On the Existence of Equilibrium in a Two-Class Economy with Production

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Thesis Submitted to the

Institute for Graduate Studies in Social Sciences

in partial fulfillment of the requirements for the degree of

Master of Arts

Boğaziçi University

2006

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10 August 2006

ABSTRACT

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This thesis considers a production economy which consists of distinct sets of workers and capitalists as agents. Both workers and capitalists provide labor for production of two distinct types of consumer goods, one of which is consumed by workers only and the other by capitalists only. Production of consumer goods involves a capacity constraint. Furthermore, firms are owned only by the capitalists. It is shown that there does not exist a Walrasian equilibrium for a large set of technological parameters with the above set of assumptions. First, an illustrative example is introduced and a condition which guarentees the non-existence of Walrasian equilibrium is derived in terms of technological parameters. Non-Walrasian equilibria are considered in this example for the cases in which there does not exist a Walrasian equilibrium in order to analyze the impact of non-existence result on efficiency. Subsequently, a generelization is carried out for the result of non-existence of Walrasian equilibrium. Non-existence of Walrasian equilibrium in this study is due to Arrow's corner which is a failure of lower hemicontinuity of the budget correspondence and upper hemicontinuity of the demand correspondence.

ÖZET

İki Sınıflı Bir Üretim Ekonomisinde Dengenin Varlığı Üzerine

Bu tezde, iktisadi ajanlar olarak birbirinden ayrı kümeler halinde ifade edilmiş işçi ve kapitalistlerden oluşan bir üretim ekonomisi incelenmiştir. Biri sadece işçiler, bir diğeri ise sadece kapitalistler tarafından tüketilen iki farklı tüketici malının üretimi için hem işçiler hem de kapitalistler emek sağlamaktadır. Tüketici mallarının üretiminde kapasite kısıtı bulunmaktadır. Ayrıca, firmalar sadece kapitalistlere aittir. Yukarıdaki bu varsayımlar altında, Walrasgil dengenin geniş bir teknoloji parametresi kümesinde var olmadığı gösterilmiştir. İlk olarak, açıklayıcı bir örnek sunulmuş ve Walrasgil denegenin var olmamasını garanti altına alan teknoloji parametreleri cinsinden bir koşul türetilmiştir. Walrasgil dengenin var olmamasının etkinlik üzerine olan etkilerini inceleyebilmek için Walrasgil dengenin olmadığı durumlarda Walrasgil olmayan denge bu örnekte ele alınmıştır. Ardından, Walrasgil dengenin yokluğuna dair elde edien sonuç için bir genelleştirilmeye gidilmiştir. Bu çalışmadaki Walrasgil dengenin yokluğunun sebebi, bütçe eşleşmelerinin alt-yarı süreksizliği ve talep eşleşmelerinin üst-yarı süreksizliğinde ortaya çıkan Arrow köşesidir.

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1. Introduction

Proving the existence of Walrasian equilibria under arguably general conditions is one of the most remarkable achievements of economic theory. Guaranteeing the existence of equilibria is significant as it provides a logical test of consistency for general equilibrium models involving perfect competition. Moreover, it allows stating and proving the First and the Second Welfare Theorems, which are the corner stones for arguments on the desirability of using competitive markets as an allocative mechanism.

However, competitive analysis, whenever the existence of equilibrium is guaranteed, by and large fail to explain many significant daily observations such as persistent unemployment and non-clearing product markets where firms face demand rationing. In this study, a competitive production economy with two classes of agents (named "workers" and "capitalists") is considered where the Walrasian equilibrium will not exist for a large set of economies. In those cases, the non-Walrasian equilibria will be considered for non-clearing markets. An interesting feature of the economy under consideration is that, while prices are assumed to be flexible and standard convexity assumptions are satisfied, Walrasian equilibrium still fails to exist for a large set of economies. Further, it is also shown that existence of equilibrium is compatible with less and less number of operating firms as technological progress

occurs.

Non-existence of Walrasian equilibrium is due to a property called "Arrow's corner", which is essentially a failure of lower hemicontinuity of the budget correspondence and upper hemicontinuity of the demand correspondence. Arrow's corner may arise in an economy when a condition called "interior endowments condition" is violated. This condition requires that individuals' initial endowments permit a positive amount of disposal of all goods. More specifically, each individual must own some positive amounts of all goods in the economy as initial endowments.[1]

Duffie and Sonnenschein [11] give an example for Arrow's corner leading to non-existence of Walrasian equilibrium in a very simple economy, which contains only one individual with strictly convex preferences, two goods, and no firms. Silvestre [22] also presents an example of Arrow's corner to show that violating interior endowments condition may give rise to not only non-existence of Walrasian equilibria, but also Pareto-efficient non-Walrasian equilibria.

Except the one by Silvestre [22], all economies with Arrow's corner are simple trade economies with a specific structure for preferences that is chosen exemplify how crucial interior endowments condition is for the existence of Walrasian equilib-

[1] The first example for non-existence of competitive equilibrium where the result is due to violation of interior endowments condition is given by Arrow [1]. Economies that do not posses competitive equilibria due to violation of interior endowments condition are said to face (suffer) *"Arrow's corner"*.

ria. This thesis presents another instance of Arrow's corner for a specific (and what we claim to be an interesting) set of economies. Specifically, agents in the economy are segmented into two distinct classes called workers and capitalists. Workers own only labor as initial endowment and firms that produce consumption goods that are owned by capitalists only. Production is made by transforming labor into consumption goods. It is assumed that there is a maximum level of possible production level for each consumption good, which is implied by technology parameters. That is, firms' production sets are not unbounded, which can be interpreted as involving capacity constraints. Further, workers and capitalists consume different goods. Finally, labor supply is assumed to increase with an increase in real wages. Under those assumptions it is shown that there does not exist a Walrasian equilibrium for a large set of technology parameters.

Note that, *Arrow's corner* in a production economy is closely related to the question of whether an equilibrium exists. Referring to *Arrow's corner,* Bryant [7] argues that,

> The impression (in some cases, the explicit instruction) may be that existence theorems for market equilibrium rest on quite general foundations when the opposite is arguably the case (p.1).

One of the basic arguments in Bryant [7] is that the interior endowments condition is restrictive. So, the range of circumstances under which market equilibrium can be expected to exist are potentially limited. Our study seems to support this

sceptical intuition.

Whenever the non-existence of Walrasian equilibrium is the case, disequilibrium analysis become relevant. This is so, because one would need to specify what the optimizing agents can do best when markets do not clear. Early work on disequilibrium economics, which is also referred to as non-Walrasian economics, aimed to explain non-clearance of markets by defining rigid prices exogenously. The motivation behind studying non-Walrasian economies is empirical evidence involving persistently high unemployment rates (Drazen [9], Bénassy [6]). Note, however, that Drazen [9] points out that,

> The main question, however, is not what happens when prices don't move, but why prices don't move. The effects of exogenous price rigidity are clear. What is necessary is a model of endogenous price behavior (p.286).

This thesis asserts that it is the non-existence of Walrasian equilibrium that leads to observance of non-clearing markets, even when prices are assumed to be fully flexible. There are also some other non-Walrasian models with flexible prices (see Bénassy [6]). But those models assume that agents are price-makers whereas price taking agents are assumed in this study.

In cases where we show market clearing prices will not exist, we turn to techniques of disequilibrium economics for analyzing optimizing behavior of agents. It turns out that non-clearing markets give rise to inefficiencies if wages are not suffi-

ciently high. In a simple example, which is a special case of our general model, it is shown that taxing profits and buying the consumption goods consumed only by workers with tax revenues, even if destroyed and not redistributed to workers, will lead to a Pareto-improvement.

We argue that the results obtained in this thesis on non-existence of Walrasian equilibria and properties of non-Walrasian equilibria relate to debates on theories of underconsumption. Bleaney [5] defines underconsumption as the state of general glut due to lack of demand for consumption goods. Further, Bleaney [5] provides a categorization of theories of underconsumption. In the first category, high motivation for saving is the reason for lack of demand for consumption goods. For detailed analyses of this type of underconsumption, see Kalecki [15], Baran and Sweezy [4], Goldstein [14], Sherman [21]. The second category identifies inadequate income ("poverty") of large number of consumers ("masses") as a reason for lack of demand for consumption goods. According to this approach, "capitalism" impoverishes masses. That is, the market system may lead to a situation where the demand for consumption goods is not sufficient compared to the amount supplied by the producers. The results obtained in our model seem to fit this second category of underconsumption theories. In our model, the non-existence of equilibrium is associated with over supply of consumption goods at prices that clear labor markets.

The rest of this thesis is organized as follows. Chapter 1 presents a simple

example of a production economy with distinct workers and capitalists. We demonstrate all of our results in this simple setting. Chapter 2 offers a generalization of the non-existence of equilibrium result. Chapter 3 shows how Arrow's corner obtains in the model studied by showing that workers' budget correspondences are not lower hemicontinuous and their demand correspondences are not upper hemicontiuous. Chapter 4 discusses assumptions of the study and concludes. Proofs of lemmata and propositions are given in separate appendices.

2. An Illustrative Example

In this chapter, basic results are illustrated in a very simple example. Let us introduce the economy very briefly, even though it will be defined more formally with details below. A production economy, where agents consist of two workers and two capitalists, is considered. Each firm owner owns one firm and finance her consumption by the profits of that firm. Those two firms produce two distinct goods which we will call as bread and cake. Bread is consumed by workers and cake is consumed by firm owners. Production is made by transforming labor into consumption goods. There is a maximum level of production for each consumption good, implied by technology parameters.

We will show that Walrasian equilibrium prices do not exist if labor is sufficiently productive or output maximizing level of labor is sufficiently low. Further, it will be shown that non-Walrasian equilibria can not be efficient in Pareto terms, if technology parameters guarantee non-existence of Walrasian equilibrium and wages are not sufficiently high. Moreover, it will also be shown that, assuming bread production is rationed, taxing profits and buying bread with tax revenues implies a Pareto-improvement. Interestingly, this result of Pareto-improvement, which is obtained by taxing profits, does not require that bread is redistributed to workers. Finally, we will argue that the number of firms compatible with the existence

of Walrasian equilibria decreases with an increase in productivity if productivity is sufficiently high already to assure the non-existence result.

2.1. The Model

The production economy studied below involves two consumption goods, each produced using a different type of labor. Good X is produced using labor of type 1, and Good Y is produced using labor of type 2.

There are two types of individuals. Members of the first type, the "workers", own labor only and live on wage income. The second type of individuals, the "firm owners", own firms and they derive income only from profits of the firm. Workers consume only Good X (call it "bread") and firm owners consume only Good Y (call it "cake").

There are two workers, Worker 1 and Worker 2. The workers can either consume their labor endowment as leisure or supply it to earn income that will finance their consumption. The utility function representing workers' preferences is taken as

$$u_i(x_i, l_i) = x_i - \tfrac{1}{2} l_i^2 \quad i = 1, 2 \ ,$$

where x_i denotes the quantity of Good X consumed and l_i denotes the quantity of labor supplied by Worker i.

There are two firm owners, Firm Owner 1 owns Firm 1, which produces Good X using labor of type 1. Firm Owner 2 owns Firm 2, which produces Good Y using labor of type 2. Labor of type 1 is supplied only by Worker 1 and labor of type 2 is supplied only by Worker 2. The production function for Firm j is taken to be

$$q_j\left(l_j\right)=\alpha\left(\overline{L}_j l_j-\frac{1}{2}l_j^2\right),\quad j=1,2,$$

where $\overline{L}_j>0$ and $\alpha>0$ are constants. Note that production functions are strictly increasing until $l_j=\overline{L}_j$ after which point it is strictly decreasing. Hence each firm can produce up to a certain amount. That is in a sense, they face a "capacity constraint", implied by $\overline{L}_j$. Both firms maximize profits. Both firm owners finance their consumption by the profits of the firm they own. Let $v_i\left(y_i\right)=y_i$ be the utility function for Firm Owner i, where y_i is the quantity of Good Y consumed by Firm Owner i.

Both the workers and the firm owners are assumed to be price takers. Let p_i denote the price of Good i, $i=X,Y$, and w_i denote the wage rate for labor of type i, $i=1,2$. Let $p=(p_1,p_2,w_1,w_2)$ denote the price vector, with $p_1=1$, implying that Good X is taken as the numeraire good.

The maximization problem for Worker i, $i = 1, 2$, is given by

$$\max_{x_i, l_i} u_i(x_i, l_i) = x_i - \tfrac{1}{2} l_i^2 \quad s.t. \quad l_i w_i \geq x_i. \tag{2.1}$$

The maximization problem for Firm j, j = 1,2 is given by

$$\max_{l_j} \pi_j(l_j) = p_j q_j(l_j) - w_j l_j. \tag{2.2}$$

For te sake of completeness, we define:

Definition 1 *Let* $\left(x_i^d(p), l_i^s(p)\right)$ *denote the solution of (2.1) for each Worker* i*. Further, let* $l_j^d(p)$ *denote the solution of (2.2) for each Firm* j*. Finally, call the solution to utility maximization of firm owners as* $y_i^d(p)$ *for each Firm Owner* i*. Then* p^* *is a Walrasian equilibrium price vector iff* p^* *satisfies:*

$$\begin{aligned} \textstyle\sum x_i^d(p^*) &= q_1(l_1^d(p^*)) \\ \textstyle\sum y_i^d(p^*) &= q_2(l_2^d(p^*)) \\ l_1^d(p^*) &= l_1^s(p^*) \\ l_2^d(p^*) &= l_2^s(p^*) \end{aligned} \tag{2.3}$$

2.2. Non-existence of Walrasian Equilibrium

Now, it will be shown that there does not exist Walrasian equilibrium for sufficiently high α in the simple economy defined above. The idea is as follows: market clearing employment levels are bounded with $\overline{L}_j$s. So, $\overline{L}_j$s constitute an

upper bound of workers' total income. This bounded total income of workers, which is also bread demand, needs to be equated to bread supply of Firm 1 in a Walrasian equilibrium. However, bread supply of Firm 1 is unbounded in α. Thus, bread supply will eventually be more than the upper bound of bread demand if α is high enough.

To show that is really the case, first we need to solve the above model in order to obtain Walrasian equilibria. This task requires solving for each individual's and each firm's optimization problem. But note that, one of the markets can be omitted while calculating Walrasian equilibrium prices. That is due to Walras law. The omitted market is chosen to be Good Y (cake) market. Thus, we do not need to solve for firm owners' utility maximization problems. Consider laborers' problem where (2.1) simply gives out,

$$l_i^s = w_i \tag{2.4}$$

where l_i^s is labor supply of Worker i. Firm j's profit maximization problem yields,

$$p_j \alpha \left(\overline{L}_j - l_j^d\right) - w_j = 0 \tag{2.5}$$

where l_j^d is the profit maximizing level of employment for Firm j. Note that clearing labor market requires $l_j^d = l_i^s$. Using this equality, (2.4) and (2.5), equilibrium level

of employment and wage rate for Firm 1 appears to be

$$l_1^* = w_1^* = \frac{\alpha \overline{L}_1}{1+\alpha}. \tag{2.6}$$

The market clearing condition for Good X market is

$$\begin{array}{ccc} \underbrace{w_1 l_1 + w_2 l_2} & - & \underbrace{q_1\left(l_1\right)} = 0. \\ X \text{ (bread) demand} & & X \text{ (bread) supply} \end{array} \tag{2.7}$$

Since $l_2^s = w_2$ for Worker 2 and $l_2^d < \overline{L}_2$ for Firm 2, (2.7) requires

$$\left(q_1\left(l_1\right) - w_1 l_1\right)^{1/2} < \overline{L}_2.$$

Now, plug (2.6) into the above inequality to obtain

$$\alpha \overline{L}_1 \left(A - A^2 \left(\frac{1}{2} + \frac{1}{\alpha} \right) \right)^{1/2} < \overline{L}_2$$

where $A = 1 \backslash \left(1+\alpha\right)$. This last inequality says that "bread supply is not more than bread demand whenever labor markets are clear". Since this inequality is necessary to obtain markets' clearing with optimizing behavior of agents, there will not be a Walrasian equilibrium whenever it is violated. Therefore,

$$\alpha \overline{L}_1 \left(A - A^2 \left(\frac{1}{2} + \frac{1}{\alpha} \right) \right)^{1/2} \geq \overline{L}_2 \tag{2.8}$$

suffices for the non-existence of Walrasian equilibrium. Evidently, sufficiently high α satisfies (2.8). Note that bread supply is larger than bread demand if (2.8) holds. Indeed, wages, which clear labor markets, are not sufficiently high to cover all produced bread which is supplied according to profit maximizing behavior of Firm 1 whenever (2.8) is satisfied.

2.3. Non-Walrasian Equilibrium

Above, it is shown that there does not exist a Walrasian equilibrium under some conditions. In fact, a high productivity is sufficient. But it is also necessary to show what the optimizing agents can do best whenever it is assured that there is no Walrasian equilibrium due to non-clearing markets. In other words, we need to apply non-Walrasian techniques. So, we define *D-equilibrium* due to Drèze [10], adopting it to our example with production.[2]

Definition 2 *A D-equilibrium for a given set of prices $\widehat{p}$ is defined by the quantity*

[2]For an exposition of non-Walrasian equilibrium's definition with more details, see Bénassy [6].

constraint vectors θ, δ, γ *and* φ *satisfying*

$$\begin{aligned} \sum x_i^d(\widehat{p}) &= q_1(l_1^d(\widehat{p})) \\ \sum y_i^d(\widehat{p}) &= q_2(l_2^d(\widehat{p})) \\ l_1^d(\widehat{p}) &= l_1^s(\widehat{p}) \\ l_2^d(\widehat{p}) &= l_2^s(\widehat{p}) \end{aligned} \tag{2.9}$$

where $\left(x_i^d, l_i^s\right), i = 1, 2$ *solves*

$$\max_{x_i, l_i} u_i(x_i, l_i) = x_i - \tfrac{1}{2} l_i^2 \quad s.t. \quad l_i w_i \geq x_i, \quad \theta_i \geq x_i, \quad \delta_i \geq l_i \tag{2.10}$$

and $y_i^d, i = 1, 2$ *solves*

$$\max_{y_i} v_i(y_i) = y_i \quad s.t. \quad \pi_i \geq \widehat{p}_2 y_i, \quad \gamma_i \geq y_i \tag{2.11}$$

and $l_j^d, j = 1, 2$ *solves*

$$\max_{l_j} \pi_j(l_j) = \widehat{p}_j q_j(l_j) - \widehat{w}_j l_j \quad s.t. \quad \varphi_j \geq l_j \tag{2.12}$$

Vectors of quantity constraints θ, δ, γ and φ, which are upper bounds of demand and supply, function as rationing schemes. By defining *D-equilibrium*, efficiency properties of non-Walrasian equilibria can be investigated. It is widely known that Walrasian equilibria are Pareto efficient by the First Welfare Theorem. On the other hand, efficiency properties of non-Walrasian equilibria is not straight forward. This subject is studied by Schmeidler [20], Nayak [19] for exchange economies and Silvestre

[22] for production economies. Those studies argue that non-Walrasian equilibria are generally Pareto-inefficient. The model in the above model is concordant with these observations. Proposition 2 puts forward that idea formally. But first, a helpful lemma should be introduced.

Lemma 1 *Let (2.8) hold. If labor supply of Worker 2 is not rationed then Firm 1 or Worker 1 is rationed.*

Proof. See Appendix A. ■

Proposition 2 *Let (2.8) hold. If labor supply of Worker 2 is not rationed then non-Walrasian equilibria are Pareto-inefficient.*

Proof. See Appendix B. ■

This proposition also suggests that there are also *D-equilibria* which are Pareto-efficient. Such equilibria can be obtained by setting high wages for labor of Worker 2 so that demand gap for bread is filled. But this increase in wages of Worker 2 will go along with the willing to supply labor more than Firm 2 is willing to hire. Thus, rationing schemes become active and labor supply of Worker 2 becomes rationed. On the other hand, if over-supply of bread is rationed by rationing directly supply of bread, which is in fact equivalent to rationing Firm 1's labor demand,

then equilibrium will be inefficient. Now, we will concentrate on inefficient market outcomes below.

2.3.1. Taxing profits as a remedy for inefficiencies

As it is mentioned previously, taxing profits may obtain a cure for inefficiency which is faced when there does not exist a Walrasian equilibrium. What we mean with the phrase of "a cure for inefficiency" is any Pareto-improvement implied by taxing profits. But more than that, redistributing tax revenues is not even necessary to obtain such a Pareto-improvement.

Proposition 3 *Let (2.8) hold and consider a D-equilibrium where Worker 1 and Firm 1 are rationed. Then applying lump-sum taxes on Firm 1 and buying Good X (bread) with tax revenues creates Pareto-improvement in a new D-equilibrium.*

Proof. See Appendix C. ■

2.3.2. On the number of firms compatible with the existence of Walrasian equilibria

Non-Walrasian equilibria's inefficiency is not the only point we make. In fact, it will be shown that the number of bread firms compatible with the existence of Walrasian equilibria decrease as productivity increases. In order to illustrate this idea, we need to work with larger number of firms and workers. So, Firm 1 and Worker

1 will be replicated by N. That is, number of firms which produce Good X (bread) is N and each firm employes one worker. Thus, there are N number of workers who supply labor which is used to produce Good X. Under these assumptions, it can be shown that $N \to 0$ as $\alpha \to \infty$.

Proposition 4 *Replicate Firm* 1 *and Worker* 1 *by* N. *Assume that each firm employes one worker. Then, for a sufficiently high* α, *the number of operating firms decrease as* α *increase if Walrasian equilibrium survives . In fact,* $N \to 0$ *as* $\alpha \to \infty$ *in Walrasian equilibrium.*

Proof. See the Appendix D. ■

3. Generalizing Non-Existence Results

In the previous chapter, we obtained that sufficiently high productivity implies non-existence of Walrasian equilibrium in a very simple production economy. Using this property, inefficiency and monopolization results are derived. One can reasonably argue that those results are due to the specifics of explicit preferences, production functions and/or simplicity of the example. In this chapter, we aim to extend our results in a more general setup. In order to reach this goal, we will confine ourselves to generalizing the main argument of this study: non-existence of Walrasian equilibrium. The basic approach is to keep the essence of the model but replace the explicitly specified components, such as production functions, preferences, etc., with standard assumptions of continuity and convexity.

So, there are arbitrary but finite number of workers, firm owners and firms in the below model. Set of firm owners and workers are disjoint. Production is made by transforming labor into consumption goods. It is assumed that there is a maximum level of possible production level for each consumption good which is implied by technology parameters. Level of labor that maximizes output will be referred as "labor constraints". As it is argued previously, this can be interpreted as capacity constraints. Each individual has a convex preference relation. Further, we assume capitalists and firm owners consume different goods. As initial endowments,

workers own labor but not the consumption goods that firm owners consume. Firm owners' initial endowments can be any good. That is, firm owners may have labor endowment. But firm owners necessarily own a positive profit share of some firms and workers do not have any ownership in the firms. Finally, labor supply is assumed to increase with an increase in real wages. Under those assumptions, it is shown that Walrasian equilibrium does not exist for a set of input levels that maximizes output. Further, it is shown that labor constraints, which assure the non-existence result, expands as productivity of labor increases. Moreover, the set of technology parameters, which do not permit a Walrasian equilibrium, is Lebesgue measure of infinity. By technology parameters, we mean productivity and labor constraints.

3.1. The Model

Set of individuals is $\mathcal{H} = \mathcal{W} \cup \mathcal{C}$ where $\mathcal{H} = \{1, ..., H\}$ and $\mathcal{W} \cup \mathcal{C} = \emptyset$. Consider $\mathcal{W}$ as the set of workers and $\mathcal{C}$ as the set of capitalists where workers and capitalist are distinct set of individuals. Set of goods is $\mathcal{G} = \{1, ..., n\}$ which is composed of three distinct subsets so that $\mathcal{G} = \mathcal{I} \cup \mathcal{J} \cup \mathcal{E}$ where $\mathcal{I}$ is the set of produced goods that workers consume, $\mathcal{J}$ is the set of goods that capitalists consume, and $\mathcal{E}$ is the set of endowments that are used as input only. Elements of $\mathcal{E}$ are non-produced goods and thought to be different types of labor. The first good is in $\mathcal{I}$, formally $1 \in \mathcal{I}$. The set of firms is $\mathcal{F} = \{1, ..., F\}$.

Consumption set of consumer h, $h \in \mathcal{H}$, is the nonnegative orthant $X_h = \mathbb{R}^n_+$. Each consumer h, $h \in \mathcal{H}$, has a continuous, monotonic, convex, rational preference relation $\succsim_h$ which is represented by a utility function, $u_h : X_h \to \mathbb{R}$. It is assumed that workers and capitalists consume different produced goods.

Assumption 1 Let $x, x' \in X_\omega$ satisfying $x_j \geq x'_j$ for some $j \in \mathcal{J}$ and $x_i = x'_i, \forall i \in \mathcal{I}$ where $\omega \in \mathcal{W}$. Then $u_\omega(x) = u_\omega(x')$ is the case. In a similar fashion, let $x, x' \in X_c$ satisfying $x_i \geq x'_i$ for some $i \in \mathcal{I}$ and $x_j = x'_j, \forall j \in \mathcal{J}$ where $c \in \mathcal{C}$. Then $u_c(x) = u_c(x')$ is the case.

Each consumer h, $h \in \mathcal{H}$ is endowed with an initial vector of commodities $e_h \in \mathbb{R}^n_+$, a vector $\alpha_h \in \mathbb{R}^F_+$ of profit shares $\left(\sum_{h \in H} \alpha_{hf} = 1,\ f \in \mathcal{F}\right)$. We also assume that workers neither own a profit share nor a good that capitalists consume as initial endowment.

Assumption 2 If $\omega \in \mathcal{W}$ then $\alpha_\omega = 0$ and $e_{\omega j} = 0$, $\forall j \in \mathcal{J}$ where j is the j^{th} element of endowment vector e_ω. If $c \in \mathcal{C}$ then $\alpha_{cf} > 0$ for some $f \in \mathcal{F}$ where f is the f^{th} element of profit share vector α_c. That is, capitalist necessarily own some profit share.

Firm f's production set $Y_f(\theta) \subset \mathbb{R}^n$ is strictly convex and continuous where $\theta \in \Theta = \mathbb{R}_{++}$ is the level of technology. If $y > 0$ then $y \notin Y_f$ and $0 \in Y_f$. Graph of

the output vector's boundary is denoted by $q : \mathbb{R}_+^{|\mathcal{E}|} \times \Theta \to \mathbb{R}_+^{|\mathcal{I} \cup \mathcal{J}|}$ which is nothing but the production function. The set of inputs which is used to produce good $i, i \in \mathcal{I} \cup \mathcal{J}$ is denoted by $\mathcal{E}_i \subset \mathcal{E}$. Therefore, i^{th} component of q, the production function of good i is $q_i : \mathbb{R}_+^{|\mathcal{E}_i|} \times \Theta \to \mathbb{R}_+$. Further, it is assumed that each q_i has a maximum respect to input.

Assumption 3 For all $i \in I$, there exists $\bar{l}_i \in \mathbb{R}_+^{|\mathcal{E}_i|}$ such that $q_i\left(\bar{l}_i\right) \geq q_i\left(l_i\right), \forall l_i \in \mathbb{R}_+^{|\mathcal{E}_i|}$.

This assumption can be interpreted as firms have capacity constraints. Maximizing level of input is connoted by a bar sign, i.e. $\bar{l}_i$. Let $\bar{l}_i \in \overline{L}_i = \mathbb{R}_+^{|\mathcal{E}_i|}$, where $i \in I \cup J$. Moreover, we assume q_i is a smooth differentiable function (but possibly its maximum is an exception) such that $\frac{\partial q_i(l)}{\partial l_j} > 0$ where $l < \bar{l}_i$ for all $i \in \mathcal{I} \cup \mathcal{J}$ and $j \in \mathcal{E}_i$ so that capacity constraints may give rise to kinks. Strict convexity of Y_f implies that each q_i is strictly concave which can be defined more formally as,

$$q_i\left(l\right) - q_i\left(l'\right) < \left(l - l'\right) \nabla q_i\left(l'\right) \text{ with } l, l' < \bar{l}_i$$

where l and l' are two different level of inputs and $\bar{l}_i$ is the vector of maximizing arguments of q_i for each i. Regarding the technology parameter, we have the following assumption.

Assumption 4 $\theta \in \Theta$ is a given parameter for firms and

$$\frac{\partial^2 q_i(l,\theta)}{\partial l_j \partial \theta} > 0,\ l \leq \bar{l}_i$$

for all goods where l_j is any input that is used to produce good i.

Therefore, technology is assumed to increase marginal productivity of labor. The set of prices is $P = \left\{p \in \mathbb{R}^n_+ \mid p_1 = 1\right\}$ where p_i is the price of good $i, i \in \mathcal{G}$. So, good 1 is chosen to be the mumeriare good.

Remark 1 *Price of l_j will be denoted by w_j instead of p_j to stress that it is the wage rate of input (labor) j.*

Each firm $f,\ f \in \mathcal{F}$ is assumed to solve

$$\max\ p.y,\ y \in Y_f.$$

We denote the solution as y^{f*}. Each individual h, $h \in \mathcal{H}$ is assumed to solve

$$\max\ u_h(x_h) \quad \text{s.t. } x_h \in \beta_h(p)$$

where $\beta_h(p)$ is the budget set of individual h and it is defined as,

$$\beta_h(p) = \left\{x_h \in X_h \mid p.e_h + \sum_{f \in \mathcal{F}} \alpha_{hf}.p.y^{f*} \geq p.x_h\right\}.$$

We denote the solution as x_h^* and call it consumption demand. So, all components of the economy is complete and now we are able to define Walrasian equilibrium.

Definition 3 *For the economy described above, a set of vectors,*

$\left(x_1^*, ..., x_s^*, y^{f*}, ..., y^{f*}, p^*\right)$ *is said to be a Walrasian equilibrium if and only if*

$$z\left(p^*\right) = \sum_{h \in \mathcal{H}} x_h^* - \sum_{f \in \mathcal{F}} y^{f*} - \sum_{h \in \mathcal{H}} e_h = 0.$$

Definition 4 *(Gross Substitution Property for Labor) Let $p', p'' \in P$ such that $p_i' > p_i''$ for some $i \in \mathcal{I}$ and all other elements be equal. Then*

$$\textstyle\sum_{\omega \in \mathcal{W}} x_{\omega j}^*\left(p'\right) \geq \sum_{\omega \in \mathcal{W}} x_{\omega j}^*\left(p''\right) \quad \textit{where} \quad j \in \mathcal{E}$$

must be the case for "Gross Substitution Property for Labor" to hold.

"Gross substitution property for labor"(GSL) implies that if the purchasing power of a type of labor increases then its total supply increases. That ensures standard positively sloped supply relation for labor markets in the aggregate. Note that GSL may hold even if there are some laborers whose preferences yield $x_{\omega j}\left(p'\right) < x_{hj}\left(p''\right)$ where $p', p'' \in P$ are defined as in the above definition.

Now, we can present our main result of non-existence of Walrasian equilibrium.

Proposition 5 *Let GSL hold for all $j \in \mathcal{E}$. Then, there exists $\bar{l}_i \in \overline{L}_i$ for each $i \in \mathcal{I} \cup \mathcal{J}$ for a given $\theta \in \Theta$ such that there does not exist a Walrasian equilibrium.*

Proof. See Appendix E. ■

Although, the proof of Proposition 5 is long and involved, it has a very basic intuition. Note that any $\bar{l}_i$ defines the maximum level of equilibrium employment for inputs that is used to produce good i. A smaller $\bar{l}_i$ implies lower employment which in return implies that equilibirum wage will fall and hence workers consume less of produced goods.[3] That leads to lack of demand for goods that workers consume. Equivalently, this observation can also be described as over supply of goods that workers consume. If wages increase to overcome this problem, this will lead to oversupply of labor.

Note the above proposition applies for the illustrative example studied in Chapter 2. This can be seen directly by considering (2.8) which is the condition of non-existence of Walrasian equilibrium. Indeed, taking productivity given, there is always a pair of labor constraints, $\left(\overline{L}_1, \overline{L}_2\right)$, that will ensure non-existence of Walrasian equilibrium. But note that (2.8) also implies the converse, in the sense that taking labor constraints given, there always exist a level of productivity to ensure non-existence of equilibrium. Similarly, it will be shown that set of input constraints, which implies non existence, gets larger with an increase in productivity in this general setup.

[3] Less employment implies lowering wages is a consequence of GSL.

Consider the proposition below.

Proposition 6 *Let* $\overline{L} = \times \left\{\overline{L}_i\right\}_{i \in \mathcal{I} \cup \mathcal{J}}$. *Define*

$$\begin{aligned} \mathcal{L}(\theta) = \quad & \Big\{ \bar{l} \in \overline{L} \mid \forall p \in P \ \textit{either} \sum_{i \in \mathcal{I}} z_i(p) > 0 \ \textit{or for some } j \in \mathcal{J} \\ & \exists k \in \mathcal{E}_j \ s.t \sum_{\omega \in \mathcal{W}} e_{\omega k} - \sum_{\omega \in \mathcal{W}} x_{\omega k}(p) > F\bar{l}_k, \ \theta \in \Theta \Big\} \end{aligned} \tag{3.1}$$

Then, $\mathcal{L}(\theta)$ *expands with an increase in* $\theta \in \Theta$.

Remark 2 *Note that* $\mathcal{L}(\theta)$ *is a set of input (labor) constraints that imply non-existence of Walrasian equilibrium for given* $\theta \in \Theta$.

Proof. See Appendix F. ■

This argument shows that it becomes more possible to observe non-existence of Walrasian equilibrium as technological progress occurs. Further, the definition of $\mathcal{L}(\theta)$ implies that if a vector of input (labor) constraints imply non existence of Walrasian equilibrium then a smaller vector of input (labor) constraints also imply the non-existence. That is, if $\bar{l} \in \mathcal{L}(\theta)$ then $\underline{l} \in \mathcal{L}(\theta)$ where $\bar{l} > \underline{l}$. Hence, $\mathcal{L}(\theta)$ is not a set of Lebesgue measure zero. Moreover, if a vector of input (labor) constraints imply non existence of Walrasian equilibrium, increase in productivity can not help Walrasian equilibrium to re-appear. Formally, if $\bar{l} \in \mathcal{L}(\theta')$ then $\bar{l} \in \mathcal{L}(\theta'')$ where

$\theta'' > \theta'$. Therefore, the set $\overline{\mathcal{L}}$ is measure set of infinity in $\overline{L} \times \Theta$ space where

$$\overline{\mathcal{L}} = \{\mathcal{L}(\theta) \times \theta \mid \theta \in \Theta\}.$$

4. Arrow's Corner and Discontinuity

Non-existence results derived previously are direct consequences of a discontinuity problem which is known as Arrow's Corner. Arrow's Corner is a failure of lower hemicontinuity of the budget correspondence and of upper hemicontinuity of the demand correspondence. It will be illustrated below that workers' budget and demand correspondences are subject to this kind of a failure of discontinuity.

Define the budget set of Worker i, as

$$\beta(p) = \{x \in X \mid p.e \geq p.x, \|x\| \leq c\}$$

where $X = \mathbb{R}_+^{n-1} \times \{0\}$ is the consumption space and $e \in \mathbb{R}_+^{n-1} \times \{0\}$ is the endowment vector of Worker i. Prices are elements of a unit simplex, thus, $\sum p = 1, p \in \mathbb{R}_+^n$. Finally, c is the artificial bound for the budget set such that $\|e\| < c$.

To see the failure of lower hemicontinuity of the budget correspondence, consider the sequence $p_v = \left(\frac{1}{v}, ..., 1 - \frac{n-1}{v}\right)$ which satisfies $p_v \to p_0 = (0, ..., 1)$. Thus,

$$\beta_i(p_v) = \{x \in X \mid \sum e \geq \sum x\}$$
$$\beta_i(p_0) = \{x \in X \mid \|x\| \leq c\}$$

which enable us to observe that $\beta(p_v) = \beta(p_{v+1})$ for every v which in turn implies

that $\beta(p_v)$ is independent of p_v. Call $\beta(p_v)$ as $\overline{\beta}$. Obviously, $\overline{\beta}$ is closed. Therefore, for any sequence $b_v \in \overline{\beta}$, it must be the case that $b_v \to b_0 \in \overline{\beta}$. So there is no sequence $b_v \in \overline{\beta}$, which may yield $b_v \to x^0 \in \text{cls}[\beta_i(p_0)]$ where cls$[\cdot]$ is closure of a set. That is so, because cls$[\beta_i(p_0)]$ lies outside of $\overline{\beta}$. Hence, we conclude $\beta(p)$ is not lower hemicontinuous.

Now we introduce $D(p)$, the demand correspondence of Worker i, and show that it is not upper hemicontinuous. Define,

$$D(p) = \{x \mid x \in \beta(p), x \succsim_i y \; \forall y \in \beta(p)\}$$

where $\succsim_i$ is the monotonic, continuous preference relation of Worker i. Again consider the sequence $p_v = \left(\frac{1}{v}, ..., 1 - \frac{n-1}{v}\right)$. Call $d_v = D(p_v)$. By definition, $d_v \to d_0 \in \overline{\beta}$. But note that since $\succsim_i$ is monotonic, $D(p_0) = \{c_1, ..., c_{n-1}, 0\}$ which again lies outside $\overline{\beta}$. Therefore it is true for all $d_v \in \overline{\beta}$ that $d_v \to d_0 \in \overline{\beta}$ but $d_0 \notin D(p_0)$. So we conclude that $D(p)$ is not upper hemicontinuous.

Those results implies the following: discontinuity of workers' demand correspondences can be solved by distributing profit shares of cake producing firms. Because, owning a profit share of a cake producing firm means that $c_n > 0$ which will guarantee that $D(p)$ is upper upper hemicontinuous. This is due to the fact that owning some positive amount of everything guarantees both $\beta(p)$ and $D(p)$ are

continuous.[4]

[4]For the validity of this final argument, see Arrow and Debreu [2].

5. Discussion and Concluding Remarks

This study presented a formal model where non-existence of general equilibrium prices is obtained under standard convexity assumptions and flexible prices. The economy studied consists of two classes of agents, workers and capitalists, and all agents are price takers.

From mathematical point of view, the non-existence of equilibrium prices is due to discontinuities of budget sets and demand correspondences that arise in the model studied (despite the fact that all preferences and production sets are continuous). The discontinuities involved are of the type labelled as "Arrow's corner" in the general equilibrium literature.

From economic intuition point of view, it is the existence of two separate classes of agents that lead to non-existence of equilibrium prices in the model. That is, the existence of poor (workers) and rich (capitalists) as two distinct set of agents lies behind the non-existence of Walrasian equilibrium. We conceptualize the difference between the structures of classes as emanating from the difference in terms of profit shares and initial endowments that agents in each class hold. This method of defining the class structure is how our model differs from a standard general equilibrium model. The interior endowments condition that is sufficient to avoid Arrow's corner

in a given economy and, hence, to guarantee existence of general equilibrium prices, does not allow a class structure of the type we employ in our model .

In this setup we first demostrate our main results using a simple illustrative example with explicit preferences and production technologies. We show how the non-existence of equilibrium is related to the specific features of the model employed and also demonstrate how a notion of non-Walrasian equilibrium can be employed to demonstrate emergence of non-clearing markets that give rise to inefficiencies. An interesting feature of the non-existence result we obtain is that it is conditional on the production technology being sufficiently developed.

We then proceed to show how the results obtained in the illustrative example can be generalized. We show that the non-existence result of our model still holds in a more general setup. Further, we also show that the set of technological parameters under which non-existence of equilibrium is obtained is not a small set (i.e., not a set of measure zero). In fact, this set is very large.

In one set of results we show that income distribution policies may be useful for not only achieving a more equitable distribution of resources, but also for pure efficiency purposes. The non-Walrasian equilibrium outcome that will be obtained when Walrasian equilibrium prices do not exist can be improved upon by government intervention. As such, the results we obtain also shed some light on the

Marxian/Keynesian underconsumption debate.

The assumptions that we employ in our model require some further comments. The non-existence result obtained for the Walrasian equilibrium is a consequence of several assumptions employed that do not appear in the standard literature. For instance it is assumed here that laborers and capitalists consume different goods. Casual observation will verify that assumption, especially if goods are considered as different whenever their quality types are different. In fact, corner solution (consuming none of some goods) seem to be crucial for optimization problems of individuals in real life. On this subject, Arrow [1] argues that

> If commodities are defined sharply, so that, for example, different types of bread are distinguished as different commodities it is empirically obvious that most individuals consume nothing of at least one commodity. Indeed, for any individual it is quite likely that the commodities on the market of which he consumes nothing exceed the number which he uses some degree (p. 509).

Using this observation, the distinction among workers and capitalists in terms preferences for our model is just internalizing the simple fact that rich and poor people may be consuming similar goods but those are totally distinct in terms of quality. That is, a worker and a capitalist may be identical in terms of preferences but distinction of income levels enforces totally different consumption decisions. Such a corner solution due to quality differences, which we all face in daily life, is simply obtained by assumption in our model. So, this assumption of distinct preferences for workers and firm owners can be tamed by defining goods sharply in terms of quality.

Additionally, there is also a distinction in terms of initial endowments and profit shares in our model. It is the most crucial assumption of our model. Because this distinction gives rise to the unique difference between standard equilibrium models and this study. In fact, Arrow's corner is a consequence of this assumption. Empirically, the distinction in terms of initial endowments and profit shares is an obvious fact and theoretically, it is violating "interior endowments condition" which says "all individuals own some positive amount of everything". Arrow and Debreu [2] regard the assumption of interior endowments as "clearly unrealistic," as do Arrow and Hahn [3]. Moore [18] remarks that the assumption "seems inconsistent with a developed economy and the reality of specialization and the necessity for exchange", whereas Debreu [8] characterizes the assumption as "overly strong." Geanakoplos and Polemarchakis [13] note that the assumption is "strong (but standard)." Ellickson [12] characterizes it as "very strong," a view that Mas-Colell, Whinston, and Green [16] endorse. The distinction in profit shares is studied by Matsuyama [17] in a theoretical growth model, where "a society inhabited by inherently identical households may be endogenously split into the rich bourgeoisie and the poor proletariat (p. 2)". Thus, distinction in profit shares in our model has also some theoretical basis.

We also assumed "gross substitution property" for labor markets. In general equilibrium models where existence of equilibrium is assured, neither stability nor uniqueness is guaranteed even for very simple exchange economies without the gross

substitution property.

Finally, capacity constraints in production are assumed in our model. It is a weakness of the model that capacity levels are not choice variables for firms. Nonetheless, we expect that if, in order to endogenize the capacity levels, we take the set of initial endowments as the source of capital supply the results we obtain will not be affected as long as the static nature of the model is kept. However, endogenizing the choice of capital by accumulation decisions of firms in a dynamic structure would likely to produce significantly different results. A dynamic model would potentially reveal more about the relation between Walrasian equilibrium's existence and Arrow's corner; however, such a model is beyond the scope of this study and will constitute an important subject for future research.

APPENDIX A:

In this appendix, it will be shown that if labor supply of Worker 2 is not rationed then Firm 1 or Worker 1 is rationed in a D-equilibrium whenever (2.8) holds. Suppose not. Then, there can exist a *D-equilibrium* where Firm 1, Worker 1 and Worker 2 are not rationed. Consider any simultaneous solution to (2.9), (2.10), (2.11) and (2.12) which characterizes any element of a *D-equilibria* set which must satisfy,

$$\frac{\partial \pi_1\left(l_1^*\right)}{\partial l_1} = \mu, \qquad \mu \geq 0$$

$$\frac{\partial u_1(l_1^* w_1, l_1^*)}{\partial l_1} = \tau, \quad \tau \geq 0$$

where τ and μ are Lagrange multipliers and l_1^* is the equilibrium level of employment. They also constitute an index of rationing: if $\tau = \theta = 0$ then neither Firm 1 nor Worker 1 is rationed. Further, if labor supply of Worker 2 is not rationed then

$w_2 = l_2^* < \overline{L}_2$ must be the case. Therefore, $\tau = \theta = 0$ implies that

$$q_1\left(l_1^*\right) - w_1^* l_1^* - w_2^* l_2^* = 0$$

$$\left(q_1\left(l_1^*\right) - w_1^* l_1^*\right)^2 < \overline{L}_2$$

$$\alpha \overline{L}_1 \left(A - A^2\left(\tfrac{1}{2} + \tfrac{1}{\alpha}\right)\right)^{1/2} < \overline{L}_2$$

must hold where $A = 1 \backslash (1 + \alpha)$. This contradicts the condition for the non-existence of Walrasian equilibrium, which is given by (2.8).

APPENDIX B:

In this appendix, it will be shown that some of the *D-equilibria* of the simple economy defined above are Pareto-inefficient. We will follow the same approach of Bénassy [6]. The set of Pareto-efficient allocations is given by the solution set of the problem,

$$\max \sum_i \xi_i u_i(l_i w_i, l_i) + \sum_i \Delta_i v_i(\tfrac{\pi_i}{p_2}) \qquad \text{s.t} \tag{B.1}$$

$$\begin{aligned} z_1 &= \sum x_i^d(p) - q_1(l_1^d(p)) = 0 \\ z_2 &= \sum y_i^d(p) - q_2(l_2^d(p)) = 0 \\ l_1^d(p) &= l_1^s(p) \\ l_2^d(p) &= l_2^s(p) \end{aligned}$$

where ξ_is and Δ_is are welfare weights which are non-negative by definition. Setting $l_i^d = l_i^s = l_i^*, i = 1, 2$ implies

$$\xi_i \frac{\partial u_i(l_i^* w_i, l_i^*)}{\partial l_i} + \Delta_i \frac{\partial v_i(\frac{\pi_i(l_i^*)}{p_2})}{\partial l_i} + \lambda_i \frac{\partial z_i\,(l_i^*)}{\partial l_i} + \lambda_j \frac{\partial z_j\,(l_j^*)}{\partial l_i} = 0 \tag{B.2}$$

and $j \neq i$ as the condition for (B.1)'s solution in l_i where $z_i\,(\cdot)$ is the standard excess demand function and λs are Lagrange multipliers for the constraints which are in

fact market clearing conditions. First of all, note that

$$\frac{\partial v_1(\frac{\pi_1(l_1^*)}{p_2})}{\partial l_1} = \frac{\partial u_1(l_1^* w_1, l_1^*)}{\partial l_1} = 0 \tag{B.3}$$

must be the case for any Pareto-efficient allocation. Suppose not. Then, (B.2) implies

$$\xi_1 (w_1 - l_1) + \Delta_1 \left(\left[\frac{\partial q_1}{\partial l_1} - w_1 \right] \frac{1}{p_2} \right) + \lambda_1 \left(w_1 - \frac{\partial q_1}{\partial l_1} \right) + \lambda_2 \left(\left[\frac{\partial q_1}{\partial l_1} - w_1 \right] \frac{1}{p_2} \right) = 0$$

for $i = 1$. Recall that, either $w_1 - l_1 \geq 0$ and/or $\frac{\partial q_1}{\partial l_1} - w_1 \geq 0$ is strict by Lemma 1. That gives,

$$-\lambda_1 + \frac{\lambda_2}{p_2} < 0 \tag{B.4}$$

for the Pareto efficient allocation problem's first order condition. However, for $i = 2$ and $j = 1$ (B.2) gives

$$\xi_2 (w_2 - l_2) + \Delta_2 \left(\frac{\partial q_2}{\partial l_2} - \frac{w_2}{p_2} \right) + \lambda_1 w_2 + \lambda_2 \left(-\frac{w_2}{p_2} \right) = 0.$$

This gives out

$$\lambda_1 - \frac{\lambda_2}{p_2} \leq 0$$

which contradicts $(B.4)$. So, we showed that (B.3) must be the case for Pareto-efficient allocations which can not hold whenever Walrasian equilibria do not exist

and Worker 2 is not rationed by Lemma 1.

APPENDIX C:

First let us introduce taxation scheme. There is a lump-sum tax collected from Firm 1's profits. Those tax revenues are used to buy Good X (bread) but those bread are not distributed. We assume (2.8) holds; Worker 1 and Firm 1 are rationed. Call the level of collected taxes as T. Then Firm 1's profits and market clearing condition for Good X market become,

$$\begin{gathered}\pi_1(l_1(\widehat{p})) = q_1(l_1(\widehat{p})) - \widehat{w}_1 l_1(\widehat{p}) - T \\ T + \widehat{w}_1 l_1(\widehat{p}) + \widehat{w}_2 l_1(\widehat{p}) - q_1(l_1(\widehat{p})) = 0\end{gathered} \tag{C.1}$$

respectively in a D-equilibrium. Let $T = 0$ and consider a marginal change in T such that prices do not move but adjusting rationing schemes obtain a new D-equilibrium. Since prices do not respond to a marginal change in T,

$$\begin{aligned}\tfrac{du_2}{dT} &= 0 \\ \tfrac{dv_2}{dT} &= 0 \\ \tfrac{du_1}{dT} &= \left(\widehat{w}_1 - \widehat{l_1}\right)\tfrac{dl_1}{dT} \\ \tfrac{dv_1}{dT} &= 0\end{aligned}$$

If one can show that $\frac{dl_1}{dT} > 0$, then the proof is complete since $\widehat{w}_1 - \widehat{l_1} > 0$ which is due to the assumption that Worker 1 is rationed. Applying implicit differentiation

on (C.1) yields

$$\frac{dl_1}{dT} = \frac{1}{\frac{\partial q_1}{\partial l_1} - \widehat{w}_1} = \frac{1}{\mu} > 0 \tag{C.2}$$

where $\mu > 0$ is the costate variable associated with the rationing scheme of Firm 1.

APPENDIX D:

In this appendix, it will be shown that number of operating firms (firms which are lucky to face any positive level of demand) is a decreasing function of productivity parameter α under Walrasian equilibrium when α is sufficiently high. Firm 1 and Worker 1 are replicated by N to obtain multiplicity of firms and workers. Denoting Walrasian equilibrium allocations and prices with a superscript of star, market clearing condition becomes

$$N\pi_1^* - w_2^* l_2^* = 0 \tag{D.1}$$

when Firm 1 is replicated by N. Now, we will solve for $N\pi^*$and further show that it is unbounded in α. However, $w_2^* l_2^*$ is bounded above with $\overline{L}_2^2$. This will imply $N \to 0$ as $\alpha \to \infty$ due to market clearing condition.

First, we need to obtain market clearing conditions for labor market of bread production. Thus, consider first order conditions of optimality for each firm and worker in aggregate. Those are explicitly,

$$F\alpha\left(\overline{L}_1 - l_1^d\right) = wF \quad \text{(aggregate labor demand relation)}$$

$$Nl_1^s = wN \quad \text{(aggregate labor supply relation)}$$

Solving these two relations for w_1^* and l_1^* gives,

$$\begin{aligned} w_1^* &= \tfrac{\alpha \overline{L}_1}{1+\alpha} \\ \\ l_1^* &= \tfrac{\alpha \overline{L}_1}{1+\alpha}, \end{aligned}$$

so that we conclude

$$\begin{aligned} \pi_1^* &= q_1\left(l_1^*\right) - w_1^* l_1^* \\ &= \left(\alpha \overline{L}_1\right)^2 \left(A - A^2\left(\tfrac{1}{2} + \tfrac{1}{\alpha}\right)\right) \\ A &= \tfrac{1}{1+\alpha} \end{aligned} \tag{D.2}$$

which implies $\pi_1^* \to \infty$ as $\alpha \to \infty$. However, $N\pi_1^*$ must be bounded since $N\pi_1^*$ is equal to $w_2^* l_2^*$, which is bounded above with $\overline{L}_2^2$, in Walrasian equilibrium.This can occur only if $N \to 0$ as $\alpha \to \infty$.

APPENDIX E:

The main proposition of this study, non existence of Walrasian equilibrium, is proven in this appendix. Since the proof is quite long, it is partitioned into smaller lemmata in order to obtain more tractable arguments.

Lemma 7 *Let* $f : \mathbb{R}^n \to \mathbb{R}_+$ *be a strictly concave function and* $f \in C^1$. *Define* $0 \in A \subset \mathbb{R}^n$ *as a compact set. If* $\frac{\partial f(x_0)}{\partial x_i} > 0, i = \{1, ..., n\}$ $\forall x_0 \in A$ *then*

(i) $\nexists i$ *s.t.* $\frac{\partial f(x_0)}{\partial x_i}$ *is arbitrarily small for* $x_0 \in A \subset \mathbb{R}^n$.

(ii) *If* $f(0) = 0$ *then* $f(x_0) - x_0.\nabla f(x_0) > 0, \forall x_0 \in A \subset \mathbb{R}^n$ *s.t* $f(x_0) > 0$.

(iii) *As* $f(x^v) \to 0$, $v = 1, 2, ...$ $\exists i$ *s.t* $\frac{\partial f\left(x^{v+1}\right)}{\partial x_i} > \frac{\partial f(x^v)}{\partial x_i}$.

(iv) *Let* $f(0) = 0$. *If* $\forall \epsilon > 0$ *there is a sufficiently high* v *such that* $f(x^v) - x^v.\nabla f(x^v) < \epsilon$, $v = 1, 2...$ *and* $x^v \in A$ *then* $x^v \to 0$.

Proof. (For part (i)) Since $A \subset \mathbb{R}^n$ is a compact set and $f \in C^1$, each $\partial f \backslash \partial x_i$ has a maximum and a minimum. If $\partial f(x_0) \backslash \partial x_i$ can be made arbitrarily small for $x_0 \in A \subset \mathbb{R}^n$, this implies $\exists x'$ satisfying $\partial f(x') \backslash \partial x_i = 0$, $x' \in A \subset \mathbb{R}^n$ which contradicts $\partial f(x_0) \backslash \partial x_i > 0$, $\forall x_0 \in A$. ■

Proof. (For part (ii)) Since $f : \mathbb{R}^n \to \mathbb{R}_+$ is a strictly concave function,

$$f(x_1) - f(x_0) < (x_1 - x_0) \nabla f(x_0) \tag{E.1}$$

Choose $x_1 = 0$ for (E.1) implies $f(x_0) - x_0 . \nabla f(x_0) > 0$ as desired. ■

Proof. (For part (iii)) Let $f(x^v) \to 0$, $v = 1, 2, ...$ Apply the definition of strict concavity to obtain,

$$f(x^v) - f(x^{v+1}) < (x^v - x^{v+1}) \nabla f(x^{v+1})$$

$$f(x^{v+1}) - f(x^v) < (x^{v+1} - x^v) \nabla f(x^v)$$

$$(x^v - x^{v+1}) \nabla f(x^{v+1}) > (x^v - x^{v+1}) \nabla f(x^v).$$

If $\nabla f(x^{v+1}) < \nabla f(x^v)$ then it must be the case that $(x^v - x^{v+1})$ must contain negative elements. That is, there is x_i^v as an increasing sequence. Now apply (ii) and $f(x^v) \to 0$ to obtain

$$x^{v+1} \nabla f(x^v) < \epsilon$$

where ϵ can be made arbitrarily small. Apply (i) which gives, $x^{v+1} \to 0$. But this contradicts our previous observation that there is x_i^v as an increasing sequence. ■

Proof. (For part (iv)) Since $\forall \epsilon > 0$, there is a sufficiently high v such that

$$\pi^v = f(x^v) - x^v.\nabla f(x^v) < \epsilon, v = 1, 2...$$

It must be the case that π^v converges to zero, that is $\pi^v \to \pi^0 = 0$. Further, $x^v \to x^0 \in A$. Since $x^v \in A$ where A is compact, applying part (ii) yields $x^0 = 0$. ■

Lemma 8 *Let $u_h : X_h \subset \mathbb{R}^n \to \mathbb{R}$ represent a continuous, strictly monotone and convex preference relation $\succsim_h$ for some individual h whose budget set is*

$$\beta_h(p) = \{x_h \in X_h \mid M_h \geq p.x_h\} \tag{E.2}$$

where M_h is the level of wealth of individual h. Then x^*_{hj} is unbounded if there are $i,j \in \{1, ..., n\}$ such that $\frac{p_j}{p_i} \to 0$, $x^*_{hi} > c$ where $c > 0$ is a given constant.

Proof. For notational clarity, we suppress h. Assume $\frac{p_j}{p_i} \to 0$ for some $i,j \in \{1, ..., n\}$ and $x^*_i > c$ where $c > 0$ is a given constant. Let $x^* \sim \widetilde{x} \in X$. Since x^* is utility maximizing, $p.x^* \leq p.\widetilde{x}$ is the case which is equivalent to $(p_i)^{-1} p.x^* \leq (p_i)^{-1} p.\widetilde{x}$. Thus,

$$(p_i)^{-1} \left(p_{-j}x^*_{-j} + p_j x^*_j\right) \leq (p_i)^{-1} \left(p_{-j}\widetilde{x}_{-j} + p_j \widetilde{x}_j\right) \tag{E.3}$$

where $-j$ defines the vector excluding the j^{th} element. Now choose

$$\widetilde{x}_{-j} = \left(x_1^*, ..., x_i^* - \frac{c}{2}, ..., x_n^*\right).$$

So the above inequality becomes,

$$\frac{c}{2} + \frac{p_j}{p_i} x_j^* \leq \frac{p_j}{p_i} \widetilde{x}_j \tag{E.4}$$

Note that, if x_j^* is bounded $\widetilde{x}_j$ is unbounded since $\frac{p_j}{p_i} \to 0$. Therefore, one can find another x' such that $x' \succ x^*$ and satisfying $p.x' \leq p.x^*$ using convexity and continuity of $\succsim$ whenever $\widetilde{x}_j$ is unbounded (which is due to boundedness of x_j^*). That contradicts x^* is utility maximizing. ■

Lemma 9 *Let $\forall i \in \mathcal{I}$, $\bar{l}_i$ be sufficiently large so that each firm $f's$ optimization problem yield an interior solution for the production of every good i at Walrasian equilibrium prices. Then, if GSL holds*

(i) p_i^* *is bounded above $\forall i \in \mathcal{I}$ for a given $\theta \in \Theta$,*

(ii) $\exists i \in \mathcal{I}$ *such that,* $p_i^* \left(q_i\left(l^*\right) - \nabla q_i\left(l^*\right) l^*\right) > c$
where $c \in \mathbb{R}_{++}$ is a given constant,

independent from all $\bar{l}_j s$ where $j \in \mathcal{J}$ and p^* is a Walrasian equilibrium price vector.

Proof. (For part (i)) Define,

$$\left\{ \times \left\{ \bar{l}_j^v \right\}_{j \in \mathcal{J}} \right\}_{v=1}^{\infty}$$

which is a J dimensional sequence of $\bar{l}_j$s where $j \in \mathcal{J}$. Therefore, there is a Walrasian equilibrium price set for each $\times \left\{ \bar{l}_j^v \right\}_{j \in \mathcal{J}}$ which is the v^{th} element of the labor constraint sequence. Thus, one can construct another Walrasian price vector sequence whose v^{th} element associates with the v^{th} element of the labor constraint sequence. Now, suppose there exists a subsequence

$$p^{*,v} \in P, v = 1, 2, ... \text{ such that } p_i^{*,v} \to \infty \text{ for some } i \in \mathcal{I}.$$

Due to Lemma 7, for all $i \in I$, it is true that $\nexists j \in \mathcal{E}_i$, such that $\frac{\partial q_i(l^{*,v})}{\partial l_j}$ is arbitrarily small . Thus,

$$p_i^{*,v} \to \infty, i \in \mathcal{I} \text{ implies that } p_i^{*,v} \frac{\partial q_i \left(l^{*,v} \right)}{\partial l_j} = w_j^{*,v} \to \infty, \forall j \in \mathcal{E}_i.$$

Moreover, $q_i \left(l^{*,v} \right)$ can not be arbitrarily small if $p_i^{*,v} \to \infty$. Because, if $q_i \left(l^{*,v} \right) < \epsilon$ for large v, this would imply that

$$\nabla q_i \left(l^{*,v} \right) l^{*,v} < \epsilon$$

due to the fact that

$$q_i\left(l^{*,v}\right) - \nabla q_i\left(l^{*,v}\right) l^{*,v} > 0$$

which is shown in Lemma 7. Apply result (iii) of Lemma 7 which gives

$$\exists j \in \mathcal{E}_i \text{ such that } \frac{\partial q_i(l^{*,v})}{\partial l_j}, v = 1, 2...$$

is an increasing sequence. Therefore $\| \ l^{*,v} \| < \delta$ for large v where $\delta > 0$ is arbitrarily small since $\nabla q_i\left(l^{*,v}\right) l^{*,v} < \epsilon$. Now choose

$$\arg\max_{i \in I} \left\{ \frac{p_i^{*,v}}{p_i^{*,v-1}} \right\} = k.$$

Then $\exists j \in \mathcal{E}_k$ such that

$$\frac{w_j^{*,v}}{p_i^{*,v}} \geq \frac{w_j^{*,v-1}}{p_i^{*,v-1}}, \forall i \in I$$

because $w_j^{*,v} = p_k^{*,v} \frac{\partial q_k(l^{*,v})}{\partial l_j}$. So that $l_j^{*,v}, j \in \mathcal{E}_k$ is increasing which contradicts $\| \ l^{*,v} \| < \delta$ where $\delta > 0$ is arbitrarily small.

Now we obtained the following result: $\exists k \in \mathcal{I}$ such that $p_k^{*,v} \to \infty$ where $q_k\left(l^{*,v}\right)$ is significantly away from zero. Now apply Lemma 8 to obtain that existence of such a $k \in \mathcal{I}$ contradicts that the sequence $p^{*,v} \in P, v = 1, 2, ...$ such that $p_i^{*,v} \to \infty$ for some $i \in \mathcal{I}$ constitutes a sequence of Walrasian equilibrium prices. ■

Proof. (For part (ii)) Suppose not. Note that, one can construct a $\{p^{*,v}\}_{v=1}^{\infty}$ whose v^{th} element associates with the v^{th} element of the labor constraint sequence as in the previous proof. Further $p^{*,v} \in P$ satisfies $p_i^{*,v}\left(q_i\left(l^{*,v}\right) - \nabla q_i\left(l^{*,v}\right) l^{*,v}\right) < \epsilon \ \forall \ i \in \mathcal{I}$ and $\forall \epsilon > 0$ for large v. Then either $p_i^{*,v}$ and/or $q_i\left(l^{*,v}\right) - \nabla q_i\left(l^{*,v}\right) l^{*,v}$ is arbitrarily small for all $i \in \mathcal{I}$. Since $p_1 = 1$ then it must be the case that $l_j^{*,v} \to 0, \forall j \in \mathcal{E}_1$. So, $x_j^{*,v}$ is away from zero for some $j \in \mathcal{E}_1 \subset \mathcal{E}$. Thus, $\forall i \in \mathcal{I} \setminus \{1\}$, it can not to be the case that $p_i^{*,v} \to 0$ because $p_i^{*,v} \to 0$ makes Lemma 8 applicable. To see that, apply result (i) of Lemma 7 which yields $w_j^{*,v}, j \in \mathcal{E}_1$ is away from zero. Therefore,

$$\frac{p_i^{*,v}}{w_j^{*,v}} \to 0 \text{ if } p_i^{*,v} \to 0.$$

Then $x_i^{*,v}$ would be unbounded due to Lemma 8 which contradicts market clearing. So, for all $i \in \mathcal{I}$, $l_j^{*,v} \to 0 \ \forall j \in \mathcal{E}_i$. Apply result (iii) of Lemma 7 to see that $\forall i \in I, \exists j \in \mathcal{E}_i$ such that marginal productivity of labor is increasing in v. Now choose

$$\arg\max_{i \in I} \left\{ \frac{p_i^{*,v}}{p_i^{*,v-1}} \right\} = k.$$

Then $\exists j \in \mathcal{E}_k$ such that

$$\frac{w_j^{*,v}}{p_i^{*,v}} \geq \frac{w_j^{*,v-1}}{p_i^{*,v-1}}, \forall i \in I$$

because $w_j^{*,v} = p_k^{*,v} \frac{\partial q_k(l^{*,v})}{\partial l_j}$. So, $l_j^{*,v}, j \in \mathcal{E}_k$ is increasing for some j due to GSL. This

contradicts , $l_j^{*,v} \to 0$ for all $i \in I$ and $\forall j \in \mathcal{E}_i$. ■

Proposition 10 *Let GSL hold for all $j \in \mathcal{E}$. Then, there exists $\bar{l}_i \in \overline{L}_i$ for each $i \in \mathcal{I} \cup \mathcal{J}$ for a given $\theta \in \Theta$ such that there does not exist a Walrasian equilibrium.*

Proof. Since good $i, i \in \mathcal{I}$ is only consumed by workers, market clearing requires

$$\begin{aligned} \sum_{f\in\mathcal{F}}\sum_{i\in\mathcal{I}} p_i^* y_i^{f*} + \sum_{i\in\mathcal{I}}\sum_{h\in\mathcal{H}} p_i^* e_{hi} &\leq \sum_{j\in\mathcal{G}}\sum_{\omega\in\mathcal{W}} p_j e_{\omega j} - \sum_{j\in\mathcal{E}}\sum_{\omega\in\mathcal{W}} p_j x_{\omega j} \\ &\leq \sum_{j\in\mathcal{I}}\sum_{\omega\in\mathcal{W}} p_j e_{\omega j} + \sum_{j\in\mathcal{E}}\sum_{\omega\in\mathcal{W}} p_j e_{\omega j} \qquad \text{(E.5)} \\ &\quad - \sum_{j\in\mathcal{E}}\sum_{\omega\in\mathcal{W}} p_j x_{\omega j} \end{aligned}$$

which implies

$$\begin{aligned} \textstyle\sum_{f\in\mathcal{F}}\sum_{i\in\mathcal{I}} p_i^* y_i^{f*} &\leq \textstyle\sum_{j\in\mathcal{E}} w_j^* l_j^* \\ \textstyle\sum_{f\in\mathcal{F}}\sum_{i\in\mathcal{I}} \left(p_i^* y_i^{f*} - \sum_{j\in\mathcal{E}_i} w_j^* l_j^*\right) &\leq \textstyle\sum_{j\in\mathcal{J}}\sum_{k\in\mathcal{E}_j} w_k^* l_k^* \qquad \text{(E.6)} \\ &\leq F.\textstyle\sum_{j\in\mathcal{J}} w^* \bar{l}_j \end{aligned}$$

where w^* is the vector of equilibrium wages. Result (ii) of Lemma 9 implies that left hand side of (E.6) can not be arbitrarily small if $\bar{l}_i$ is sufficiently large $\forall i \in \mathcal{I}$. Now, choose $\bar{l}_j, \forall j \in \mathcal{J}$ arbitrarily small such that $\exists k \in \mathcal{E}$ where w_k^* is unbounded

above. But result (i) of Lemma 9 implies l_k^* must increase for such k due to GSL. This contradicts $\bar{l}_j, \forall j \in \mathcal{J}$ is small. That is, excess supply of labor is inevitable for a set of sufficiently low $\bar{l}_j$. ■

APPENDIX F:

First of all, observe that $\mathcal{L}(\theta)$, which is defined in Proposition 6, is non-empty due to Proposition 5. Now, let $\theta'' > \theta'$ and $\bar{l}_{\theta'} \in \mathcal{L}(\theta')$. We need to show $\mathcal{L}(\theta') \subset \mathcal{L}(\theta'')$. For a given $p \in P$, $\sum_{i \in I} z_i(p)$ will only increase if θ increases. To see that, apply implicit function theorem to first order conditions of each firm $i, i \in I$. That yields,

$$\frac{\partial l_j}{\partial \theta} = -\frac{\frac{\partial^2 q_i(l)}{\partial l_j \partial \theta}}{\frac{\partial^2 q_i}{\partial l_j^2}} > 0$$

which implies that total production of each good i increases with an increase in θ. Further consumption plans of individuals would not alter with given prices. So, if $\sum_{i \in I} z_i(p) > 0$ then an increase in θ will not help. Now, we only need to show

$$\sum_{\omega \in \mathcal{W}} e_{\omega k} - \sum_{\omega \in \mathcal{W}} x_{\omega k}(p) > F\bar{l}_k$$

will not change its sign when θ increases. In fact, above inequality is not a function of θ but only a function of prices, endowments and preferences. Thus, $z_k(p) > 0$, which is implied by the above inequality, will not change with an increase in θ. So $\mathcal{L}(\theta') \subset \mathcal{L}(\theta'')$. Obviously $\mathcal{L}(\theta')$ is a proper subset of $\mathcal{L}(\theta'')$.

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