed measures the average expansion of the infinitesimally nearby surrounding geodesics. If Θ is negative the congruence is converging and if it is positive the congruence is diverging. So a trapped surface is a set of points on which both ingoing and outgoing null geodesics (light rays) converge, as we proceed into the future.

Trapped surfaces arise when a gravitational collapse of a localized body (e.g. a star) to within its Schwarzschild radius takes place, provided that the deviation of the model from spherical symmetry is negligible. Thus in a space-time satisfying the conditions 1-2-3, a singularity ensues as a result of gravitational collapse.

In particular when the trapped surface contains a single point p the theorem remains valid provided that future null cone of p can encounter sufficient matter that it starts converging, i.e. $\{p\}$ is future trapped. In the time reversed form, if the past null geodesics through p sweep out a region with sufficient matter such that the expansion

changes sign somewhere to the past of p (i.e. $\{p\}$ is past trapped), the theorem still applies. This condition seems to be satisfied in our universe by taking p as the Earth at the present epoch [1]. Another example of a trapped surface is a compact space-like hypersurface, hence the theorem applies to closed universe models. There may also be different situations of physical interest in which trapped surfaces arise.

1.3. Cosmic Censorship Conjecture

Hawking-Penrose singularity theorems [1] imply the formation of space-time singularities as a result of gravitational collapse in which a trapped surface develops. In an asymptotically predictable space-time which satisfies the energy condition (1.1), a trapped surface $\mathcal{T}$ is entirely contained within the *black hole* region $\mathcal{B}$; $\mathcal{T} \subset \mathcal{B}$. If there is a well defined external infinity $\mathcal{I}^+$ the black hole region of the space-time is defined by

$$\mathcal{B} = M - J^-(\mathcal{I}^+) \tag{1.6}$$

with an *event horizon* that constitutes the boundary of $\mathcal{B}$ in M.

$$\mathcal{H} = \partial\mathcal{B} = \dot{J}^-(\mathcal{I}^+) \cap M \tag{1.7}$$

Therefore if the singularity has formed in the way prescribed by Hawking and Penrose *it is hidden behind an event horizon*, so it is invisible to the rest of the space time. In the opposite case where the space-time contains a singularity that lies to the past of $\mathcal{I}^+$, so time-like curves may be drawn into the past that terminate on the singularity, the singularity is said to be *naked*. Whether a singularity is *naked* or *clothed* is crucial to preserve causal structure.

In the presence of naked singularities – which are not hidden behind an event horizon– initial conditions on a Cauchy surface become undefined since the surface necessarily intersects the singularity, thus asymptotic predictability is disabled. In

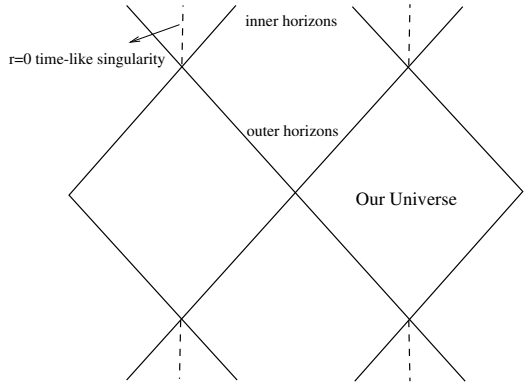


Figure 1.3. Penrose diagram of a Kerr-Newman black hole. Singularity is hidden behind an event horizon.

Section 1.1 we have seen that causal behaviour is characterized by prohibiting the existence of closed time-like or causal curves. Frank J. Tipler [2] proved that for asymptotically flat space-times, closed time-like curves which violate causality, can only evolve in the regions with naked singularities. It becomes impossible to predict the behaviour of space-time in the causal future of a singularity.

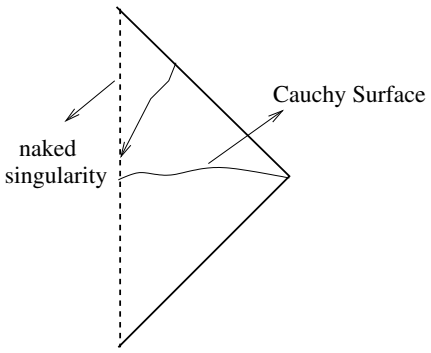


Figure 1.4. Penrose diagram of a space-time involving a naked singularity.

It is an open question whether physically realistic collapse situations may arise without trapped surfaces, resulting in singularities that are not necessarily hidden behind an event horizon. For example the theorem does not apply in the presence of a positive cosmological constant since (1.2) does not imply (1.1) via Einstein's equations, so it is unknown how the collapse will develop in that case. For that reason Penrose proposed the *cosmic censorship conjecture* (CCC) in 1969 [3] , to avoid the theoretical problems that could arise due to the inevitable formation of singularities in gravita-

tional collapse, thereby preserving the deterministic nature of general relativity. In its original form –which was later called the *weak* CCC as opposed to the strong one – CCC asserts that "naked singularities" can not evolve starting from nonsingular initial data. The gravitational collapse of a body always ends up in a black hole rather than a naked singularity so all singularities are hidden within the event horizons of black holes; i.e. they are invisible to distant observers. Distant observers neither encounter any singularities nor any effects propagating out of singularities. Conjecturing singularities to be hidden behind event horizons without any access to distant observers, enables the specification of a well defined initial value problem. In that way we can preserve causal structure despite the existence of singularities.

Penrose [4] later suggested that if the weak CCC is correct its validity may depend on a more general fundamental property of Einstein's equations. He put forward a stronger version of the conjecture –*strong* CCC– which claims that no time-like singularities can evolve, all physically reasonable space-times are globally hyperbolic. The strong version of the CCC drops away the "distant" proviso; no singularity is ever visible to any observer.

On the other hand, we know from the singularity theorems that singularities are a rather generic occurrence in general relativity. For instance, we can obtain a singularity from regular and generic initial data just by concentrating a sufficient amount of matter into a given compact region of space. The statement of cosmic censorship then implies that one cannot destroy a black hole (i.e. its horizon) just by throwing in more "reasonable" matter.

As reasonable as it may seem, a concrete proof of CCC has been elusive, so one way to test the validity of the conjecture is to challenge its seemingly weak spots by constructing thought ("gedanken") experiments. In these experiments one envisages a black hole absorbing some particles or fields coming from infinity. The no-hair theorem [5] in classical general relativity which states that stationary, asymptotically flat space-times are uniquely parametrized by three parameters (Mass M, charge Q, and angular momentum per unit mass a), guarantees that once the particles/fields are ab-

sorbed/reflected, the space-time will settle to another black hole with new values of M, Q, and a. The existence of the event horizon, which discriminates black holes and naked singularities, depends on an inequality involving these three parameters:

$$M^2 \geq Q^2 + a^2. \tag{1.8}$$

In other words, such a space-time, which is described by the Kerr-Newman metric, corresponds to a black hole, if (1.8) is satisfied; but to a naked singularity if it is violated; in the borderline case, i.e. when (1.8) is saturated, the space-time is said to describe an extremal, or a critical black hole. We construct our thought experiments to check if we can push the initially nonsingular space-time satisfying (1.8) beyond the extremal limit, so that the final space-time violates (1.8), and thus describes a naked singularity. The changes in the Kerr-Newman parameters is infinitesimal, therefore one usually prefers to start the thought experiment from conditions infinitesimally close to where we would like to push the system, i.e. the extremal black hole.

The first such thought experiment was constructed by Wald, who showed [6] that one cannot overcharge and/or overspin an extreme black hole [i.e. one saturating the inequality (1.8)] by throwing in test point particles with electric charge and/or spin. Needham [7] verified Wald's result and argued that cosmic censorship is violated when the the parameters of the particle are comparable to those of the hole. Wald's argument was generalized (for spinless test particles) to the case of the dyonic black holes by Hiscock [8] and independently, by Semiz [9]

The problem of trying to overcharge a maximally charged black hole with another kind of charge was revisited by Bekenstein & Rosenzweig [10], where nonzero size of the particle —classical and quantum— is pointed out and argued to save CCC. Hod [11] considered lowering the differently-charged particle slowly towards the horizon (to minimize its energy), and argued that the finite size of the particle helps preserving the horizon in this case too, aided by the polarization of the black hole.

Thought experiments that do not use the electric black hole-magnetic test par-

ticle combination and duality-rotated (see Section 2.2) cases include that of Ford & Roman [12], who considered sending a flux of negative energy down an extreme Reissner-Nordström (R-N) black hole by using a quantized massless scalar field. These negative energy fluxes are allowed in Quantum Field Theory, but the authors find that the exposure of the naked singularity can be only temporary ("Cosmic Flashing") and that the effect probably gets swamped by quantum fluctuations. Jensen [13], analyzed the effect of the collapse of a spherical domain-wall with negative energy onto an extreme Reissner-Nordström (R-N) black hole and decided that the WCCC remains valid in this case. Hubeny [14] argued that one can overcharge a R-N black hole by starting not from extreme, but slightly sub-extreme condition, i.e one may "jump over" the extreme condition. Hod [15] made the same claim for general black holes while pointing out that curved-space modifications to particle self-energy may be relevant. The concept of jumping over the extreme condition was also treated by Jacobson & Sotiriou [16]. Hod & Piran [17] suggested that the WCCC can be violated classically, but may be saved by Quantum Gravity. de Felice & Yu [18] calculated that a naked singularity can be produced by throwing a fast-spinning flat disk into an extreme R-N black hole. More recently, Matsas *et al.* [19, 20] made the same claim for quantum tunneling of low-energy, high-angular-momentum massless scalar particles; and the claim was disputed by Hod [21, 22]. Richartz and Saa [23, 24] also claimed that naked singularities can form by quantum effects, and overcharging a near extreme black hole is possible by quantum tunneling. Bouhmadi-López *et al.* [25] generalized Wald's thought experiment to higher-dimensional black holes and black rings. Saa and Santarelli [26] suggested that a nearly extremal Kerr-Newman black hole can be destroyed by test particles. Isoyama *et al.* [27] found that overcharging a R-N black hole is not possible considering back-reaction effects. Zimmerman *et al.* [28] also treated electromagnetic self-force as a cosmic censor. Hod [29] also challenged cosmic censorship with shells. Gao and Zhang [30] claimed that an extremal black hole can also be overspun considering higher order terms.

There are fewer examples that test the validity of CCC in case of fields. Semiz [31] considered a gedanken experiment with a dyonic Kerr-Newman black hole and a massive complex scalar test field coupled to electromagnetism; using conservation laws

and equations of motion for the test field he found that an extremal black hole can not be overcharged or overspun by sending in test scalar fields. Toth [32] attacked the same problem and derived the same result using conservation laws and the null energy condition.

1.4. Outline of the Thesis

In this thesis, we focus on gedanken experiments to test the validity of weak CCC. We briefly review previous works that are conceptually important and/or that we often refer to. We introduce original thought experiments involving fields, starting from both extremal and subextremal black holes, and test the validity of CCC in each case.

In Chapter 2 we review Wald's attempt to overcharge/overspin an extremal black hole by sending test bodies, its generalization to dyonic black holes –which can also have magnetic charge– by Hiscock and Semiz, and the test of CCC with scalar fields and Kerr-Newman black holes by Semiz.

In Chapter 3 we review spinor algebra and Newman-Penrose formalism, and study Teukolsky's derivations of perturbations of a Kerr black hole. Next we construct an original thought experiment with electromagnetic test fields. Using Teukolsky's solutions for electromagnetic test field in Kerr space-time, we show –for the first time– that one can not overspin an extremal Kerr black hole by sending in electromagnetic test fields.

In Chapter 4 we consider gedanken experiments that start with nearly extremal black holes instead of extremal ones. After a review of particle cases, we construct another original experiment with integer spin test fields (scalar, electromagnetic, or gravitational) and a subextremal black hole. We show that it is possible to overspin a subextremal black hole by sending in integer spin test fields as long as radiative and self-force effects are neglected. Finally we consider the case of half integer spin. We show that a neutrino field, i.e. a massless spin 1/2 field, can be used to turn a black

hole into a naked singularity, in a generic violation of CCC.

2. GEDANKEN EXPERIMENTS WITH PARTICLES AND FIELDS

In this chapter we review previous gedanken experiments for testing the validity of CCC by using particles or fields, to give a general opinion of the methods used in constructing the experiments. We will describe the pioneering work of Wald [6] and its generalization to dyonic black holes by Hiscock [8] and Semiz [9] as examples of the cases involving test particles. Next we present the gedanken experiment of Semiz [31] involving scalar fields and Kerr-Newman black holes, which we often refer to in the coming chapters.

2.1. Wald's Experiment with Test Particles

In 1974, Wald considered an extremal Kerr-Newman black hole, i.e., one for which (1.8) is saturated ($M^2 = Q^2 + a^2$), and supposed that the black hole captures a small body of energy E, angular momentum L in the direction of the angular momentum of the black hole, and charge e. To be able to treat the particle as a test body whose contribution to the geometry is negligible, we assume that $e \ll Q$, $L \ll Ma$, and $E \ll M$.

The idea of the gedanken experiment is to check if we can have the black hole capture a small body so that the resulting configuration violates (1.8) ($M^2 < Q^2 + a^2$), and the space-time represents a naked singularity instead of a black hole. The condition for the final black hole to violate (1.8) is

$$\delta(M^2) < \delta(Q^2 + a^2) \tag{2.1}$$

The change in the mass of the black hole δM is the energy of the incoming particle as

long as radiation effects are neglected. This gives

$$\delta M = E < \frac{eQM + aL}{M^2 + a^2} \tag{2.2}$$

Thus, if the black hole captures a small body satisfying (2.2) it becomes a naked singularity.

2.1.1. Particle Orbits

In Boyer-Lindquist coordinates, the Kerr-Newman metric is

$$\begin{aligned} ds^2 = g_{\mu\nu}dx^\mu dx^\nu &= \frac{a^2\sin^2\theta - \Delta}{\rho^2}dt^2 + \frac{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}{\rho^2}\sin^2\theta d\phi^2 \\ &+ 2\frac{\Delta - (r^2+a^2)}{\rho^2}a\sin^2\theta dt d\phi + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 \end{aligned} \tag{2.3}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{2.4}$$

$$\Delta = r^2 - 2Mr + a^2 + Q^{\,2} \tag{2.5}$$

and the horizon is at

$$r = r_+ = M + \sqrt{M^2 - a^2 - Q^{\,2}} \tag{2.6}$$

It should be noted that $r = r_+$ is where g_{rr} changes sign: The radial coordinate r goes over from being a spacelike coordinate outside, to being a timelike one inside. The

electromagnetic field tensor of the Kerr-Newman black hole is

$$\begin{aligned} F &= \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu \\ &= Q\rho^{-4}\{(r^2 - a^2\cos^2\theta)dr \wedge dt + (2a^2 r\sin\theta\cos\theta)dt \wedge d\theta\} \\ &+ Q\rho^{-4}(r^2 - a^2\cos^2\theta)a\sin^2\theta d\phi \wedge dr \\ &+ Q\rho^{-4}(r^2 + a^2)2ar\sin\theta\cos\theta d\theta \wedge d\phi \end{aligned} \tag{2.7}$$

which can be derived from the vector potential

$$A = -\frac{Qr}{\rho^2}(dt - a\sin^2\theta d\phi) \tag{2.8}$$

The equations of motion of a test particle of mass m and charge q are:

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} = \frac{e}{m}F^{\mu\nu}\frac{dx_\nu}{ds} \tag{2.9}$$

which can be derived from the Lagrangian

$$\mathcal{L} = \frac{1}{2}mg_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + eA_\mu\frac{dx^\mu}{ds} \tag{2.10}$$

The constants of motion in stationary and axisymmetric space-times including Kerr-Newman geometry are

$$-E = p_t = \frac{\partial\mathcal{L}}{\partial\dot{t}} = mg_{t\mu}\frac{dx^\mu}{ds} + eA_t \tag{2.11}$$

$$L = p_\phi = \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = mg_{\phi\mu}\frac{dx^\mu}{ds} + eA_\phi \tag{2.12}$$

Since the 4-velocity of the particle is a timelike unit vector, we have

$$-1 = g_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{2.13}$$

Combining this with Equations (2.11)-(2.12), and similarly defined other momenta,

$$\begin{aligned} -1 &= \frac{1}{m^2}\left\{g^{tt}(-E-eA_t)^2+2g^{t\phi}(-E-eA_t)(p_\phi-eA_\phi)\right. \\ &\quad \left.+g^{\phi\phi}(p_\phi-eA_\phi)^2+g^{rr}{p_r}^2+g^{\theta\theta}{p_\theta}^2\right\} \end{aligned} \tag{2.14}$$

This is a quadratic equation for E, with solution

$$\begin{aligned} E = &- eA_t+\frac{1}{g^{tt}}\{g^{t\phi}(p_\phi-eA_\phi)-[(p_\phi-eA_\phi)^2(g^{t\phi})^2 \\ &- g^{tt}g^{\phi\phi}(p_\phi-eA_\phi)^2-g^{tt}(g^{rr}{p_r}^2+g^{\theta\theta}{p_\theta}^2+m^2)]^{1/2}\} \end{aligned} \tag{2.15}$$

The sign of the square root in (2.15) is chosen to make $dt/ds > 0$ outside the horizon for all trajectories. Therefore,

$$E \geq -eA_t+\frac{1}{g^{tt}}g^{t\phi}(p_\phi-eA_\phi) \tag{2.16}$$

If the black hole captures the particle, at one point r must be equal to r_+, the horizon radius ($r_+ = M+\sqrt{M^2-a^2-Q^2} = M$). (2.16) can be evaluated there, resulting

$$E \geq \frac{eQ_eM+aL}{M^2+a^2} \tag{2.17}$$

Thus any particle which enters the black hole must satisfy (2.17). In that case Equation (2.2) which is the condition to violate CCC can not be satisfied. Here we find that particles with enough charge or angular momentum to violate CCC miss the black hole so an extremal Kerr-Newman black hole can not be overcharged or overspun.

2.2. Generalization to Dyonic Case

Wald's result was expended to dyonic black holes which can have magnetic charge Q_m in addition to electric charge Q_e, by Hiscock [8] and Semiz [9].

Maxwell's equations can be generalized to include the possibility of magnetic

charges:

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J_e^\alpha \tag{2.18}$$

$$\tilde{F}^{\alpha\beta}{}_{,\beta} = -4\pi J_m^\alpha \tag{2.19}$$

$$ma^\alpha = (q_e F^{\alpha\beta} - q_m \tilde{F}^{\alpha\beta})u_\beta \tag{2.20}$$

where F is the electromagnetic field tensor and $\tilde{F}$ its dual, defined by

$$\tilde{F}_{\alpha\beta} = \frac{1}{2}\epsilon_{\alpha\beta\gamma\delta}F^{\gamma\delta} \tag{2.21}$$

with $\epsilon_{\alpha\beta\gamma\delta}$ the Levi-Civita tensor, which in flat space reduces to the completely antisymmetric symbol $[\alpha\beta\gamma\delta]$ such that $[0123] = +1$; if cartesian coordinates are used. J_e and J_m are the electric and magnetic four-currents. Equation (2.20) gives the (generalized) Lorentz force acting on a test particle with mass m, electric and magnetic charges q_e and q_m, moving with 4-velocity u, in a field F.

The set of Equations (2.18), (2.19) and (2.20) are invariant under the duality transformation by angle χ:

$$F^{\alpha\beta} \to F'^{\alpha\beta} = F^{\alpha\beta}\cos\chi - \tilde{F}^{\alpha\beta}\sin\chi \tag{2.22}$$

$$J_e^\alpha \to J_e'^\alpha = J_e^\alpha\cos\chi + J_m^\alpha\sin\chi \tag{2.23}$$

$$J_m^\alpha \to J_m'^\alpha = J_e^\alpha\cos\chi - J_e^\alpha\sin\chi \tag{2.24}$$

Since they still satisfy the —generalized— Maxwell's Equations, the transformed fields also correspond to a valid physical situation. Since the Lorentz force stays the same, any motion stays the same, therefore the sources of the field are not mechanically disturbed by the changed fields and all the change in them can be accounted for by the transformation, completing the circle. We can say that electrodynamics is invariant under the duality transformation.

The energy-momentum tensor of the electromagnetic field is also left invariant

by the duality transformation. This can best be seen by writing $T^{\alpha\beta}$ as

$$T^{\alpha\beta} = \frac{1}{8\pi}(F^{\alpha\sigma}F_{\sigma}{}^{\beta} + \tilde{F}^{\alpha\sigma}\tilde{F}_{\sigma}{}^{\beta}) \tag{2.25}$$

The metric of the dyonic black hole is the Kerr-Newman metric with the replacement $Q^2 \to Q_e{}^2 + Q_m{}^2$. The transformation (2.22) gives

$$F_{\alpha\beta} = Q_e f_{\alpha\beta} + Q_m \tilde{f}_{\alpha\beta} \tag{2.26}$$

where $f_{\alpha\beta}$ is the Kerr-Newman $F_{\alpha\beta}/Q$ and $\tilde{f}_{\alpha\beta}$ the dual, defines as in Equation (2.21). The fields can be derived from the vector potential

$$A_t = -Q_e\frac{r}{\rho^2} + Q_m\frac{a\cos\theta}{\rho^2} \tag{2.27}$$

$$A_r = A_\theta = 0 \tag{2.28}$$

$$A_\phi = Q_e\frac{ar\sin^2\theta}{\rho^2} + Q_m[\pm 1 - \cos\theta\frac{r^2+a^2}{\rho^2}] \tag{2.29}$$

Before tackling the problem of a test particle with (m,q_e, q_m) moving in the field of a black hole with (M,a,Q_e,Q_m), let us note a simplification: A duality transformation with $\tan\chi = q_m/q_e$ will map the problem into a test particle with (m, $q_e^{'} = e = \sqrt{q_e{}^2 + q_m^2}$, $q_m^{'} = 0$) and a black hole with (M, $Q_e^{'} = (Q_e q_e + Q_m q_m)/e$), $Q_m^{'} = (Q_m q_e - Q_e q_m)/e$), *without changing the motion of the test particle* (both metric and Lorentz force invariant). Therefore, dropping primes, we can solve the problem of a purely electric test particle, *without any loss of generality.* The constant of motion are:

$$p_t = -E \tag{2.30}$$

$$p_\phi = L_\infty + eQ_m(\pm 1 - \cos\theta_\infty) \tag{2.31}$$

where E is the energy, L_∞ and $-eQ_m\cos\theta_\infty$ are the z-components of the (mechanical)

angular momentum of the particle and the angular momentum of the electromagnetic field respectively, when the particle is at infinity. Proceeding in the same way as Wald we find that if a particle is captured by the black hole, it satisfies

$$E \geq \frac{eQ_eM + a(L_\infty - eQ_m \cos\theta_\infty)}{M^2 + a^2} \tag{2.32}$$

On the other hand the condition for violating (1.8) becomes

$$\delta M \geq \frac{eQ_eM + a\delta(L_\infty - eQ_m \cos\theta_\infty)}{M^2 + a^2} \tag{2.33}$$

Equations (2.32) and (2.33) can not be both satisfied. Therefore we conclude that therefore a violation of $M^2 \geq {Q_e}^2 + {Q_m}^2 + a^2$ cannot be obtained by shooting in a test particle, even a magnetic or dyonic one.

2.3. Scalar Field and Cosmic Censorship

In this section we review a gedanken experiment of Semiz [31], in which a weak wave packet of the complex massive scalar field interacts with a four-parameter (mass, angular momentum, electric and magnetic charges) Kerr-Newman black hole. He calculates the changes of the black hole parameters by first constructing expressions for fluxes of charge, angular momentum and energy carried by the scalar field directly and by its accompanying electromagnetic field, then evaluates them to the lowest non-trivial order in a perturbation expansion around the initial Kerr-Newman configuration. In this thought experiment we investigate if the inequality (1.8), therefore the Weak Cosmic Censorship Conjecture can be violated for any combination of black hole parameters.

To calculate the changes in the black hole parameters, we start from the associated conservation laws. The energy and angular momentum conservation laws are consequences of space-time symmetries and locally take the form of continuity equa-

tions

$$(T^{\mu\nu}X_{\nu})_{;\mu} = 0 \tag{2.34}$$

where $T^{\mu\nu}$ is the energy-momentum tensor for test particles or fields and X^{ν} is the Killing vector generating the symmetry.

We also have the current continuity equation

$$(\sqrt{-g}\, j^{\nu})_{,\nu} = 0 \tag{2.35}$$

where j^{ν} is the current density four-vector.

The Killing vectors of Kerr-Newman space-time are $\partial/\partial x^0$ and $\partial/\partial x^3$. For the first of these, integrating (2.34) over a spacelike volume we get

$$\frac{d}{dx^0}\int_V (\sqrt{-g}\, T_0{}^0)\, d^3x = -\int_V (\sqrt{-g}\, T_0{}^i)_{,i}\, d^3x = -\int_S \sqrt{-g}\, T_0{}^i dS_i \tag{2.36}$$

where S is the surface (boundary) of the volume V.

The mass of a physical system is defined only in the asymptotically flat part of space at infinity, if it exists; and the energy density is $-T_0{}^0$ [33]. Therefore we have

$$\left(\frac{dM}{dt}\right)_{\text{b.h}} = \int_{S_\infty} \sqrt{-g}\, T_0{}^1 d\theta d\phi \tag{2.37}$$

where the label b.h. stands for black hole, S_∞ is the spherical surface as $r \to \infty$. Similarly, Equations (2.34) and (2.35) lead to

$$\left(\frac{dL}{dt}\right)_{\text{b.h}} = -\int_{S_\infty} \sqrt{-g}\, T_3{}^1 d\theta d\phi \tag{2.38}$$

$$\left(\frac{dQ_e}{dt}\right)_{\text{b.h}} = -\int_{S_\infty} \sqrt{-g}\, j^1 d\theta d\phi \tag{2.39}$$

To evaluate $T_{\mu\nu}$ for our problem, we start from the Lagrangian of the combined scalar and electromagnetic fields :

$$\mathcal{L} = \frac{1}{8\pi}[-g^{\alpha\beta}(\partial_\alpha - ieA_\alpha)\psi^*(\partial_\beta + ieA_\beta)\psi - \mu^2\psi^*\psi] - \frac{1}{16\pi}F_{\alpha\beta}F^{\alpha\beta} \tag{2.40}$$

The energy-momentum tensor is defined by [33]

$$T_{\mu\nu} = -2\frac{\delta\mathcal{L}}{\delta g^{\mu\nu}} + g_{\mu\nu}\mathcal{L} \tag{2.41}$$

hence

$$T_\mu{}^\nu = \frac{1}{8\pi}[\mathcal{D}_\mu\psi^*\mathcal{D}^\nu\psi + \mathcal{D}^{*\nu}\psi^*\mathcal{D}^*_\mu\psi] + \frac{1}{4\pi}F_{\mu\beta}F^{\nu\beta} + \delta_\mu{}^\nu\mathcal{L}. \tag{2.42}$$

We define $\mathcal{D}_\mu \equiv \partial_\mu - ieA_\mu$, $\mathcal{D}^\nu \equiv \partial^\nu + ieA^\nu$, and $*$ denotes complex conjugate. Also, the current density j^μ is given by

$$j^\mu = \frac{ie}{8\pi}[\psi^*\mathcal{D}^\mu\psi - \psi\mathcal{D}^{*\mu}\psi^*]. \tag{2.43}$$

We assume deviations from background are small, ("test field") and expand the fields in some small parameter ϵ. The background is the dyonic Kerr-Newman spacetime, therefore for the zeroth order fields we have $\psi^{(0)} = 0$, $F^{(0)}_{\mu\nu} \neq 0 \Longrightarrow A^{(0)}_\mu \neq 0$. We adopt the notation $\mathcal{D}^{(n)}_\mu \equiv \partial_\mu - ieA^{(n)}_\mu$.

From the Lagrangian (2.40) we derive the equations of motion for the fields. The equation of motion for Ψ gives to first order (the zeroth order part has no content, $0 = 0$)

$$\frac{1}{\sqrt{-g}}\mathcal{D}^{*(0)}_\mu[\sqrt{-g}\mathcal{D}^{\mu(0)}]\psi^{(1)} = \mu^2\psi^{(1)} \tag{2.44}$$

Similarly, the equation of motion for $F^{\mu\nu}$ gives to zeroth and first orders

$$F^{(0)\alpha\beta}{}_{;\beta} = 0, \qquad F^{(1)\alpha\beta}{}_{;\beta} = 0 \tag{2.45}$$

and to second order

$$F^{(2)\alpha\beta}{}_{;\beta} = 4\pi \left\{ \frac{ie}{8\pi} [\psi^{*(1)} \mathcal{D}^{\alpha(0)} \psi^{(1)} - \psi^{(1)} \mathcal{D}^{*\alpha(0)} \psi^{*(1)}] \right\} = 4\pi j^{(2)\alpha} \tag{2.46}$$

We require there to be no photons. Since there is no source for the electromagnetic field to zeroth or first order, this requirement, together with boundary conditions at infinity, makes $F^{(0)\alpha\beta} + F^{(1)\alpha\beta}$ the dyonic Kerr-Newman solution. Since $F^{(0)\alpha\beta}$ is the background, and therefore already is Kerr-Newman, we have $F^{(1)\alpha\beta} = 0 \Longrightarrow A^{(1)}_{\mu} = 0$ and $F^{(2)\alpha\beta}{}_{;\beta} = 4\pi j^{(2)\alpha}$, with $j^{(2)\alpha}$ is made up of $\psi^{(1)}$'s, as above.

In fact the solution for the test electromagnetic field $F^{(1)\alpha\beta}$ in Kerr-Newman space-time exists. In the next chapter we will introduce the solution and evaluate CCC for the test electromagnetic field interacting with a Kerr black hole, for the first time.

The lowest-order contributions to ${T_0}^1$, ${T_3}^1$, and j^1 are second order in ϵ: Terms with ψ, ψ^* and their derivatives are obvious. FF terms also vanish to zeroth order, a result one can verify by calculation and understand physically: There are no energy, or angular momentum flows on the exact Kerr-Newman metric. FF terms also vanish to first order, because $F^{(1)\alpha\beta} = 0$. Therefore the lowest order contributions to dM/dt, dL/dt, and dQ_e/dt [given in Equations (2.37-2.39)] are second order in ϵ.

2.3.1. Treating Separated Equations

As shown in [34], Equation (2.44) can be separated in Boyer-Lindquist coordinates by

$$\psi^{(1)} = R(r)\Theta(\theta)e^{-i\omega t}e^{i(m\mp eQ_m)\phi}. \tag{2.47}$$

The angular equation constitutes a Sturm-Liouville problem, therefore the solutions form a complete set and can be orthonormalized. Labeling the eigenvalue λ by l, we choose

$$\int Y^*_{ql'm'}(a,\mu,\omega,\theta,\phi)Y_{qlm}(a,\mu,\omega,\theta,\phi)d\Omega = \delta_{ll'}\delta_{mm'} \tag{2.48}$$

where

$$Y_{qlm}(a,\mu,\omega,\theta,\phi) = \Theta_{qlm}(a,\mu,\omega,\theta)e^{i(m\mp eQ_m)\phi} \tag{2.49}$$

are functions called *monopole spheroidal harmonics* [34].

By switching the independent variable to the 'tortoise' coordinate r^* defined by

$$\frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta} \Rightarrow \Delta\frac{d}{dr} = (r^2+a^2)\frac{d}{dr^*} \tag{2.50}$$

and making a substitution

$$R(r) = \frac{U(r)}{\sqrt{r^2+a^2}} \tag{2.51}$$

for the dependent variable, for the radial equation near the horizon ($r \to r_+$, $r^* \to -\infty$) we get

$$\frac{d^2U}{dr^{*2}} + \left[\frac{(r_+^2+a^2)\omega + eQ_e r_+ - am}{r_+^2+a^2}\right]^2 U = 0 = \frac{d^2U}{dr^{*2}} + \bar{\omega}^2 U = 0 \tag{2.52}$$

(where $\bar{\omega}$ has been defined in the equation) and near infinity (as $r^* \to r \to \infty$) it becomes

$$\frac{d^2U}{dr^{*2}} + (\omega^2 - \mu^2)U = 0 \tag{2.53}$$

That is, in terms of U and r^*, the radial problem at a given frequency looks like a one-dimensional Schrödinger scattering problem with two (different) constant potentials at two ends and a complicated potential well in the middle.

The solutions for U near the horizon are, from Equation (2.52),

$$U = e^{\pm i\bar{\omega}r^*} \tag{2.54}$$

and near infinity, from Equation (2.53),

$$U = e^{\pm ikr^*} \text{ for } \omega^2 > \mu^2 \text{ , where } k^2 = \omega^2 - \mu^2 \tag{2.55}$$

and

$$U = e^{\pm \kappa r^*} \text{ for } \omega^2 < \mu^2 \text{ , where } \kappa^2 = \mu^2 - \omega^2 \tag{2.56}$$

For $\omega^2 < \mu^2$, the exponentially growing solution $e^{\kappa r^*}$ is unphysical, and the solution $e^{-\kappa r^*}$ corresponds to a bound particle and vanishes exponentially at infinity, and therefore will not contribute to our integrals for dM/dt, dQ/dt, dL/dt.

For $\omega^2 > \mu^2$, the outgoing solution is unphysical at the future event horizon: It varies infinitely fast as seen by an ingoing, future-directed observer. We are left with the ingoing solution:

$$\lim_{r \to r_+} U_{\omega lm}(r^*) = B_{\omega lm} e^{-i\bar{\omega}r^*} \tag{2.57}$$

and for the same mode,

$$\lim_{r\to\infty} U_{\omega lm}(r^*) = e^{-ikr^*} + A_{\omega lm}e^{ikr^*} \tag{2.58}$$

where the solution has been normalized such that the coefficient of the first term on the right-hand side of the last equation is unity. In other words, this corresponds to a wave of unit amplitude coming in from infinity, being transmitted into the black hole with amplitude B and reflected back to infinity with amplitude A.

Since the radial equation is real, the complex conjugate of any solution is also a solution, and since it doesn't involve first derivatives, the Wronskian of the two solutions U and U^*

$$W = U\frac{dU^*}{dr^*} - U^*\frac{dU}{dr^*} \tag{2.59}$$

is constant; therefore its values at the two limits can be put equal to each other, giving after a few lines

$$\lim_{r\to r_+} W = \bar{\omega}BB^* = \lim_{r\to\infty} W = k(1 - AA^*) \tag{2.60}$$

where the labels ωlm are implied for the relevant variables in the last two equations.

2.3.2. Testing the Cosmic Censorship Conjecture

To test the inequality (1.8), let us define a cosmic censorship indicator,

$$CCC = M^2 - (Q^2 + a^2) \tag{2.61}$$

Then, using $dQ_m/dt = 0$ and $a = L/M$, and implicitly defining dC/dt,

$$\delta(CCC) = \int \frac{dC}{dt} dt = \int \frac{2}{M} \left\{ (M^2 + a^2) \frac{dM}{dt} - M Q_e \frac{dQ_e}{dt} - a \frac{dL}{dt} \right\} dt. \tag{2.62}$$

Recalling the previous results, we have to second order in ϵ

$$\frac{dC}{dt} = \int_{S_\infty} d\Omega [(M^2 + a^2) T_0{}^1 + M Q_e j^1 + a T_3{}^1] \tag{2.63}$$

where $d\Omega = \sqrt{-g} d\theta d\phi$. In the pure electromagnetic contribution to dC/dt (the part coming from the $F_{\mu\beta} F^{\nu\beta}$ term in $T_\mu{}^\nu$, see Equation (2.42)) we write $F_{\mu\beta}$ in terms of A_μ:

$$\left(\frac{dC}{dt} \right)_{\text{em}} = \frac{1}{2\pi M} \int_{S_\infty} d\Omega [(M^2 + a^2)(A_{\alpha,0} - A_{0,\alpha}) + a(A_{\alpha,3} - (A_3 \mp Q_m)_{,\alpha})] F^{1\alpha} \tag{2.64}$$

where the $\mp Q_m$ was added to A_3 for later convenience. After extensive manipulation we get

$$\delta(CCC)_{\text{em}} = \frac{2a}{M} \int_{-\infty}^{\infty} \int_{S_\infty} d\Omega (A_3^{(0)} \mp Q_m) j^{(2)1} \, dt \tag{2.65}$$

Now substituting Equations (2.42) and (2.43) into the remaining terms of $\delta(CCC)$, and combining (2.65) with the second term of (2.63),

$$\begin{aligned} \delta(CCC) = \frac{2}{M} \int_{-\infty}^{\infty} \int_{S_\infty} d\Omega \Bigg\{ \Bigg[& \frac{(M^2 + a^2)}{8\pi} \mathcal{D}_0 \psi^* \mathcal{D}^1 \psi \\ & + \frac{a}{8\pi} \mathcal{D}_3 \psi^* \mathcal{D}^1 \psi + \frac{M Q_e + a(A_3 \mp Q_m)}{8\pi} ie \psi^* \mathcal{D}^1 \psi \Bigg] + \text{c.c.} \Bigg\} dt \end{aligned} \tag{2.66}$$

where c.c. denotes the complex conjugate of the expression that precedes it and the A's are all zeroth order now. Then we observe that $A^1 = 0$; factor out $\partial^1 \psi$ and note

that the A_3 terms cancel. So

$$\begin{aligned}\delta(CCC) &= \frac{1}{4\pi M}\int_{-\infty}^{\infty}\int_{S_\infty} d\Omega\,\big\{[(M^2+a^2)\mathcal{D}_0 \\ &\quad +a\partial_3 + ie(MQ_e \mp aQ_m)]\psi^*\partial^1\psi + \text{c.c.}\big\}\,dt \end{aligned} \tag{2.67}$$

Since $e^{-i\omega t}$ and the monopole spheroidal harmonics $Y_{qlm}(a,\mu,\omega,\theta,\phi)$ form complete sets, we can expand the wavefunction ψ in terms of the eigenfunctions:

$$\psi = \sum_{l,m}\int d\omega\, f_{lm}(\omega)e^{-i\omega t}Y_{qlm}(a,\mu,\omega,\theta,\phi)R_{lm}(a,\mu,\omega,r) \tag{2.68}$$

where $f_{lm}(\omega)$ are arbitrary coefficients. Since the integrals are taken on S_∞, we can use the asymptotic form of the radial function at infinity, $R_{lm}(a,\mu,\omega,r)\to 1/r(e^{-ikr}+A_{\omega lm}e^{ikr})$. Now we substitute the expression for ψ and its complex conjugate in (2.67). We first take the time-integral, which gives a δ-function in ω and ω', allowing the ω'-integral to be done. Then the ϕ and θ-integrals can be done (They could not be done previously, because the orthogonality relation (2.48) for the angular wavefunctions is only valid if the frequencies in the functions $\Theta_{qlm}(a,\mu,\omega,\theta)$ and $\Theta^*_{ql'm'}(a,\mu,\omega',\theta)$ are the same), giving $\delta_{ll'}\delta_{mm'}$ and thereby allowing the l' and m' sums to be done:

$$\delta(CCC) = \frac{1}{2\pi M}\int d\omega\sum_{l,m}|f|^2k[(M^2+a^2)\omega - am + eQ_eM](1-|A|^2) \tag{2.69}$$

where $|f|^2\equiv f_{lm}(\omega)f^*_{lm}(\omega)$ and $|A|^2\equiv A_{\omega lm}A^*_{\omega lm}$. Using the Wronskian relation (2.60),

$$\begin{aligned}\delta(CCC) &= \frac{1}{2\pi M}\int d\omega\sum_{l,m}|f|^2[(M^2+a^2)\omega - am + eQ_eM]\bar{\omega}|B|^2 \\ &= \frac{M^2+a^2}{2\pi M}\int d\omega\sum_{l,m}|f|^2[\omega + \frac{eQ_eM-am}{M^2+a^2}] \\ &\quad \times[\omega + \frac{eQ_er_+ - am}{r_+^2+a^2}]|B|^2 \end{aligned} \tag{2.70}$$

where $|B|^2 \equiv B_{\omega lm} B^*_{\omega lm}$. For the extreme black hole, $r_+ \to M$, therefore

$$\delta(CCC)_+ = \frac{M^2 + a^2}{2\pi M} \int d\omega \sum_{l,m} |f|^2 \bar{\omega}_+^2 |B|^2 \tag{2.71}$$

where the subscript $\{+\}$ denotes that we evaluate at $r_+ = M$.

Expression (2.71) is strictly nonnegative. Only if all $\bar{\omega}_+ = 0$ vanish can expression (2.71) be zero, therefore, the second order not dominant. Except for this very limited case the expression (2.71) for $\delta(CCC)_+$ will be the dominant one, and its strict positivity means that M^2 increases faster than $(Q^2 + a^2)$ to the dominant (second) order in the field, meaning that to this order and with the very limited, inconclusive exception noted, the Cosmic Censorship Conjecture cannot be violated by adding charge and/or angular momentum to an extreme black hole via a classical Klein-Gordon test field.

3. ELECTROMAGNETIC FIELD AND COSMIC CENSORSHIP

In this chapter we review spinor calculus and Newman Penrose formalism. We introduce Teukolsky's solutions for perturbations of black holes. Using these solutions we construct an original thought experiment involving an extremal Kerr black hole and a test electromagnetic field, and check the validity of cosmic censorship conjecture in this case.

3.1. Spinor Calculus and Newman Penrose Formalism

In this section we present a general review of spinor algebra and Newman Penrose formalism which can handle fields of any spin in curved spaces, as opposed to tensors which can only represent integer-spin fields. Teukolsky [39] uses this formalism to give a unified treatment of massless scalar, electromagnetic and neutrino field perturbations of Kerr black holes, which will be very useful for testing the cosmic censorship conjecture. We mainly follow John Stewart's book "Advanced General Relativity" [35] to introduce spinor algebra.

3.1.1. Symplectic Linear Structure and Spin Basis

Let S be a two dimensional vector space over $\mathcal{C}$. A symplectic linear structure on S is a non-degenerate bilinear two-form called the skew scalar product, which is denoted by $[\xi, \eta] = -[\eta, \xi]$. A two form on S is non-degenerate if $[\xi, \eta] = 0$ for all η in S implies $\xi = 0$. The space S endowed with a symplectic structure is called a symplectic vector space. A linear transformation $Q : S \to S$ is called symplectic if it preserves the skew-scalar product $[\xi, \eta]$. The set of symplectic transformations form the symplectic group of appropriate dimension.

It is well known that no non-zero vector is orthogonal to itself with the stan-

dard Euclidean scalar product, and every null vector is orthogonal to itself with the Lorentzian scalar product. However, as far as the skew-scalar product is concerned every vector is orthogonal to itself. The set of vectors orthogonal to a non-zero vector ξ consists precisely of vectors proportional to ξ. If there existed a vector η orthogonal to ξ and not proportional to it, the two vectors would constitute a basis for S and every vector would be orthogonal to ξ, which would contradict the non-degeneracy condition.

We need to construct a standard basis to do calculations. Let o be an arbitrary vector in S, and let i be any non-parallel vector, which means $[o, i] \neq 0$. We may normalize i to set $[o, i] = 1$. Clearly i is not unique since we may add an arbitrary multiple of o without disturbing the normalization. Given a non-zero vector o and i chosen to set $[o, i] = 1$, (o, i) constitutes a spin basis for S. We may define the components of arbitrary vectors with respect to the spin basis. If ξ is in S, the components of ξ are ξ^A (where $A = 0, 1$), and with respect to the basis

$$\xi = \xi^0 o + \xi^1 i \tag{3.1}$$

By the definition of the spin basis

$$o^A = (1, 0) \ , \ i^A = (0, 1) \tag{3.2}$$

The dual space of S is denoted by S^*. The skew-scalar product defines a natural isomorphism between two spaces. To each element ξ of S we associate $[\xi,]$ which is an element of S^*. We shall identify these isomorphic partners by the same Greek kernel so that $\xi^A \in S$ and $\xi_A \in S^*$, where suffix identifies to which space the element belongs. Since ξ_A is the isomorphic partner of ξ^A defined by $[\xi,]$, $\xi_A \eta^A$ is the same as $[\xi, \eta] \in \mathcal{C}$.

The symplectic structure can be identified with an element of $S^* \times S^*$ which will be denoted by $\epsilon_{AB} = -\epsilon_{BA}$, so that $[\xi, \eta] \equiv \epsilon_{AB} \xi^A \eta^B$. The condition that (o, i) is a

spin basis for S gives

$$\epsilon_{AB}o^A o^B = \epsilon_{AB}i^A i^B = 0,\ \epsilon_{AB}o^A i^B = 1 \tag{3.3}$$

With respect to the components in this basis we have

$$\epsilon_{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.4}$$

The inverse of this matrix is identified with an element of $S \times S$, and defined with a conventional factor of -1.

$$\epsilon^{AB} = -(\epsilon^{-1})^{AB} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.5}$$

ϵ_{AB} can be regarded as an index lowering operator by the convention on the kernel letter of dual elements.

$$\begin{aligned} & [\xi, \eta] = (\epsilon_{AB}\xi^A)\eta^B \equiv \xi_B \eta^B \\ \Rightarrow \quad & \epsilon_{AB}\xi^A = \xi_B \end{aligned} \tag{3.6}$$

Note the position of indices. Since ϵ_{AB} is non-singular we have

$$\epsilon_{AB}(\epsilon^{-1})^{BC}\epsilon = \delta_A{}^C \tag{3.7}$$

where $\delta_A{}^C$ is the spinor Kronecker delta. Multiplying both sides by ξ^A

$$\begin{aligned} & \xi^A \epsilon_{AB}(\epsilon^{-1})^{BC}\epsilon = \xi^A \delta_A{}^C \\ & \xi_B(\epsilon^{-1})^{BC} = \xi^C \\ \Rightarrow \quad & -\epsilon^{BC}\xi_B = \epsilon^{CB}\xi_B = \xi^C \end{aligned} \tag{3.8}$$

We have established the method of using the symplectic structure to raise and lower

the indices of spinors. One has to be careful with the order of indices unlike the use of metric in the conventional tensor analysis. Let us also note that ϵ is usually used as the kernel letter for Kronecker delta.

$$\epsilon_C{}^A = \delta_C{}^A = -\epsilon^A{}_C \tag{3.9}$$

Next we consider a linear transformation applied to a spin basis.

$$\tilde{o}^A = \alpha o^A + \beta i^A, \quad \tilde{i}^A = \gamma o^A + \delta i^A \tag{3.10}$$

where $\alpha, \beta, \gamma, \delta \in \mathcal{C}$. The same equations hold with indices lowered so $(\tilde{o}, \tilde{i})$ forms a spin basis if and only if $\alpha\gamma - \beta\delta = 1$. So the transformation matrix is in $SL(2, \mathcal{C})$. We deduced that the condition that a linear transformation preserves a spin basis is precisely the condition that it be a symplectomorphism, which proves that $Sp(2)$ is isomorphic to $SL(2, \mathcal{C})$.

3.1.1.1. Symmetrization and Antisymmetrization. The usual symmetrization ( ) and antisymmetrization [ ] operations can be applied to spinor suffices. Since the dimension of S is two, $\tau_{\ldots[ABC]\ldots} = 0$ for any multivalent spinor τ, for at least two of the bracketed indices must be equal. In particular we have the Jacobi identity

$$\epsilon_{A[B}\epsilon_{CD]} = 0 = \epsilon_{AB}\epsilon_{CD} + \epsilon_{AC}\epsilon_{DB} + \epsilon_{AD}\epsilon_{BC} \tag{3.11}$$

Here we set a lemma. Let $\tau_{\ldots CD\ldots}$ be a multivalent spinor. Then

$$\tau_{\ldots AB\ldots} = \tau_{\ldots(AB)\ldots} + \frac{1}{2}\epsilon_{AB}\tau_{\ldots C}{}^C{}_{\ldots} \tag{3.12}$$

When τ has valence two, the proof is straightforward. Raising the CD indices of Jacobi identity and multiplying with τ_{CD} gives

$$\begin{aligned} & \epsilon_{AB}\tau_C{}^C - \tau_{AB} + \tau_{BA} = 0 \\ \Rightarrow \quad & \tau_{[AB]} = \frac{1}{2}\epsilon_{AB}\tau_C{}^C \end{aligned} \tag{3.13}$$

Since $\tau_{AB} = \tau_{(AB)} + \tau_{[AB]}$ the lemma is justified for the case of valence two. An elegant proof for the general case is given by Penrose and Rindler [36]. Now we set a more general theorem.

Theorem 3.1.1. *Any spinor $\tau_{A\ldots F}$ is the sum of the totally symmetric spinor $\tau_{(A\ldots F)}$ and outer products of ϵ's with totally symmetric spinors of lower valence.*

Proof. The result is obvious for valence 2, we proceed by induction for the the proof for valence n. We define an equivalence relation $\sim$ between two spinors α, β of the same valence. If the difference between α and β is a sum of terms, each of which is an outer product of ϵ with a spinor of lower valence, we say $\alpha \sim \beta$. Now

$$n\tau_{(ABC\ldots F)} = \tau_{A(BC\ldots F)} + \tau_{B(AC\ldots F)} + \tau_{F(BC\ldots A)} \tag{3.14}$$

Consider the difference between the first and any other term on the right $\tau_{A(BC\ldots F)} - \tau_{B(AC\ldots F)}$. This is clearly skew-symmetric in AB so by lemma (3.12) can be written as ϵ_{AB} times a spinor of valence $(n-2)$. Applying this to each term on the right after the first we find that all n terms on the right are equivalent to the first so that $\tau_{(ABC\ldots F)} \sim \tau_{A(BC\ldots F)}$. If the result is true for valence $(n-1)$ then it is true for valence n, which proves the result. Thus only symmetric spinors matter, or more precisely every irreducible representation of $SL(2,\mathcal{C})$ can be realized by symplectomorphisms acting on symmetric spinors. □

3.1.2. Spinors and Vectors

In this section we are going to show how to identify the set of Hermitian spinors with the tangent space (at a point) of a 4 dimensional manifold M. This is the crucial feature which enables the use of spinors in general relativity. We start by defining the complex conjugate of the space S.

One might be misled by the fact that S is a vector space over $\mathcal{C}$ to conclude that the complex conjugate of a spinor in S is an element of S. However, this is not the case. Let us denote the operation of complex conjugation by a bar. If α, β are in S, and $c \in \mathcal{C}$ the conjugate of $\alpha + c\beta$ is $\bar{\alpha} + \bar{c}\bar{\beta}$ and not $\bar{\alpha} + c\bar{\beta}$, so complex conjugation is an anti-isomorphism rather than an isomorphism. Also if complex conjugation were an isomorphism real and imaginary valence one spinors could be distinguished. But in the discussion of transformations between spin basis we have seen that any valence one spinor can be mapped to another by a symplectomorphism, so that a real spinor would be isomorphic to an imaginary one. Therefore complex conjugation must be regarded as an anti-isomorphism from S to a new vector space $\bar{S}$. The complex conjugate of α^A is denoted by $\bar{\alpha}^{A'}$. Both the kernel letter acquires a bar and the suffix gains a prime. Though there is a clear redundancy, this notation is standard. Complex conjugation applied twice gives the identity map so $\overline{\bar{S}} = S$ and $\overline{\bar{\alpha}^{A'}} = \alpha^A$. The symplectic structure of S maps to a symplectic structure for $\bar{S}$, so the dual space of $\bar{S}$, denoted by $\bar{S}^*$, is the complex conjugate of S^*. The notation for $\overline{\epsilon_{AB}}$ is $\epsilon_{A'B'}$ and not $\bar{\epsilon}_{A'B'}$. We may now build a tensor algebra out of S, S^*, $\bar{S}$, $\bar{S}^*$. Because S and S^* are regarded as the same vector space the order of unprimed indices matter.

$$\tau^{AB}{}_{CD} \neq \tau^{A}{}_{C}{}^{B}{}_{D} \tag{3.15}$$

Similarly

$$\tau^{A'B'}{}_{C'D'} \neq \tau^{A'}{}_{C'}{}^{B'}{}_{D'} \tag{3.16}$$

However S and $\bar{S}$ are different vector spaces and we do not need to distinguish between $S \times \bar{S}$ and $\bar{S} \times S$. Thus all primed and unprimed indices can be shuffled through each other.

$$\tau_{AB'} = \tau_{B'A} \tag{3.17}$$

In that case we will not be able to treat $\tau_{AB'}$ as a 2×2 matrix. The only exception will be the Infeld-van der Waerden symbols which will be defined soon.

A hermitian spinor τ is defined to be one for which $\bar{\tau} = \tau$. For this to make sense τ must have as many primed as unprimes indices and relative positions of the unprimed indices must be the same as the relative positions of the primed ones. The simplest example is $\tau^{AA'} \in S \times \bar{S}$. Let (o, i) and $(\bar{o}, \bar{i})$ be spin bases for S and S^*. Then there exist scalars ξ, η, ζ, σ such that

$$\tau^{AA'} = \xi o^A \bar{o}^{A'} + \eta i^A \bar{i}^{A'} + \zeta o^A \bar{i}^{A'} + \sigma i^A \bar{o}^{A'} \tag{3.18}$$

Requiring $\tau^{AA'}$ to be hermitian makes ξ and η real, and ζ and σ are complex conjugates of each other. Thus the set of hermitian spinors forms a real vector space of dimension 4. In spinor analysis we are going to identify this with $T_p(M)$, the tangent space at a point in a 4-manifold M. Similarly the set of hermitian spinors $\tau_{AA'}$ forms a real vector space dual to the one above, which will form the cotangent space $T_p^*(M)$. This identification is denoted by relabelling $AA' \mapsto a$. Every spin basis defines a tetrad of vectors.

$$l^a = o^A \bar{o}^{A'} \quad n^a = i^A \bar{i}^{A'} \quad m^a = o^A \bar{i}^{A'} \quad \bar{m}^a = i^A \bar{o}^{A'} \tag{3.19}$$

l and n are real while m and $\bar{m}$ are complex conjugates. It is straightforward to take the contractions.

$$l_a n^a = n_a l^a = -m_a \bar{m}^a = -\bar{m}_a m^a = 1 \tag{3.20}$$

while all other contractions vanish. The set of 4 vectors constructed above is called the Newman-Penrose null tetrad. This is usually abbreviated to NP null tetrad or just null tetrad.

ϵ_{AB} and its complex conjugate can be used to construct a fundamental hermitian spinor.

$$g_{ABA'B'} = \epsilon_{AB}\epsilon_{A'B'} \tag{3.21}$$

which has symmetric tensor equivalents g_{ab}, g^{ab}. Using

$$\epsilon_{AB} = o_A i_B - i_A o_B \tag{3.22}$$

one derives

$$g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)} \tag{3.23}$$

g has all the properties of a metric tensor. Thus

$$g^{ab}l_b = l^a,\ g_{ab}l^b = l_a,\ g^{ab}g_{bc} = \delta_c{}^a \tag{3.24}$$

Let us define a tetrad of vectors

$$\begin{aligned} e_{\hat{0}} &= 2^{-1/2}(l+n), \qquad & e_{\hat{1}} &= 2^{-1/2}(m+\bar{m}) \\ e_{\hat{2}} &= 2^{-1/2}\mathrm{i}(\mathrm{m}-\bar{\mathrm{m}}), \qquad & e_{\hat{3}} &= 2^{-1/2}(l-n) \end{aligned} \tag{3.25}$$

$(e_{\hat{i}})$ form an orthonormal tetrad with $g_{\hat{a}\hat{b}} = \mathrm{diag}(1,-1,-1,-1)$. So the existence of a spinor structure forces the space-time to have one time-like and three space-like dimensions.

We can spell out the correspondence $\tau^{AA'} \mapsto \tau^a$ explicitly by writing

$$\epsilon_{AB}\epsilon_{A'B'} = g_{\hat{a}\hat{b}}\sigma^{\hat{a}}{}_{AA'}\sigma^{\hat{b}}{}_{BB'} \tag{3.26}$$

where the σ's are the Infeld-van der Waerden symbols. To be able to treat σ's as matrices, index shuffling is not allowed, the primed index always comes last. A suitable choice is

$$\sigma_{\hat{a}}{}^{AA'} = 2^{-1/2}\sigma_{\hat{a}} \; ; \; a = 0, 1, 2, 3 \tag{3.27}$$

where $\sigma_{\hat{a}}$ are the Pauli matrices. Raising and lowering indices give

$$\sigma^{\hat{a}}{}_{AA'} = \lambda_{\hat{a}}2^{-1/2}\sigma_{\hat{a}} \quad ; \quad \lambda_{\hat{a}} = (1, 1, -1, 1) \tag{3.28}$$

If u is any spacetime vector we may write

$$u^{\hat{a}} = u^{AA'}\sigma^{\hat{a}}{}_{AA'}. \tag{3.29}$$

We have seen that every univalent spinor κ^A defines a real null vector $k^a = \kappa^A\bar{\kappa}^{A'}$. In order to improve our space-time interpretation of spinors we first generalize this to state that every non-vanishing real null vector κ^a can be written in the form

$$k^a = \pm\kappa^A\bar{\kappa}^{A'} \tag{3.30}$$

However, $e^{i\theta}\kappa^A$ defines the same null vector for real θ so there is some additional phase information in κ. To extract this information we complete κ to a spin basis (κ, μ) where μ is unique up to an additive multiple of κ. Now we define a unit spacelike vector orthogonal to k^a

$$s^a = 2^{-1/2}(\kappa^A\bar{\mu}^{A'} + \mu^A\bar{\kappa}^{A'})$$

The 2-plane in the tangent space spanned by k^a, s^a is unique. we define another unit spacelike vector orthogonal to k^a

$$t^a = 2^{-1/2}\mathrm{i}(\kappa^{\mathrm{A}}\bar{\mu}^{\mathrm{A}'} - \mu^{\mathrm{A}}\bar{\kappa}^{\mathrm{A}'})$$

s, t span a spacelike 2-surface orthogonal to k. Now we make a phase change $\kappa \mapsto e^{i\theta}\kappa$, since (κ, μ) form a spin basis, $\kappa_A\mu^A = 1$, therefore $\mu \mapsto e^{-i\theta}\mu$. From the definitions it follows that

$$s^a \mapsto \cos 2\theta s^a + \sin 2\theta t^a \quad t^a \mapsto -\sin 2\theta s^a + \cos 2\theta t^a$$

Penrose has suggested geometric interpretation of a univalent spinor as a flag. The flagpole direction of κ^A is defined to be parallel to the direction of k^a, while the flag itself lies in the two-plane spanned by k^a and s^a. Thus the flagpole lies in the plane of the flag. Changing the phase of κ^A by θ leaves the flagpole invariant but rotates the flag plane around the pole by an angle 2θ as measured in the space-like two-plane orthogonal to the pole spanned by s^a and t^a. In that case a phase change by π leaves the flag plane invariant but reverses the sign of κ^A.

We can also define spinor fields in a region of spacetime provided that it fulfills some requirements. Equation (3.30) implies that we have a consistent method for distinguishing two possibilities; a null vector is said to be future pointing if + occurs in the decomposition and past pointing otherwise. Thus our spacetime must be time orientable. Further the effect of a phase change on a flag can be used to define a space orientation in the spacetime: the flag plane rotation corresponding to $\theta > 0$ is defined to be right-handed. Then the orthonormal tetrad has the standard orientation, i.e., $e_{\hat{0}}$ is future pointing and $e_{\hat{1}}, e_{\hat{2}}, e_{\hat{3}}$ form a right handed triad of vectors. Thus orientability of spacetime is a necessary condition for the existence of spinor fields. In a non-compact spacetime the existence of 4 continuous vector fields constituting a Minkowski tetrad at each point is not only necessary but also sufficient for the existence of spinor fields.

3.1.3. Spinor Analysis and Newman Penrose Formalism

Let θ, ϕ, and ψ be spinor fields such that θ and ϕ have the same valence. The spinor covariant derivative is defined axiomatically as a map $\nabla_x = \nabla_{XX'} : \theta_{\dots} \mapsto \theta_{\dots;XX'}$ such that

(i) $\nabla_{\mathrm{x}}(\theta + \phi) = \nabla_{\mathrm{x}}\theta + \nabla_{\mathrm{x}}\phi$

(ii) $\nabla_{\mathrm{x}}(\theta\psi) = (\nabla_{\mathrm{x}}\theta)\psi + \theta(\nabla_{\mathrm{x}}\psi)$

(iii) $\psi = \nabla_{\mathrm{x}}\theta \Rightarrow \bar{\psi} = \nabla_{\mathrm{x}}\bar{\theta}$

(iv) $\nabla_{\mathrm{x}}\epsilon^{\mathrm{AB}} = \nabla^{\mathrm{x}}\epsilon^{\mathrm{AB}} = 0$

(v) ∇_{x} commutes with index substitutions not involving X, X'

(vi) $\nabla_{\mathrm{x}}\nabla_{\mathrm{y}}\mathrm{f} = \nabla_{\mathrm{y}}\nabla_{\mathrm{x}}\mathrm{f}$ for all scalars f

(vii) for any derivation D there exists a spinor $\xi^{\mathrm{XX}'}$ such that $D\psi = \xi^{XX'}\nabla_{XX'}\psi$ for all ψ

where a derivation refers to linear map acting on spinor fields which obeys Leibniz rule and annihilates constant scalar fields. Apart from (vii) all of these conditions are conventional. This is needed to ensure that $\xi^{XX'}$ is a (possibly complex) tangent vector in the spacetime so we can identify the 4 dimensional vector space characterized by a hermitian spinor with the tangent space $T_p(M)$ at a point $p \in M$, and its dual with the cotangent space $T_p^*(M)$. Condition (vi) is equivalent to the absence of torsion. The proof that ∇_x exists and unique can be found in Penrose and Rindler [36]. Two useful properties of the spinor covariant derivative are worth noting. First if $\nabla_x\kappa^A = 0$ then the corresponding flagpole and flagplane are paralelly transported. Second, condition (iv) implies that $\nabla_x g_{ab} = 0$ so that, acting on tensors ∇_x is the covariant derivative compatible with the metric.

In 1962, Newman and Penrose [37] have developed the idea of adapting tetrads of null vectors to spinors. In this way they were able to translate the abstract spin basis (o, i) to real spacetime to perform calculations. As we saw in Equation (3.19) (o, i) induce four null vectors with l, n real and $m, \bar{m}$ complex conjugates. The directional

derivatives along these directions are denoted by conventional symbols

$$D = l^a \nabla_a, \quad \Delta = n^a \nabla_a, \quad \delta = m^a \nabla_a, \quad \bar{\delta} = \bar{m}^a \nabla_a \tag{3.31}$$

∇_a can be expressed as a linear combination of these operators.

$$\begin{aligned} \nabla_a &= g_a{}^b \nabla_b \\ &= (n_a l^b + l_a n^b - \bar{m}_a m^b - m_a \bar{m}^b) \nabla_b \\ &= n_a D + l_a \Delta - \bar{m}_a \delta - m_a \bar{\delta} \end{aligned} \tag{3.32}$$

One can replace ∇_a by (3.32) and convert all remaining tensor equations to sets of scalar ones by contraction with the NP null tetrad. Although this usually leads to a vast set of equations the fact that only scalars are involved and that most problems involve discrete symmetries makes them easier to handle.

Let us consider the Ricci rotation coefficients [42] defined by:

$$\Gamma_{\hat{a}\hat{b}\hat{c}} = e_{\hat{a}a} \nabla_{\hat{c}} e_{\hat{b}}{}^a \equiv e_{\hat{a}a} e_{\hat{c}}{}^c \nabla_{\hat{c}} e_{\hat{b}}{}^a = -e_{\hat{b}}{}^a \nabla_{\hat{c}} e_{\hat{a}a} \tag{3.33}$$

The analogue in NP formalism takes the form

$$\alpha^A \nabla \beta_A \tag{3.34}$$

where α and β are one of (o, i) and ∇ is one of D, Δ, δ, $\bar{\delta}$. Since $o_A i^A = 1$, we have $o^A \nabla i_A = i^A \nabla o_A$, so there are three independent choices for α and β. The fact there are four choices for ∇ leads to 12 complex rotation coefficients corresponding to the 24 real coefficients. these coefficients are also known as NP spin coefficients and defined

as follows:

$$\begin{aligned} Do_A &= \epsilon o_A - \kappa i_A \quad & Di_A &= \pi o_A - \epsilon i_A \\ \Delta o_A &= \gamma o_A - \tau i_A \quad & \Delta i_A &= \nu o_A - \gamma i_A \\ \delta o_A &= \beta o_A - \sigma i_A \quad & \delta i_A &= \mu o_A - \beta i_A \\ \bar{\delta} o_A &= \alpha o_A - \rho i_A \quad & \bar{\delta} i_A &= \lambda o_A - \alpha i_A \end{aligned} \tag{3.35}$$

From (3.35) one directly reads that $o^A Do_A = \kappa, o^A Di_A = \epsilon ...$ which is the definition found in most books. The definition (3.35) enables us to easily raise and lower indices and take complex conjugates. Note that we could have also made a vectorial description of NP spin coefficients. Since $l^a \nabla l_a = 0$ there are only 6 choices of vectors which leads to 24 real coefficients. To see the relation between two representations consider

$$\begin{aligned} \kappa &= m^a D l_a \\ &= o^A \bar{i}^{A'} D(o_A \bar{o}_{A'}) \\ &= o^A o_A \bar{i}^{A'} D\bar{o}_{A'} + o^A \bar{o}_{A'} \bar{i}^{A'} D o_A \\ &= o^A D o_A \end{aligned} \tag{3.36}$$

This can be generalized as follows

$$\begin{aligned} o^A \nabla o_A &= m^a \nabla l_a \\ o^A \nabla i_A &= 1/2(n^a \nabla l_a - \bar{m}^a \nabla m_a) \\ i^A \nabla i_A &= -\bar{m}^a \nabla n_a \end{aligned} \tag{3.37}$$

With this description NP formalism can be used in practical calculations without any reference to its spinorial origin.

Since we are only dealing with scalars for any NP quantitiy ϕ, $\nabla_{[a} \nabla_{b]} \phi = 0$. Let

us multiply this from the left by $l^a m^b$.

$$\begin{aligned} & m^b D\nabla_b\phi = l^a\delta\nabla_a\phi = 0 \\ \Rightarrow\quad & D\delta\phi - (Dm^b)\nabla_b\phi - \delta D\phi + (\delta l^a)\nabla_a\phi = 0 \end{aligned} \tag{3.38}$$

where

$$\begin{aligned} Dm^b &= D(o^{B}\bar{i}^{B'}) \\ &= \epsilon o^{B}\bar{i}^{B'} - \kappa i^{B}\bar{i}^{B'} + \bar{\pi}o^{B}\bar{o}^{B'} - \bar{\epsilon}o^{B}\bar{i}^{B'} \\ &= \bar{\pi}l^b - \kappa n^b + (\epsilon - \bar{\epsilon})m^b \end{aligned} \tag{3.39}$$

Similarly

$$\delta l^a = (\bar{\alpha} + \beta)l^a - \bar{\rho}m^a - \sigma\bar{m}^a \tag{3.40}$$

This leads to a commutation relation in D and δ. In total there are four commutation relations.

$$\begin{aligned} (\delta D - D\delta)\psi &= [(\bar{\alpha} + \beta\bar{\pi})D + \kappa\Delta(\bar{\rho} + \epsilon - \bar{\epsilon})\delta - \sigma\bar{\delta}]\psi \\ (\Delta D - D\Delta)\psi &= [(\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})]\psi \\ (\delta\Delta - \Delta\delta)\psi &= [-\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + (\mu - \gamma + \bar{\gamma})\delta + \bar{\lambda}\bar{\delta}]\psi \\ (\bar{\delta}\delta - \delta\bar{\delta})\psi &= [(\bar{\mu} - \mu)D(\rho \bar{-} \rho\Delta + (\alpha - \bar{\beta})\delta - (\bar{\alpha}_{\beta})\bar{\delta}]\psi \end{aligned} \tag{3.41}$$

A common practical problem is the computation of NP scalars. Suppose we have a coordinate chart and NP tetrad of vectors of a given spacetime. The components of the tetrad vectors are known functions so it is straightforward to compute the covector components and the metric g_{ab}. However since the description of the NP scalars involve covariant derivatives, (for example $\kappa = m^a Dl_a = m^a l^b \nabla_b l_a$), it looks as though it is necessary to compute Christoffel symbols from g_{ab}. Fortunately this is not the case.

Using a procedure which was explicitly described by Cocke [38]

$$\begin{aligned} \kappa &= m^a D l_a = m^a D l_a - l^a \delta l_a \\ &= 2 l^a m^b \nabla_{[a} l_{b]} \end{aligned} \tag{3.42}$$

Similar expressions can be obtained for all 12 NP scalars where only partial derivatives are needed. We proceed with a theorem:

Theorem 3.1.2. *Let $\tau_{AB...C}$ be totally symmetric. Then there exist univalent spinors $\alpha_A, \beta_B, ..., \gamma_C$ such that*

$$\tau_{AB...C} = \alpha_{(A}, \beta_B, ..., \gamma_{C)} \tag{3.43}$$

$\alpha, \beta, ..., \gamma$ are called the principal spinors of τ. The corresponding real null vectors are called the principal null vectors of τ.

Proof. Let τ have valence n and let $\xi^A = (x, y)$. Define

$$\tau(\xi) = \tau_{AB...C} \xi^A \xi^B ... \xi^C$$

This is a homogeneous polynomial of degree n in complex x, y which can be factorized completely.

$$\tau(\xi) = (\alpha_0 x - \alpha_1 y)(\beta_0 x - \beta_1 y)...(\gamma_0 x - \gamma_1 y)$$

which proves the result since x and y are arbitrary. Notice that neither the order nor possible factors $\alpha, \beta, ..., \gamma$ are specifiable. We may decompose ϕ_{AB} using (3.43).

$$\phi_{AB} = \alpha_{(A} \beta_{B)} \tag{3.44}$$

If α and β are proportional then α is called a repeated principal spinor of ϕ, and ϕ is said to be algebraically special or null, or of type N. The corresponding real null

vector $\alpha_a = \alpha_A \bar{\alpha}_{A'}$ is called a repeated principal null direction. If α and β are not proportional then ϕ is said to be algebraically general or type 1. □

3.2. Electromagnetic Field in Newman Penrose Formalism

In this section we introduce the description of electromagnetism in spinor language. We define the NP scalars of electromagnetism and derive Maxwell equations.

Let us first introduce the totally antisymmetric Maxwell tensor F_{ab} with its spinor equivalent $F_{ABA'B'}$. We define

$$\phi_{AB} \equiv \frac{1}{2} F_{ABC'}{}^{C'} = \phi_{BA} \tag{3.45}$$

Using (3.12) we have

$$F_{ABA'B'} = F_{AB(A'B')} + \phi_{AB}\epsilon_{A'B'} \tag{3.46}$$

and a second application gives

$$F_{ABA'B'} = F_{(AB)(A'B')} + \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\bar{\phi}_{A'B'} \tag{3.47}$$

The first term on the right hand side vanishes because of the fact that F is antisymmetric, so

$$F_{ABA'B'} = \phi_{AB}\epsilon_{A'B'} + \epsilon_{AB}\bar{\phi}_{A'B'} \tag{3.48}$$

We define the dual of F as $F^*_{ab} = (1/2)\epsilon_{ab}{}^{cd}F_{cd}$. Then using

$$\epsilon_{abcd} = \epsilon_{ABCDA'B'C'D'} = i(\epsilon_{AB}\epsilon_{CD}\epsilon_{A'C'}\epsilon_{B'D'} - \epsilon_{AC}\epsilon_{BD}\epsilon_{A'B'}\epsilon_{C'D'})$$

we derive

$$F_{ab} + iF^*_{ab} = 2\phi_{AB}\epsilon_{A'B'} \tag{3.49}$$

F_{ab} and ϕ_{AB} are fully equivalent in this context. Consider a spin 1 zero rest mass field in vacuum, i.e. electromagnetism. Maxwell's equations are equivalent to

$$\nabla^a(F_{ab} + iF^*{}_{ab}) = 0 \tag{3.50}$$

By means of Equation (3.49) the form of Maxwell equations in NP formalism is attained.

$$\nabla^{AA'}\phi_{AB} = 0 \tag{3.51}$$

There are four complex equations here by $A', B = 0, 1$, corresponding to eight real Maxwell equations. To derive the explicit forms of the equations we first note that the symmetric valence 2 spinors ϕ_{AB} generates 3 complex scalars via

$$\phi_0 = \phi_{AB}o^A o^B, \quad \phi_1 = \phi_{AB}o^A i^B, \quad \phi_2 = \phi_{AB}i^A i^B \tag{3.52}$$

The NP description of electromagnetism is given in terms of the scalars (3.52). Recalling $\epsilon^{AB} = o^A i^B - i^A o^B$ one obtains

$$\phi_{AB} = \phi_2 o_A o_B - 2\phi_1 o_{(A} i_{B)} + \phi_0 i_A i_B \tag{3.53}$$

One can give an equivalent description in terms of the null tetrad (3.19) induced by the basis.

$$\begin{aligned} &\phi_0 = F_{\mu\nu}l^\mu m^\nu \ , \ \phi_1 = \frac{1}{2}F_{\mu\nu}(l^\mu n^\nu + m^{*\mu}m^\nu) \\ &\phi_2 = F_{\mu\nu}m^{*\mu}n^\nu \end{aligned} \tag{3.54}$$

These relations can be inverted to give

$$F_{\mu\nu} = 2[\phi_1(n_{[\mu}l_{\nu]} + m_{[\mu}m^*_{\nu]}) + \phi_2 l_{[\mu}m_{\nu]} + \phi_0 m^*_{[\mu}n_{\nu]}] + \text{c.c.} \tag{3.55}$$

NP scalars of electromagnetism can be related to the components of electric and magnetic field using (3.49).

$$\begin{aligned} E_{\hat{3}} - iB_{\hat{3}} &= (F + iF^*)_{ab}n^a l^b = 2\phi_{AB}\epsilon_{A'B'}i^A\bar{i}^{A'}o^B bar o^{B'} \\ &= -2\phi_{AB}i^A o^B = -2\phi_1 \end{aligned} \tag{3.56}$$

Similarly $E_{\hat{1}} - iB_{\hat{1}} = \phi_0 - \phi_2$ and $E_{\hat{2}} - iB_{\hat{2}} = i(\phi_0 + \phi_2)$. Now we can expand the Maxwell Equations (3.51) to derive the explicit forms.

$$\nabla^{AA'}\phi_{AB} = \epsilon^{AC}\nabla_{AA'}\phi_{CB} = (o^A i^C - i^A o^C)\nabla_{AA'}\phi_{CB} = 0 \tag{3.57}$$

We may multiply (3.57) by $\bar{o}^{A'}o^B$ from the left

$$(o^A\bar{o}^{A'}o^B i^C - i^A\bar{o}^{A'}o^B o^C)\nabla_{AA'}\phi_{BC} = 0 \tag{3.58}$$

expanding ϕ_{BC}

$$(o^B i^C D - o^B o^C\bar{\delta})[\phi_0 i_B i_C - \phi_1 o_B i_C - \phi_1 i_B o_C + \phi_2 o_B o_C] = 0 \tag{3.59}$$

Acting with the derivative operators we derive our first maxwell equation. Next multiplying (3.57) with $\bar{o}^{A'}i^B$, $\bar{i}^{A'}o^B$, and $\bar{i}^{A'}i^B$ respectively we derive the corresponding maxwell equations.

$$(D - 2\rho)\phi_1 - (\bar{\delta} + \pi - 2\alpha)\phi_0 + \kappa\phi_2 = 0 \tag{3.60}$$

$$(D - \rho + 2\epsilon)\phi_2 - (\bar{\delta} + 2\pi)\phi_1 + \lambda\phi_0 = 0 \tag{3.61}$$

$$(\delta - 2\tau)\phi_1 - (\Delta + \mu - 2\gamma)\phi_0 - \sigma\phi_2 = 0 \tag{3.62}$$

$$(\delta - \tau + 2\beta)\phi_2 - (\Delta + 2\mu)\phi_1 - \nu\phi_0 = 0 \tag{3.63}$$

Hence we have derived the complete set of source-free Maxwell equations in spinor language.

3.3. Perturbations of a Kerr Black Hole

The spacetime of a Kerr black hole is stationary and axisymmetric with corresponding Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$. For that reason the time and ϕ dependence of a general perturbation is naturally expected to be in the form $e^{-i\omega t+im\phi}$, where m is an integer. However, the dependence of the amplitude of these fields on the remaining coordinates r and θ was not expected to be separable. Yet, Teukolsky achieved this further separation in his celebrated series of papers [39, 40], where he decouples Maxwell's equations and gravitational equations to combine them into a single master equation which also describes neutrino field perturbations. He derives linear equations which are separated into ordinary differential equations with simple asymptotic solutions. In his pioneering work he also lays the mathematical background for the stability of the black hole, tidal friction effects [41] and superradiant scattering of the integer spin test fields [40]. Chandrasekhar [42] states that, this unexpected separability of the equations of perturbations of Kerr black holes was a sort of "open sesame" for unlocking further doors.

The mathematical technique underlying his work is to linearize Einstein and Maxwell equations around a known stationary black hole solution. Rather than treating the black hole as a fixed geometrical background, the linearized equations provide dynamical freedom of interacting with gravitational, electromagnetic and scalar waves that would result in changes of black hole parameters such as mass and angular momentum. The fundamental perturbation equations to be derived are separable partial differential equations whose independent variables are certain decoupled components of the Weyl tensor (in the gravitational case), or of the electromagnetic field tensor. For both gravitational and electromagnetic perturbations it can be proved that the decoupled components contain complete information about all nontrivial features of the full perturbing field. For the electromagnetic case – which we will focus on – the result is due to Fackerell and Ipser [43].

Obtaining linearized perturbation equations involves considerable algebraic complexity even in the Schwarzschild case. The background metric in the Kerr case is more complicated and replacement of spherical symmetry by axial symmetry prevents us from making a separation into spherical harmonics; we may end up with partial differential equations in r and θ instead of ordinary differential equations in r. Fortunately, there is an alternative approach provided by the NP formalism. To do perturbation theory in this formalism, one specifies the perturbed geometry by $l = l^A + l^B, \quad n = n^A + n^B$, where A, B denote the background and the perturbation respectively. The Schwarzschild and Kerr metrics are similar from the NP point of view. Using this similarity Teukolsky derives decoupled equations for electromagnetic test fields in Kerr spacetime. Then he writes the equations for gravitational perturbations, electromagnetic test field and neutrino test field as a single master equations, which can be separated.

3.3.1. Test Electromagnetic Field in Kerr Spacetime

The derivation of the decoupled equations of electromagnetic test field applies to any Type D background metric. The Schwarzschild and Kerr solutions are both of this type. Choosing the l and n vectors of the unperturbed tetrad along the repeated principal null directions of the Weyl tensor we have

$$\kappa = \sigma = \nu = \lambda = 0 \tag{3.64}$$

for four of the twelve NP spin coefficients. In the gedanken experiments to evaluate the validity of Cosmic Censorship Conjecture we use the test field approximation. Since the amplitude of the electromagnetic stress-energy, is second order in the electromagnetic field, the change in the background geometry caused by the electromagnetic perturbation is also second order. Thus in Maxwell's equations the change in the geometry can be neglected to first order. We rewrite the Maxwell's Equations (3.60-3.63) in any

Type D space-time with no sources, imposing (3.64)

$$(D-2\rho)\phi_1-(\delta^*+\pi-2\alpha)\phi_0=0 \tag{3.65}$$

$$(\delta-2\tau)\phi_1-(\Delta+\mu-2\gamma)\phi_0=0 \tag{3.66}$$

$$(D-\rho+2\epsilon)\phi_2-(\delta^*+2\pi)\phi_1=0 \tag{3.67}$$

$$(\delta-\tau+2\beta)\phi_2-(\Delta+2\mu)\phi_1=0 \tag{3.68}$$

where ϕ's are the first order test fields. Let us introduce a commutation relation which holds for any Type D metric.

$$\begin{aligned}&[D-(p+1)\epsilon+\epsilon^*+q\rho-\rho^*](\delta-p\beta+q\tau)\\&-[\delta-(p+1)\beta-\alpha^*+\pi^*+q\tau](D-p\epsilon+q\rho)=0\end{aligned} \tag{3.69}$$

where p and q are any two constants. We act on Equation (3.65) with $(\delta-\beta-\alpha^*-2\tau+\pi^*)$, and on Equation (3.66) with $(D-\epsilon-\epsilon^*-2\rho-\rho^*)$, and subtract one equation from the other. Using identity (3.69) with $p=0$ and $q=-2$ we observe that ϕ_1 terms disappear, leaving a decoupled equation for ϕ_0:

$$\begin{aligned}&[(D-\epsilon+\epsilon^*-2\rho+\rho^*)(\Delta+\mu-2\gamma)\\&-(\delta-\beta-\alpha^*-2\tau+\pi^*)(\delta^*+\pi-2\alpha)]\phi_0=0\end{aligned} \tag{3.70}$$

By interchanging l and n, and m and $\bar{m}$, or by using (3.67) and (3.68), we derive an analogous decoupled equation for ϕ_2:

$$\begin{aligned}&[(\Delta+\gamma-\gamma^*+2\mu+\mu^*)(D-\rho+2\epsilon)\\&-(\delta^*+\alpha+\beta^*+2\pi^-\tau^*)(\delta^-\tau+2\beta)]\phi_2=0\end{aligned} \tag{3.71}$$

An analogous equation for ϕ_1 can also be found, but this equation does not appear to be separable in Kerr case. We need to emphasize that we have a decoupled set of equations valid in any spacetime of Type D. The next step is to write out the equations

in the Kerr metric. We use an NP tetrad of the form:

$$l^\mu = [(r^2+a^2)/\Delta, 1, 0, a/\Delta], \quad n^\mu = [(r^2+a^2, -\Delta, 0, a]/(2\Sigma)$$
$$m^\mu = [ia\sin\theta, 0, 1, i/\sin\theta]/[\sqrt{2}(r+ia\cos\theta)] \tag{3.72}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$ which should not be confused with the NP derivative operator Δ. (3.72) induces Kerr metric by (3.23). The nonvanishing spin coefficients are

$$\rho = -1/(r - \mathrm{ia}\cos\theta, \quad \beta = -\rho^*\cot\theta/2\sqrt{2}, \quad \pi = \mathrm{ia}\rho^2\sin\theta/\sqrt{2}$$
$$\tau = \mathrm{ia}\rho\rho^*\sin\theta/\sqrt{2}, \quad \mu = \rho^2\rho^*\Delta/2, \quad \gamma = \mu + \rho\rho^*(\mathrm{r} - \mathrm{M})/2,$$
$$\alpha = \pi - \beta^* \tag{3.73}$$

Note that $\epsilon = 0$ in addition to $\kappa, \sigma, \lambda, \nu$. Using these we combine (3.70) and (3.71) to derive Teukolsky's master equation:

$$\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\sin^2\theta\right]\frac{\partial^2\psi}{\partial t^2} + \frac{4Mar}{\Delta}\frac{\partial^2\psi}{\partial t\partial\psi} + \left[\frac{a^2}{\Delta} - \frac{1}{\sin^2\theta}\right]\frac{\partial^2\psi}{\partial^2\phi}$$
$$-\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial\psi}{\partial r}\right) - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) - 2s\left[\frac{a(r-M)}{\Delta} + \frac{i\cos\theta}{\sin^2\theta}\right]\frac{\partial\psi}{\partial\phi}$$
$$-2s\left[\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right]\frac{\partial\psi}{\partial t} + (s^2\cot^2\theta - s)\psi = 0 \tag{3.74}$$

Actually, this equation turns out to be more general, and applies for massless scalar, neutrino, electromagnetic and gravitational perturbations. The perturbations are characterized by s, called the spin weight of the field. The value $s = 0$ gives the equation for a test scalar field, $s = \pm 1/2$ for a test neutrino field, $s = \pm 1$ for a test electromagnetic field, and $s = \pm 2$, for gravitational perturbations. The field quantities that satisfy (3.74) with the corresponding spin weights is given below in table (3.3.1). Here

Table 3.1. Field quantities and spin weights.

s	0	1/2	-1/2	1	-1	2	-2
ψ	Φ	χ_0	$\rho^{-1}\chi_1$	ϕ_0	$\rho^{-2}\phi_2$	ψ_0	$\rho^{-4}\psi_4$

χ_0 and χ_1 are the components of valence 1 spinor Φ_A in the dyad legs, which satisfies the massless Dirac equation $\nabla_{AA'}\Phi^A = 0$. Ψ_0 and Ψ_4 are two of the five scalars generated by the valence 4 spinor Ψ_{ABCD} which satisfies the spinor analogue of Bianchi identities in vacuum $\nabla^{DD'}\Psi_{ABCD} = 0$. They are defined by $\Psi_0 = \Psi_{ABCD}o^A o^B o^C o^D$ and $\Psi_4 = \Psi_{ABCD}i^A i^B i^C i^D$.

The master equation can be separated in the form

$$\psi = e^{-i\omega t}e^{im\phi}S(\theta)R(r) \tag{3.75}$$

Then the radial equation is

$$\Delta^{-s}\frac{\partial}{\partial r}\left(\Delta^{s+1}\frac{\partial R}{\partial r}\right) + \left(\frac{K^2 - 2is(r-M)K}{\Delta} + 4is\omega r - \lambda\right)R = 0 \tag{3.76}$$

where $K \equiv (r^2 + a^2)\omega - am$ and $\lambda \equiv A + a^2\omega^2 - 2am\omega$. The angular equation takes the form

$$\begin{aligned}&\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right)\\ &+\left(a^2\omega^2\cos^2\theta - \frac{m^2}{\sin^2\theta} - 2a\omega s\cos\theta - \frac{2ms\cos\theta}{\sin^2\theta} - s^2\cot^2\theta + s + A\right)S = 0\end{aligned} \tag{3.77}$$

Equation (3.77) constitutes a Sturm-Liouville eigenvalue problem for the separation constant $A = {}_sA^m_l(a\omega)$, together with boundary conditions of regularity at $\theta = 0$ and π. For fixed s, m, and $a\omega$ we label the eigenvalues by l. The smallest eigenvalue is $l = \max(|\mathrm{m}|, |\mathrm{s}|)$. From Sturm-Liouville theory the eigenfunctions ${}_sS^m_l$ are complete and orthogonal on $0 \le \theta \le \pi$ for each s, m, and $a\omega$.

$$\int_0^\pi {}_sS^2_{lm}(\theta)\sin\theta d\theta = 1 \tag{3.78}$$

When $s = 0$ the eigenfunctions are spheroidal wave functions $S^m_l(-a^2\omega^2\cos\theta)$. When $a\omega = 0$ the eigenfunctions are spin-weighted spherical harmonics ${}_sY^m_l = {}_sS^m_l e^{im\phi}$,

and $A = (l-s)(l+s+1)$. In the general case the eigenfunctions are referred to as spin-weighted spheroidal harmonics.

To discuss the boundary conditions and asymptotic solutions for the radial function, it is useful to make the transformation

$$Y = \Delta^{s/2}(r^2+a^2)^{1/2}R, \quad dr^*/dr = (r^2+a^2)/\Delta \tag{3.79}$$

where r^* is the tortoise coordinate. Recall that $r^* \to \infty$ as $r \to \infty$ and $r^* \to -\infty$ as the horizon is approached. Then the radial equation becomes

$$Y_{,r^*r^*} + \left[\frac{K^2 - 2is(r-M)K + \Delta(4ir\omega s - \lambda)}{(r^2+a^2)^2} - G^2 - G_{,r^*}\right] Y = 0 \tag{3.80}$$

where $G = s(r-m)/(r^2+a^2) + r\Delta/(r^2+a^2)^2$ and a comma denotes partial differentiation. As $r \to \infty$ and $r^* \to \infty$, (3.80) becomes

$$Y_{,r^*r^*} + (\omega^2 + 2i\omega s/r)Y \approx 0 \tag{3.81}$$

with asymptotic solutions $Y \sim r^{\pm s}e^{\mp i\omega r*}$, i.e., $R \sim e^{-i\omega r*}/r$ and $e^{i\omega r*}/r^{2s+1}$. This corresponds to

$$\begin{aligned} &\phi_2 \sim e^{i\omega r*}/r, \quad \phi_0 \sim e^{i\omega r*}/r^3 \text{ outgoing waves} \\ &\phi_0 \sim e^{-i\omega r*}/r, \quad \phi_2 \sim e^{-i\omega r*}/r^3 \text{ ingoing waves} \end{aligned} \tag{3.82}$$

The event horizon is at $r = r_+ \quad (r^* \to \infty)$, the larger root of $\Delta = 0$. Near the event horizon (3.80) becomes

$$Y_{,r^*r^*} + \left[\frac{k^2 - 2\mathrm{is(r_+ - M)k}}{2Mr_+} - \frac{s^2(r_+-M)^2}{(2Mr_+)^2}\right] Y \approx 0 \tag{3.83}$$

where $k = w - m\Omega$, and $\Omega = a/2Mr_+$ is the rotational frequency of the black hole.

The asymptotic solutions near the horizon are

$$Y \sim e^{\pm i[k-is(r_+-M)/(2Mr_+)]r*} \sim \Delta^{\pm s/2} e^{\pm ikr*}$$
$$\Rightarrow R \sim e^{ikr*} \quad \text{or} \quad R \sim \Delta^{-s} e^{-ikr*} \tag{3.84}$$

The correct boundary condition at the horizon can be formulated by requiring that a physically well-behaved observer should see nonspecial fields. Here nonspecial refers to neither singular nor identically zero, nor varying infinitely fast. Equivalently, we may demand that the radial group velocity of a wave packet as measured by a physically well behaved observer be negative, which means signals can travel into the hole but can not come out. Therefore the asymptotic solution near the horizon with the correct boundary condition is

$$R \sim \Delta^{-s} e^{-ikr*} \tag{3.85}$$

3.3.2. Relations Between Decoupled Variables

In this section we present a further analysis of electromgnetic field variables ϕ_0 and ϕ_2 and find the relations between their relative amplitudes. The following identity for the Kerr metric can be used to eliminate ϕ_1 from (3.65) and (3.67).

$$\begin{aligned}
&[\bar{\delta} - (p+1)\alpha - (q+1)\beta^* + (m+1)\pi][D - p\epsilon - q\epsilon^*(m+p)\rho]\psi \\
&-[D - (p-1)\epsilon - (q+1)\epsilon^* - (m+p+1)\rho][\bar{\delta} - p\alpha - q\beta^* + mp]\psi \\
&= p\psi(D + \epsilon - \epsilon^*)\pi
\end{aligned} \tag{3.86}$$

where ψ is any NP quantity and p, q, m are arbitrary constants. Operate on Equation (3.65) with $(\bar{\delta}+3\pi-\beta^*-\alpha)$ and on Equation (3.67) with $(D-3\rho+\epsilon-\epsilon^*)$ and substract one equation from the other. The identity (3.86) with $p = q = 0$ and $m = 2$ yields

$$(\bar{\delta} + 3\pi - \beta^* - \alpha)(\bar{\delta} + \pi - 2\alpha)\phi_0 = (D - 3\rho + \epsilon - \epsilon^*)(D - \rho + 2\epsilon)\phi_2 \tag{3.87}$$

By interchanging $l \leftrightarrow n$, $m \leftrightarrow \bar{m}$, or using Equations (3.66) and (3.68) gives

$$(\delta - 3\tau + \beta + \alpha^*)(\delta - \tau + 2\beta)\phi_2 = (\Delta + 3\mu - \gamma + \gamma^*)(\Delta + \mu - 2\gamma)\phi_0 \tag{3.88}$$

We expand the solutions for ϕ_0 and ϕ_2.

$$\begin{aligned}\phi_0 &= \int d\omega e^{-i\omega t} \sum_{l,m} e^{im\phi}({}_1S_{lm}(\theta, a\omega))({}_1R_{wlm}(r)) \\ \rho^{-2}\phi_2 &= \int d\omega e^{-i\omega t} \sum_{l,m} e^{im\phi}({}_{-1}S_{lm}(\theta, a\omega))({}_{-1}R_{wlm}(r))\end{aligned} \tag{3.89}$$

Next we substitute in (3.87). Amazingly, the equation separates, giving

$$\left(\partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta\right)\left(\partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + \cot\theta\right)S_1 = S_{-1} \tag{3.90}$$

$$(\partial_r - iK/\Delta)(\partial_r - iK/\Delta)R_{-1} = R_1 \tag{3.91}$$

Here R_s is proportional to ${}_sR_{wlm}$ and S_s is proportional to ${}_sS_{lm}$. Suppose S_1 is normalized to be equal to ${}_1S_{lm}$. Then one can verify from Equation (3.90) that S_{-1} satisfies Teukolsky's angular equation (3.77) with $s = -1$, so $S_{-1} = B({}_{-1}S_{lm})$, where B is some normalization constant. Rewriting Equation (3.90) in the form

$$\mathcal{L}_0\mathcal{L}_1({}_1S_{lm}) = B({}_{-1}S_{lm}) \tag{3.92}$$

where

$$\mathcal{L}_n = \partial_\theta + \frac{m}{\sin\theta} - a\omega\sin\theta + n\cot\theta \tag{3.93}$$

We get

$$\begin{aligned} B^2 &= B^2 \int_0^\pi ({}_{-1}S_{lm})^2 \sin\theta d\theta \\ &= \int_0^\pi (\mathcal{L}_0\mathcal{L}_1({}_1S_{lm}))^2 \sin\theta d\theta \\ &= \int_0^\pi ({}_1S_{lm})(\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger\mathcal{L}_0\mathcal{L}_1({}_1S_{lm})) \sin\theta d\theta \end{aligned} \tag{3.94}$$

where $\mathcal{L}_n^\dagger \equiv \mathcal{L}_n(-\omega, -m)$. One can reduce the fourth order operator acting on $({}_1S_{lm})$ to a first order operator by substituting the expression for $d^2({}_1S_{lm})/d\theta^2$ by the angular equation (3.77). The result is that the first derivative terms vanish and the coefficient of $({}_1S_{lm})$ is a constant. Since the integral of $({}_1S_{lm})^2$ is unity, we find that

$$B^2 = (E + a^3\omega^2 - 2a\omega m)^2 + 4ma\omega - 4a^2\omega^2 \tag{3.95}$$

The sign of B can be found by comparing with the spherical case, $a\omega = 0$. This comparison shows that we must take the positive square root in Equation (3.95).

We can also substitute the expansion (3.89) into (3.88) instead of (3.87) to obtain a complementary set of equations.

$$\mathcal{L}_0^\dagger\mathcal{L}_1^\dagger S_{-1} = B^2 S_1 \tag{3.96}$$

$$\mathcal{D}^\dagger\mathcal{D}^\dagger\Delta R_1 = 2B^2\Delta^{-1}R_{-1} \tag{3.97}$$

where $\mathcal{D}^\dagger \equiv \partial_r + iK/\Delta$. Now we introduce Teukolsky's normalizations for solutions at $r \to \infty$

$$\begin{aligned} \phi_0 &\sim e^{-i\omega t}e^{im\phi}({}_1S_{lm}(\theta))(Y_{\rm in}e^{-i\omega r*}/r + Y_{\rm out}e^{i\omega r*}/r^3) \\ \rho^{-2}\phi_2 &\sim e^{-i\omega t}e^{im\phi}({}_{-1}S_{lm}(\theta))(Z_{\rm in}e^{-i\omega r*}/r + Z_{\rm out}re^{-i\omega r*}) \end{aligned} \tag{3.98}$$

Substituting these in (3.90) and (3.91) we find that

$$BY_{\rm in} = -8w^2 Z_{\rm in} \tag{3.99}$$

while Equations (3.96) and (3.97) give

$$-2w^2 Y_{\rm out} = BZ_{\rm out} \tag{3.100}$$

On the horizon an electromagnetic perturbation has the form

$$\begin{aligned} \phi_0 &\sim e^{-i\omega t} e^{im\phi} ({}_1S_{lm}(\theta)) Y_{\rm hole} \Delta^{-1} e^{-ikr*} \\ \rho^{-2}\phi_2 &\sim e^{-i\omega t} e^{im\phi} ({}_{-1}S_{lm}(\theta)) Z_{\rm hole} \Delta e^{-ikr*} \end{aligned} \tag{3.101}$$

Equations (3.90) and (3.91) can be used to relate $Y_{\rm hole}$ and $Z_{\rm hole}$

$$\begin{aligned} BY_{\rm hole} &= -32ikM^2 r_+^2(-ik + 2\epsilon) Z_{\rm hole} \\ &= 16kMr_+[-2kMr_+ - i(r_+ - M)] Z_{\rm hole} \end{aligned} \tag{3.102}$$

where $\epsilon \equiv (M^2 - a^2)^{1/2}/4Mr_+$. Teukolsky's radial equation (3.80) can be written in the form

$$Y_{,r^*r^*} + VY = 0 \tag{3.103}$$

by letting

$$\begin{aligned} V &= [K^2 - 2isK(r-M) + \Delta(4irws - Q) - s^2(M^2 - a^2)]/(r^2 + a^2)^2 \\ &\quad -\Delta(2Mr^3 + a^2r^2 - 4Mra^2 + a^4)/(r^2 + a^2)^4 \end{aligned} \tag{3.104}$$

where $K = (r^2 + a^2)w - am$ and $Q = E + a^2w^2 - 2awm$. The Wronskian of any two solutions of (3.103) is conserved. $V(r, w, m, l, s, a) = V^*(r, w, m, l, -s, a)$ so two

linearly independent solutions of (3.103) are $Y(s)$ and $Y^*(-s)$. Thus

$$[Y(s)_{,r^*}Y^*(-s) - Y(s)Y^*(-s)_{,r^*}]_{r=r_+} = [Y(s)_{,r^*}Y^*(-s) - Y(s)Y^*(-s)_{,r^*}]_{r=\infty} \quad (3.105)$$

Applying (3.105) to electromagnetic perturbations, we get

$$\frac{iB|Y_{\text{hole}}|^2}{8kMr_+} = \frac{iB|Y_{\text{in}}|^2}{4w} - \frac{4iw^3|Y_{\text{out}}|^2}{B} \quad (3.106)$$

(3.100) and (3.106) lead to

$$\frac{|Y_{\text{in}}|^2}{4} - |Z_{\text{out}}|^2 = \frac{w|Y_{\text{hole}}|^2}{8kMr_+} \quad (3.107)$$

3.4. Testing the Cosmic Censorship

In this section we construct an original thought experiment where we send test electromagnetic test fields to a Kerr black hole to check if CCC is violated as a result of the interaction. For this purpose we use Teukolsky's solutions for test electromagnetic field in Kerr space-time, and the conservation laws induced by the energy momentum tensor and the Killing vectors of the space-time.

We start by definitions. Let X be a smooth vector field on a Riemannian manifold M. Let ϕ_t denote the flow of X so that ϕ_t is a diffeomorphism for each t. Let τ be a smooth covariant tensor field on M. τ is invariant under the flow of X if

$${\phi_t}^*(\tau_{\phi_t(p)}) = \tau_p \quad (3.108)$$

Here the superscript $*$ denotes pullback. Let (M, g) be a Riemannian manifold. A smooth vector field X on M is called a Killing field if g is invariant under the flow of

X. By definition of Lie derivatives

$$(\mathcal{L}_X \tau)_p = \lim_{t \to 0} \frac{{\phi_t}^*(\tau_{\phi_t(p)}) - \tau_p}{t} \tag{3.109}$$

X is a Killing field if and only if

$$\mathcal{L}_X g = 0 \tag{3.110}$$

On a coordinate basis let

$$X = X^\sigma \frac{\partial}{\partial \sigma}, \quad g = g_{\alpha\beta} dx^\alpha \otimes dx^\beta \tag{3.111}$$

where $\otimes$ denotes tensor product. Then

$$\mathcal{L}_X g = X^\sigma g_{\alpha\beta,\sigma} dx^\alpha \otimes dx^\beta + g_{\alpha\beta}(X^\alpha_{,\mu} dx^\mu) \otimes dx^\beta + g_{\alpha\beta} dx^\alpha \otimes (X^\beta_{,\nu} dx^\nu) \tag{3.112}$$

where a comma denotes partial derivative. Let us take a component of (3.112)

$$(\mathcal{L}_X g)_{ab} = X^k g_{ab,k} + X^l_{,a} g_{lb} + g_{am} X^m_{,b} \tag{3.113}$$

Note that

$$X^l_{,a} g_{lb} = (X^l g_{lb})_{,a} - X^l (g_{lb,a}) = X_{b,a} - X^k (g_{kb,a})$$

Similarly

$$g_{am} X^m_{,b} = X_{a,b} - X^k (g_{ak,b})$$

We can rewrite (3.113)

$$\begin{aligned}(\mathcal{L}_X g)_{ab} &= X_{a,b} + X_{b,a} - (X_m g^{km})\{-g_{ab,k} + g_{kb,a} + g_{ak,b}\} \\ &= \nabla_a X_b + \nabla_b X_a \end{aligned} \tag{3.114}$$

Then the condition that X is a Killing field can also be written as

$$\nabla_{(a} X_{b)} = 0 \tag{3.115}$$

Now suppose T^{ac} is the energy-momentum tensor of a continuous distribution of matter satisfying $\nabla_c T^{ac} = 0$. The symmetry of T^{ac} and (3.115) lead to:

$$\nabla_a (T^{ac} X_c) = 0 \tag{3.116}$$

Kerr metric is static and axisymmetric with corresponding Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$. These give the energy and angular momentum fluxes, respectively.

$$\left(\frac{dM}{dt}\right)_{\text{b.h}} = -\int_{S_\infty} \sqrt{-g}\, T^1{}_0 d\theta d\phi \tag{3.117}$$

$$\left(\frac{dL}{dt}\right)_{\text{b.h}} = \int_{S_\infty} \sqrt{-g}\, T^1{}_3 d\theta d\phi \tag{3.118}$$

Note that Kerr metric induced by the tetrad (3.72) has signature $(+,---)$, whereas the metric in Section 2.3 has signature $(-,+,+,+)$. For that reason the fluxes have opposite signs. To test the validity of cosmic censorship let us re-define the indicator of Section 2.3 for the neutral case,

$$CCC = M^2 - a^2 \tag{3.119}$$

Then, using $a = L/M$, and implicitly defining dC/dt,

$$\delta(CCC) = \int \frac{dC}{dt} dt = \int \frac{2}{M} \left\{ (M^2 + a^2) \frac{dM}{dt} - a \frac{dL}{dt} \right\} dt \tag{3.120}$$

Using (3.117) and (3.118)

$$\frac{dC}{dt} = \int_{S_\infty} \sqrt{-g} [(M^2 + a^2)(-T^1{}_0) - aT^1{}_3] d\theta \, d\phi \tag{3.121}$$

We use the NP tetrad (3.72) to derive

$$T_{ab} l^a l^b \frac{\Delta^2}{4\Sigma^2} - T_{ab} n^a n^b = \frac{\Delta}{\Sigma^2} (r^2 + a^2) T_{10} + \frac{a\Delta}{\Sigma^2} T_{13} \tag{3.122}$$

This expression is similar to (3.121), except for the difference of r and M, which suggests evaluating the two expressions at the horizon of extremal black holes, where they will be equal. Note that (3.122) is valid everywhere and for every field in Kerr space-time.

The energy momentum tensor in terms of electromagnetic scalars is given by [39]

$$\begin{aligned} 4\pi T_{\mu\nu} \;=\; & \{\phi_0 \phi_0^* n_\mu n_\nu + 2\phi_1 \phi_1^* [l_{(\mu} n_{\nu)} + m_{(\mu} m_{\nu)}^*] + \phi_2 \phi_2^* l_\mu l_\nu \\ & -4\phi_1 \phi_0^* n_{(\mu} m_{\nu)} - 4\phi_2 \phi_1^* [l_{(\mu} m_{\nu)} + 2\phi_2 \phi_0^* m_\mu m_\nu \} \\ & +\text{c.c.} \end{aligned} \tag{3.123}$$

Since any NP tetrad satisfies $\mathbf{l}.\mathbf{n} = 1$, $\mathbf{m}.\mathbf{m}^* = -1$ and all other dot products equal to zero, (3.123) implies that $T_{ab} l^a l^b = (1/2\pi)|\phi_0|^2$ and $T_{ab} n^a n^b = (1/2\pi)|\phi_2|^2$. Using this and the following relation valid in Kerr space-time we will be able to evaluate $\delta(CCC)$ though we do not have an explicit solution for ϕ_1. Let us substitute the expressions for $T_{ab} l^a l^b$ and $T_{ab} n^a n^b$ into (3.122)and raise indices:

$$\frac{\Sigma}{2\pi} \left(|\phi_0|^2 \frac{\Delta^2}{4\Sigma^2} - |\phi_2|^2 \right) = -(r^2 + a^2) T^1{}_0 - aT^1{}_3 \tag{3.124}$$

(3.124) is valid everywhere but only for electromagnetic field in Kerr space-time. We can use (3.124) to evaluate dC/dt for a surface at the horizon, for extremal black holes. In the neutral case of Kerr black holes where there is no zeroth order interaction of the test electromagnetic field with the electromagnetic field of the hole, no contribution to the mass (energy) and angular momentum exists outside the horizon. Then the time integrated fluxes through surfaces at the horizon and at infinity are equal and we can evaluate $\delta(CCC)$ at the horizon. Let us first write the general solution near the horizon in the form of an expansion:

$$\begin{aligned}\phi_0 &= \int dw e^{-iwt} \sum_{l,m} e^{im\phi}{}_1S_{lm}(\theta, aw) Y_{\text{hole}}(\Delta^{-1}) e^{-ikr*} \\ \phi_2 &= \int dw e^{-iwt} \sum_{l,m} e^{im\phi}{}_{-1}S_{lm}(\theta, aw) \rho^2 Z_{\text{hole}} \Delta e^{-ikr*}\end{aligned} \tag{3.125}$$

then we define

$$\begin{aligned}\phi_0 &\equiv \Delta^{-1}\psi_0 \\ \phi_2 &\equiv \rho^2\Delta\psi_2\end{aligned} \tag{3.126}$$

Substituting these expressions in (3.124), near the horizon we have

$$\frac{1}{2\pi\Sigma}\left(\frac{|\psi_0|^2}{4} - \Delta^2|\psi_2|^2\right) = -(r^2+a^2)T^1{}_0 - aT^1{}_3 \tag{3.127}$$

where we have used $\rho^2\rho^{*2} = 1/\Sigma^2$. Now we evaluate (3.127) at the horizon ($\Delta \to 0$) for extremal black holes ($r_+ = M$)

$$\frac{|\psi_0|^2}{8\pi\Sigma} = -(M^2+a^2)T^1{}_0 - aT^1{}_3 \tag{3.128}$$

where we know we can drop the Δ^2-term and keep the other since ψ_0 and ψ_2 are regular by (3.125) and (3.126). The right hand side of (3.128) is the integrand in (3.121). Let

us evaluate Equation (3.121) at the horizon

$$\frac{dC'}{dt} = \int_{S_{\rm H}} \sqrt{-g} \left[\frac{|\psi_0|^2}{8\pi\Sigma} \right] d\theta \; d\phi \tag{3.129}$$

and

$$\delta(CCC) = \int \int_{S_{\rm H}} \sqrt{-g} \left[\frac{|\psi_0|^2}{8\pi\Sigma} \right] d\theta \; d\phi dt \tag{3.130}$$

As argued above, $\delta(CCC)$ can be calculated either at infinity or at the horizon for the Kerr black hole.The expression (3.130) is strictly positive. Thus, one can not overspin an extremal Kerr black hole by sending in electromagnetic test fields; CCC is not violated for this case.

4. NEARLY EXTREMAL BLACK HOLES AND COSMIC CENSORSHIP

In the previous chapters we have reviewed many attempts to construct gedanken experiments to violate cosmic censorship. In all these works including our analysis of the electromagnetic field the authors start out the thought experiment from an extremal black hole which saturates the main inequality (1.8). Apparently such processes do not lead to destroying the horizon. However, in 1999 Veronika E. Hubeny [14] constructed a thought experiment starting with a nearly extremal black hole instead of an extremal one, and somewhat surprisingly she managed to jump over extremality to overcharge the black hole, by letting it capture a particle with appropriate parameters. Later in 2009 Ted Jacobson and Thomas P. Sotiriou [16] applied a similar procedure to overspin a nearly extremal Kerr black hole. Though the validity of both of these results is constrained to the test particle approximation, they are very interesting examples of gedanken experiments to test cosmic censorship. We were also motivated by both Hubeny and Jacobson-Sotiriou to construct an experiment where we try to overspin a Kerr black hole by sending in integer spin test fields instead of particles [44].

In this chapter we first review previous attempts to overspin and overcharge black holes with test bodies. Next we present our analysis involving test fields.

4.1. Overcharging a Nearly Extremal Black Hole

This section consists of a review of Hubeny's work where she starts with a black hole arbitrarily close to, but not exactly at, extremality and presents a mechanism in which a black hole does "capture" a particle such that the parameters of the final configuration correspond to a naked singularity. Though the "test" process, i.e. neglecting back-reaction effects, can manage to make the particle fall into the singularity and overcharge the black hole, the implications about cosmic censorship cannot be easily assessed without first estimating the back-reaction effects. She also attempts to

do so in this work.

The general idea for "overcharging" a black hole is the following: We start with a near-extremal Reissner-Nordström black hole with charge Q and mass M. The goal is to send in (radially) a charged particle with charge q, rest mass m, and energy E, such that $m < E < q \ll Q < M$. First we require that the particle falls past the radial coordinate, r_+, corresponding to the initial horizon of the black hole. Then we check if it is possible for the final configuration to exceed extremality, $Q + q > M + E$. For the time being we assume that the particle may be treated as a test particle; that is, the "back-reaction" effects are negligible.

Then, after the end of such a process, the solution should approach a space-time described by Reissner-Nordström metric with $M_{\text{final}} \leq M + E < Q + q \equiv Q_{\text{final}}$, which corresponds to a naked singularity rather than a black hole. We write $M_{\text{final}} \leq M + E$ in order to include the possibility that some of the mass is carried away by gravitational radiation before the system settles down to its final state. In such a case, our process seems to "destroy the horizon of a black hole", in violation of cosmic censorship. In Chapter 2 we have seen that one cannot destroy a black hole using this process in a quasi-stationary manner, that is, by allowing the black hole mass and charge to change just by infinitesimal amounts. Namely, if one considers strictly test processes, to overcharge a black hole, one must first reach extremality. But once extremality is reached, any further changes would at best maintain it. The crucial point in this process is that the change in black hole's parameters is finite (albeit small), so that we are not constrained to starting with an extremal black hole in order to overcharge it.

We start by considering the motion of a test particle of charge q, mass m, and four-velocity u^a, in a fixed background Reissner-Nordström space-time with parameters M and Q, and an electrostatic potential A_a. In the ingoing Eddington coordinates, (v, r, θ, φ), the Reissner–Nordström metric is

$$ds^2 = -g(r)\, dv^2 + 2\, dv\, dr + r^2\, d\Omega^2 \tag{4.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2$ and $g(r) \equiv 1 - 2M/r + Q^2/r^2$. For $Q \leq M$, the metric describes a black hole with charge Q and mass M. In this case we can write $g(r) = (r - r_+)(r - r_-)/r^2$ with the event and Cauchy horizon given by $r_\pm = M \pm \sqrt{M^2 - Q^2}$. Hence $r_- \leq Q \leq M \leq r_+$. Extremality corresponds to $Q = M$, whereas for $Q > M$, we have a naked singularity. The electrostatic potential A_a has the only nonvanishing component $A_v(r) = -Q/r$.

The equation of motion for a test charge is given by

$$u^a \nabla_a u^b = \frac{q}{m} F^b{}_c\, u^c \tag{4.2}$$

with $F_{ab} \equiv 2\nabla_{[a} A_{b]}$. For radial trajectories, we can write $u^a = \dot{v}(\partial/\partial v)^a + \dot{r}(\partial/\partial r)^a$ where dot denotes $d/d\tau$ and τ is the affine parameter along the worldline. Since $(\partial/\partial v)^a$ is a Killing field, the particle's "energy", $E \equiv -(\partial/\partial v)^a\,(m u_a + q A_a) = m\,[g(r)\,\dot{v} - \dot{r}] - qA_v$ is a constant of motion along the particle's worldline. The particle follows a time-like trajectory, so that $-1 = u^a u_a = -g(r)\,\dot{v}^2 + 2\dot{v}\dot{r} = -1/m(E + qA_v)\dot{v} + \dot{v}\dot{r}$. Eliminating $\dot{v}$ from these expressions and solving for $\dot{r}$, we obtain the radial equation of motion in a particularly useful form:

$$\dot{r}^2 = \frac{1}{m^2}\left[E + qA_v(r)\right]^2 - g(r). \tag{4.3}$$

Since $\dot{r} = 0$ corresponds to a turning point, the particle "falls in" if $\dot{r}^2$ never vanishes outside the horizon,

$$\dot{r}^2 > 0 \qquad \forall\ r \geq r_+. \tag{4.4}$$

Thus, if we find suitable parameters m, E, and q, such that Equation (4.4) is satisfied, our first requirement will hold. Since $g(r)$ is positive outside the horizon, Equations (4.3) and (4.4) imply that $\left[E + qA_v(r)\right]^2 > m^2\, g(r) > 0 \ \ \forall\ r \geq r_+$. This imposes the necessary condition on the energy: $E > -q\, A_v(r) = qQ/r$. (Note that this relation also

follows from the above definition of the energy, since $g(r)\,\dot{v} - \dot{r} > 0$ initially.) We can simplify this condition by noting that $-q\,A_v(r)$ attains its maximal value at $r = r_+$. Replacing r with r_+ above, we have

$$E > \frac{qQ}{r_+}. \tag{4.5}$$

So the condition that the the particle falls past the horizon provides a lower bound on E. We could have alternatively derived (4.5) from Needham's condition [7] for a particle that enters the black hole:

$$dM - \Omega dJ - \Phi dQ > 0 \tag{4.6}$$

where $\Omega = a/(r_+^2 + a^2)$ is the angular velocity of the hole and $\Phi = Qr_+/(r_+^2 + a^2)$ is the electrostatic potential of the horizon. dM, dJ, and dQ are the changes in the parameters of the black hole. For an incident test body $dM = E$ and $dQ = q$, so for the Reissner-Nordström case ($a = 0$), (4.6) exactly gives (4.5).

The condition that the final configuration defines a naked singularity gives an upper bound for E.

$$E < Q + q - M. \tag{4.7}$$

Thus, in order to find E satisfying both conditions, q must satisfy the inequality $qQ/r_+ < Q + q - M$, or

$$q > r_+ \left(\frac{M - Q}{r_+ - Q}\right) = \frac{r_+ - Q}{2}. \tag{4.8}$$

Note that if we start with an extremal black hole, $Q = M = r_+$, then we obtain $E > q$ from Equation (4.5), and $E < q$ from Equation (4.7), which means that we cannot overcharge an extremal black hole. On the other hand, for a non-extremal

black hole, we can always find sufficiently large q such that there exists a finite range of allowed energies E.

To address the condition the particle passes through the horizon, we must return to Equation (4.3). Since $g(r)$ is bounded and $[E + qA_v(r)]^2 > 0 \ \ \forall\, r \geq r_+$, we can always find m such that $\dot{r}^2 > 0$. In particular, we must pick $m < (E - qQ/r)/\sqrt{g(r)} \ \forall\, r \geq r_+$. Analytically, the RHS attains its minimum at

$$\bar{r}_m = Q\left(\frac{Mq - QE}{Qq - ME}\right) > r_+, \tag{4.9}$$

where the inequality follows from Equation (4.5). Upon substitution, we find that m must satisfy the bound

$$m < Q\sqrt{\frac{-E^2 + 2(MEq)/Q - q^2}{M^2 - Q^2}} \tag{4.10}$$

Note that the positivity of the radical follows automatically from the above conditions:

$$\begin{aligned} -E^2 + 2\frac{M}{Q}Eq - q^2 &= 2Eq\left(\frac{M}{Q} - 1\right) - (q - E)^2 \\ &> (M - Q)^2\left[2r_+\frac{M - Q}{(r_+ - Q)^2} - 1\right] = 0. \end{aligned} \tag{4.11}$$

To summarize, given the background spacetime with parameters (Q, M), we can choose the parameters m, E, and q as follows:

- Pick q which satisfies Equation (4.8) with the given values of Q and M.
- Select E satisfying Equations (4.5) and (4.7) with the chosen q, Q, and M.
- Finally, choose m according to Equation (4.10) using the other parameters.

This prescription guarantees that if the particle follows the test charge equation of motion, then it falls past the horizon and "overcharges" the black hole. Also note that the parameters necessary for this process are not too restricted: q is bounded only from

below, the range of allowed values of E grows linearly with q, and m is bounded only from above.These can be seen by characterizing the black hole following in terms of a parameter ε describing its deviation from extremality (working to second order in ε):

$$M \equiv 1 \tag{4.12}$$

$$Q \equiv 1 - 2\varepsilon^2 \implies r_+ = 1 + 2\varepsilon \tag{4.13}$$

$$q > \varepsilon + \varepsilon^2 \implies q \equiv a\varepsilon \quad \text{with} \quad a > 1 \tag{4.14}$$

$$a\varepsilon - 2a\varepsilon^2 < E < a\varepsilon - 2\varepsilon^2 \implies E \equiv a\varepsilon - 2b\varepsilon^2 \quad \text{with} \quad 1 < b < a \tag{4.15}$$

$$m < \sqrt{a^2 - b^2}\varepsilon \implies m \equiv c\varepsilon \quad \text{with} \quad c < \sqrt{a^2 - b^2} \tag{4.16}$$

where we set $M \equiv 1$. All other quantities are then treated as dimensionless; and the linear relationship between E and q can be seen.

For a numerical example, consider a black hole with $Q = 0.99999$, i.e. $M - Q = 10^{-5}$. Suppose the particle has charge $q = 3 \times 10^{-3}$, mass $m = 1.8 \times 10^{-3}$, and is thrown radially inward with energy $E = 2.9897 \times 10^{-3}$. In terms of the ε-expansion notation used above, we have: $\varepsilon \approx 0.0022$, $a \approx 1.34$, $b \approx 1.03$, and $c \approx 0.805$. (As one may check explicitly, the conditions imposed in Equations (4.14)–(4.16) are all satisfied.) We then have the following results:

First, the test particle equation of motion yields $\dot{r}^2 > 0 \ \ \forall r$; in fact, $\dot{r}^2(r_m) \approx 2.36 \times 10^{-6}$, i.e. the particle is captured by the black hole.

Second, the final total mass is $M_{\text{final}} \leq M + E = 1.0029897$, whereas the final total charge is $Q_{\text{final}} = Q + q = 1.0029900$. Thus, the charge of the final configuration is greater than the mass, $M_{\text{final}} - Q_{\text{final}} \leq -3 \times 10^{-7}$. Although the process does not overcharge the black hole significantly, this was to be expected from the fact that we consider only "nearly test" processes.

Hubeny also considers back-reaction effects. There are several different meanings of back-reaction, depending on the context. First, the presence of the particle modifies

the space-time metric, so that the approximation of the particle moving on a fixed background space-time need no longer be valid. Also there exist corrections to the test particle equation of motion due to its self field effects, and the particle loses energy to electromagnetic and gravitational radiation as it falls towards the black hole. She estimates that the first order correction to the equation of motion indicates that overcharging will no longer hold, because the particle will bounce. However, since this conclusion depends crucially on the energy loss near the horizon, where the linearized approximation may not be valid, and since our further calculation suggests that increasing the initial energy of the particle may allow the particle to fall past the horizon, we can not reach a definite answer.

To summarize, Hubeny proves that it might be possible to overcharge a nearly extremal Reissner-Nordström black hole. We need a more detailed analysis of back-reaction effects to reach a conclusion whether or not cosmic censorship prevails.

4.2. Overspinning a Nearly Extremal Black Hole

Similar to Hubeny's analysis in the previous section, Jacobson and Sotiriou considered an initial Kerr black hole that is nearly but not quite extremal, and examined whether one can "jump over" the extremal limit in the test body approximation. They also ignored radiation and/or self-force effects, which might prevent the overspinning. In this section we are going to review their analysis.

In order to be able to treat the body as a test body, we shall assume its energy δE and angular momentum δJ, defined with respect to the Killing vectors of the black hole, are small compared to those of the black hole,

$$\delta E \ll M, \qquad \delta J \ll J \tag{4.17}$$

Note that the black hole must start out very close to extremal if the small perturbation caused by the body is to have any chance of pushing it over the extremal limit.

We assume that the angular momentum of the body is aligned with the spin of the black hole, in order to maximize the angular momentum added. The question, then, is whether δE and δJ can be chosen such that the body falls into the black hole, with negligible corrections to the test body approximation, such that the resulting configuration has too much angular momentum to be a black hole

$$J + \delta J > (M + \delta E)^2 \tag{4.18}$$

This sets the lower bound on the required angular momentum carried by the body to violate cosmic censorship, for a given δE:

$$\delta J > \delta J_{\min} = (M^2 - J) + 2M\delta E + \delta E^2 \tag{4.19}$$

Since we are assuming $\delta E \ll M$, it might seem that the δE^2 term may as well just be neglected at this stage, but in fact that term imposes an upper bound on δE and δJ.

Equation (4.19) yields

$$\delta J/\delta E^2 > 2M/\delta E \gg 1 \tag{4.20}$$

An upper bound on the angular momentum of the body comes from the requirement that it does indeed cross the horizon to end up in the black hole. Wald derived such a bound by analyzing the trajectories using the geodesic equations for the case of orbital angular momentum and spin respectively. JS use another path to the same results which does not actually require the geodesic equations.

If the body falls across the horizon, then the flux of energy and angular momentum into the black hole are related to the stress energy tensor of the body via

$$\delta E - \Omega_H \delta J = \int T_{ab} \chi^a d\Sigma^b. \tag{4.21}$$

Here $\Omega_H = a/2Mr_+$ is the angular velocity of the horizon, where $a = J/M$ the specific angular momentum and $r_+ = M + (M^2 - a^2)^{1/2}$ is the horizon radius in Boyer-Lindquist coordinates. The vector $\chi^a = \partial_t^a + \Omega_H \partial_\phi^a$ is the horizon-generating Killing vector, and $d\Sigma^b$ is the horizon surface element. Both χ^a and $d\Sigma^b$ are parallel to the null generator of the horizon, so the null energy condition on the matter of which the body is composed implies

$$\delta E > \Omega_H \delta J, \tag{4.22}$$

We point out that this is also Needham's condition [7] for a particle to fall through the horizon with $Q = 0$, i.e. for Kerr black holes.

The upper bound on δE sets a lower bound on δJ.

$$\delta J < \delta J_{\max} = \frac{2Mr_+}{a} \delta E. \tag{4.23}$$

This is derived requiring that the body falls across the horizon starting from some point outside, although in general it is in a bound orbit that does not come from spatial infinity.

As long as $\delta J_{\min} < \delta J_{\max}$ for some δE, there will be values of δJ satisfying both inequalities (4.19) and (4.23). If the black hole starts out extremal ($J = M^2$, so $a = M = r_+$), then $\delta J_{\min} = 2M\delta E + \delta E^2$ is never less than $\delta J_{\max} = 2M\delta E$, so it is impossible to over-spin an extremal black hole. In the sub-extremal case, however, overspinning can be achieved. To find the allowed range we make an expansion in the

small dimensionless quantity $\epsilon \ll 1$ defined by

$$J/M^2 = a/M = 1 - 2\epsilon^2 \tag{4.24}$$

(Hubeny used a similar parameter to analyze the charged case.) Also at this stage we adopt units with $M = 1$, which will keep the expressions simpler. Then we have

$$\delta J_{\rm min} = 2\epsilon^2 + 2\delta E + \delta E^2 \tag{4.25}$$

$$\delta J_{\rm max} = (2 + 4\epsilon)\delta E \tag{4.26}$$

where terms of order $O(\epsilon^2 \delta E)$ have been dropped in (4.26). The allowed range of δE lies where the difference

$$\delta J_{\rm max} - \delta J_{\rm min} = 4\epsilon\delta E - \delta E^2 - 2\epsilon^2 \tag{4.27}$$

is positive, i. e.

$$(2 - \sqrt{2})\epsilon < \delta E < (2 + \sqrt{2})\epsilon \tag{4.28}$$

We have found an allowed range of δE for a given δJ in the range $(\delta J_{\rm min}, \delta J_{\rm max})$ to overspin a nearly extremal Kerr black hole. In particular, δE must be of order ϵ, which is consistent with the requirement (4.17) that the body make only a small perturbation. For a given δE, the allowed values of δJ are near $2\delta E$, so we must have

$$\delta J \sim \delta E. \tag{4.29}$$

Note that the *width* (4.27) of the allowed range of δJ is only of order $\epsilon^2 \ll \epsilon$.

As already mentioned, the black hole must start out very nearly extremal, but now we can be somewhat more quantitative. According to (4.17) and (4.28) we must have $\epsilon \ll 1$, and $a - 1 = 2\epsilon^2$ is parametrically smaller. For example, if $\epsilon = 10^{-2}$, then

the initial black hole must have $a = 0.9998$. For a thought experiment we can imagine even smaller values of ϵ. We conclude that, if the body can be treated as a point test particle, the black hole can indeed be over-spun.

Later Barausse et. al. considered back reaction, radiative and self-force effects for this case, and concluded that conservative self-force may prevent the particle from being captured [45, 46]. They used MiSaTaQuWa equations [47, 48] for gravitational self force which are derived for a test body of mass m approximated by a black hole.

4.3. Integer Spin Test Fields and Cosmic Censorship

We have seen that, starting with a slightly sub-extremal Kerr black hole instead of an extremal one, it is possible to overspin a black hole past the extremal limit and turn it into a naked singularity by sending test bodies, if one neglects radiative and self-force effects. We were motivated by these works to construct an experiment where we try to overspin a nearly extremal Kerr black hole by sending integer spin test fields instead of particles [44]. In this work we show that:
(i) An extremal Kerr black hole can not be overspun as a result of the interaction with massless integer spin test fields (scalar, electromagnetic, or gravitational).
(ii) Overspinning can be achieved if we start with a nearly extremal black hole instead.
(iii) For the scalar field, the argument applies to more general black holes, and also allows use of a more general field configuration.
Our analysis also neglects radiative and self-force effects.

Let us rewrite our main inequality for Kerr black holes ($Q = 0$)

$$M^2 \geq a^2. \tag{4.30}$$

A space-time described by the Kerr metric corresponds to a black hole, if (4.30) is satisfied, but to a naked singularity if it is violated; in the borderline case, i.e. when (4.30) is saturated, the spacetime is said to describe an extremal, or a critical black hole. We construct our thought experiments to check if we can push the initially

nonsingular space-time satisfying (4.30) beyond the extremal limit, so that the final space-time violates (4.30), and describes a naked singularity.

4.3.1. Test Field Modes on the Kerr Metric

We consider the scattering of free test fields from a Kerr black hole. We use massless fields of spin 0,1,2 (scalar, electromagnetic and gravitational) that satisfy Teukolsky Equation (3.74) for perturbations of a Kerr black hole [39]. Our analysis also neglects radiative and self-force effects. Restricting ourselves to neutral black holes we avoid the first order interaction of the test electromagnetic field with the electromagnetic field of the black hole, so we can treat the electromagnetic field on equal footing with scalar and gravitational fields. The thought experiment consists of a packet of waves (so that the field energy is finite) of dominant frequency ω and dominant azimuthal wave number m, scattering from the black hole: Initially, at $t \to -\infty$ there is no field, just the black hole. Then the field comes in from infinity, interacts with the black hole, being partially "transmitted" and partially reflected back to infinity. The interaction changes the mass and angular momentum of the central object. As $t \to \infty$, the field decays away leaving behind (by the no-hair theorem) another Kerr space-time, but with new values of M and a.

An incident wave mode can be written in Boyer-Lindquist coordinates as

$$\Psi(r, \theta, \phi, t) = f(\theta, r) e^{im\phi} e^{-i\omega t} \tag{4.31}$$

Due to the boundary conditions the function f describes only ingoing waves at the horizon, and both ingoing and outgoing waves at infinity.

For scalar waves the stress energy tensor is

$$T_{\mu\nu} = \Psi_{,\mu}\Psi_{,\nu} - \frac{1}{2} g_{\mu\nu} \Psi_{,\alpha}\Psi^{,\alpha}. \tag{4.32}$$

Since the Kerr space-time admits two Killing vectors $\partial/\partial t$ and $\partial/\partial\phi$, the net radial flux

of energy and the net radial flux of angular momentum are given by surface integrals of $-T^r{}_t$ and $T^r{}_\phi$ respectively. Therefore for scalar waves one can show that the ratio of the net flux of angular momentum to the net flux of energy across any sphere centered at the black hole is just m/ω, by comparing $-T^r{}_t$ and $T^r{}_\phi$. Bekenstein states that the same ratio holds for electromagnetic and gravitational waves by his general argument which regards the waves as a composition of many quanta with energy $\hbar\omega$ and angular momentum $\hbar m$ far from the black hole [49]:

$$dJ = (m/\omega)dE \tag{4.33}$$

where $dE = dM$ for the black hole.

4.3.2. The Subextremal Kerr Black Hole and Cosmic Censorship

We are searching for the range of frequencies that can lead to a violation of (1.8) *for a given* δE, that is, for a given change in black hole mass. Let us recast the condition for WCCC violation in terms of angular momentum,

$$J + \delta J > (M + \delta E)^2 \tag{4.34}$$

and use the same parametrization for closeness to extremality as JS, so that we can compare with their results:

$$J/M^2 = a/M = 1 - 2\epsilon^2, \tag{4.35}$$

where $\epsilon \ll 1$ is implied. This turns (4.34) into

$$\delta J > 2\epsilon^2 M^2 + 2M\delta E + \delta E^2. \tag{4.36}$$

Since we are dealing with waves of a given angular wave number and (dominant) frequency, we can use (4.33) to convert this to

$$0 > 2\epsilon^2 M^2 + 2M\left(1 - \frac{\omega_0}{\omega}\right)\delta E + \delta E^2 \tag{4.37}$$

where have defined

$$\omega_0 = \frac{m}{2M}. \tag{4.38}$$

Conventionally, ω is taken to be positive without loss of generality. If $(\omega - \omega_0)$ is positive, we need negative δE to satisfy the inequality (4.37); i.e. the field must be *amplified* as is scatters off the black hole. This is the well-known effect of *superradiance* [50], occuring for $0 < \omega < \omega_{\rm sl}$. This amplification occurs for scalar, electromagnetic and gravitational waves, that is, for integer-spin fields, and for all three types, the limiting frequency is given by [51]

$$\omega_{\rm sl} = m\Omega = \frac{ma}{r_+^2 + a^2} = \frac{ma}{2Mr_+} = \omega_0 \frac{(1 - 2\epsilon^2)}{(1 + 2\epsilon\sqrt{1 - \epsilon^2})} \tag{4.39}$$

where Ω is the rotational frequency of the black hole, and m is positive; that is, superradiance does not occur for negative m.

Hence we must take positive m in attempting to violate CCC [for negative m, both $(\omega - \omega_0)$ and δE will be positive]. Now, to satisfy the inequality (4.37) we must have

$$4M^2\left(1 - \frac{\omega_0}{\omega}\right)^2 - 8\epsilon^2 M^2 > 0. \tag{4.40}$$

For $\omega > \omega_0$ we get

$$1 - \frac{\omega_0}{\omega} > \sqrt{2}\,\epsilon \Longrightarrow \omega > \frac{\omega_0}{1 - \sqrt{2}\,\epsilon} \equiv \omega_2 \tag{4.41}$$

On the other hand, for $\omega < \omega_0$, (4.40) gives

$$1 - \frac{\omega_0}{\omega} < -\sqrt{2}\,\epsilon \Longrightarrow \omega < \frac{\omega_0}{1+\sqrt{2}\,\epsilon} \equiv \omega_1 \tag{4.42}$$

Note that

$$\omega_{\rm sl} \leq \omega_1 \leq \omega_0 \leq \omega_2, \tag{4.43}$$

where the inequalities are saturated for $\epsilon = 0$, i.e. the extremal black hole. In fact, the differences smoothly go to 0 in that limit.

Now, (4.41) just means $\omega > \omega_2$, since $\omega_2 \geq \omega_0$. But in this range, (4.37) cannot be satisfied, since both factors in the second term of (4.37) are positive.

Similarly, (4.42) means $\omega < \omega_1$, since $\omega_1 \leq \omega_0$. Since $\omega_{\rm sl} \leq \omega_1$, we can separate this range into two. The range $0 < \omega < \omega_{\rm sl}$ represents superradiant frequencies, hence (4.37) cannot be satisfied, since both factors in the second term are negative.

However, for $\omega_{\rm sl} < \omega < \omega_1$, it *is* possible to satisfy (4.37): The first factor in the second term is negative, the second positive. For example, we may choose

$$\begin{aligned} \bar{\omega} &= \omega_0(1 - \frac{3}{2}\epsilon), \\ \delta\bar{E} &= -M\left(1 - \frac{\omega_0}{\bar{\omega}}\right) = \frac{3\epsilon}{2-3\epsilon}M \end{aligned} \tag{4.44}$$

for ω and δE, to get

$$-\frac{\epsilon^2 M^2}{(2-3\epsilon)^2}(1 + 12\epsilon - 18\epsilon^2) \tag{4.45}$$

for the right-hand-side of (4.37). Hence this case represents a violation of the weak CCC, given our assumptions.

We conclude that the only frequency range that can violate the WCCC is $\omega_{\rm sl} < \omega < \omega_1$. The fact that the interval shrinks to zero as the extreme case is approached establishes the proof that the WCCC cannot be violated by sending scalar, electromagnetic, or gravitational test fields of a certain frequency into an extremal black hole, in a limited sense generalizing [31, 32] to all integer-spin test fields.

For subextremal black holes, we can parametrize the frequency in the above interval as $\omega = \omega_0 + (s-2)\epsilon\omega_0$ where $0 < s < 2-\sqrt{2}$ and we are working to first order in ϵ. Then, if δE is chosen in the range between

$$\delta E_{1,2} = \left[(2-s) \mp \sqrt{(2-s)^2 - 2}\right] \epsilon M, \tag{4.46}$$

the right-hand-side of (4.37) will be negative. The example above (4.45) corresponds to the choices $s = 1/2$ and $\delta E = (2-s)\epsilon M$, (the central value, to first order).

It is comforting to see that δE is of order ϵM, i.e. that $\delta E \ll M$; we also have $\delta J \ll J$; hence the test-field approximation is justified.

The only apparent problem is that at the lower end of the frequency range, that is, at $\omega = \omega_{\rm sl}$, the absorbed energy δE passes through zero (writen as function of ω for given amplitude [40], Fig.1; after all, δE is positive on one side and negative on the other), so that we would not be able to choose a nonzero δE at that frequency. But the interval of the useful frequencies is an *open interval*, and we can always avoid $\omega_{\rm sl}$, as illustrated by the example above; and of course, one example is enough.

Therefore we have shown that there exists a combination ω, δJ and δE for any integer-spin test field incident on a slightly subextremal Kerr black hole, that can overspin the black hole into a naked singularity as long as the radiative and self force effects are neglected.

4.3.3. Scalar Waves, Subextremal Black Holes and Cosmic Censorship

For scalar fields, we can actually do better, removing the constraints of neutral black holes, massless fields and a single (dominant) frequency: Let us remember Equation (2.70) in Chapter 2

$$\begin{aligned}\delta(CCC) &= \frac{1}{2\pi M}\int d\omega \sum_{l,m} |f|^2[(M^2+a^2)\omega - am + eQ_eM]\bar{\omega}|B|^2 \\ &= \frac{M^2+a^2}{2\pi M}\int d\omega \sum_{l,m} |f|^2[\omega + \frac{eQ_eM - am}{M^2+a^2}] \\ &\quad \times[\omega + \frac{eQ_er_+ - am}{r_+^2+a^2}]|B|^2 \end{aligned} \tag{4.47}$$

where e is the charge of a field quantum, Q_e the electric charge of the black hole, $B_{\omega lm}$ the amplitude at the horizon for the transformed radial wavefunction $U(r) = \sqrt{r^2+a^2}R(r)$ of the wavemode ω, l, m such that it has unit ingoing amplitude at infinity, and $f_{lm}(\omega)$ the coefficient showing that mode's contribution to the wave packet. The first observation is that there are no cross-terms, i.e. each mode's contribution can be calculated independently, and subsequently added. The second observation is that only frequencies between ω_1 and ω_2 will make negative contribution to $\delta(CCC)$, where

$$\omega_1 = \frac{am - eQ_er_+}{r_+^2+a^2} \quad \text{and} \quad \omega_2 = \frac{am - eQ_eM}{M^2+a^2}. \tag{4.48}$$

In terms of the same variables, it is also possible to calculate δM and δJ:

$$\delta M = \frac{1}{2}\int d\omega \sum_{l,m} |f|^2[\omega + \frac{eQ_er_+ - am}{r_+^2+a^2}]\,\omega\,|B|^2 \tag{4.49}$$

$$\delta J = \frac{1}{2}\int d\omega \sum_{l,m} |f|^2[\omega + \frac{eQ_er_+ - am}{r_+^2+a^2}]\,m\,|B|^2 \tag{4.50}$$

Equation (4.49) allows us to identify ω_1 as the superradiance limit; consistent

with the paragraph after (4.38). Of course, the main result of [31] is that for extremal black holes, ω_1 and ω_2 coincide, hence CCC cannot be decreased from its value of zero. But now, let us consider a slightly subextremal black hole. Such a black hole will be characterized by a small positive initial value of CCC, say $\varepsilon^2 M^2$ in a parametrization similar to that of JS. Then, to turn this black hole into a naked singularity, we need to have $\delta(CCC) < -\varepsilon^2 M^2$. This could be accomplished by a wave packet with frequencies dominant between ω_1 and ω_2, say a Gaussian distribution with peak near the average of ω_1 and ω_2 *for each* m (tails may extend outside the interval, as long as the interval is dominant). Then we can estimate for typical values of $f_{lm}(\omega)$:

$$|f_{lm}|^2 \gtrsim \frac{8\pi\varepsilon^2 M^3}{(M^2+a^2)\Delta\omega\,(\omega_2-\omega_1)^2|B|^2}\alpha_{lm} \tag{4.51}$$

where $\Delta\omega$ is the width of the frequency distribution, $|B|$ the typical magnitude for $B_{\omega lm}$, and α_{lm} a factor of order unity or smaller, describing the contribution of modes labeled by l and m to $\delta(CCC)$. But, $(\omega_2-\omega_1)$ must depend on ε, considering that it vanishes in the extremal case. We find to first order in ε

$$(\omega_2-\omega_1) \approx \frac{\varepsilon M}{(M^2+a^2)^2}[2Mam - eQ_e(M^2+a^2)]. \tag{4.52}$$

Since $\Delta\omega$ must also be similar to $(\omega_2-\omega_1)$, we have $|f_{lm}|^2 \sim 1/\varepsilon$. This seems surprising at first, but one should keep in mind that the single-mode case also corresponds to $|f_{lm}|^2 \sim \delta(\omega-\omega_0)$.

These results give us estimates for contributions of same modes to δM and δJ, again to first order in ε :

$$\delta M_{lm} \gtrsim \frac{\varepsilon\pi M(am-eQ_eM)}{am-\frac{eQ_e}{2M}(M^2-a^2)}\alpha_{lm} \tag{4.53}$$

$$\delta J_{lm} \gtrsim \frac{\varepsilon\pi M(M^2+a^2)m}{am-\frac{eQ_e}{2M}(M^2-a^2)}\alpha_{lm} \tag{4.54}$$

from which we can see that $\delta M \ll M$ and $\delta J \ll J$, confirming the validity of the test field approximation. Hence we have generalized the Hubeny – JS idea [14, 16] to general

classical black holes (not just Reissner-Nordström or Kerr, but with four parameters, including magnetic charge), using tailored wave packets of general (massive, complex) scalar test fields (but no photons). We have neither assumed single-mode perturbations, nor the Bekenstein relation (4.33) initially, but were forced to use a narrow range of frequencies for each m used [the absence of cross-terms in the integral expression (2.70) allows us to use a single positive m, if we want, approximately realizing the Bekenstein case].

We have shown that the idea of starting from slightly subextremal black holes to violate WCCC seems to work also using integer-spin test fields instead of test particles; again, if radiative and self-force effects are neglected. Starting with a Kerr black hole, and massless single-frequency-dominant fields, we can present a unified treatment for all three types of fields. In addition, for scalar fields, we can start from very general assumptions about the black hole and the field; but the requirement of WCCC violation forces the field to (a) narrow frequency range(s), after all.

4.4. Superradiance, Neutrino Fields and Cosmic Censorship

In the previous section the phenomenon of superradiance played an important role in restricting the frequency interval where WCCC can be violated. This leads to a very strong prediction about the case of spin 1/2 fields.

Let us start with an extremal black hole ($M^2 = a^2$) and try to violate (4.30). For this we require

$$\frac{\delta a}{\delta M} > 1 \tag{4.55}$$

so that

$$(M + \delta M)^2 < (a + \delta a)^2 \tag{4.56}$$

Up to first order, the condition (4.55) guarantees that (4.30) is violated for extremal

black holes. We can combine (4.33) and (4.55) with $J = aM$ to find the condition to increase a further than M.

$$\frac{\delta a}{\delta M} > 1, \quad \forall \omega < \frac{m}{M + a} \tag{4.57}$$

Let us define $\omega_c \equiv m/(M + a)$. Conventionally ω is taken to be positive without loss of generality, and since we want $\delta a/\delta M > 1$, m cannot be negative or zero. $\delta a/\delta M$ increases as ω decreases so smaller frequencies seem to be better for challenging (4.30). For a black hole, that satisfies $M^2 > a^2$ by definition, the frequency of the superradiance limit ω_{sl} is smaller than the critical value ω_c; for extremal black holes these two points coincide. We may use the small parameter ϵ to clarify this fact.

$$\omega_{sl} = \omega_0 \frac{(1 - 2\epsilon^2)}{(1 + 2\epsilon\sqrt{1 - \epsilon^2})} < \frac{m}{(M + a)} = \omega_0 \frac{1}{(1 - \epsilon^2)} \tag{4.58}$$

So by means of Equation (4.57) $\delta a/\delta M > 1$ as we approach ω_{sl} from the right (and certainly for all $\omega \leq \omega_{sl}$).

For an extremal black hole interacting with a bosonic test field, the region $\omega > \omega_c$ does not challenge (4.30) since $(\delta a/\delta M) < 1$, and having δM negative and $(\delta a/\delta M) > 1$ in the superradiant region $0 < \omega < \omega_{sl} = \omega_c$, results in a negative δa with magnitude even larger than that of δM, hence reinforce (4.30), instead of violating it. It is also not possible to decrease M and increase a since δM is negative only in the superradiant region, and superradiance occurs for positive m (δa and δM have the same sign if m/ω is positive).

However, the argument does not apply to the case involving a neutrino field, since superradiance occurs only for integer spin test fields. Neutrino fields do not exhibit superradiant behaviour; δM is always positive [42]. In the absence of superradiance, the region $\omega > \omega_c$ is still not challenging, but any field with a frequency mode below the value ω_c incident on an extremal black hole leads to overspinning the black hole into a naked singularity. For subextremal black holes, the solution of (4.40) is not

bounded from below by $\omega_{\rm sl}$, so the whole range $(0, \omega_1)$ can be used to violate WCCC. The range of δE for a chosen ω just expands as ω decreases. So WCCC can be violated for subextremal *or extremal* black holes using neutrino fields for a frequency range that is much wider than for bosonic fields.

We therefore suggest that *Superradiance is the cosmic censor in the case of fields.* Because of the absence of superradiance, *neutrino fields can "generically" violate cosmic censorship conjecture.* Among all the previous attempts to violate CCC –including our analysis of integer spin test fields and subextremal black holes– the case of neutrino fields is unique since (i) it leads to violation for even extremal black holes, even in the test field approximation, (ii) the violation of the conjecture is "generic" in the sense that the thought experiment does not require field parameters to be "fine-tuned" to match the black hole's degree of closeness to extremality; e.g. the range of frequencies that give rise to violation does not tend to a set of "measure zero".

5. CONCLUSIONS

We have seen that the formation of singularities is inevitable in a gravitational collapse, and conjecturing these singularities to be hidden behind event horizons by cosmic censorship conjecture, enables us to preserve causal structure.

The status of the CCC is currently unresolved. In the absence of a concrete proof there exists many supporting and few counterexamples. We have mentioned gedanken experiments by Ford-Roman [12], and presented detailed analysis of the works of Hubeny [14], Jacobson-Sotiriou [16], and our work involving integer-spin test fields [44] that lead to the destruction of the horizon. In addition to these, some numerical studies simulating gravitational collapse indicate the formation of naked singularities. Shapiro and Teukolsky [52] performed numerical calculations simulating the collapse of gas spheroids and concluded the singularity that forms might be naked. Their result was criticized by Wald [53]. Choptuik [54] analysed the behaviour of Einstein-Klein-Gordon solutions which are just on the verge of collapsing to a black hole and found that naked singularities may occur.

However, the examples of both gedanken experiments and numerical simulations of gravitational collapse fail to be "generic", since the initial data required for these examples is so special that it would be physically impossible to achieve, considering back reaction effects [53]. In other words all initial data giving rise to violations of the behaviour specified in the conjecture is confined to a "set of measure zero". (Unfortunately there is no precise definition of what measure or topology should be imposed on the initial data to decide that.) From this point of view these examples where dynamical evolution produces naked singularities are not *fatal* to the physical content of the conjecture. For that reason our analysis of neutrino fields uniquely differs from all previous attempts to violate CCC in the sense that it leads to a "generic" violation of the conjecture.

It has also been argued [55] that the extreme black hole, which was the starting point of many gedanken experiments, can not be reached starting from a non-extreme black hole, which seems to be in accord with the claims [56, 57] that the extreme black holes are not the limit of regular black holes; in particular, that the entropy of the extreme black hole is zero, contrary to the Bekenstein-Hawking formula.

One might also argue that a theory of quantum gravity is expected to get rid of the singularities of classical general relativity irrespective of whether they are naked or not. In that respect it does not seem to matter whether CCC holds. This may sound like a legitimate argument but one should consider the fact that a quantum theory would replace a singularity by a region of extremely high, albeit finite, curvature. If the classical singularity is not hidden behind an event horizon, the quantum region of extremely high curvature will also be visible to distant observers. Physical processes taking place in this region will not be regular and its effects may propagate to distant observers. A quantum theory is not expected to restore the horizon, if it is absent in the classical theory [58]. Therefore quantum gravity does not invalidate the question of cosmic censorship.

Let us conclude with the approach of the owner of the conjecture. Penrose himself believes that [59] the assumption might be false in suitable circumtances and the consequences of possible violations of CCC are certainly worth considering. However, he does not share the view that the occurrence of naked singularities would be disastrous for physics, since we already have the big-bang singularity which we can not avoid. On the contrary he thinks that if naked singularities occur, their effects will be accessible to observation and theories of singularities can be put to test, which will be the very reverse of disaster.

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