EXAMINING ALGEBRAIC THINKING OF SIXTH-GRADE STUDENTS IN REALISTIC MATHEMATICS EDUCATION-BASED LESSONS ENRICHED WITH THREE-ACT MATH TASKS

by

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ABSTRACT

EXAMINING ALGEBRAIC THINKING OF SIXTH-GRADE STUDENTS IN REALISTIC MATHEMATICS EDUCATION-BASED LESSONS ENRICHED WITH THREE-ACT MATH TASKS

This embedded mixed method study aims to examine the algebraic thinking of sixth-grade students in Realistic Mathematics Education (RME)-based lessons enriched with three-act math tasks and to reveal changes in students' algebra achievement and attitudes toward mathematics. The sixth-grade algebraic expressions topic in the 2018 Ministry of National Education (MoNE) curriculum was taught across five lessons by combining MoNE's objectives with Kaput's (2008) three content strands of algebra: generalized arithmetic, functional thinking, and modeling. The study, involving 46 sixth-grade students, was conducted in a public school during the spring semester of the 2022-2023 academic year. Quantitative data for the study were gathered using the Achievement Test and the Attitude Towards Mathematics Scale, while qualitative data were obtained from in-class discussions, worksheets, observations, and interviews. Qualitative data were analyzed using content analysis to examine students' algebraic thinking, while quantitative data were analyzed with the Paired Samples t-test and Wilcoxon Signed-Rank test to investigate any significant differences in students' algebra achievement and attitudes toward mathematics before and after the lessons. According to the results, the students' post-test achievement and attitude towards mathematics scores were significantly higher than the pre-test, and findings reveal how students' algebraic thinking was during and at the end of the lessons. Moreover, the study showed that these lessons enhance students' learning by sparking curiosity, emphasizing key concepts in the topic, and supporting an active learning environment.

ÖZET

ÜÇ PERDELİ MATEMATİK GÖREVLERİYLE ZENGİNLEŞTİRİLMİŞ GERÇEKÇİ MATEMATİK EĞİTİMİ TABANLI DERSLERDE ALTINCI SINIF ÖĞRENCİLERİNİN CEBİRSEL DÜŞÜNMELERİNİN İNCELENMESİ

Bu gömülü karma yöntemli araştırma, üç perdelik matematik görevleriyle zenginleştirilmiş Gerçekçi Matematik Eğitimi (GME) tabanlı derslerde altıncı sınıf öğrencilerinin cebirsel düşünmelerini incelemeyi ve öğrencilerin başarılarında ve matematiğe yönelik tutumlarındaki değişiklikleri ortaya koymayı amaçlamaktadır. 2018 Milli Eğitim Bakanlığı (MEB) müfredatındaki altıncı sınıf cebirsel ifadeler konusu, MEB'in hedefleri ile Kaput'un (2008) genelleştirilmiş aritmetik, fonksiyonel düşünme ve modelleme olan cebirin üç içerik alanı birleştirilerek, beş ders boyunca işlendi.46 altıncı sınıf öğrencisini kapsayan çalışma, 2022-2023 eğitim öğretim yılının bahar döneminde bir devlet okulunda yürütülmüştür. Nicel veriler, Başarı Testi ve Matematiğe Yönelik Tutum Ölçeği kullanılarak toplanırken, nitel veriler sınıf içi tartışmalar, çalışma kağıtları, gözlemler ve görüşmeler yoluyla elde edilmiştir. Nitel veriler, öğrencilerin cebirsel düşünmelerini incelemek için içerik analiziyle analiz edilirken, nicel veriler, derslerden önce ve sonra öğrencilerin cebir başarısı ve matematiğe yönelik tutumlarında anlamlı farklılıklar olup olmadığını araştırmak için Eşleştirilmiş Örneklem t-testi ve Wilcoxon İşaretli Sıra testi ile analiz edildi. Sonuçlara göre, öğrencilerin son test başarısı ve matematiğe yönelik tutum puanları, ön testlerden anlamlı derecede yüksektir ve bulgular, öğrencilerin cebirsel düşünmelerini dersler sırasında ve sonunda ortaya koymaktadır. Ayrıca araştırma, bu derslerin öğrencilerin öğrenmelerini, merak uyandırarak, konu içindeki temel kavramları vurgulayarak ve aktif öğrenme ortamını destekleyerek geliştirdiğini göstermiştir.

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LIST OF SYMBOLS

$=$	Equal sign
$<$	Less than
$>$	Greater than

LIST OF ACRONYMS/ABBREVIATIONS

CCSI	Common Core State Standards Initiative
G1	Group 1
G2	Group 2
G3	Group 3
G4	Group 4
G5	Group 5
G6	Group 6
G7	Group 7
G8	Group 8
H1	High Level Student 1
H2	High Level Student 2
L1	Low Level Student 1
L2	Low Level Student 2
M1	Medium Level Student 1
M2	Medium Level Student 2
MoNE	Ministry of National Education
NCTM	National Council of Teachers of Mathematics
RME	Realistic Mathematics Education
S	Student
T	Teacher

1. INTRODUCTION

Mathematics is a way of thinking, a language that allows us to articulate our world, and most importantly, a tool to solve both practical and abstract problems in our daily lives (NCTM, 2004). Real-world situations can become meaningful and connected with the help of mathematics. When students engage with mathematics through the analysis and interpretation of the world, they can see math as a tool to understand reality, making the subjects more accessible and no longer viewing it as an abstract concept (Boaler, 1993).

Algebra is an essential mathematical area that allows this engagement. It serves a similar function to writing and reading in the industrial age, and it is essential to understanding science, business, statistics, and technology, thus playing an important role in academic and career progression. (Schoenfeld, 1995). Algebra satisfies this passage with many skills, including generalization, abstraction, problem-solving, and representation (Kaput, 1999; Kriegler, 2008). It brings a way of solving real-life problems logically (Anthony *et al.*, 2012) and thinking critically (Ballew, 1967). In this respect, teaching algebra not only prepares students for higher-level mathematics but also prepares them for real life. Therefore, algebra should be taught by connecting it to real life rather than a set of procedures and rules (Kaput, 1999).

Algebra education, which is connected to real life, also meets the main goals of mathematics education, as explained in the Turkish mathematics curriculum (MoNE, 2018), which includes the ability of students to develop positive attitudes toward mathematics and to solve real-life problems by associating them with mathematics. Students' reach and use of knowledge is another main goal of mathematics education. At the same time, students should be able to communicate with each other during the problem-solving process and share their mathematical ideas (MoNE, 2018). In this

sense, educational approaches that relate algebra to real life rather than to a set of procedures and meet the main goals of mathematics education are required. Instructions suitable for these educational approaches should be prepared and implemented with appropriate materials. One of these approaches is Realistic Mathematics Education (RME), developed by Freudenthal (1971) in the Netherlands. RME is an educational approach that provides strong relationships between mathematics and real life, enables students to reach and use knowledge, and allows students to communicate with each other. In this regard, it can be an effective educational approach that can meet the main goals of mathematics education and increase students' attitudes and achievements in mathematics.

One of the essential components of the instructions based on this educational approach is the educational materials. RME lessons, which start with problems related to real life, should be enriched with appropriate materials. In this regard, three-act math tasks, developed by Meyer (2010), can be effective educational material used in RME lessons. Thus, students' attitudes toward mathematics can develop positively, and their achievement can increase. Moreover, lessons based on RME that include three-act math tasks can be enhanced by incorporating algebraic thinking components, which can assist students in utilizing and showcasing their algebraic thinking, thereby playing a vital role in students' thinking, academic performance, and attitude within a realistic learning setting. In this context, this study aims to examine the students' algebraic thinking in RME-based lessons enriched with three-act math tasks and reveal changes in students' algebra achievement and attitudes toward mathematics before and after the lessons.

2. LITERATURE REVIEW

2.1. Algebra

Algebra is a fundamental area of mathematics that plays a significant role in solving various real-life problems through logical solutions (Anthony *et al.*, 2012) and improving critical thinking (Ballew, 1967). It is also an essential learning area in mathematics education since "Algebra is a way of thinking and a set of concepts and skills that enable students to generalize, model, and analyze mathematical situations. Algebra provides a systematic way to investigate relationships; help to describe, organize and understand the world" (NCTM, 2000, p. 1). The significance of algebra as an essential area of mathematics and mathematics education has led researchers to study algebra and offer a wide range of definitions and interpretations (Bell, 1996; Howe, 2005; Kaput, 2008; Malisani and Spagnolo, 2009; Smith and Philips, 2000; Steele, 2005; Tabach and Friedlander, 2008; Usiskin, 1988). Understanding algebra's various definitions and interpretations is essential for gaining a broader perspective on its significance and effectively designing mathematics lessons that support students' learning and thinking.

Algebra has been seen as a way in order to generalize mathematical notions, which include numbers, quantities, relations, and functions (Kaput, 2008), and a way that facilitates the explanation of generalizations, relations, and procedures and allows the interpretation of real-world problems mathematically through equations and using unknowns and it is also a concept that can emerge in different fields of mathematics with its symbolism, concepts, and methods (Bell, 1996). Therefore, different perspectives emerged from the literature related to algebra. For instance, some researchers defined algebra as a symbolic relation of arithmetic and arithmetic patterns, using variables to perform arithmetic operations and create new mathematical structures representing abstract concepts (Howe, 2005; Tabach and Friedlander, 2008). Along with this, some researchers define algebra as a mathematical language with a unique structure that

supports the development of problem-solving, calculation, reasoning, and interpretation skills related to abstract models, generalization, and reasoning (Steele, 2005; Usiskin, 1988).

Moreover, algebra can be defined as the study of relationships between covariating quantities (Smith and Phillips, 2000) and the description of relationships between various quantities using symbols (Malisani and Spagnolo, 2009). Linear and nonlinear relationships can be examples of such relationships, and different types of quantity connections can be represented with tabular, graphical, and symbolic forms.

From the different definitions and statements about algebra in the literature, algebra can be defined as a way of generalizing mathematical concepts including numbers, quantities, relations, and functions (Kaput, 2008), can be symbolic relations of arithmetic and various quantities (Howe, 2005; Malisani and Spagnolo, 2009; Tabach and Friedlander, 2008), a symbolic mathematical language (Usiskin, 1988) and a way to analyze, interpret, modeled, generalize the real-world situations mathematically (Bell, 1996). Therefore, it can be concluded that algebra comprises various mathematical skills, from understanding basic arithmetic relations to dealing with higher-level mathematics. NCTM (1997) divided this broad concept into four themes: a) functions and relations, b) modeling, c) structure, and d) language and representation. Therefore, students should be able to understand patterns and relationships, make modeling to represent situations mathematically, show and understand quantitative relationships, and analyze and show the changes between different types of events (NCTM, 2000). Mathematical situations can be modeled, analyzed, and generalizable with the help of algebra, and problem situations can be solved in this way. Hence, it is crucial to give the concept of algebra the necessary importance. Besides, educators should understand how algebra changes our way of thinking. In other words, it is essential to understand the meaning of algebraic thinking; in this way, educators can design effective mathematics lessons that enhance students' way of thinking in a wide range of algebraic concepts.

2.2. Algebraic Thinking

Students can have various thinking skills to different extents. Students' ways of thinking can support their understanding of concepts and events in both mathematics and real life. They can construct better relationships between events through different ways of thinking. One of these ways is algebraic thinking, which enables students to construct more robust relationships between mathematical concepts and elevates students to a "new higher plane of thought" (Vygotsky, 1986, p. 202). In the literature, there is a distinction between algebraic thinking and algebra. Thus, this literature review section highlights this difference and the various frameworks that explain algebraic thinking.

According to many researchers, algebra is expressing relationships by using symbols; it is a symbolic relation of arithmetics, the language of mathematics, which involves engaging with unknowns, variables, patterns, and rules symbolically (Kaya, 2015; Malisani and Spagnolo, 2009; Steele, 2005; Tabach and Friedlander, 2008; Usiskin, 1988). Algebraic thinking, on the other hand, is being able to make generalizations of patterns, relations, and rules, being able to solve problems by modeling, which relies on focusing relations, thinking functionally, and understanding variables (Driscoll, 1999; Kaput, 1999; Kieran, 2004; Kriegler, 2008). Besides, algebraic thinking is associated with analyzing patterns and comprehending the relationship of variables between patterns, whereas algebra is associated with studying symbol systems (Smith, 2003). Algebraic thinking was defined by different frameworks in the literature.

Kriegler (2008) asserted that algebraic thinking and mathematical thinking are related. Mathematical thinking, which includes problem-solving and reasoning skills, can be developed with the help of algebraic thinking. A person's reasoning skills in quantitative situations and his/her algebraic ideas, which include functions and generalized arithmetic, can be developed in this way. Similarly, according to Driscoll (1999), algebraic thinking is necessary to reveal different connections between variables because it facilitates representing situations. Therefore, algebraic thinking supports

making generalizations and understanding relationships, and it is supported by certain habits of mind, which include working backward, understanding and structuring rules and patterns, and abstracting (Driscoll, 1999).

Kieran (1996) defines algebraic thinking as dealing with situations relationally by using various representations. This means that algebraic thinking is related to relational aspects of operations. According to Kieran's framework (1996), algebraic thinking includes;

- Not only focusing on the calculation of a numerical answer but also focusing on relations
- Focusing on operations and their inverses
- Problem-solving with representation and modeling
- Focusing on numbers and letters, which means working with unknowns, variables, or parameters that are represented by letters, being able to compare to equivalent expressions
- Comprehending the meaning of the equal sign.

Lins (1992) proposed a different framework for algebraic thinking, which focused on arithmetical, internal, and analytical thinking. Making arithmetical operations via numbers, modeling numbers, and solving problems by representations includes thinking arithmetically. The second characteristic, which is thinking internally, includes focusing on numerical aspects rather than associating numbers with external representations, such as geometric figures or contextualized scenarios. Thirdly, analytical thinking includes dealing with relationships, including numbers and operations. Besides, it is a feature of algebraic thinking that provides learners with the ability to analyze the given data and develop elaboration skills.

According to Van de Walle *et al.* (2011), algebraic thinking is related to generalizing arithmetic and recognizing and investigating patterns and functions. Similar points were stated by Kaput (1999) but with a different term, "algebraic reasoning", instead

of algebraic thinking. According to Kaput's (1999) framework, algebraic reasoning involves;

- Making generalizations of mathematical ideas by using arithmetic
- Making generalizations of the patterns and supporting functional thinking
- Using modeling to explain generalizations
- Extending the mathematical structures derived from the relationships

Kaput's (1999) framework gives us a valuable viewpoint on the features of algebraic reasoning. Besides, in later years, Blanton and Kaput (2005) defined algebraic thinking as a "habit of mind" that allows students to identify and express mathematical structures and relationships. According to Kaput (2008), this habit of mind is based on two main components, which are generalization and symbolization abilities, and these skills are necessary to make generalizations, represent mathematical ideas by using symbols, and solve problems (Carpenter and Levi, 2000). The symbolizing component was defined as "systematically symbolizing generalizations of regularities and constraints" (Kaput, 2008, p. 11), while the generalization component was defined as "syntactically guided reasoning and actions on generalizations expressed in conventional symbol systems" (Kaput, 2008, p. 11). According to Kaput (2008), generalization and symbolization skills comprise and take place across three content strands of algebra: generalized arithmetic, functional thinking, and modeling. The algebraic thinking components included in these content standards cannot be strictly separated from each other. Despite this, each content standard predominantly features its own unique style of algebraic thinking. For instance, the heart of generalized arithmetic is based on generalization, specifically, generalizing properties of arithmetic operations (Kaput, 2008).

Functional thinking also includes generalization but requires determining regularities and patterns in numeric relationships. Functional thinking starts with elementary patterning activities, and later in high school, it leads to understanding the concept of function. Modeling is based on the syntax of algebra, which means using unknowns or

variables and 'algebrafying' arithmetic problems (Kaput, 2008). Each of these strands was explained in the following subsections to describe algebraic thinking according to these strands.

2.2.1. Generalized Arithmetic

Radford (2012) stated that "there is something inherently arithmetic in algebra and something inherently algebraic in arithmetic" (p. 676). The generalized arithmetic strand highlights this with some essential algebraic concepts, which are numbers, operations, the equals sign, equality, inequality, equation, expression, symbols, operations, and variables (Blanton and Kaput, 2005; Blanton *et al.*, 2018; Chimoni *et al.*, 2018; Isler *et al.*, 2013; Kaput, 2008; Ralston *et al.*, 2018). These essential algebraic concepts appear in different ways throughout algebra as generalized arithmetic.

Firstly, numbers and operations can be seen in the center of the generalized arithmetic strand. Determining relationships between numbers, generalizing arithmetic operations and their properties such as associative and commutative properties of addition and multiplication, multiplication as repeated addition, and distributive property of multiplication over the addition are included in this strand (Blanton and Kaput, 2005). In that sense, Kaput (2008) stated that generalizing arithmetic operations and their properties and reasoning about this can be seen as the heart of the generalized arithmetic strand.

Students not only generalize properties of operations or string of operations (for example, x+y-y=x) but also generalize particular number properties like even and odd numbers (Bastable and Schifter, 2008; Blanton *et al.*, 2011). For example, students can generalize that the sum of two odd numbers is even, the sum of two even numbers is even, and the sum of odd and even numbers is odd with reasoning (Blanton and Kaput, 2005; Isler *et al.*, 2013).

Composing and decomposing quantities to facilitate computation and later simplifying algebraic expressions is another critical mathematical practice in the generalized arithmetic strand. Students first think deeply about computational strategies at primary grade levels and use these strategies. In later grades, students move on to simplifying algebraic expressions (Blanton *et al.*, 2011). All of these experiences are crucial for building a foundation for effectively using and working variables (Blanton *et al.*, 2011). Thus, the generalized arithmetic strand also comprises "working with strings of letters and or other symbols according to specified rules" (Kaput, 2008, p.15) and operating with symbols and unknown quantities (Chimoni *et al.*, 2018). Properties of arithmetic operations are used in the same manner in algebra. When variables are included, students can also understand some concepts like associativity and commutativity (Ralston *et al.*, 2018). These allow students to use operation properties to transform algebraic expressions (Britt and Irwin, 2011). Thus, students can work with letters, use variables, model situations by algebraic expressions, and get equivalent expressions (Blanton *et al.*, 2011). In other words, this strand comprises "building the syntactic aspect of algebra from the structure of arithmetic-building the basic idea that one can replace one expression by an equivalent one" (Kaput, 2008, p.12). For instance, students can work with variables based on arithmetic generalizations, represent these generalizations through symbols (Blanton *et al.*, 2019), and they can represent unknown or generalized quantities by using variables (Blanton *et al.*, 2018; Kieran *et al.*, 2016). Therefore, activities that include letters and symbols play a role in the development of students' algebraic thinking (Kieran, 2004). Kieran *et al.* (2016) extend this as follows: "In early algebra, generalized arithmetic not only includes number/quantity, operations, properties, equality, and related representations and diagrams, but also can include variables, expressions, and equations-depending on whether or not alphanumeric symbols have been integrated into the learning environment" (p.13). In that way, building equivalent expressions and looking at arithmetic expressions in terms of form rather than value become important with this strand (Kaput, 2008; Radford, 2022).

The meaning of a variable is also a part of the generalized arithmetic strand (Chimoni *et al.*, 2018). Blanton *et al.* (2011) mentioned variables as "versatile tools that are used to describe mathematical ideas in succinct ways" (p.32), and there are different meanings of variables (Lloyd and Herbel-Eisenmann, 2011). Variables can represent numbers in a generalized pattern, fixed but unknown numbers, varying quantities, parameters, and abstract placeholders in the algebraic process (Blanton *et al.*, 2011; Lloyd and Herbel-Eisenmann, 2011) but in every situation, "a variable represents the measure or amount of an object, not the object itself". (Blanton *et al.*, 2011, p.36). On the other hand, some students can incorrectly grasp the meaning of variables like objects or letters (Brizuela *et al.*, 2015). Then again, as an example, a variable cannot present the apple itself but can represent the number of apples, the weight, or the cost of apples (Blanton *et al.*, 2011).

Along with the variable, equal sign is another important concept in this strand (Asquit *et al.*,2007; Falkner *et al.*,1999). There are two meanings of equal sign: operational and relational meaning (Mirin, 2020). From the operational perspective, an equal sign represents the result of an operation (McNeil *et al.*, 2006). On the other hand, from a relational perspective, the equal sign does not merely state results; it states an equal relationship between two sides of the equation (Jones *et al.*, 2012). Use of such expressions, a+b=c, and such equations 2+3=5, 5+7=? support the operational meaning of equal signs; however, emphasizing intensively operational meaning in primary grade levels can prevent understanding relational meaning in later grades (McNeil *et al.*, 2006). Students who do not grasp the relational meaning of the equal sign and interpret it as "the answer comes next" find it hard to solve equations like 3x+5=x+1 (Kieran, 1980; Stacey and MacGregor, 1997b). They have difficulties transforming algebraic expressions into equivalent ones (Blanton *et al.*, 2011). For the development of algebraic thinking, the relational meaning of an equal sign should also be emphasized, and such expressions a+b=c+d and 5+7=9+3 and balance scales by associating such expressions should be used for this acquisition. (Carpenter and Levi, 2000; Kieran, 1981; Knuth *et al.*, 2006; McNeil *et al.*, 2006; Van de Walle *et al.*, 2011).

Understanding the meaning of the equal sign and variables is essential for other practices in generalized arithmetic. Solving equations, working with missing number sentences, understanding inequalities, distinguishing mathematical expressions and equations, and identifying the value of unknown quantities within equalities and inequalities by reasoning can be given as examples of other essential understandings and practices in generalized arithmetic strands (Blanton and Kaput, 2005; Blanton *et al.*,2011; Chimoni *et al.*, 2018; Isler *et al.*,2013; Kaput, 2008; Lloyd and Herbel-Eisenmann, 2011; Ralston *et al.*, 2018).

2.2.2. Functional Thinking

Functional thinking is an essential content in algebraic thinking (Stephens *et al.*, 2017; Warren and Cooper, 2006) and can be described as "representational thinking that focuses on the relationship between two (or more) varying quantities, specifically the kinds of thinking that lead from specific relationships (individual incidences) to generalizations of that relationship across instances" (Smith, 2008, p.143). Functional thinking is associated with several algebraic concepts like covariance, correspondence, and change (Chimoni *et al.*, 2018). For instance, it is associated with qualitative and quantitative changes (Warren and Cooper, 2006), focusing on arithmetic in terms of change and analyzing the relationships between varying quantities (Warren *et al.*, 2006). Concepts that are in generalized arithmetic, like variables, expressions, and equations, are also included in this strand but with different interpretations (Kieran *et al.*, 2016). To be specifically mentioned, these concepts can be used to generalize and represent relationships between quantities, to reason, and to examine functional relationships (Blanton *et al.*, 2011). In this way, functional thinking is used for identifying numerical and figural patterns (Chimoni *et al.*, 2018) and representing, justifying, and generalizing the relationships by using variables (Blanton and Kaput, 2005). During this process, three important types of relationships can be observed: recursive, covariational, and correspondence (Blanton *et al.*, 2011). In recursive patterns, students focus on changes in only one variable (Stephens *et al.*, 2017). The relationship between the independent and dependent variables is ignored and not described. On the other

hand, functional thinking also requires "understanding how quantities vary in relation to each other or covary" (Blanton *et al.*, 2011, p.52). Since recursive patterns do not involve covariation between quantities, finding recursive patterns can be viewed as a first step in understanding data (Blanton and Kaput, 2004; Blanton *et al.*, 2015). On the other hand, students also need to describe co-variational relationships and correspondence rules (Blanton *et al.*, 2011; Blanton *et al.*, 2015). Covarational relationship includes examining how two quantities change together, and correspondence rule is "a correlation between two quantities expressed as a function rule". (Blanton *et al.*, 2011, p. 53). Thus, while through co-variational thinking, students can describe the variation between two quantities with respect to each other (Confrey and Smith, 1995), correspondence rules move students' thinking in a broader aspect because it is a function rule that describes the correlation between two quantities. Therefore, identifying correspondence relationships and generalizing relationships between two quantities in this way is very important for thinking functionally, and this also allows students to find the distant terms of a pattern more easily and quickly (Blanton *et al.*, 2011; Chimoni *et al.*, 2018). Thus, students' understanding of these relationships can open the door to different practices in functional thinking. Finding close, distant terms and corresponding terms of a pattern, forming a general rule to represent a relationship and using this rule to find far values, representing this relationship through tables, graphs, or equations, and describing numeric, geometric, and figural patterns can be given as examples for functional thinking (Blanton and Kaput, 2005; Blanton *et al.*, 2015; Blanton *et al.*, 2019; Chimoni *et al.*, 2018; Kieran *et al.*, 2016; Twohill, 2013).

2.2.3. Mathematical Modeling

Mathematical modeling enables a deep understanding of mathematics through real-life connections (Zbiek, 1998) and can be defined as transforming a real-world situation into a mathematical model via tables, graphs, drawings, and mathematical expressions (Blum and Ferri, 2009; Chimoni *et al.*, 2018; Kaput, 1999; Yenilmez and Yıldız, 2019). Some practices in the modeling strand can be writing mathematical expressions that are appropriate to the context, constructing mathematical models

that are appropriate to the problem, representing relationships among variables in the problem situation by using numbers and symbols (Chimoni *et al.*, 2018), describing regularities in the problem situations through various methods (table, graph, equation, verbal, arithmetical, algebraic expressions, equation), writing and interpreting an algebraic expression, determining different ways for writing an expression, representing a problem through equation (Blanton and Kaput, 2005; Blanton *et al.*, 2019), thus, analyzing and interpreting various situation in both inside and outside of the mathematics (Kaput, 2008). Thus, it can be concluded that mathematical modeling involves an extensive area of mathematics. Concerning that, Kaput (1999) stated that mathematical modeling includes the generalization of the operation properties and number system, which are included in the generalization of the arithmetic strand, meaningful use of symbols, and studying patterns and functions, which are included in the functional thinking strand. In that sense, mathematical modeling can comprise both generalized arithmetic and functional thinking strands of algebraic thinking (Blanton and Kaput, 2005; Chimoni *et al.*, 2023; Kaput, 1999). Kieran *et al.* (2016) also stated that most of the studies integrate the third strand into the first two content strands through different mathematical problems. For instance, representing a problem situation by using variables, writing algebraic expressions or equations, and transforming an algebraic expression or equation into an equivalent form are included in both modeling and generalized arithmetic strands. Similarly, examining the relationship between two quantities and describing this relationship by variables, words, tables, graphs, or function rules comprises both functional thinking and modeling since these practices require both thinking functionally and modeling the situations. Besides, using and working with variables in both generalized arithmetic and functional thinking is a part of mathematical modeling. By considering its broad extent, mathematical modeling is a powerful tool to solve problems, understand the world better, and support the development of different mathematical skills, and it reflects mathematics's general picture (Blum and Ferri, 2009).

2.3. Importance of Algebra in Mathematics Education

Algebra plays a crucial role in enhancing students' understanding of mathematics in the future, including higher-level mathematics. Therefore, learning algebra is fundamentally significant for students (Rakes *et al.*, 2010). According to Van de Walle *et al.* (2011), elementary and middle school levels are critical for developing students' understanding of algebra. Through activities such as pattern generalization and variable usage, students develop algebraic thinking from pre-kindergarten to high school. Moreover, efficient comprehension of algebra at the middle school level enables higher scores on mathematics tests, more achievement in higher mathematics, and higher attendance of students in high school (Wang and Goldschmidt, 2003).

Furthermore, according to NCTM (2000), one of the content standards of mathematics education is "algebra" since algebra teaching and improving students' algebraic thinking is crucial. NCTM (2000) proposed that every student should learn algebra and stated that they should learn the required level of algebra from kindergarten through high school (K-12).

In light of this, the National Council of Teachers of Mathematics (NCTM, 2000) standards determined the five goals for teaching algebra. These goals are "Goal 1-understand patterns, relations, and functions; Goal 2-represent and analyze mathematical situations and structures using algebraic symbols; Goal 3-use mathematical models to represent and understand quantitative relationships, and Goal 4-analyze change in various contexts" (NCTM, 2000, p.296); thus, instructional programs should aim to provide students with these skills from pre-kindergarten to 12th grade.

At the middle school level, students should be able to define algebraic expressions understand their relations and functions, use algebraic symbols as a language tool to identify mathematical expressions, interpret the relationships between these expressions, make modeling and analyze them (NCTM, 2000).

According to the Turkish education system, numbers and operations, algebra, probability, geometry and measurement, and data analysis are the main parts of mathematics education (MoNE, 2018). Therefore, algebra is among the main parts of mathematics education.

In the Turkish education system, the algebra concept is first introduced in the sixth grade. At this grade level, students are expected to understand algebraic expressions, write algebraic expressions, and transform word sentences into algebraic expressions and vice versa. Along with this, concepts like variable, term, constant term, similar term, and coefficient are expected to be learned and used. Students also calculate a value of algebraic expression for a specific numerical value of given variables. Modeling algebraic expressions is included at this grade level as well.

At the seventh-grade level, performing addition and subtraction in algebraic expressions and multiplying an algebraic expression with a number are added to objectives. Recognizing the concept of equality and first-order equations with one unknown, writing first-order equations about a given real-life context, and solving these equations and related problems are also included in the objectives. Moreover, number patterns and creating a general rule that represents a number pattern, using variables for this aim, finding a term of a pattern, and representing patterns and relationships are expected from students.

In eighth grade, students learn a wide range of algebraic concepts, including identities, linear equations, and inequalities. They are expected to make sense of algebraic expressions and identities, factor them, and understand inequalities. The curriculum also covers multiplication among algebraic expressions and modeling identities. In addition, students learn how to represent variables that have linear relationships via tables or equations and draw graphs of linear equations. They are also expected to create and interpret equations, tables, or graphs of linear equations that relate to real-life situations. Middle school algebra finishes with the study of inequalities with one unknown in Turkey.

MoNE's (2018) curriculum highlights that algebra holds a significant role at various grade levels. As suggested by Van de Walle *et al.* (2011), it is crucial for students to acquire the necessary level of proficiency in algebraic concepts during their elementary and middle school years. Therefore, effective teaching-learning approaches with appropriately designed tasks are necessary to facilitate an efficient transition between these grade levels and promote the development of algebraic thinking.

2.4. Attitude Towards Mathematics

Attitude is a construct that has been studied in extensive areas, including psychology, sociology, and education, with various definitions and theoretical perspectives. According to Eagly and Chaiken (1993), an attitude refers to "a psychological tendency that is expressed by evaluating a particular entity with some degree of favor or disfavor" (p. 1). It can also be defined as a positive or negative tendency towards an object (Thurstone, 1931) and emotional and mental readiness that is shaped by experience and directs a person's behavior (Allport, 1935).

Attitude has three main components: cognitive, affective, and behavioral (Eagly and Chaiken, 1993). Specifically, the affective component refers to an individual's feelings or emotions toward an object or situation, while the cognitive component refers to an individual's beliefs and thoughts about an object or situation. The behavioral component refers to the individual's past behaviors associated with an object (Haddock and Maio, 2008).

Different factors can lead to forming attitudes, such as personal experiences, social influence, and cognitive processes, and attitude can influence behavior, cognition, and affect (Ajzen, 1991; Petty, 2018). In the educational context, the learning process is affected by emotions and expectations; attitude toward a lesson stems from emotions (Nazlıçiçek and Erktin, 2002), and it is an important construct that can impact academic performance and liking and disliking a subject. Due to these influences and factors, attitude has an important place in educational settings.

In mathematics education, students' interaction with mathematics and progress in mathematics has been examined in terms of various factors, including social, environmental, cognitive, and affective factors. It was proposed that "If research on learning and instruction is to maximize its impact on students and teachers, affective issues need to occupy a more central position in the minds of researchers" (McLeod, 1992, p. 575) and therefore, studies in mathematics education can be enhanced by combining affective issues with cognitive and instructional issues and these affective issues include beliefs, attitudes, and emotions. Attitude toward mathematics is among these affective factors that different scholars have extensively studied.

Attitude towards mathematics can be defined as "liking or disliking of mathematics, a tendency to engage in or avoid mathematical activities, a belief that one is good at or bad at mathematics, and a belief that mathematics is useful or useless" (Neale, 1969, p. 632). There are different components of attitude toward mathematics, such as enjoyment, self-confidence, value, and motivation (Davadas and Lay, 2017). Similarly, According to the TIMSS 2019, like, confidence, and values are three components of attitude toward mathematics (Mullis *et al.*, 2020). Students who have a positive attitude toward mathematics enjoy the subject and find mathematics valuable (Kiwanuka *et al.*, 2020). Therefore, it can affect students mathematics success (Chen *et al.*, 2018). If students have a high attitude toward mathematics, their success level is also high (Aiken, 1970; Minato and Yanese, 1984). Thus, it can be concluded that attitude is a crucial construct that affects learning mathematics. Attitude is formed by experience (Goldin *et al.*, 2020). Therefore, one of the main goals of mathematics lessons is to provide students with experiences that support their attitude toward mathematics and learning.

2.5. Realistic Mathematics Education

Realistic Mathematics Education (RME) is a teaching-learning approach developed by Freudenthal in the Netherlands, intending to realize the reform that the whole world needs in mathematics teaching and schools (Van den Heuvel-Panhuizen, 1996).

According to the view of Freudenthal (1991), there are two important points of mathematics education: mathematics should be close to real life and be a human activity. According to the first view, mathematics should be connected with real-life situations. The meaning of real life was defined as both real situations in students' minds and real in the world (Zulkardi, 2010). Therefore, it is not necessary to use everyday life situations; stories, tales, and even formal mathematics can also be used to connect with real life (Van den Heuvel-Panhuizen, 2003). According to the second view of Freudenthal (1991), human activity is important for mathematics education. In that sense, RME is based on a guided reinvention process, which means students re-invent mathematics with the guidance of teachers and instruction (Treffers, 1987; Zulkardi, 2010).

According to Freudenthal's view, real-life problems should be the starting point for reaching formal knowledge (Zulkardi, 2010). Therefore, teaching mathematics should not start with teaching formal knowledge but should start with a real-life problem. Thus, with the RME, students start to think about a problem and try to solve it. In this way, they can reach mathematical concepts and make sense of them. This process is called "mathematization" (Freudenthal, 1991; Treffers, 1991). Through mathematization, students reach mathematical knowledge by investigating a real-life problem, modeling it, and working on it. There are two types of mathematization, according to Treffers (1987), which are horizontal and vertical mathematization. Horizontal mathematization includes;

> ...identifying or describing the specific mathematics in a general context, schematizing, formulating and visualizing a problem in different ways, discovering relations, discovering regularities, recognizing isomorphic aspect in different problems, transferring a real-world problem to a mathematical problem, and transferring a real-world problem to a known mathematical problem. (Zulkardi, 2010, p.4).

In addition, vertical mathematization includes "representing a relation in a formula, proving regularities, refining, and adjusting models, using different models, combining and integrating models, formulating a mathematical model, and generalizing" (Zulkardi, 2010, p.4). According to Freudenthal (1991), these two types of mathemati-

zation cannot be separated from each other with a clear cut. Different stages of learning may involve both kinds of mathematization. His definition highlights that horizontal mathematization includes going from the world of life to the world of symbols; vertical mathematization includes moving within the world of symbols.

2.5.1. Teaching Principles of Realistic Mathematics Education

The realistic mathematics education approach provides a realistic way for an effective mathematics education. In light of this, Van den Heuvel-Panhuizen and Drijvers (2014) discussed six core principles of RME. These are the *activity principle, reality principle, level principle, intertwinement principle, interactivity principle*, and *guidance principle*. With the *activity principle*, students learn mathematics by doing mathematics. They actively participate in their learning process and put forward their own mathematical products during their learning. With the *reality principle*, students mathematize real-life context and determine important mathematical points in it. This principle can be applied by using real-life problems as a starting point for the lesson. Thus, students learn mathematical concepts by solving a real-life situation and can reach essential concepts. With the *level principle*, students pass through various levels of understanding during their learning. Through the activities, the relationships between informal mathematics and formal mathematics can be revealed. Students reach the connections between mathematical concepts and strategies. With the *intertwinement principle*, students engage with various mathematical subjects and concepts since mathematics can be seen as a whole that includes interrelated concepts. For instance, geometry, numbers, and algebra are related to each other and can be thought of together. With the *interactivity principle*, students interact with each other and share their ideas, strategies, and solution approaches. They socially construct their learning in an environment that allows communication and interaction. Thus, they can learn different approaches and thinking strategies. Thus, their level of understanding can be moved to a higher level. With the *guidance principle*, students reach mathematical knowledge through the teacher's guidance. The teacher's role is to guide, organize, and assess the teaching-learning process and to support the "guided reinvention", where

students re-invent mathematics with a similar process to how mathematics was invented (Zulkardi, 2010).

2.5.2. Designing Realistic Mathematics Education Lessons

Streefland (1991) proposed that RME lessons involve three levels: classroom level, course level, and theoretical level (as cited by Zulkardi, 2010).

The classroom level should be prepared according to the principles of RME and allow students to make horizontal mathematization (Zulkardi, 2010). Therefore, teaching materials that are related to real life can be presented to the classroom, and then RME principles can be implemented. At the classroom level, students can transform a real-life situation into a mathematical situation according to the reality principle. Starting a context-based problem is necessary for this transformation and making sense of mathematics within the real world. Thus, with the intertwining principle, students also use other concepts in mathematics; with the interactivity principle, they communicate with each other and pass various levels of thinking with the level principle. The teacher is a facilitator during this process.

Course level involves moving learning from the local level to a more general level. At this level, students can grasp the general goal of the course. Teaching materials and plans should be continued in a general way (Zulkardi, 2010).

At the theoretical level, other materials and plans in preceding levels can be continued and learning moved to a more general and theoretical way. Theoretical and generative productions are revealed at this level (Zulkardi, 2010).

2.5.3. Components of the RME Lesson Plan

De Lange (1995) pointed out four components of the RME lesson plans. These are goals, materials, activities, and assessments.

2.5.3.1. Goals. De Lange (1995) mentioned three main goals of mathematics education: higher-level, middle-level, and lower-level goals. RME goals should be middle or higher-level goals. This means that lessons should not only be based on a set of procedures and rules. Goals should allow one to connect mathematical concepts and strategies and understand the background of mathematics. Problems in RME should allow unique strategies and multiple methods to be shared and discussed. Thus, students should be able to move on to higher-level mathematics. Goals can also be flexible. This means new goals can be added, and goals can be redesigned to develop students' higher-order thinking skills.

2.5.3.2. Materials. De Lange (1995) stated that RME should start with material that includes real-life problems. This problem should be domain-specific and foster different solution strategies. Students should be able to enter the process of mathematization through these context-based problems.

2.5.3.3. Activities. Students should develop their own set of procedures and strategies to solve real-life problems through activities. To provide students with this work, teachers' should follow some steps and these are summarized as follows (Zulkardi, 2010):

- A context-based problem should be provided as a starting point of the lesson.
- When students engage with the problem, a clue should be given to provide guidance.
- Students should be encouraged to share and discuss their solutions. The focus of the discussion should be an interpretation of the problem and an evaluation of the effectiveness and efficiency of the various solution methods students used to solve it.
- Teachers should allow students to arrive at their own solutions, empowering them to explore based on their experiences and knowledge.
- Lastly, another problem within the same context should be given.

2.5.3.4. Assessment. Assessment is another important component of the RME. De Lange (1995) pointed out five principles for assessment in RME. These are:

- The main purpose of the RME is to develop student's learning. Therefore, assessment should be performed both in the unit and at the end of the unit.
- Assessment should allow students to show their learning. Therefore, problems that include multiple solution strategies can be selected.
- Assessment should include all of the goals of mathematics education, which are lower, middle, and higher.
- Tests that enable understanding of what students learn should be preferred.
- Assessment tools should be practical and appropriate to the school culture.

In the light of the literature, characteristics and important principles of RME, levels of the lesson plans based on RME, and components of the lesson plans were discussed. As the literature highlighted, meaningful materials that include real-life context are starting points for the RME lessons. Therefore, effective task selection is one of the main duties of the teacher since the teacher should "give the students a contextual problem that relates to the topic as the starting point" (as cited in Zulkardi, 2010, p.11). Selected tasks should be appropriate for the characteristics and main principles of the RME lesson. Therefore, mathematical tasks should be evaluated effectively to design an RME lesson in terms of appropriateness with the RME approach.

2.6. Mathematical Tasks

A successful mathematics teaching and learning process is performed through the active involvement of students in their learning. Students should build their own knowledge, make reasoning based on their previous learnings, communicate their mathematical ideas, and discuss mathematical problems (NCTM, 2000). It is important to give students opportunities to show the meanings and connections between mathematical concepts and real-life situations with mathematics.

Thus, students can understand the background of mathematical situations and ideas meaningfully and become successful problem solvers.

In order to provide students with an effective learning environment, the role of mathematical tasks is essential (Doyle, 1983). Tasks support the teaching-learning process and can be seen as a bridge between what students need to learn and what the teacher mentions in the lessons. A successful mathematical task should provide students with active involvement, and the teacher's role should be to direct student-centered mathematical activities. Teachers are facilitators during this process. Thus, a successful mathematics task leads to creating a constructivist environment.

Mathematical tasks have a broad explanation in the literature. Its importance has been emphasized and explained by many mathematicians. Stein *et al.* (1996) defined a mathematical task as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (p.460). Mathematical tasks attract students' attention (Doyle, 1983) and lead students to think mathematically. Students think about mathematical concepts and make connections between them through mathematical tasks. Thus, students' meaningful learning can be supported.

2.6.1. Mathematical Task Types

NCTM (1991) stated that the "central responsibility of teachers is to select and develop worthwhile tasks and materials that create opportunities for students to develop mathematical understandings, competence, interests, and dispositions" (p. 24). Therefore, teachers should design or select tasks that motivate students and make them comfortable with the subject matter (Kilpatrick *et al.*, 2002). Students' mathematical disposition and interest can be improved in this way. Along with this, students' learning can be effectively improved through mathematical tasks (Doyle, 1983). Students' mathematical thinking can be developed, and students can experience "doing mathematics" (Henningsen and Stein, 1997; NCTM, 1991).

Various mathematical task types are defined in the literature. Different mathematical tasks affect students' learning in different aspects (Doyle, 1983). Selecting an appropriate task with the aim of the lesson is crucial. One of the main classifications of mathematical tasks is made by Stein and Smith (1998) according to the cognitive demand of the tasks. According to this classification, mathematical tasks are divided into two main categories: higher-level demand and lower-level demand tasks. Lower-level demand tasks are far from the conceptual understanding and connection between mathematical situations and underlying meanings. These tasks require little cognitive demand to complete successfully. Memorization tasks and procedure without connection tasks are two types of lower-level demand tasks. To complete memorization tasks, students need to memorize previous learnings without any mathematical connection and without using any computation. With procedures without connection tasks, students use pre-learned procedures and do not need to understand the mathematical background of the procedure.

On the other hand, higher-level demand tasks require conceptual understanding and mathematical reasoning. Procedures with connection and doing mathematics tasks are two types of higher-level demand tasks. The procedure with connection tasks requires conceptual understanding during the use of procedures. Students need to reason mathematically and connect mathematical ideas. Doing mathematics tasks requires students to use higher-order thinking skills rather than using algorithmics. Students can discover and comprehend mathematical concepts, connections, and relationships.

Mathematical task types are also classified as mathematically rich tasks and non-mathematically-rich tasks according to teaching purpose (Yeo, 2007). While mathematically rich tasks aim for students to learn new mathematical content, non- mathematically rich tasks aim for students to practice pre-taught content rather than learn new content. Mathematical-rich tasks can be divided into analytical tasks and synthesis tasks.

Analytical tasks support the development of students' analytical skills and include problem-solving tasks, investigative tasks, and guided discovery tasks. Synthesis tasks aim to develop students' creative thinking skills and include problem-posing tasks. Non-mathematically rich tasks include procedural tasks and word problems.

2.6.2. Three Act Math Tasks

In today's mathematics education, it is crucial to discover mathematical situations, relate mathematics to real-life situations, and pose and solve problems. Mathematical tasks that involve all these experiences are essential for making sense of mathematics and improving mathematical skills. In recent years, a mathematical task type that provides students with multiple mathematical experiences has been developed by Meyer (2011): Three Act Task. Three-act tasks were introduced by Meyer (2010) in his TED talk and adapted to the primary and middle school curriculum by Fletcher (2016) and other mathematicians. These tasks have become increasingly popular on different platforms, such as the Internet and television, and have already begun taking place in some countries' school textbooks (Lomax *et al.*, 2017). Thanks to three-act tasks, students use and develop their mathematical abilities during their learning. These mathematical abilities might include problem-posing, problem-solving, mathematical estimation, and mathematical modeling skills (Lewis and Colonnese, 2021; Lomax *et al.*,2017). Students reveal these abilities and develop them through real-life problems. With the guidance of their teachers, they ask questions about context-based situations and solve real-life problems by discovering the mathematical ideas behind them. Thus, they can reach new mathematical situations and construct their learning. In this respect, three-act tasks may support students in learning a new concept and support the development of analytical and creative thinking skills. Therefore, they can be considered in the class of mathematically rich tasks according to the Yeo (2007) classification. They also fit the definition of higher-level cognitive tasks in terms of cognitive load according to the Stein and Smith (1998) classification since they require understanding mathematical situations, their background, and connections between them. Students can experience "doing mathematics" in this way.

Three-Act Tasks have three stages: Act 1, Act 2, and Act 3. Through these stages, students can understand and mathematize a real-life problem, ask mathematical questions about context, find the mathematics, and learn how to use mathematics in real life. Thus, they can reach important mathematical concepts, strategies, and ideas by solving a real-life problem.

In the first stage, the teacher presents a real-life situation through a video or a photograph. Then, students discuss what they remember and wonder about the video or photograph. According to the goal of the lesson, teachers can guide the discussion, foster mathematical features of thoughts, encourage students to pose problems about the subject and reveal their creative thinking. Generative questions such as "What if? What else?" can be used for this aim (Lewis and Colonnese, 2021). Then, a central mathematics question is determined based on class discussion. Thus, the real-life mathematics situation is converted into a mathematical problem according to the students' wonderings.

During the second act, students work on solving the mathematical problem determined in Act 1. During their studies for the solution, they think about the required knowledge to solve the problem and what they need. To encourage students to think, the teacher can ask questions like "What do you need to solve the problem? What information should be given to reach the answer?". Thinking about the necessary information for the answer can support students' mathematical reasoning. Teachers give necessary information or resources. Concrete manipulatives might also be provided, and students can be encouraged to use mathematical modeling to make sense of the situation and solve the problem. Thus, students can model mathematical situations and use mathematical knowledge by constructing mathematical relations to reach the answer. In this way, a student can reach important mathematical ideas and concepts. Their mathematical abilities, such as problem-solving and mathematical modeling, are also supported.

During the third act, students discuss their findings and solution strategies. Teachers can guide mathematical discussions and encourage students to think about mathematical relationships. Students compare their answers and think about different solution strategies. Mathematical concepts or situations students learn can be highlighted through a discussion environment. In the end, the teacher presents a video or photograph of the solution. If students find a different answer, the reason for this can be discussed. In successful three-act tasks, the third act allows students to extend their ideas rather than show the solution (Hallman-Thrasher *et al.*, 2018). Students should broaden their thoughts through act three.

2.6.3. Evaluation of Three Act Math Tasks in terms of RME Principles

RME principles, defined by Van den Heuvel-Panhuizen and Drijvers (2014), are one of the most important considerations for preparing RME lesson material. In that sense, three-act math tasks should be evaluated according to these principles.

One of the essential principles of the RME is "reality". RME lessons start with real-life problems and continue with the process of horizontal and vertical mathematization. Students make sense of the problem and mathematize it by determining its important mathematical aspects. Three-act math tasks start with real-life situations. Students are expected to determine the mathematical points of the situations, create mathematical problems based on real life, and solve these problems. With three-act tasks, students make the mathematization process in this way. They mathematize real-life situations, solve problems, and go through horizontal and vertical mathematization.

Another principle of RME is the "activity" principle. Students should actively participate in the learning process, put forward their products, and learn mathematics by doing mathematics. With three-act math tasks, students are actively involved in all stages, which are in Act 1, Act 2, and Act 3. For instance, in Act 1, students do mathematics by asking mathematical problems, finding mathematical aspects of the

context, and making mathematization. Posed questions can be a part of a product of students in Act 1. In Act 2, they can do mathematics by modeling real-life situations and finding the connection between the real-life problem and the mathematical clue given by the teacher. In Act 3, they show their products, share their ideas and strategies, and actively participate in discussions based on a mathematical problem.

The level principle of RME explains that students pass through various levels of thinking. Three-act math tasks foster students' different levels of thinking. They can see connections between formal mathematics and informal mathematics. Thus, they can apply mathematics, passing through various thinking levels. In this way, students can reveal important concepts and strategies.

It is known that real-life situations involve different parts of mathematics, such as proportion and geometry. The intertwinement principle of RME defines this connection between mathematical subjects. Since three-act math tasks are based on real life, they also have this connection.

Three-act math tasks also fit the interactivity principle of RME since students interact with each other, show their products, and share their ideas, strategies, and solution approaches.

Teachers' role in three-act math tasks is to foster the mathematical aspect of students' thinking, giving them clues for solving the real-life problem in Act 2, guiding them, and supporting them to reach the knowledge and solution. The guidance principle of RME also explains the roles of teachers in the same manner. Thus, the teacher can support "guided reinvention" with three-act math tasks in RME.

When all these are considered, it can be concluded that three-act math tasks are suitable for the principles of RME defined by Van den Heuvel-Panhuizen and Drijvers (2014) and, therefore, can be used as material in the RME lessons.

2.7. Related Studies

2.7.1. Related Studies about RME

In this section, studies related to RME were presented. According to the literature, the RME approach was used to investigate various variables like instructional efficiency, students' achievement about a subject, students' attitudes towards mathematics, and views about the approach.

Rasmussen and King (2000) conducted a study to investigate teaching differential equations with the RME approach. Data was collected through videos, written documents of students, and interviews. According to the findings, students developed their own methods and created their ideas to understand the subject conceptually. Thus, it was shown that the lessons based on RME increased the effectiveness of the teaching-learning process.

A similar study was conducted by Kwon (2002) to investigate the RME approach in teaching differential equations. The participants were 43 university students. According to the results, meaningful learning occurred by connecting the subject with real-life situations. It was concluded that the RME approach is useful for teaching differential equations and broadening students' perspectives at both primary and university levels.

Besides, Miller (2015) examined the effect of the RME approach on teaching algebraic expressions. The participants of this qualitative study were 5th-grade students. As a result, it was found that the RME approach contributed to the teaching-learning process positively.

Similarly, Özkan (2019) conducted a study with 6th-grade students on teaching algebra based on the RME approach. Algebra achievements of students and permanence were investigated in this experimental study. According to the result of the study, the experimental group's scores are significantly higher than the control group's.

Altaylı (2012) examined the proportional reasoning and mathematics achievement of seventh-grade students with RME-based instruction. According to the result of this quasi-experimental research, the experimental group's achievement was greater than the control group. Therefore, it was concluded that RME increases the efficiency of teaching and learning.

As the studies show, RME is a powerful approach to teaching and learning a subject (Altaylı, 2012; Kwon, 2002; Miller, 2015; Özkan, 2019; Rasmussen and King, 2000). On the other hand, students' views about RME and whether RME increases students' attitudes towards mathematics have become a topic of curiosity and worth researching. Therefore, there are different studies related to this issue as well (Arseven, 2010; Fauzan *et al.*, 2012; Üzel and Uyangör, 2006; Zakaria and Syamaun, 2017).

In the study conducted by Fauzan *et al.* (2002), the RME approach was used in teaching the subject of "area and perimeter". Data were collected from the control and experimental groups. According to the result, RME is an effective approach to teaching the subject, and interviews revealed that students liked the RME approach. According to students' expressions, they are more active and creative with this approach. Teachers who served as observers also acknowledged positive changes in students' behavior after engaging in RME-based lessons.

Üzel and Uyangör (2006) examined the attitudes toward mathematics of seventh-grade students, using instruction based on RME. According to the study result, the attitudes of the experimental group who was taught with the RME approach were significantly higher than the attitudes of the control group.

Arseven (2010) used activities based on RME and examined students' achievement, problem-solving skills, and attitudes toward mathematics. According to the study result, the experimental group performed significantly better than the control group.

Another study was conducted by Zakaria and Syamaun (2017) to investigate the effect of RME on mathematics achievement and attitudes towards mathematics. According to the result, the attitudes of the experimental group did not significantly differ. On the other hand, achievement significantly differed in favor of the experimental group.

2.7.2. Related Studies about Three Act Math Tasks

There is a limited number of studies related to three-act tasks, and they were conducted by college or university faculty members. Different grade levels, implementation procedures, and the effect of the teaching-learning process were examined.

Lomax *et al.* (2017) conducted a study with second graders. They integrated a three-act math task in a teacher's lesson as a problem-solving activity. In their study, students were encouraged to model real-life contexts to make sense of the problem. Firstly, a real-life scenario was presented to students. The teacher noted the students' wonderings and what they remembered. Then, students were expected to pose mathematical questions about their wonderings. Among the posed questions, the local question was determined with the teacher's guidance. In the second act, the teacher observed different solution strategies and approaches. Students used different mathematical models like number lines, cubes, and equations. Solution strategies and different models were discussed and compared in the third act. Students had a chance to learn the different representations of a real-life scenario mathematically and understand the different approaches for the solution. After the study, three important distinctions between three-act tasks and other mathematical activities were noted. Firstly, three-act math tasks focus on students' ideas and are based on students' knowledge of real life.

All stages of the task are student-centered. In other words, during the problem-posing, problem-solving, modeling, and discussion, students are active, and their ideas are centered. Secondly, all students have a chance to participate in the activity since different models, representations, sounds, and images are used, and different entry points are provided for all students' participation in some way. Thirdly, three-act math tasks provide young students with doing mathematics as CCSSI (2010) described. In other words, students understand and represent a real-life problem mathematically by making sense of its background. They also can have a chance to choose the information and tools they need and model the mathematical situation. Finally, they can present their reasoning and justification. During this time, they also analyze other approaches and strategies.

Flynn (2017) conducted a study with fourth graders by taking the role of a teacher. Pre-made three-act tasks from online blog sites were used. Findings about the implementation and administration of three-act tasks in classrooms were obtained. Accordingly, three-act math tasks allow students to solve mathematics problems more meaningfully rather than following memorized procedures and focusing on the answer. It was noted that three-act tasks are mostly based on interacting with students, creating a classroom environment based on reasoning and questioning, and giving students a sense of ownership in their learning. The teacher's role should be to provide these three characteristics of three-act math tasks.

Hallman- Thrasher *et al.* (2018), a group of teachers, used three-act math tasks in their study. They noted that the tasks are high cognitive demand tasks (Stein *et al.*, 2009), require students' active engagement, and the goal is to provide students with conceptual understanding. The participants of the study were sixth, seventh, and eight-grade students. They implemented three act-tasks in the pre-algebra and first-year algebra classes. Students engage with mathematical modeling by sketching a time graph of a real-life scenario that was presented in a video format. The findings showed that presenting a real-life scenario with the video format and additional provided materials supported students' understanding and reasoning. It was also concluded that

since mathematics is more than what mathematics textbooks include, using a real-life video pushes them to find important mathematical details in the scenario. Students had opportunities to reason about the mathematics behind a real-life context. They also made mathematical modeling for the situations. It was also found that using visuals increased students' motivation to pose their own questions. Different representations like graphs and tables, as well as making a connection between them, supported the discussion environment as well. They also noted that students' participation was sustained through teacher-student and student-student dialogue. Students generated and validated their own ideas, and the teachers' role was to listen, observe, and support the connection between the ideas.

Lewis and Colonnese (2021) conducted a study with second-grade gifted children. Students' problem-posing skills and mathematical creativity were examined through three-act math tasks. In order to encourage students to pose problems, generative questions and manipulatives were used. Also, during Act Three, a sequel, which means creating a new question based on information from ACT 1 or based on their solution, was used. It was noted that three-act tasks support students' problem-posing and mathematical creativity. Using concrete manipulatives, mathematical modelings, sequels, and generative questions supported this process. It was also noted that the three-act math task framework fits the problem-posing in mathematics education.

Frabasilio (2022) examined seventh-grade students' adaptive reasoning indicators and how they adapt their reasoning. Three-act math tasks were used to investigate this. The result showed that students adapt their reasoning mostly by relationships and connections or justifications. According to themes, students had different adaptive reasoning strategies. Some of these strategies are communicating through arguments, justifications, drawn erasures, and arrows. Interaction between students led to effective adaptive reasoning strategies. Findings suggested that three-act math tasks provide students with adaptive reasoning skills as a part of mathematical proficiency.

Unlike all these studies, Buluklu (2021) conducted a study with pre-service middle school mathematics teachers rather than students. Pre-service teachers were asked to prepare three-act math tasks. Then, the cognitive level of preparation and implementation processes was investigated. How the cognitive level of the tasks changed during preparation and implementation was observed. It was found that implementing low cognitive level tasks has a low cognitive level, and implementing high cognitive level tasks has a high cognitive level. This research study supported pre-service teachers to prepare and implement three-act math tasks.

Similarly, Zhou and Hansen (2022) conducted a study with pre-service teachers. The aim was to investigate the effect of lessons with three-act math tasks on the mathematical modeling process. Results showed that pre-service teachers use three-act math tasks in order to form modeling problems, but they have difficulty forming these problems for low-grade levels (grades 1-3). The second conclusion of their study is that even though pre-service teachers form mathematical modeling problems, these problems did not include all sub-processes of the modeling cycle.

3. SIGNIFICANCE OF THE STUDY

The literature review focused on the principles and characteristics of the RME approach and demonstrated that this approach is a valuable method to teach mathematics topics, such as algebra, geometry, and linear equations (Fauzan *et al.*, 2002; Kwon, 2002; Özkan, 2019; Rasmussen and King, 2000; Zakaria and Syamaun, 2017). It also demonstrated that RME positively increased students' attitudes towards mathematics (Arseven, 2010; Üzel and Uyangör, 2006). However, there is a lack of research that investigates the RME approach with a specific material. Thus, studies lack an explanation of the materials that can be used in the RME lessons, although they demonstrated the efficiency of the RME in mathematics education. Since one of the main components of the RME lessons is the material, it is crucial to conduct studies to investigate the RME with the materials used. In this research, one of the new task types, the "Three-Act Math Task", developed by Meyer (2011), was selected for this purpose. As evaluated before, three-act math tasks are appropriate materials in terms of the principles of RME. On the other hand, the efficiency and applicability of using these materials in RME lessons should be demonstrated.

Another point worth mentioning is that, as the literature pointed out, three-act math tasks provide students with different mathematical skills, including reasoning, modeling, problem-solving, and problem-posing (Frabasilio, 2022; Hallman-Thrasher *et al.*, 2018; Lewis and Colonnese, 2021; Lomax *et al.*, 2017). Therefore, three-act math tasks have significant potential to develop students' mathematical competencies. On the other hand, studies that integrate such important material into lessons are still few, and integrating three-act math tasks into lessons is crucial. Therefore, it is expected that this research study will shed light on how three-act math tasks can be integrated into the lessons according to the principles of the RME approach, which is one of the significant educational approaches.

It is thought that this study will contribute significantly to the literature in terms of implementing the RME approach with a specific type of task, examining the potential of this implementation, and thus, providing a new lens to educators and practitioners. This potential can be examined through teaching algebra, which is seen as an abstract mathematics subject and, therefore, should not be far from real life. Kaput (1999) stated that algebra should not be away from real life. Three-act math tasks have a strong connection with real life (Meyer, 2011) and might have great potential to teach algebra in RME lessons. In this regard, it is crucial to teach algebraic expressions and variables to students in the early stages based on real-life scenarios. Therefore, this study will be one of the first studies on how a specific material, three-act math tasks, can enrich the RME approach and put it into practice on the subject of algebra.

As mentioned in the literature, three-act math tasks support students' mathematical competencies. Additionally, an important finding from reviewing various conceptual frameworks on algebraic thinking suggests that students' algebraic thinking may be a fundamental element of many mathematical competencies. In this sense, three-act math tasks with the RME approach can be crucial in revealing students' algebraic thinking. According to Kaput (2008), there are three content strands of algebra: generalized arithmetic, functional thinking, and modeling. These content strands include some crucial components, such as the meaning of variables, generalization of arithmetic operations and their properties, understanding functional relationships among quantities, and mathematical modeling (Blanton and Kaput, 2005; Blanton *et al.*, 2011; Chimoni *et al.*, 2018; Isler *et al.*, 2013; Kaput, 2008). In this regard, learning algebra in the framework of Kaput's (2008) content strands is very important to support students' algebraic thinking and prepare them for real-life and higher-level mathematics in a better way.

Therefore, preparing three-act math tasks according to algebraic thinking content strands can support this situation, and examining students' algebraic thinking may shed light on educators on how this new approach can be combined with algebraic thinking content strands. The potential of three-act tasks to reveal mathematical

skills and how they reveal algebraic thinking not only demonstrates the importance of the study but also provides insight into future research regarding effective instructional design.

With the structure of three-act math tasks, a real-life scenario is mathematized by students, becomes a problem, and then students work on this problem (Meyer, 2011). If these tasks are used in the RME approach, students can learn new topics and concepts. Students, for instance, can identify variables within a context, ask questions, determine relationships between variables, and solve a problem. Thus, they can construct their learning with the help of the RME-based lessons enriched with three-act math tasks. Therefore, lessons prepared with the RME approach enriched with three-act math tasks can create a difference in students' achievement.

Besides, attitude is constructed with experience (Goldin *et al.*, 2016), and enjoyment and value are some essential components of attitude toward mathematics (Davadas and Lay, 2017). Three-act math tasks can spark students' enjoyment and make learning math more valuable by making the students notice and wonder about essential details in the subject. Besides, these lessons will have a strong connection with real life that is familiar to the students, and students will have a chance to experience learning mathematics in a meaningful manner. Thus, lessons based on RME and three-act math tasks might positively influence students' attitudes toward mathematics.

As a result, this study can pave the way to using three-act math tasks with the RME approach and reveal students' achievement, attitudes toward mathematics and changes on these, and students' algebraic thinking. Hence, the study can contribute significantly to math education by enhancing the effectiveness and efficiency of the RME approach with three-act math tasks, thus offering valuable insights and tools for educators and instructional designers.

4. STATEMENT OF THE PROBLEM

The purpose of this embedded mixed design study is to examine how students think algebraically during and at the end of the RME-based lessons enriched with three-act math tasks and to investigate whether these lessons change the sixth-grade students' algebra achievement and attitudes toward mathematics. For this purpose, the researcher prepared lesson plans for the sixth-grade students based on the Realistic Mathematics Education (RME) approach and enhanced these lesson plans with three-act math tasks.

4.1. Research Questions

The research questions of the study are:

(i) Is there a statistically significant difference between sixth-grade students' achievement scores, as measured by the "Achievement Test", before and after attending RME-based lessons enriched with three-act math tasks?

(ii) Is there a statistically significant difference between sixth-grade students' mathematics attitude scores, as measured by the "Mathematics Attitude Scale", before and after attending RME-based lessons enriched with three-act math tasks?

(iii) How do sixth-grade students think algebraically during and at the end of the RME-based lessons enriched with three-act math tasks?

4.2. Research Hypothesis

The null hypotheses for this research study are:

(i) $H_0 1$: There is no statistically significant difference between sixth-grade students' achievement scores, as measured by the "Achievement Test", before and after attending RME-based lessons enriched with three-act math tasks.

(ii) $H_0 2$: There is no statistically significant difference between sixth-grade students' mathematics attitude scores, as measured by the "Mathematics Attitude Scale", before and after attending RME-based lessons enriched with three-act math tasks.

4.3. Variables

The independent variable of this study is RME-based lessons enriched with three-act math tasks. The dependent variables of this study are algebra achievement and students' attitudes towards mathematics.

4.3.1. RME-Based Lessons

It is a type of instruction based on the Realistic Mathematics Education approach developed by Freudenthal in the Netherlands, starting with real-life problems (Zulkardi, 2010), including horizontal and vertical mathematization (Treffers, 1987), and guided reinvention process that students reinvent mathematics (Freudenthal, 1991; Treffers, 1991).

RME-based lessons enriched with three-act math tasks are based on the Realistic Mathematics Education approach and consist of three-act math tasks, developed by Meyer (2010), that include three stages: discussions based on what students notice and wonder in a real-life scenario and determining a focus problem (Act-1), thinking about the information needed to solve the problem and working on the solution with the clue (Act-2), and lastly discussion and revealing the solution (Act-3).

4.3.2. Algebra Achievement

Algebra achievement in this study is the achievement scores of students as measured by the "Algebra Achievement Test" developed by Özkan (2019), which consists of eight open-ended and five multiple-choice questions based on MoNE (2018) objectives.

4.3.3. Mathematics Attitude

Mathematics Attitudes in this study are the scores of students as measured by the "Mathematics Attitude Scale" developed by Nazlıçiçek and Erktin (2002) that is a 5 point liker type (1=never, 2=rarely, 3=sometimes, 4=usually, and 5=always) and includes the subdimension of the perceived mathematics achievement by students, perceived benefits of mathematics by students, and interest in math lessons.

4.3.4. Algebraic Thinking

Algebraic thinking is students' generalization and symbolization abilities that take place across three content strands of algebra: generalized arithmetic, functional thinking, and modeling (Kaput, 2008). It was examined through worksheets, observations, interviews, and classroom discussions.

5. METHODOLOGY

This study aims to examine how students think algebraically during and at the end of the RME-based lessons enriched with three-act math tasks and to investigate the effect of lessons on sixth-grade students' algebra achievement and attitudes towards mathematics. In this section, the study's methodology was presented under the subtitles of research design, participants, data collection tools, implementation and data collection procedure, and data analysis procedure.

5.1. Research Design

In this study, an embedded mixed-method design, which includes both quantitative and qualitative data collection techniques, was used. In the embedded mixed study, quantitative and qualitative data are collected for one study, and the researcher analyzes these two data separately, which address different research questions (Creswell, 2015). The quantitative part of the study employed a one-group pretest-posttest design to investigate the effect of RME-based lessons enriched with three-act math tasks on sixth grade students' algebra achievement and attitudes toward mathematics. The study's dependent variables are students' algebra achievement and attitude toward mathematics, while the independent variable is the RME-based lessons enriched with three-act math tasks. In the qualitative part of the study, students' algebraic thinking was investigated both during and at the end of the implementation. This approach enabled the examination of students' algebraic thinking in a detailed way, both with the activities during the instruction and with the questions at the end of the implementation.

5.2. Participants

Participants of this study are sixth-grade students in the spring term of the 2022-2023 academic year in a public school in the Akdeniz district of Mersin. The purposeful

sampling technique was used for sample selection since participants were sixth-grade students and would soon start a new algebra topic. Besides, the school environment in which they study is suitable for the research in terms of the classroom size and technological equipment. There are 48 students who participated in the lessons, but one student participated only in the pretest, and one student participated only in the post-test due to absence. Therefore, the study's participants are 46 students (24 female and 22 male). Besides, to analyze students' algebraic thinking during the lesson, eight groups (4 students in each) from the lessons were selected according to their attendance at the lessons. Along with this, six students were selected for the interview to examine students' algebraic thinking at the end of the lessons by considering some criteria explained in the procedure of the data collection section.

5.3. Data Collection Tools

In this research, the "Algebra Achievement Test", developed by Özkan (2019), and the "Mathematics Attitude Scale", developed by Nazlıçiçek and Erktin (2002) were used to collect quantitative data. Qualitative data sources comprise students' worksheets, interviews, and audio records during the instruction.

5.3.1. Mathematics Attitude Scale

The "Mathematics Attitude Scale" scale was developed by Nazlıçiçek and Erktin (2002). It is a 5-point Likert and consists of 20 questions, eight of which are negative and 12 of which are positive. The answers to the scale include 1=never, 2=rarely, 3=sometimes, 4=usually, and 5=always. The scale includes three subscales. These are the perceived mathematics achievement by students, perceived benefits of mathematics by students, and interest in math lessons.

The Cronbach's Alpha reliability coefficient of the scale was calculated as .84 by Nazlıçiçek and Erktin (2002). For the current study, the Cronbach's Alpha reliability coefficient was calculated as 0.865 for the pre-test and 0.853 for the post-test. Since

the Cronbach Alpha coefficient of the scale is above 0.7, it can be considered reliable (Pallant, 2016).

Nazlıçiçek and Erktin (2002) also explained the scale's validity. For the content validity of the scale, factorial analysis of variances was examined (Nazlıçiçek and Erktin, 2002). When the items are categorized into three subscales, 52 percent of the variance was explained. Besides, the correlation coefficient between students' attitudes towards mathematics and their math achievement was calculated as .36 with a 0.01 significance level, which shows the validity of the scale (Nazlıçiçek and Erktin, 2002). According to Nazlıçiçek and Erktin (2002), this value is statistically meaningful level and consistent with the previous studies.

5.3.2. Achievement Test

An achievement test on the topic of algebra in sixth grade was developed by Özkan (2019) to investigate the effect of the RME approach on sixth-grade students' achievements. During the test preparation, expert opinions were taken, and the questions were based on the sixth-grade algebra objectives (Özkan, 2019). Along with this, a pilot study was conducted to ensure validity and reliability with a total of 140 students in four different schools, and some questions were eliminated after the pilot study (Özkan, 2019). The final form of achievement test consists of 13 questions, eight open-ended and five multiple choices. A rubric developed by Özkan (2019) was used for scoring.

Questions comprise all the sixth-grade algebra objectives expressed in the MoNE (2018) curriculum (Özkan, 2019). Thus, the content validity of the achievement test was provided. On the other hand, since this achievement test was prepared during the transition period to the new curriculum, the participants in Özkan's (2019) study had prior knowledge of how to operate with integers, while the participants in this study did not have this prior knowledge. Therefore, the researcher changed the expression "(-3) + 7" in the 6th question of the achievement test to 7 - 3" and the expression

"sum of coefficients" in the 4th question to "coefficients" by consulting her advisor. In addition, the researcher consulted three academicians in the mathematics education department in order to use this achievement test and took their opinions to determine the appropriateness of the test for the study. Thus, the validity of the test was also provided for this study. The Cronbach's Alpha reliability coefficient of the test was calculated as 0.711; therefore, it is a reliable test for the current study.

5.3.3. Worksheets

Worksheets were developed by the researcher to examine students' algebraic thinking during the lesson. While preparing worksheets, both MoNE (2018) objectives regarding algebra topics and literature related to algebraic thinking were considered. Specifically, according to Kaput (2008), three content strands of algebra, which are generalized arithmetic, functional thinking, and modeling, were considered during the preparation. These content strands were merged with the MoNE (2018) objectives in order to satisfy these as well. Both three-act math tasks and the follow-up activities include worksheets. Thus, worksheets aimed to examine students' algebraic thinking in detail. While preparing the worksheets, the researcher consulted her advisor and took suggestions. In addition to the opinion of the researcher's advisor, opinions were also obtained from a mathematics teacher, an academician, and a doctoral candidate in the field of mathematics education. Besides, the instruction was piloted with five sixth-grade students, and worksheets were evaluated during the pilot study. In the light of expert opinions and pilot study, worksheets were arranged and finalized. During the lessons, students were divided into groups, which included four students each, and worksheets were conducted as group work. Group work supported the communicative and collaborative environment of the classroom.

5.3.4. Interview

In order to examine students' algebraic thinking at the end of the lesson, the researcher prepared three interview questions by consulting her advisor. The researcher

prepared the interview questions considering literature related to algebraic thinking and MoNE (2018) objectives. Interview questions focused on three algebra content strands that Kaput (2008) defined: generalized arithmetic, functional thinking, and modeling. A detailed examination of the literature regarding these content strands was conducted during the preparation. After the preparation, expert opinions were also obtained from two academicians, a doctoral candidate working in the field of mathematics education and a mathematics teacher who worked at the school where the research was conducted. Questions were piloted with five sixth-grade students. Interview questions were improved and finalized after the expert opinions and the pilot study.

The first question of the interview includes three items and focuses on two content strands of algebra: generalized arithmetic and modeling. Thus, students are expected to transform a real-world situation into a mathematical model, write equivalent algebraic expressions by using operation properties, use the value of variables for finding an unknown value in the context, and interpret the meaning of the variables.

The second interview question includes five items and focuses on the functional thinking strand, which means students are expected to find a relationship between two variables that are represented in a graph, find the close and distant term of the pattern, formulate a general rule of a pattern algebraically and use this rule to find a distant term.

The third interview question includes three items and focuses on the modeling strand. Thus, students are expected to transform a context-based situation into a mathematical model. It also includes generalized arithmetic strands, and students are expected to use the value of variables to find an unknown value in the context and find more than one way by combining different variables to represent a specific value in the context.

In order to conduct the interviews, six voluntary students were selected after the post-tests by considering their achievement test results and communication skills. The think-aloud strategy was used while conducting interviews. Thinking aloud is simultaneously talking about thoughts while performing a task (Ericsson and Simon, 1993). Thus, it allows educators and researchers to examine students' thinking processes. While students talked about their thoughts while solving the problems, the researcher took an audio recording and transcribed it later for analysis.

5.4. Procedure

This section explains the research procedure, including the data collection procedure and preparation and implementation of the instructional design.

5.4.1. Procedure of Data Collection

Firstly, permissions were taken from the Bogazici University Ethical Committee, the Ministry of National Education, the school administration, and students' parents after providing them with necessary information about the research. Then, data was collected in the spring semester of the 2022/2023 academic year. "Achievement Test", which was developed by Özkan (2019), and the "Mathematics Attitude Scale", which was developed by Nazlıçiçek and Erktin (2002), were conducted as a pre-test before the instructions and as a post-test after the instructions. Tests were conducted using paper-pencil methods and lasted one lesson hour in total. Between pre-tests and post-tests, the topic of algebraic expressions was taught to the students with five consecutive RME-based lessons enriched with three-act math tasks. Qualitative data was collected through worksheets, observation notes, and audio-recorded classroom discussions to investigate students' algebraic thinking during the instruction. After the instructions, an interview that included three mathematics questions on algebra was conducted with six selected students to examine their algebraic thinking at the end of the implementation. Volunteer students were selected according to their achievement test results and communication skills.

According to the achievement test, two were at a high level, two were at a medium level, and two were at a low level students. High-level students were students who had higher-than-average achievement, and their scores were more than one standard deviation above the mean. Low-level students were students who had lower-than-average achievement. Their scores were less than one standard deviation below the mean. Medium-level students were almost at class average. Interviews with each student lasted approximately 30 minutes. While conducting the interview, students solved mathematics questions with a think-aloud procedure, and the researcher recorded their voices. The procedure of data collection is presented in Table 5.1 below.

Table 5.1. Procedure of data collection.

Before Instructions	**Instructions**	**After Instructions**
Achievement Test Mathematics Attitude Scale	Worksheets Observation Notes Audio records of classroom discussions	Achievement Test Mathematics Attitude Scale Interview

5.4.2. Procedure of Instructional Design and Implementation

Under the purpose of the research, the researcher prepared lesson plans based on the RME approach and enriched these lesson plans with three-act math tasks. Before preparing the lesson plans, the researcher examined the literature in detail. Essential properties and principles of the RME approach, properties of three-act math tasks, and properties of questions that comprise three strands of algebra that are generalized arithmetic, functional thinking, and modeling were examined in a detailed way. In light of the literature examination, the researcher started preparing three-act math tasks, worksheets, and lesson plans that include these worksheets and tasks. During this process, the researcher met with her advisor regularly and took suggestions to improve the instructions. Thus, it was aimed to ensure that the three-act math tasks and worksheets were suitable to grade level and covered both MoNE (2018) objectives and algebra content strands defined by Kaput (2008). Besides, under the principles of the RME, an evaluation rubric for lesson plans was created by the researcher by consulting her advisor. Then, expert opinions from two academicians, one mathematics teacher,

and one doctoral candidate were taken. Experts evaluated the lesson plans according to the RME approach and expressed their opinions about whether the three-act math tasks and worksheets covered both the MoNE (2018) objectives and algebra content strands by giving necessary suggestions. In addition, lesson plans were piloted with five sixth-grade students, and necessary adjustments were made by considering pilot implementation and experts' suggestions.

According to the annual plan, there were two weeks and 10 hours for algebra the topic. Therefore, the implementation procedure was in accordance with this, which means instructions took 10 lesson hours and were completed in two weeks. Each instruction lasted two lesson hours (80 min). Instructions were implemented by the researcher. In this way, the researcher had a teacher role as well. Necessary permissions were obtained from the classroom teacher and school principals for this. The classroom's mathematics teacher was also in the class, but s(he) was only an observer. Table 5.2 below describes each instruction in terms of time, relevant algebra content strands focused on, and the objectives of the Ministry of National Education (2018).

Table 5.2. Lessons' algebra content strands and MoNE (2018) objectives.

Lessons	**Time**	**Algebra Content Strands**	**MoNE (2018) Objective**
1st Lesson	80 mins	Generalized Arithmetic Modeling	Students will be able to write an algebraic expression appropriate to a verbally given situation and a verbal situation appropriate to a given algebraic expression.
2nd Lesson	80 mins	Generalized Arithmetic Modeling	
3rd Lesson	80 mins	Generalized Arithmetic	Students will be able to calculate the value of the algebraic expression for different natural number values that the variable will take.
4th Lesson	80 mins	Functional Thinking	
5th Lesson	80 mins	Modeling	Students will be able to explain the meaning of simple algebraic expressions (with modeling studies).

5.5. Data Analysis

Owing to the study's embedded mixed design, quantitative and qualitative data analysis methods were used. Quantitative analysis methods were used to analyze the first and second research questions via using the SPSS program version 27.0. Qualitative analysis techniques were used to analyze the third research question. Further details about the analysis methods are provided in the following sections.

5.5.1. Analysis of Achievement Test Data

For the first research question, the research hypothesis was analyzed to determine if there is a statistically significant difference between sixth-grade students' achievement scores, as measured by the "Achievement Test", before and after attending RME-based lessons enriched with three-act math tasks. Since the mean scores of a single group were analyzed at two different times (before and after the treatment), the paired sample t-test should be used. For this aim, assumptions of the paired sample t-test were assessed first.

There are three main assumptions of paired sample t-test. The first assumption is the normal distribution of the dependent variables. The normal distribution of difference scores should be checked for paired sample t-tests (Pallant, 2016). According to Tabachnich and Fidell (2013), the normality assumption refers to the requirement that the data should follow a normal distribution, which means that the distribution of the data should be symmetrical and bell-shaped. One way to assess normality is by examining skewness and kurtosis values. Skewness measures the symmetry of the distribution, while kurtosis measures the "peakedness" of the distribution (Tabachnich and Fidell, 2013). Skewness and kurtosis values between -1.5 and +1.5 indicate a normal distribution (Tabachnich and Fidell, 2013). Besides, the Shapiro-Wilk test can be assessed to check normality in datasets with less than 50 sample sizes. If the test results are not statistically significant, with a p-value greater than 0.05, the null hypothesis is retained, indicating that the normality assumption holds. Conversely,

if the p-value is less than 0.05, the null hypothesis is rejected, indicating that the normality assumption is not met (Field, 2009). Therefore, the Shapiro-Wilk test was used, and skewness and kurtosis values were checked to assess the normality. Table 5.3 shows the results of the test of normality for difference scores. It can be concluded that difference scores are normally distributed since the significance value of Shapiro-Wilk is found as .623, $p > .05$.

Table 5.3. The test of normality for difference scores of the achievement.

	Shapiro-Wilk		
	Statistic	**df**	**Sig.**
Difference	0.980	46	0.623

Skewness and kurtosis values were also examined in order to support normality assumptions. Skewness and kurtosis values regarding pre-test, post-test, and difference scores were presented in Table 5.4. Since Skewness and kurtosis values are between -1.5 and +1.5, the normality assumption is not violated (Tabachnich and Fidell, 2013).

Table 5.4. Skewness and kurtosis values of the achievement scores.

	Skewness	**Kurtosis**
Difference	0.352	0.-177
Pre-Test	0.885	0.693
Post-Test	0.041	-1.150

The second assumption of the paired sample t-test is independence. It requires that each paired observation is independent of the others, meaning that the outcome of one observation should not be affected by another (Field, 2009; Pallant, 2016). Since the achievement test was conducted two different times and the outcomes of the tests were not affected by each other, this assumption was also met.

Furthermore, the third assumption is about the level of measurement, which means that the dependent variable must be measured using a continuous scale at the

interval or ratio level rather than through discrete categories (Pallant, 2016). Since the achievement scores of participants are continuous, this assumption was also met.

If one of these assumptions is not met, the non-parametric alternative of the paired sample t-test, which is the Wilcoxon Signed-Rank Test, should be used (Field, 2009). On the other hand, since all assumptions are met, a paired sample t-test was conducted to analyze data.

5.5.2. Analysis of Mathematics Attitude Scale Data

For the second research question, the research hypothesis was analyzed to determine if there is a statistically significant difference between sixth-grade students' mathematics attitude scores, as measured by the "Mathematics Attitude Scale", before and after attending lessons based on the RME approach enriched with three-act math tasks. A paired sample t-test is an appropriate method for analyzing this since there is one group at two different times. In this regard, assumptions were analyzed first.

The normal distribution of difference scores was analyzed through the normality test, as well as Skewness and Kurtosis values. Table 5.5 shows the normality test result, and Table 5.6 shows the Skewness and Kurtosis values of pre-test, post-test, and difference scores.

Table 5.5. The test of normality for difference scores of the mathematics attitude.

	Shapiro-Wilk		
	Statistic	**df**	**Sig.**
Difference	0.935	46	0.013

Table 5.6. Skewness and kurtosis values of the mathematics attitude scores.

	Skewness	**Kurtosis**
Difference	1.088	1.958
Pre-Test	-0.882	1.108
Post-Test	-1.057	1.516

As shown in Table 5.5, the significance value of the normality test was less than .05, which means the significance value of the Shapiro-Wilk test is .013, and the normality assumption is violated, $p < 0.05$. Moreover, as depicted in Table 5.6, kurtosis values of difference scores and post-test scores are not in the -1.5 and +1.5 range. Therefore, this can also support the violation of normality assumptions.

Since the normality assumption of the paired sample t-test was not met, the Wilcoxon Signed-Rank Test, which is a non-parametric alternative of the paired sample t-test, should be used (Pallant, 2016). For this aim, two main assumptions of the Wilcoxon Signed-Rank Test should be met.

The first assumption is random sampling. The sample should be selected randomly. Second, paired observations should be independent of each other (Pallant, 2016). When these assumptions were examined, two assumptions, random sampling and independence of paired observations, were met. Therefore, the Wilcoxon Signed-Rank Test is decided to be conducted to analyze data.

5.5.3. Analysis of Students' Algebraic Thinking

For the third research question, in order to examine sixth-grade students' algebraic thinking during and after RME-based lessons enriched with three-act math tasks, data obtained from the worksheets that were conducted during the lessons, recorded classroom discussions, and the interview were analyzed. Therefore, qualitative analysis was divided into two parts. In the first part, data from the lessons were analyzed. During this process, classroom discussions were transcribed.

Besides, in the lessons, worksheets were conducted in groups, which included four students each. For analysis, eight groups who attended all classes regularly were determined. Group names were labeled as G1, G2, G3, G4, G5, G6, G7 and G8. Their worksheets were analyzed. In the second part of the qualitative data analysis, data from the interview with six selected students were analyzed. Firstly, voice records were transcribed. Then, students' names were labeled according to their achievement test results, meaning higher-level students were labeled as H1 and H2, medium-level students as M1 and M2, and lower-level students as L1 and L2.

The content analysis method was used to analyze the data. With content analysis, similar data are combined and interpreted around themes and categories; thus, interviews and written content can be systematically analyzed (Yıldırım and Şimşek, 2013).

Students' algebraic thinking in these two parts, which are in lessons and interviews, was analyzed according to the framework created by arranging the common algebraic thinking indicators emerging in the literature around Kaput's (2008) three algebra content strands. Therefore, the themes of the qualitative analysis comprise three algebra content strands: Generalized Arithmetic, Functional Thinking, and Modeling.

The categories, formed by combining the algebraic thinking indicators mentioned in the literature review with those expected from students during the lesson, are algebraic thinking indicators that reflect students' algebraic thinking while achieving the lesson objectives during the lesson and while interviewing at the end of the lesson.

The codes were obtained from the data and paired with each category. Table 5.7 shows the framework that shows the algebraic thinking indicators around three content strands of algebra and thus guides the qualitative analysis.

Table 5.7. Algebraic thinking indicators around three content strands of algebra.

Generalized Arithmetic	**Functional Thinking**	**Modeling**
Writing a general statement to express an operation that includes an unknown .	Identifying the relationship in a pattern and describing it with words.	Transforming a real-life (or context-based) situation into a mathematical model.
Explaining the meaning of variable.	Finding close term of the pattern	
Using equal sign relationally.	Finding the distant Term of the Pattern	
Using the relationship between operation properties to form equivalent algebraic expressions.	Finding the corresponding terms of the pattern.	
	Formulating a general rule of a pattern algebraically.	
	Using the general rule of a pattern to find the distant term.	
	Drawing a graph or a table to represent relationships in a pattern.	

6. RESULTS

This study aimed to investigate whether lessons based on the RME approach enriched with three-act math tasks change the sixth-grade students' algebra achievement and attitudes toward mathematics significantly and to examine students' algebraic thinking during and at the end of the lessons. For this purpose, the first and the second research questions were analyzed quantitatively, and the third research question was analyzed qualitatively. The results of the analyses were presented in the following sub-sections.

6.1. Achievement of Students

In this study, the first research question is: "*Is there a statistically significant difference between sixth-grade students' achievement scores, as measured by the 'Achievement Test', before and after attending RME-based lessons enriched with three-act math tasks?*"

In order to examine whether there is a statistically significant difference between the pre-and post-test achievement scores of sixth-grade students who attended the RME-based lessons enriched with three-act math tasks, a paired sample t-test was conducted. The paired sample statistics results are presented in Table 6.1, while the paired sample t-test results are displayed in Table 6.2.

Table 6.1. Paired samples statistics.

	Mean	**N**	**Std. Deviation**	**Std. Error Mean**
Pre Test	17.250	46	10.1575	1.4976
Post Test	69.261	46	17.8334	2.6294

Table 6.2. Paired samples test.

	Mean	Std. Deviation	Std. Error Mean	t	df	Sig (2 tailed)
Post Test-Pre Test	52.0109	14.4724	2.1338	24.374	45	0.000

As tables shows; post test achievement scores (M = 69.26, SD = 17.83) is statistically higher than pre-test achievement scores (M =17.25, SD = 10.15) with a mean difference 52.01, t(45) = 24.37, p < .05, d = 3.59. According to Cohen (1988), the effect size (d= 3.59) shows a very large effect size.

6.2. Students' Attitude Towards Mathematics

This study's second research question is: "*Is there a statistically significant difference between sixth-grade students' mathematics attitude scores, as measured by the 'Mathematics Attitude Scale,' before and after attending RME-based lessons enriched with three-act math tasks?*"

To examine this research question, the Wilcoxon Signed-Rank Test was conducted. Table 6.3 below shows descriptive statistics related to pre-test and post-test scores of the Mathematics Attitude Scale.

Table 6.3. Descriptive statistics of the Mathematics Attitude Scale.

	N	Mean	Std. Deviation	Minimum	Maximum
Pretest	46	77.5000	11.70897	40.00	97.00
Posttest	46	82.1304	9.99246	51.00	96.00

As seen in the table above, the mean score of the post-test (M = 82.13, SD = 9.99) is higher than the mean score of the pre-test (M = 77.50, SD = 11.70). In order to examine whether the result is significant or not, the result of the Wilcoxon Signed Rank Test was analyzed. Table 6.4 and Table 6.5 shows the results of the Wilcoxon Signed Rank Test.

Table 6.4. Wilcoxon signed rank test results-ranks.

		N	Mean Rank	Sum of Ranks
Posttest - Pretest	**Negative Ranks**	13	14.65	190.50
	Positive Ranks	28	23.95	670.50
	Ties	5		
	Total	46		

Table 6.5. Wilcoxon signed rank test results-test statistics.

	Posttest - Pretest
Z	-3.112
Asymp. Sig. (2-tailed)	0.002

Wilcoxon Signed Rank Test Results showed a significant increase in mathematics attitude scores after the intervention compared to before the intervention, z = -3.112, $p < 0.05$. Specifically, 28 of the students have higher, 5 of the students have equal, and 13 of the students have lower mean rank on mathematics attitude scores compared to before the intervention. The effect size was calculated as r = -0.32, and it is a medium effect size.

6.3. Students' Algebraic Thinking

The third research question in this study is: "*How do sixth-grade students think algebraically during and at the end of the RME-based lessons enriched with three-act math tasks?*". The qualitative analysis was divided into two parts to answer this research question. In the first part, the data obtained from the lessons, which were in-class discussions, observation notes, and worksheets, were analyzed. In the second part, the data obtained from the interviews at the end of the lesson was analyzed. Analyses were conducted according to the three content strands of algebra: generalized arithmetic, functional thinking, and modeling. Categories for each content strand and details of the analysis framework were previously defined in the data analysis section (Table 5.7).

6.3.1. Students' Algebraic Thinking During the Instruction

The lesson's data comprise the worksheet in both three-act math tasks and follow-up activities, observation notes, and in-class discussion transcripts. Eight groups who attended classes regularly were selected, and their data was examined. These groups were labeled as G1, G2, G3, G4, G5, G6, G7 and G8 .

6.3.1.1. First Lesson. The objective of the first lesson, according to the MoNE (2018), is to write an algebraic expression appropriate to a verbally given situation and write a verbal situation appropriate to a given algebraic expression. The lesson prepared by considering this objective was examined under the generalized arithmetic and modeling strands under the categories of writing a general statement to express operations that include an unknown and transforming a real-world situation into a mathematical model.

In parallel with the main principle of realistic mathematics education, the lesson begins with a real-life problem, and students are expected to discover how to write a general statement to express an operation that includes unknown by mathematizing the real-life problem. In other words, they discover mathematics from a real-life situation and move from the real-life to the symbolic world of mathematics. This real-life problem was presented via a three-act math task, and students worked on creating a general statement to express an operation that includes an unknown quantity.

In the first act, students were shown a video of three tangerines being taken from a basket, and then they were asked what they were wondering and what they noticed. Then, they were asked to pose a mathematical problem. First, students worked as a group, and then the group ideas were discussed in the classroom.

In this discussion, students stated that they noticed three tangerines in the basket and wondered why three tangerines were taken from the basket, in the beginning, how many tangerines were in the basket, etc. To illustrate this, dialogue between students and teachers was given as follows.

T: What do you notice?
S1: There are tangerines.
T: What do you wonder?
S1: Why are there three tangerines?
S2: Why were they taken from the basket?
T: What do you notice?
S2: There are tangerines.
T: What do you wonder?
S2: Why are three tangerines taken from the basket?
S3: In the beginning, how many tangerines were in the basket?
T: Okay, I will ask you a question. What is the changing quantity in this basket?
S4: The number of tangerines in the basket decreases.
T: Okay, by looking at this video, what kind of mathematical problem can you pose?
S5: Three tangerines are taken from the basket.
T: But It should be mathematics problem, let's ask something.
S6: If we remove three tangerines from 10 tangerines, how many tangerines will remain?
S2: If we remove thee tangerines from 20 tangerines, how many tangerines will remain?
T:Okay, everyone, can you write your math questions on the worksheet?

5 minutes later, students' problems were shared as follows.

T: Is everyone finished? Now I will ask each group one by one. First group, what is your problem?
S1: Three tangerines were removed from a basket, and 27 tangerines were left in the basket. How many tangerines are in the basket at the beginning ?
T: Let's take the second group's question.
S2: There are 15 tangerines in a basket. If three tangerines were taken, how many tangerines were left in the basket?
T: Okay, let's take yours question.
S3: There are 18 tangerines in a basket. If three tangerines were taken, how many tangerines were left in the basket?

When worksheets were examined, other groups wrote similar questions. Students posed questions by using numbers. On the other hand, third group (G3) uses the term "unknown" in the question. This group question is "There are an unknown number of tangerines in a basket. How many tangerines will remain if we take three tangerines out of this basket?"

After students pose problems, to increase curiosity, the teacher wants students to estimate the number of tangerines left in the basket after three tangerines were taken. Different answers came up from students like 3, 10, 25. Then, the teacher asked a question in the classroom.

> T: Okay, I will ask you a question. What do you think what is the unknown from this video?
> S5: The number of tangerines in the basket is unknown.
> T: Okay, there is one more unknown in this question.
> S4: Why are three tangerines taken from the basket?
> S6: How many tangerines are left in the basket?
> T: Yes, we do not know the number of tangerines left in the basket after three tangerines were taken. Now, we will form a mathematical expression to express the number of tangerines left in the basket after three tangerines were taken from the basket. So, our focus question is, "How can we mathematically express the number of tangerines left in the basket after three tangerines are taken from the basket?". You can write our focus question and try to solve it.

During the second act, students worked on solving the main question which is *"How can we mathematically express the number of tangerines left in the basket after three tangerines are taken from the basket?"*

The students said that they needed to know how many tangerines were in the basket at the beginning to solve the problem. On the other hand, the teacher said that the total number of tangerines in the basket is unknown. The teacher distributes a worksheet to the students to give them a clue necessary for Act 2 and the guidance principle of RME. With this worksheet, students write how they would express the number of tangerines left in the basket if they knew the number of tangerines in the basket. For example, they are asked what they would write if five tangerines were in the basket. Students answered as 5-3. Students were asked what they would write if there were 15, 40, 1000, and 100,000 tangerines, respectively. All groups' answers were the same manner; they wrote 15-3, 40-3, 1000-3, 1000.000-3.

The teacher then asked the class, *"Suppose there are so many tangerines that you don't know the number; that is, the number of tangerines is unknown; how can*

we express tangerines left in the basket in general?". Then, students, as groups, wrote their own solutions on the worksheet.

The worksheets of eight different groups were examined. Their expression types were analyzed under the categories of "Writing a general statement to express an operation that includes unknown" and "Transforming a real-world (or context-based) situation into a mathematical model". Their expression types for the first category are given in the table with emerged codes from the data (Table 6.6). Each group wrote different and more than one statement to express the tangerines left in the basket. In each group's worksheet, there were five or six statements. Some examples were given in students' worksheets to clarify the description of each code. For instance, for the first code in the table, two examples were given in groups' worksheets, but the other statements they wrote were similar. In other words, they used different letters, but the expression format was the same, which means they subtracted "3" from a letter.

Table 6.6. Algebraic thinking indicators of groups regarding generalized arithmetic in the first lesson.

Writing a general statement to express an operation that includes an unknown			
Codes	**Description**	**Example**	**Groups**
General statements using letters	Students write general statements using letters.	x-3 a-3	G1, G5
Non-general statements using numbers	Students use arbitrary numbers to express operations; therefore, their statements are not general.	50-3 70-3	G4
Both general statements using letters and non-general statements using numbers	Students both write general statements using letters and non-general statements using arbitrary numbers.	60-3 b-3	G2 , G3
General statements using letters and shapes	Students use letters and shapes and write general statements.	p-3 □ -3	G6, G7
General statements using words	Students write general statements using the term "unknown"	Unknown number -3	G8

As Table 6.6 shows, their statements consist of numbers, shapes, letters, and the term “unknown”. According to worksheets, there are two groups out of eight groups who used only letters in the expression. Therefore, it can be said that these groups (G1 and G5) were able to write a general statement to express an operation that includes an unknown, and their expressions were algebraic. Also, two groups (G6 and G7) used letters and shapes in their statements. It can be said that these groups (G6 and G7) were able to write general statements and write algebraic expressions in some examples. Specifically, G7 first used shapes and then used letters; on the other hand, the expressions of G6 did not have a specific order. Besides, two groups (G2 and G3) used letters and numbers in their expression. When their worksheets were examined, both groups first used numbers in their expression. Then, their other expressions were formed by letters. In other words, they transformed their numeric expression into general form by using letters later. Therefore, it can be said that these groups (G2 and G3) were also able to write general expressions of an operation that includes unknown since they transformed their numeric expression into the algebraic form. However, one group (G4) used only numbers in their expression. They used arbitrary numbers for this. Therefore, they are far from writing a general statement. Besides, one group (G8) used the term “unknown” in their expression. They wrote their expression as the “unknown number – 3”. Even if they could not write algebraic expressions, they used a crucial term and wrote a general statement. Thus, in the ”Generalized Arithmetic” perspective, seven groups (G1, G2, G3, G5, G6, G7, G8) out of eight groups were able to write a general statement to express an operation that includes unknown. One group (G4) used arbitrary numbers in their expression, and their work is not an example of writing a general statement.

The groups’ expression regarding the modeling theme was analyzed under the category of “Transforming a real-life (or context-based) situation into a mathematical model”. Table 6.7 shows the codes and related groups as the following.

Table 6.7. Algebraic thinking indicators of groups regarding modeling in the first lesson.

Transforming a real-life (or-context based) situation into a mathematical model			
Codes	**Description**	**Example**	**Groups**
Appropriate algebraic expression	Students correctly transform real-life (or context-based) situations into an algebraic expression.	a-3 c-3 x-3	G1, G2, G3, G5, G6, G7
Appropriate expression with shapes	Students correctly transform real-life (or context-based) situations into expressions that include shapes for variables.	□ **-3**	G6, G7
Appropriate expression with words	Students correctly transform real-life (or-context based) situation into an expression that includes words.	"unknown number" -3	G8
No modeling	Modeling is not observed, students write expressions with numbers.	50-3 70-3	G4, G2, G3

In terms of the "Modeling" perspective, among the seven groups who can write a general expression of the operation, all groups' worksheets (G1, G2, G3, G5, G6, G7, and G8) include examples of modeling since they used letters, shapes, or words to transform a real-world situation into a mathematical model. Two groups (G1 and G5) among these seven groups used only letters in their expression and formed only algebraic expressions. Two groups (G6 and G7) not only used letters but also used shapes in their expressions. On the other hand, in two groups' worksheets (G2 and G3), expressions that cannot be regarded as mathematical models were also observed, although they used letters in some statements they wrote. In other words, these groups (G2 and G3) also wrote the expressions with numbers, which means not a mathematical model. One group (G8) transformed a real-world situation into a general expression by using a word that is the term "unknown". Thus, all groups who could write a general

statement could also transform a real-world situation into a mathematical model by using various methods. On the other hand, one group's worksheet (G4) does not include an example of modeling since this group used only numbers in their expressions.

In the third act, groups shared their expressions. The teacher asked them to discuss which expressions could be a general statement of this operation that includes an unknown. According to the discussion, the class concluded that expressions using shapes, letters, and words are general expressions to express an operation that includes an unknown quantity. Later, a video of the solution for the focus question was shown in class. Then, it was explained that expressions containing at least one unknown and operation are called algebraic expressions. The teacher explained that although these general expressions students formed can contain shapes and words, algebraic expressions generally contain "letters", and these letters are called "variables". Then, a worksheet that included examples of converting verbal situations to algebraic expressions and vice versa was solved by classroom discussion, and the lesson ended.

6.3.1.2. Second Lesson. The second lesson objective, according to the MoNE (2018), is the same as the first lesson's objective, which is writing an algebraic expression appropriate to a verbally given situation and a verbal situation appropriate to a given algebraic expression. This objective was included in the lesson via activities with two themes: Generalized Arithmetic and Modeling, which are two content strands of Algebra, according to Kaput (2008). The lesson started with a brief review of what was learned in the previous lesson. Then, the introduction part of the lesson started with a three-act math task.

In the first act, students watched a video that showed two glasses of bulgur being placed in a bowl. Students were then asked what they noticed and wondered. The discussions in the classroom and their worksheet show that they noticed there were two glasses of bulgur potted in the bowl, and they wondered why the bulgur spilled in the bowl, the amount of bulgur, and the mass of the bowl. Then, the teacher asked students to pose mathematical questions. In general, they created mathematical

questions related to the amount of bulgur in the bowl. The worksheets for G5 in Figure 6.1 and G6 in Figure 6.2 are given to illustrate this.

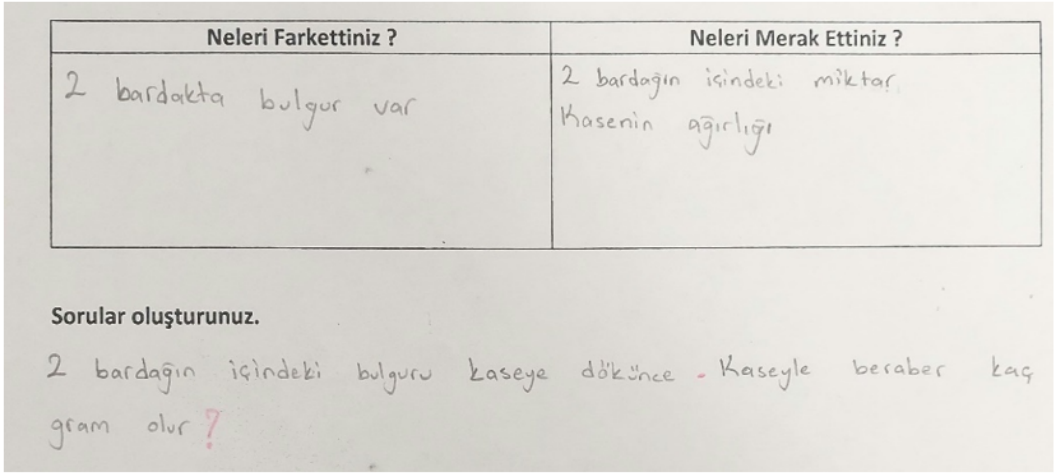


Figure 6.1. Worksheet of G5 in the ACT 1 of the second lesson.

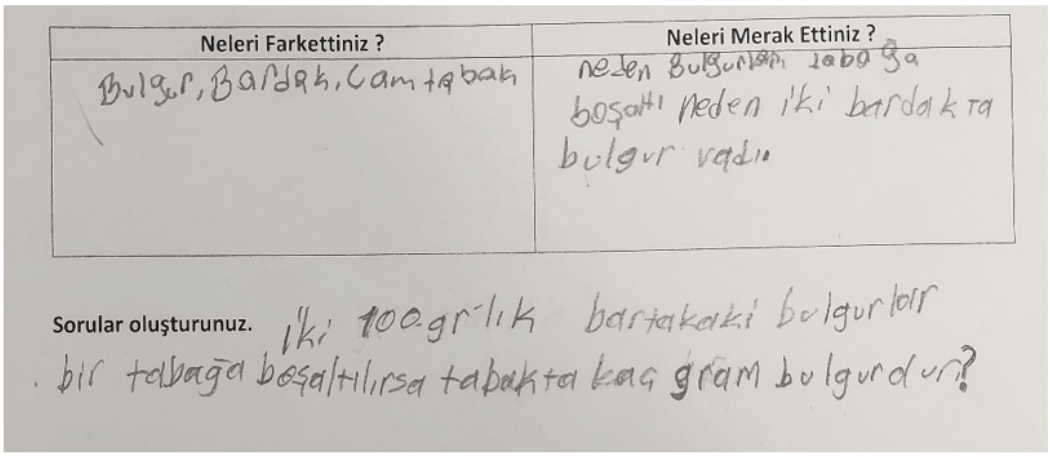


Figure 6.2. Worksheet of G6 in the ACT 1 of the second lesson.

After that, the teacher asked students' estimation of how much bulgur could be in the bowl. Different answers came up as 100gr, 150gr, and 500gr. Then, the focus question was determined after the dialogue below.

T: What do we know in this video?
S9: We know that there are two glasses of bulgur in a bowl.
T: Well, what we don't know in this video ?
S20: The number of bulgur.
S12: How many grams of bulgur is.

Then, the teacher wrote the focus question “After pouring two glasses of bulgur into the bowl, how can we mathematically express how much bulgur is in the bowl”.

During the second act, the teacher asked them what information they should know to solve the problem. They stated that they should know how much bulgur in one glass. Then, the teacher showed Figure 6.3 as a clue and distributed a worksheet that included how they expressed how much bulgur in the bowl if they knew how much bulgur in each glass.



Figure 6.3. Clue for main question.

Later, students worked on these worksheets. In this worksheet, there were questions that asked to express both additively and multiplicatively how much bulgur would be in the glass if we assumed that there were 20 grams, 26 grams, 50 grams, and 100 grams of bulgur in a glass, respectively. All groups answered these questions correctly. For example, if it was 20 grams, they expressed it as 20+20 additively and 2.20 multiplicatively. These worksheets and students’ answers in the discussion below show that students could think both additively and multiplicatively to find the total bulgur in the bowl with the known values.

> T: Everyone, if there is 20 gr bulgur in one glass, how can we find how much bulgur in this bowl?

S5: We can find it by summation.
S2: We can find it by multiplication.
T: Yes, we can find it by using summation and multiplication.

Then, they moved on to the other part, where the total amount of bulgur in a glass was unknown, and created their own expressions. According to their worksheets, groups' expression types were given in Table 6.8 under the generalized arithmetic and Table 6.9 under modeling. Students wrote various expressions in their worksheets. Some examples were given from their worksheets. Their expressions were similar to the given examples.

Table 6.8. Algebraic thinking indicators of groups regarding generalized arithmetic in the second lesson.

Writing a general statement to express an operation that includes an unknown			
Codes	**Examples of additive**	**Examples of multiplicative**	**Groups**
General statements using letters	x + x a + a	2x 2a	G1, G2, G3, G4, G5, G7
General statements using shapes	□ □ +□ □	No Answer	G6
Incorrect general statements that do not fit the context.	? + 20 y + 20 y + m	?.20 y.20 2xym	G8, G7
Using the relationship between operation properties to form equivalent algebraic expressions.			
Codes	**Description**	**Examples of Additive**	**Examples of Multiplicative**	**Groups**
Multiplication as repeated addition	Students use the relationship between multiplication and addition, thus write equivalent forms.	x + x a + a	2x 2a	G1, G2, G3, G4, G5, G7
Only repeated addition without equivalent form	Students write expressions with addition and do not write equivalent forms.	□ □ +□ □	No Answer	G6
Incorrect equivalent form.	Students attempt to write an equivalent form of an algebraic expression but incorrectly write.	? + 20 y + 20 y+m	?.20 y.20 2xym	G7, G8

Table 6.9. Algebraic thinking indicators of groups regarding modeling in the second lesson.

Transforming a real-life (or-context based) situation into a mathematical model			
Codes	**Description**	**Examples**	**Groups**
Appropriate algebraic expression	Students correctly transform real-life (or context-based) situations into an algebraic expression.	x + x, 2x a + a, 2a	G1, G2, G3, G4, G5, G7
Appropriate expression with shapes	Students correctly transform real-life (or context-based) situations into expressions that include shapes for variables.	□ □ + □ □	G6
Inappropriate expression with symbols and letters	Students transform real-life (or context-based) situations into expressions by using letters and symbols, but the expressions are not appropriate to the context.	?+20, ?.20 y+20, y.20 y+m, 2xym	G7, G8

As shown in Table 6.8 and Table 6.9 above, six groups (G1, G2, G3, G4, G5, G7) out of eight groups wrote correct general expressions both additively and multiplicatively, and their expressions were algebraic. Therefore, these groups were able to establish a relationship between multiplication and addition and use the relationship between operation properties to form equivalent algebraic expressions. They could also transform a real-world situation into a mathematical expression in a correct way by using letters. On the other hand, among these six groups, one group (G7) had an incorrect expression as well. This group (G7) wrote all the expressions correctly except "y + m".

In this example (y+m), students used different letters to represent equal amounts. This group also wrote the multiplicative form of this as 2xym, suggesting that they

could not use operation properties to form equivalent algebraic expressions when the variables are different. Their other expressions were regarded as correct in both additive and multiplicative perspectives.

One group (G8) transformed a real-world situation into a mathematical model using symbols and letters. On the other hand, their expressions were incorrect since they wrote ? +20 and y + 20 to express the total grams of bulgur as if the amount of one of the glasses was known. They also multiplied the terms of their expressions like ?.20 and y.20. Therefore, this group could not use operation properties to form equivalent expressions and could not write correct general statements in both additive and multiplicative perspectives.

One group (G6) used shapes to write a general statement; therefore, this group could transform a real-world situation into a mathematical model using only shapes. On the other hand, there is no evidence of using operation properties to form equivalent expressions since there is no finding about multiplicative examples.

During the third act, the teacher asked students their solutions. Students shared their solutions like x+x, 2x, and y+y. From this, the concept of "coefficient", "term" and "similar term" was explained, and some examples were given. Besides, the teacher explained that shapes and other symbols are not commonly used when writing algebraic expressions, then showed the solution to the focus question via a video, which showed how to write the algebraic expression for two glasses of bulgur in the bowl from an additive and multiplicative perspective. Then, there was a classroom discussion as follows.

T: What is the relationship between addition and multiplication?
S1: For example, when someone asks 2+2, we can write 2.2.
T: Okay, other answers?
S2: For example, we can write 5.3 instead of 5+5+5.
S5: The shortcut to addition is multiplication.
S17: Exponent numbers can be an example.
T: But exponent numbers are repeated multiplication. For example, you can write a.a.a = a^3

As seen in the above, there was a misconception regarded repeated addition as exponents. On the other hand, students' worksheets and the majority of discussions show that most of the students make correct relationships between addition and multiplication and use operation properties to form equivalent expressions. Discussion at the end of the lesson supports this, as seen below.

T: How can you write x+x as multiplication?
S1: 2 times x.
T: Okay, what is the coefficient here?
S1: Here, 2.
T: Okay, there is another question. How can you write x+x+x+x as multiplication?
S2: 4 times x.
S1: 4 times x.
T: Why?
S1: There is no need to long way.
S2: Because there are 4 "x".

In the second 40 minute part of the lesson, a worksheet containing a cafe menu design activity was distributed to students. Students created their own productions as groups thanks to the cafe design activity. The prices of the food on the menu in this cafe are unknown. Using the concept of "variable" learned before, students used variables instead of unknown prices.

Then, they created double and triple menus by writing algebraic expressions. During this activity, the teacher wants students to transform their algebraic expressions into equivalent forms by using operation properties. In other words, students are expected to write more than one algebraic expression that is equivalent.

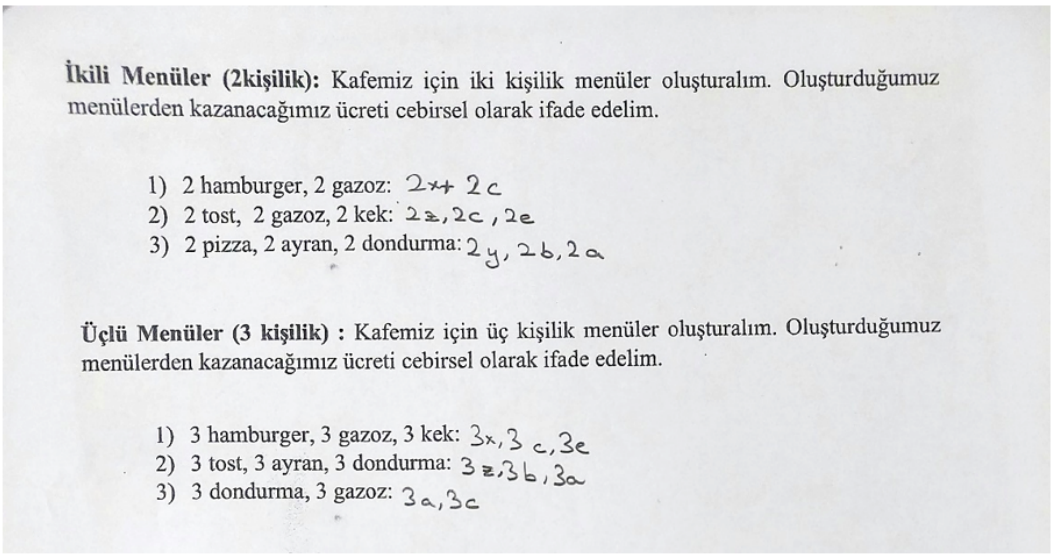
The menu design worksheet of the first group (G1) is given in Figure 6.4 as an example. Other groups have the same worksheets.



Figure 6.4. First group's menu design worksheet.

They used variables for the price of the hamburger as "x", pizza as "y", toast as "z", ayran as "b", soda as "c", cake as "e", and ice cream as "a". Then, they created double and triple menus, as seen in Figure 6.5 below. Therefore, the work of this group is an example of modeling since they transformed a context-based situation into a mathematical expression by using letters. The teacher wanted students to write their expressions by considering operation properties in multiple ways. On the other hand, this group wrote their expression by multiplication only. Therefore, no evidence exists of using the relationship between operation properties to form equivalent algebraic

expressions. Besides, except for the first expression, there is no summation symbol between prices to represent the total income gained. They wrote general expressions to express the price of each food separately. Therefore, other expressions, except the first example, are not general expressions that will express the total income but general expressions that express the price of each food separately.



İkili Menüler (2kişilik): Kafemiz için iki kişilik menüler oluşturalım. Oluşturduğumuz menülerden kazanacağımız ücreti cebirsel olarak ifade edelim.

1) 2 hamburger, 2 gazoz: 2x + 2c
2) 2 tost, 2 gazoz, 2 kek: 2z, 2c, 2e
3) 2 pizza, 2 ayran, 2 dondurma: 2y, 2b, 2a

Üçlü Menüler (3 kişilik) : Kafemiz için üç kişilik menüler oluşturalım. Oluşturduğumuz menülerden kazanacağımız ücreti cebirsel olarak ifade edelim.

1) 3 hamburger, 3 gazoz, 3 kek: 3x, 3c, 3e
2) 3 tost, 3 ayran, 3 dondurma: 3z, 3b, 3a
3) 3 dondurma, 3 gazoz: 3a, 3c

Figure 6.5. Representation for double and triple menus' income of G1.

G2 used the initial letters of food names as the variable. Their worksheet is presented in Figure 6.6. They transformed a context-based situation into algebraic expressions using letters, and their work is a mathematical model and general statement to represent total income from the menu. On the other hand, they used only addition to represent the income from the menus. There is no other way like multiplication. Therefore, no evidence exists of using operation properties to form an equivalent algebraic expression. Besides, they made a mistake since they used both uppercase and lowercase letters to represent the same variable. The teacher then explained that uppercase and lowercase letters are not used to represent the same variable in algebraic expressions and that variables in algebraic expressions are usually represented by lowercase letters.

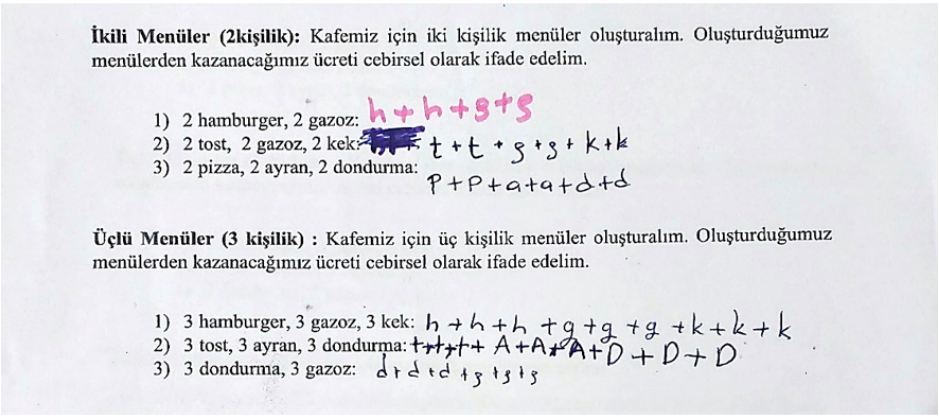


Figure 6.6. Representation for double and triple menus' income of G2.

In contrast to G2, G3 did not use addition and only used the multiplicative method in their worksheet, as seen in Figure 6.7 below. On the other hand, both groups are similar in using the initial letters of food names as variables. Their work can be regarded as modeling since they wrote algebraic expressions to represent the total income from the menu. Besides, their work is a general expression to represent total income. On the other hand, since there is no different way to represent the total income as an algebraic expression, no evidence exists to use the relationship between operation properties to form equivalent algebraic expressions.

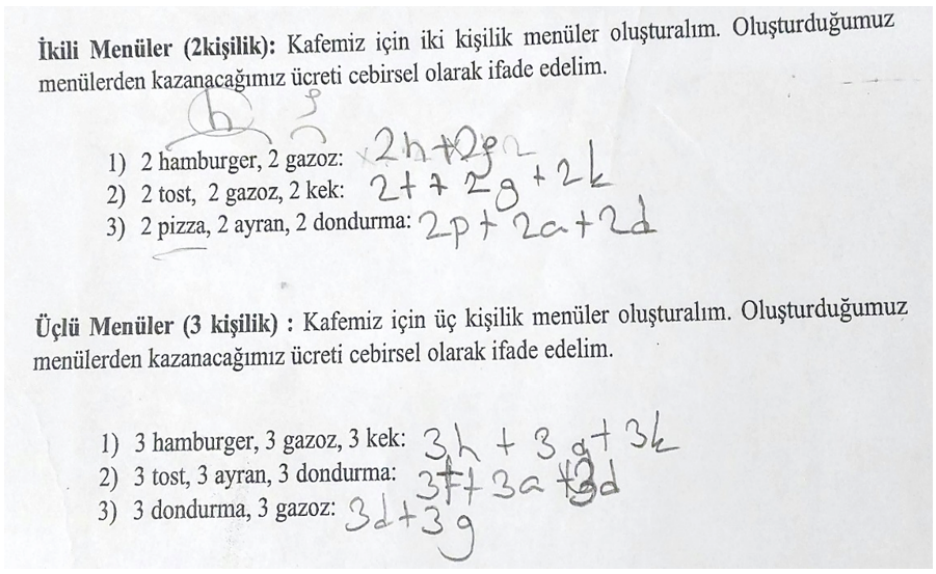


Figure 6.7. Representation for double and triple menus' income of G3.

Different from the first three groups (G1, G2, and G3), G4 wrote the total income from the menus in two different ways. They used both multiplication and addition in their worksheet, as seen in Figure 6.8 below. Similar to G2 and G3, they used initial letters of food names as variables. Besides, the equal sign between expressions shows that they used equal sign relationally, which means they could have an understanding that the expressions on both sides of the equal sign are equivalent rather than an operational understanding that "the answer comes next". Their studies can be regarded as general expressions to express the total income, and they translated a situation into an algebraic expression using letters and wrote equivalent algebraic expressions using the relationship between operation properties.

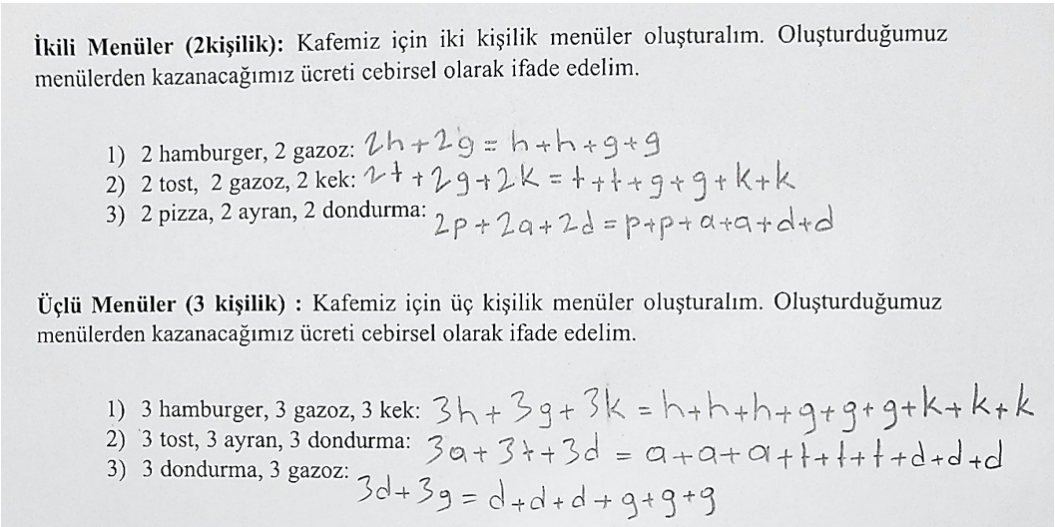


Figure 6.8. Representation for double and triple menus' income of G4.

When the fifth group (G5) worksheet was examined, it was seen that they used three different methods to represent the total income from the menus. As seen in Figure 6.9 below, they used multiplication, repeated addition, and the distributive property of multiplication over addition. Therefore, unlike the first four groups (G1, G2, G3, and G4), this group wrote equivalent algebraic expressions using the relationship among three different operations properties. The equal sign between the expressions shows they used equal signs relationally. Therefore, they could have a relational understanding of equal signs rather than an operational understanding, which is "the answer comes next". Similar to G2, G3, and G4, this group also used the initial letters

of the food names to represent variables. Therefore, this group wrote three different general statements to represent the total income from the menu, and their works can be examples of modeling since they used letters and wrote algebraic expressions to transform a situation into a mathematical model. Besides, they used relationships between operation properties to form equivalent algebraic expressions.

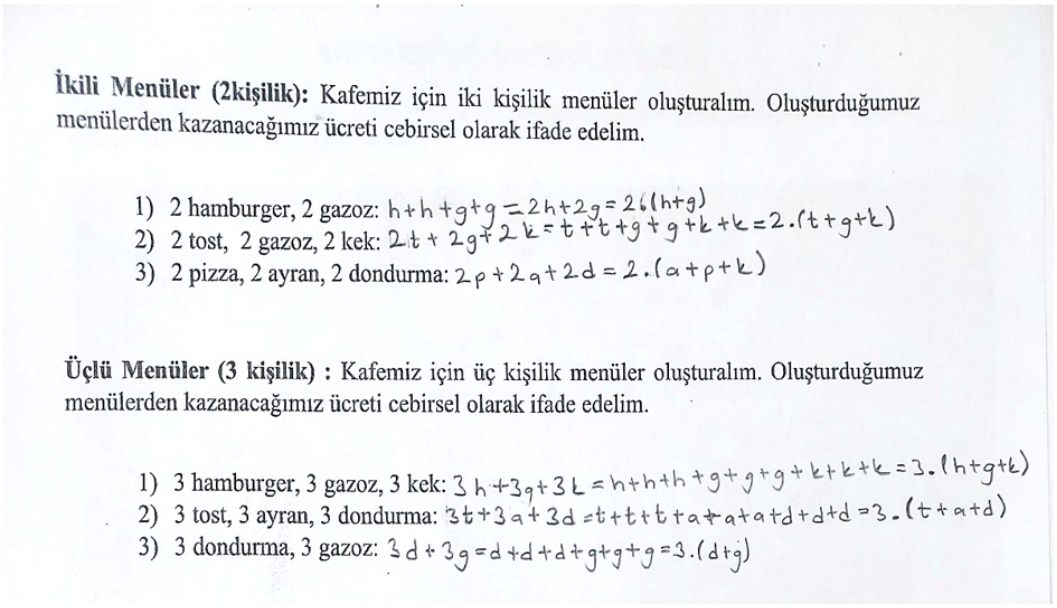
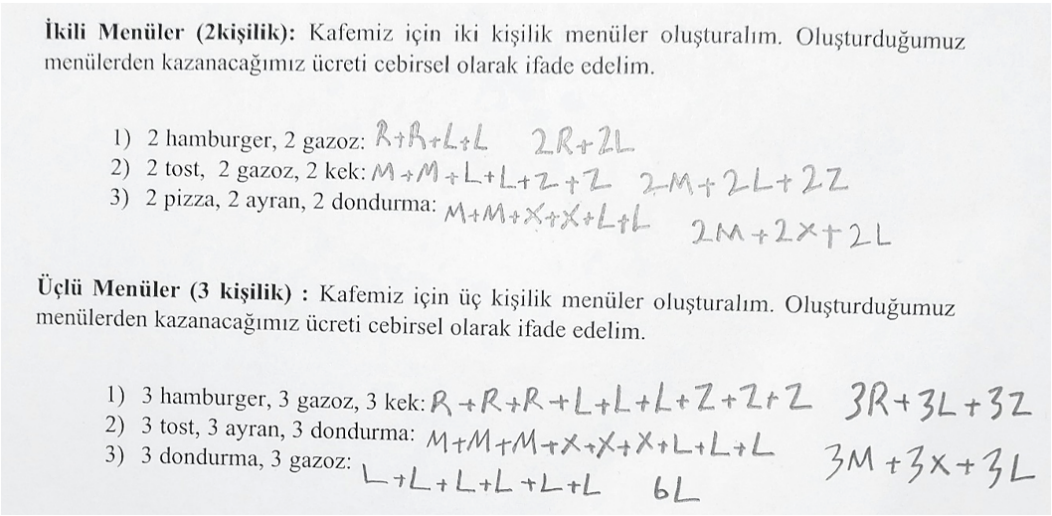


Figure 6.9. Representation for double and triple menus' income of G5.

G6 also transformed a context-based situation into a mathematical model by representing variables with letters and writing algebraic expressions. They used relationships between operation properties to form equivalent algebraic expressions. The Figure 6.10 below shows their works. Their worksheet shows that they used both additive and multiplicative methods in their expression, like G4. Unlike G4, they did not use the equal signs to represent two expressions' equivalence. Different from G2, G3, G4 and G5, this group did not use the initial letters of the food names as the variable.

They attached the variable "R" for the price of the hamburger, "L" for the price of the soda, "M" for the price of the toast, "Z" for the price of the cake, "M" for the price of the pizza, "X" for the price of the ayran, "L" for the price of the ice cream. They used the same variables to represent the price of the toast and pizza and the price

of the soda and ice cream. Besides, like G2, they used uppercase letters to represent variables. The teacher later explained that algebraic expressions are generally formed by lowercase rather than uppercase letters, and there should be different variables to represent the cost of the different foods.


İkili Menüler (2kişilik): Kafemiz için iki kişilik menüler oluşturalım. Oluşturduğumuz menülerden kazanacağımız ücreti cebirsel olarak ifade edelim.

1) 2 hamburger, 2 gazoz: R+R+L+L 2R+2L
2) 2 tost, 2 gazoz, 2 kek: M+M+L+L+Z+Z 2M+2L+2Z
3) 2 pizza, 2 ayran, 2 dondurma: M+M+X+X+L+L 2M+2X+2L

Üçlü Menüler (3 kişilik) : Kafemiz için üç kişilik menüler oluşturalım. Oluşturduğumuz menülerden kazanacağımız ücreti cebirsel olarak ifade edelim.

1) 3 hamburger, 3 gazoz, 3 kek: R+R+R+L+L+L+Z+Z+Z 3R+3L+3Z
2) 3 tost, 3 ayran, 3 dondurma: M+M+M+X+X+X+L+L+L 3M+3X+3L
3) 3 dondurma, 3 gazoz: L+L+L+L+L+L 6L

Figure 6.10. Representation for double and triple menus' income of G6.

G7 also transformed a context-based situation into a mathematical model by using letters to represent variables as seen in the Figure 6.11. On the other hand, they used only the multiplicative method in their expressions like G3 and G1. Besides, there is no sum symbol between the prices of the different foods to represent the total income gained.

Therefore, their expressions do not state the total cost; instead, they state each food's cost separately, and there is no evidence of using relationships between operation properties to form equivalent algebraic expressions. Besides, they used multiplication symbols. The teacher later explained that multiplication symbols generally are not used in algebraic expressions. Like G6 and G1, this group did not use the initial letters of the food names as the variable.

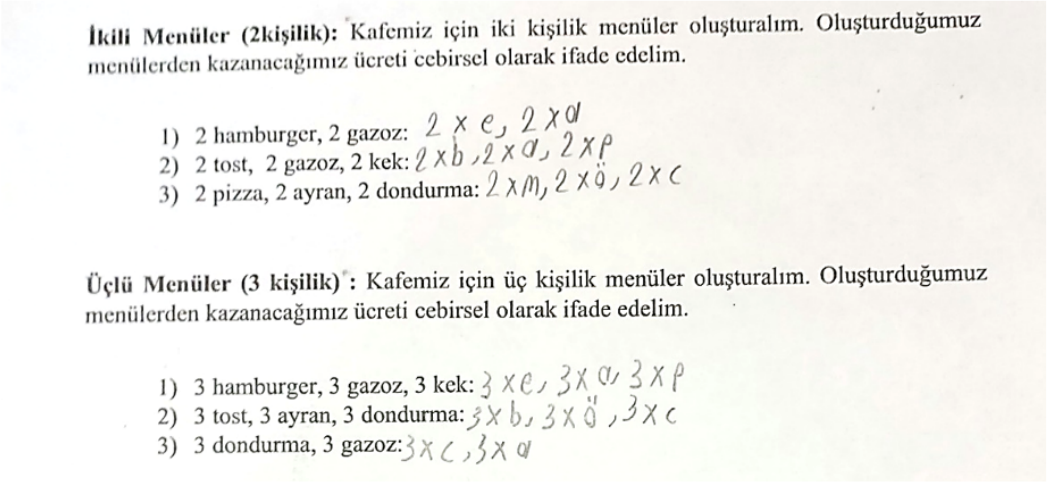


Figure 6.11. Representation for double and triple menus' income of G7.

Similar to G3, G7, and G1; G8 used only the multiplicative method in their expressions, as seen in Figure 6.12. Therefore, no evidence exists of using operation properties to form equivalent algebraic expressions. They model context-based situations by writing algebraic expressions, and they did not use the initial letters of the food names as the variable similar to Groups G7, G1, and G6.

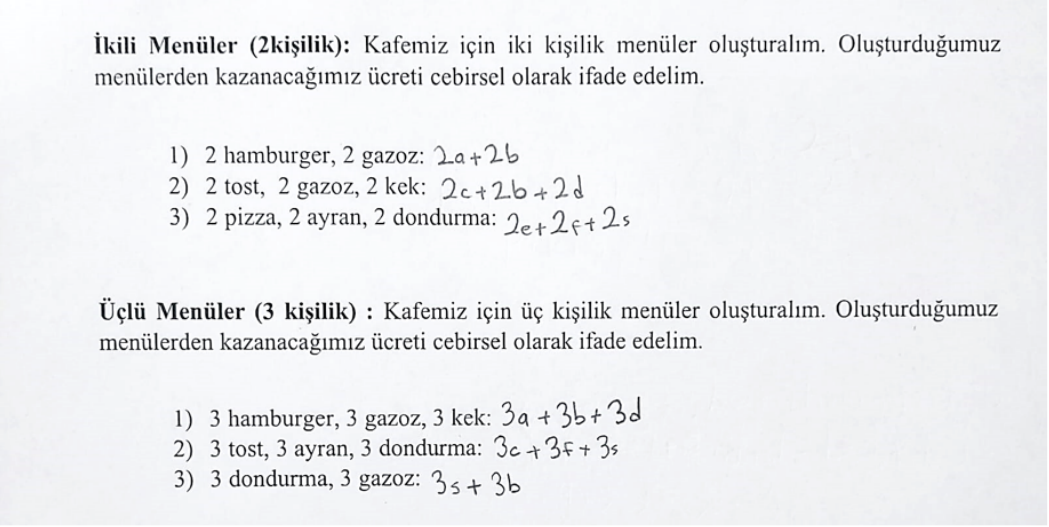


Figure 6.12. Representation for double and triple menus' income of G8.

After students finished their work, the teacher re-explained the concept of the "term", "similar term", and "coefficient" for algebraic expressions by giving examples from their worksheets. Besides, different examples regarded "term" and "similar

term" were given, and students' expressions were discussed around these concepts and properties of operations they used.

Table 6.10 below summarizes the groups' algebraic thinking indicators in terms of generalized arithmetic and modeling. All groups were able to write a general statement to express an operation that includes unknown, and their expressions were algebraic. Therefore, at the same time, all groups transformed a context-based situation into a mathematical model by using letters to represent variables. On the other hand, there is no finding in worksheets of G1, G2, G3, G7 and G8 regarding utilizing a relationship between operation properties to form equivalent algebraic expressions since they used only one way to represent the total income gained from the menus. In contrast, groups G4, G5, and G6 showed evidence of using a relationship between operation properties to get equivalent algebraic expressions since they represent total income in multiple ways. Specifically, groups G4 and G6 made a relationship between multiplication and addition, and G5 wrote three different equivalent expressions using multiplication, addition, and the distributive property of multiplication over addition. Their work (G5) formed evidence of making a relationship among these three operation properties. Besides, among groups who were able to write equivalent algebraic expressions, G4 and G5 used equal signs relationally by representing the equivalence of equivalent algebraic expressions.

Table 6.10. Algebraic thinking indicators regarding both themes.

Using the relationship between operation properties to form equivalent algebraic expressions.			
Codes	**Description**	**Example**	**Groups**
Only repeated addition without equivalent form	Students write expressions with addition and do not write equivalent forms.	h+h+h+h x+x+x	G2

Table 6.10 Algebraic thinking indicators regarding both themes. (cont.)

Codes	**Description**	**Example**	**Groups**
Only multiplication without equivalent form	Students write expressions with only multiplication and do not write equivalent forms.	4h, 4xh, 3x	G1, G3, G7, G8
Multiplication as repeated addition	Students use the relationship between multiplication and addition, thus write equivalent forms.	h+h+h+h, 4h x+x+x, 3x	G4, G6
Multiplication as repeated addition and Distributive Property	Students use the relationship between multiplication and addition use distributive property, and write equivalent forms.	3d+3g d+d+d+g+g+g 3(d+g)	G5
Using the equal sign relationally.			
Codes	**Description**	**Example**	**Groups**
Using	Students use equal sign relationally.	3d+3g=d+d+d+g+g+g	G4, G5
No Equal Sign.	There is no equal sign	3d+3g, d+d+d+g+g+g	G1, G2, G3, G6, G7, G8
Transforming a real-life (or-context based) situation into a mathematical model			
Codes	**Description**	**Example**	**Groups**
Appropriate algebraic expression	Students correctly transform real-life (or context-based) situations into an algebraic expression.	h+h+a+a 2h+2a	G1, G3, G4, G5, G7, G8
Partially appropriate algebraic expression	Students correctly transform real-life (or context-based) situations into an algebraic expression	Represent the price of the toast and pizza with same variables.	G2, G6

Table 6.10 Algebraic thinking indicators regarding both themes. (cont.)

Codes	Description	Example	Groups
	but use the same letters for different variables.		

6.3.1.3. Third Lesson. According to MoNE (2018), the third lesson objective is calculating the value of the algebraic expression for different natural number values that the variable will take. Thus, via the three-act math task and activities, it was aimed to provide students with the knowledge that numbers can take variables' place and that it is possible to find the value of the algebraic expression with these number values. This objective was combined with the generalized arithmetic theme.

A meaningful introduction to the lesson was provided with a three-act math task that allowed the transition from real life to the world of mathematics. In the first act of the three-act math task, Figure 6.13 below was shown to the students, and then they were asked what they noticed and wondered.



Figure 6.13. Photo for ACT 1.

Students stated that they noticed different foods like hamburgers, pasta, and cola, and each letter represented food. They also stated that they wondered about the price of foods and orders, grams of pasta, etc. Then, the teacher wanted the students to pose a mathematics question regarding this photo. Some examples from the groups' answers were given in Figure 6.14 and Figure 6.15. Other groups wrote similar answers in their worksheets.

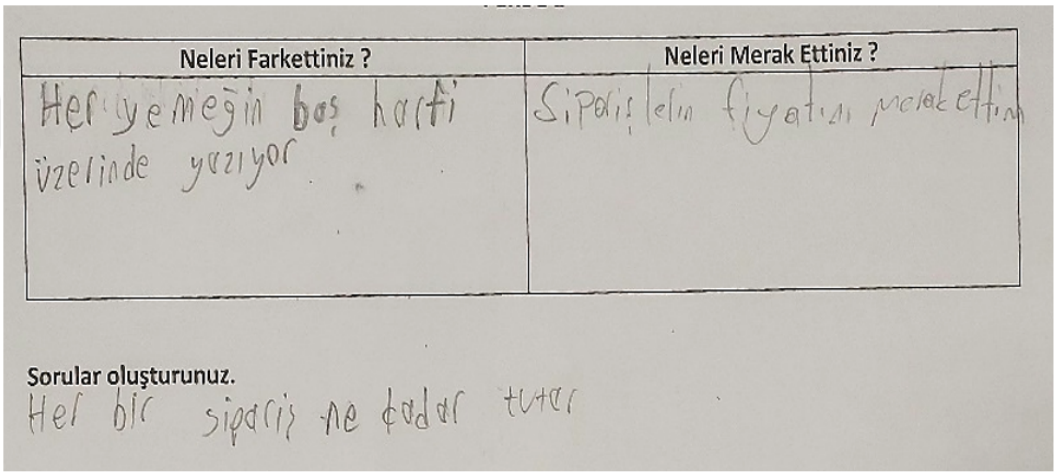


Figure 6.14. Answers of G6 in ACT 1.

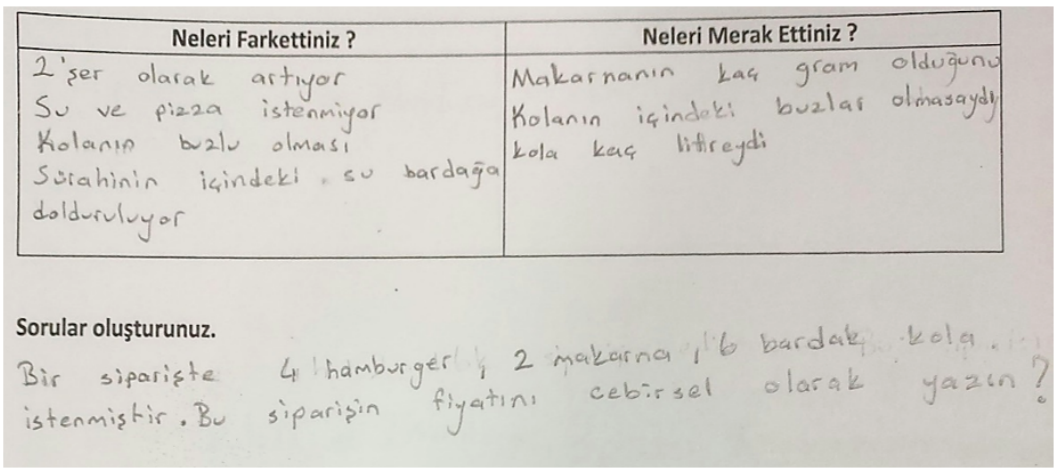


Figure 6.15. Answers of G5 in ACT 1.

After students shared what they noticed and wondered, the focus question was determined as "How much will the order cost?". Then, the teacher wanted them to estimate the cost of the order. Different answers came up as 200, 350 and 120 liras.

During the second act, the teacher asked students which information they needed to know to calculate the order cost. The dialogue between students and the teacher is given below.

> T: Yes, there is an order note. We will research the cost of the order. Which information we need to know to calculate this?
> S17: Price of the foods.
> S15: Prices
> T: Yes, we should know the prices. If we know the price of the hamburger, pasta, and cola, we can calculate the cost of the order.
> Class: Yes.

Then, the teacher showed a photograph as a clue. In this photograph, there is a menu, and each variable represents the price of different foods. Students looked at the menu and started to work on calculating the cost of the order. The teacher distributed a worksheet for their work, and in this worksheet, there was an additional question that asked students to transform the algebraic expression into equivalent forms. The teacher indicated they should write equivalent algebraic expressions using operation properties they knew.

The worksheet of G1 is shown in Figure 6.16. They calculated the cost of the order by multiplying the product price by the number of products separately. As shown in the photograph, they transformed the algebraic expression into an equivalent by making a repeated addition. Therefore, it was observed that this group used a relationship between multiplication and addition to form equivalent algebraic expressions.

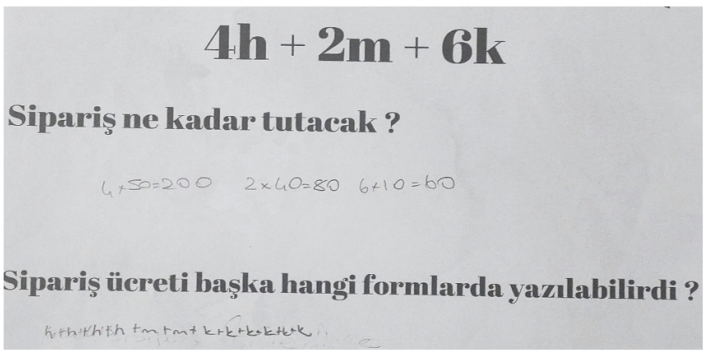


Figure 6.16. The worksheet of G1 in ACT 2.

The worksheet of G2 is presented in Figure 6.17. They calculated the cost of the order by calculating each food's price differently and then by adding them. During their calculation, instead of multiplication, they made repeated additions. They also transformed the algebraic expression into an equivalent by making repeated additions but did not sum their representations. Although they did not write the total cost of the order, it was observed that this group established a relationship between multiplication and addition in algebraic expressions and used this relationship to form equivalent algebraic expressions. They also used the equal sign relationally to represent the equivalence of expressions. Besides, it was observed that this group wrote one more additional food cost in their expression that is P+P+P+P+P+P+P=7P, and wrote uppercase letters in their expression that are not usual.

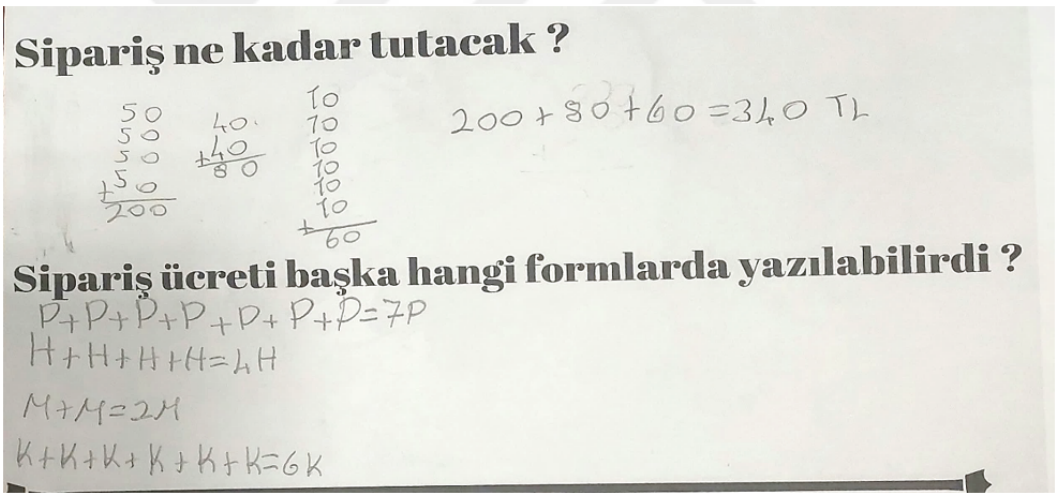


Figure 6.17. The worksheet of G2 in ACT 2.

G3 used the commutative property of addition to write equivalent form, as shown in Figure 6.18. They changed the order of the terms. Thus, It was observed that they understood that in algebraic expressions, the result does not change when the place of the collected terms is changed. For this reason, they made sense of the commutative property of addition in algebraic expressions. Besides, they correctly calculated the order cost. This showed that they recognized that variables stand for the numbers, and they made operations.

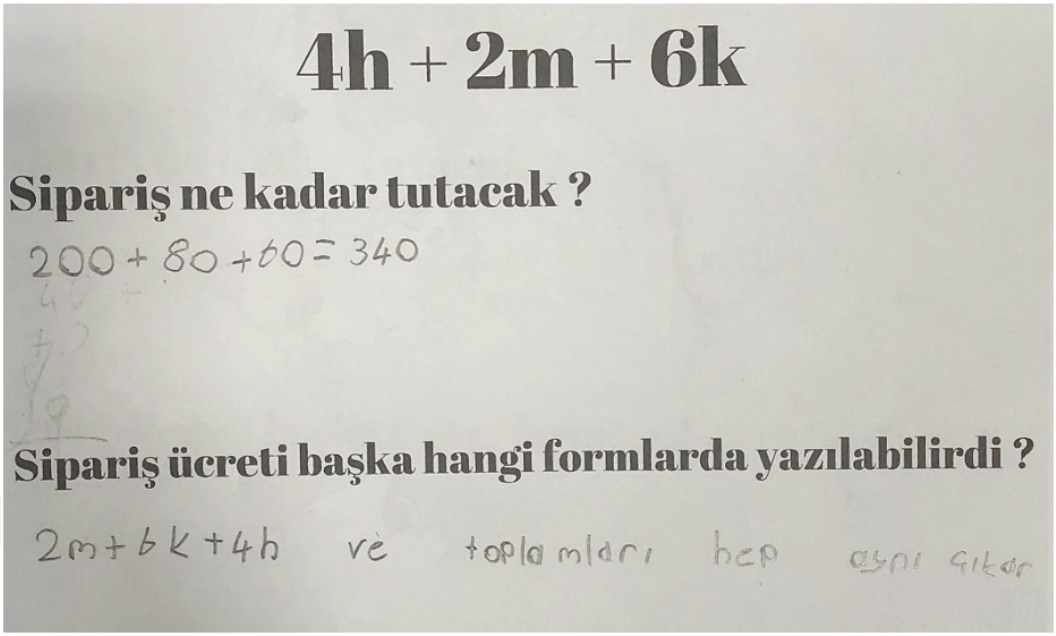


Figure 6.18. The worksheet of G3 in ACT 2.

G4 calculated the order cost similarly to G3. Both groups summated the cost of four hamburgers, two pasta, and six colas. G4 wrote equivalent expressions using both repeated addition form and commutative property of addition. Therefore, it was observed that this group established a relationship between multiplication and addition and was able to use the commutative property of addition in algebraic expressions. Their worksheet is presented in Figure 6.19 below.

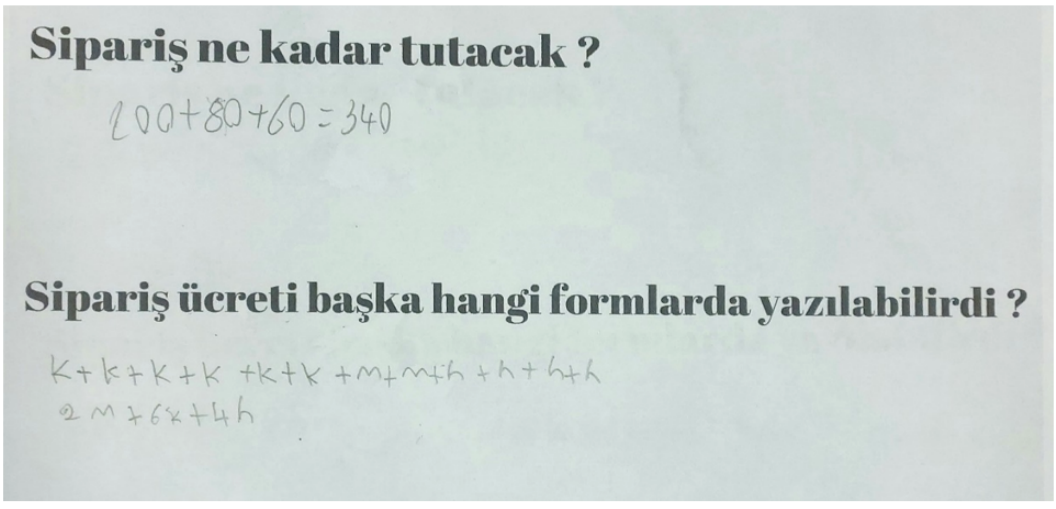


Figure 6.19. The worksheet of G4 in ACT 2.

G5 used a similar method to G3 and G4 to calculate the cost of the order. In other words, they summated the cost of four hamburgers, two pasta, and six colas. They transformed the order cost into equivalent by making repeated additions, and this shows that they established a relationship between multiplication and addition even though they did not sum the cost of three foods after they wrote them with the addition method. They also used the equal sign relationally to represent the equivalence of expressions that are on both sides. They also changed the place of the collected terms, and this shows that they used the commutative property of addition in algebraic expressions. Their worksheet is presented in Figure 6.20 below.

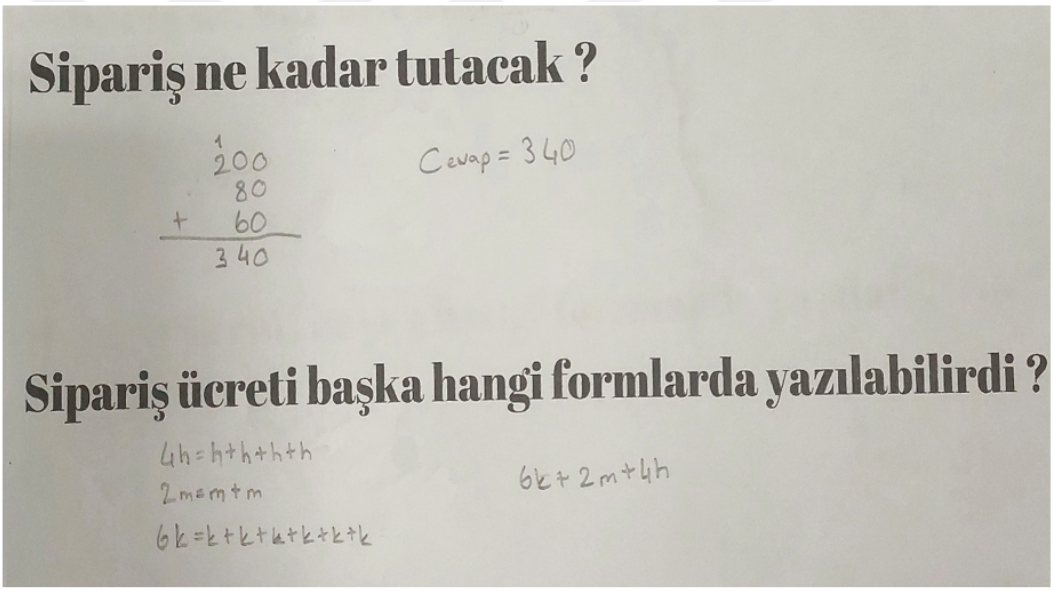


Figure 6.20. The worksheet of G5 in ACT 2.

G6 used a similar way to calculate the cost of the order. They summed the cost of four hamburgers, two pastas, and six colas, as seen in Figure 6.21. Their worksheet shows that they used the commutative property of addition since they changed the place of collected terms when writing equivalent algebraic expressions. They also wrote their expression in the equation format using the equal sign, which suggests their relational understanding of the equal sign.

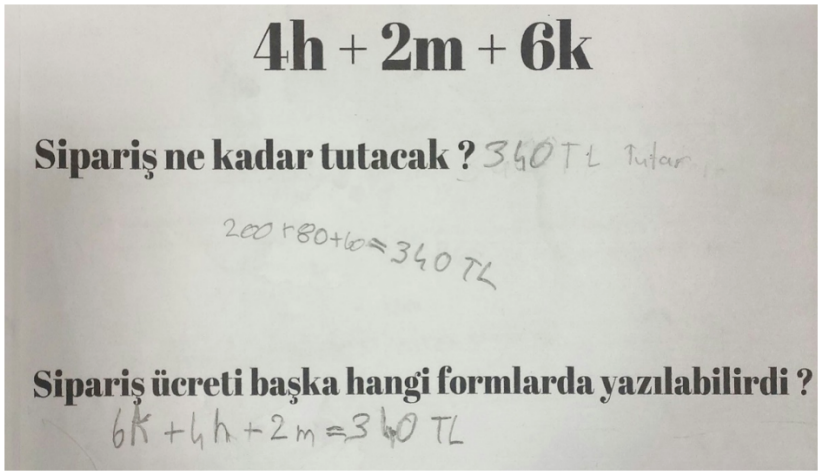


Figure 6.21. The worksheet of G6 in ACT 2.

When the worksheet of G7 was examined, it was seen that they used repeated addition when they calculated the cost of the order and wrote the equivalent form. Therefore, this group could establish a relationship between multiplication and addition in algebraic expressions. It was also clearly seen that they placed the numbers instead of variables and calculated the cost in this way. They also wrote their expression by using the equal sign in an equation format; this suggests their relational understanding of the equal sign. Their worksheet is presented in Figure 6.22.

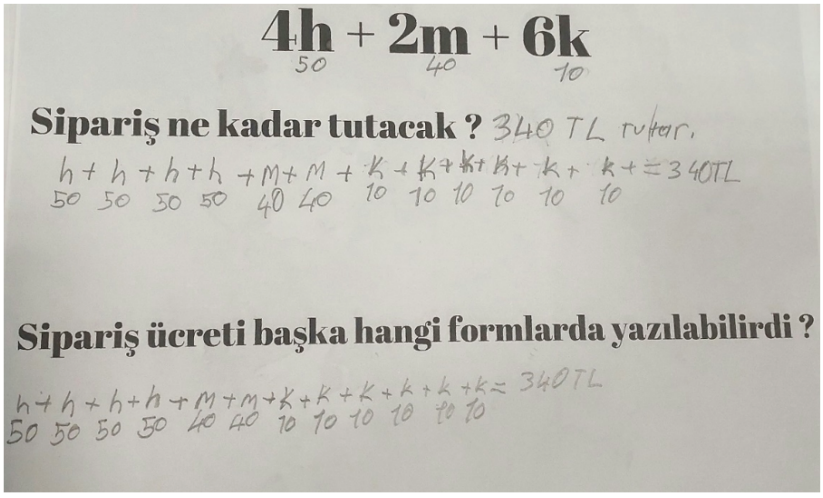


Figure 6.22. The worksheet of G7 in ACT 2.

The worksheet of G8 is presented in Figure 6.23. When they calculated the cost of the order, they first multiplied the number of foods and their price separately for each food; then they added the results. Their equivalent algebraic expression shows that they established a relationship between multiplication and addition.

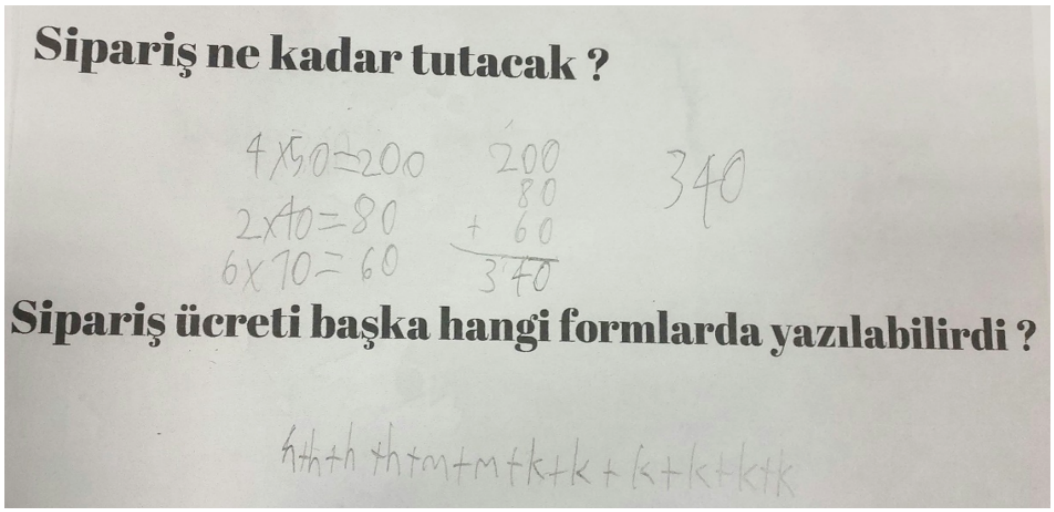


Figure 6.23. The worksheet of G8 in ACT 2.

Table 6.11 below summarizes the groups' algebraic thinking indicators in terms of generalized arithmetic theme. During the second act of the task, all groups recognized that variables stand for numbers and correctly calculated the algebraic expression's value by making operations.

In the second question of the task, all students used the relationship between operation properties to form equivalent algebraic expressions. Specifically, G1, G2, G7, and G8 used repeated addition, G3 and G6 used commutative property, and G4 and G5 used both repeated addition and commutative property to form equivalent algebraic expressions. Besides, G2, G5, G6, and G7 used equal signs relationally by forming equality.

Table 6.11. Algebraic thinking indicators of groups regarding generalized arithmetic during ACT 2 in the third lesson.

Using the relationship between operation properties to form equivalent algebraic expressions.			
Codes	**Description**	**Example**	**Groups**
Multiplication as repeated addition	Students use the relationship between multiplication and addition, thus write equivalent forms.	Writing 4h as h+h+h+h, 2m as m+m	G1, G2, G7, G8
Commutative Property	Students write an equivalent form of an algebraic expression using commutative property.	Changing the places of collected terms	G3, G6
Multiplication as repeated addition and Commutative Property	Students write equivalent forms of an algebraic expression using both commutative property and the relationship between multiplication and addition.	Writing 4h as h+h+h+h, 2m as m+m and changing the places of collected terms	G4, G5
Using Equal Sign relationally.			
Codes	**Description**	**Example**	**Groups**
Using	Students use equal sign relationally.	3d+3g=d+d+d+g+g+g	G2, G5, G6, G7
No Equal Sign.	There is no equal sign	3d+3g, d+d+d+g+g+g	G1, G3, G4, G8

In the third act, groups shared their solutions, and the solution was revealed with a photograph that showed the answer to the focus question. Different examples were given on how to find the value of an algebraic expression for different number values. Then, operation properties were discussed by giving examples from the equivalent expressions they wrote. Meanwhile, the concepts of the "term", "coefficient", and "like term" were explained and exemplified from the algebraic expressions in the task.

In the second part of the lesson, one more additional worksheet was distributed to the classroom. This worksheet has four similar questions, which include calculating the cost of orders and transforming algebraic expressions into equivalent forms. The worksheet of G1 is presented in Figure 6.24. In all four questions, they calculated the cost of the order and found the correct results. They used both multiplication and addition during their calculation. Their way is based on multiplication since they calculated the total cost of each different type of food by multiplying the quantity of food and the price of one food. Then, they added the cost of different foods to find the total order cost. Their transformations of the equivalent expressions show that they made a relationship between multiplication and addition. In addition, their ability to translate algebraic expressions containing distributive properties into equivalent forms shows that they made sense of the distributive property of addition over multiplication in algebraic expressions.

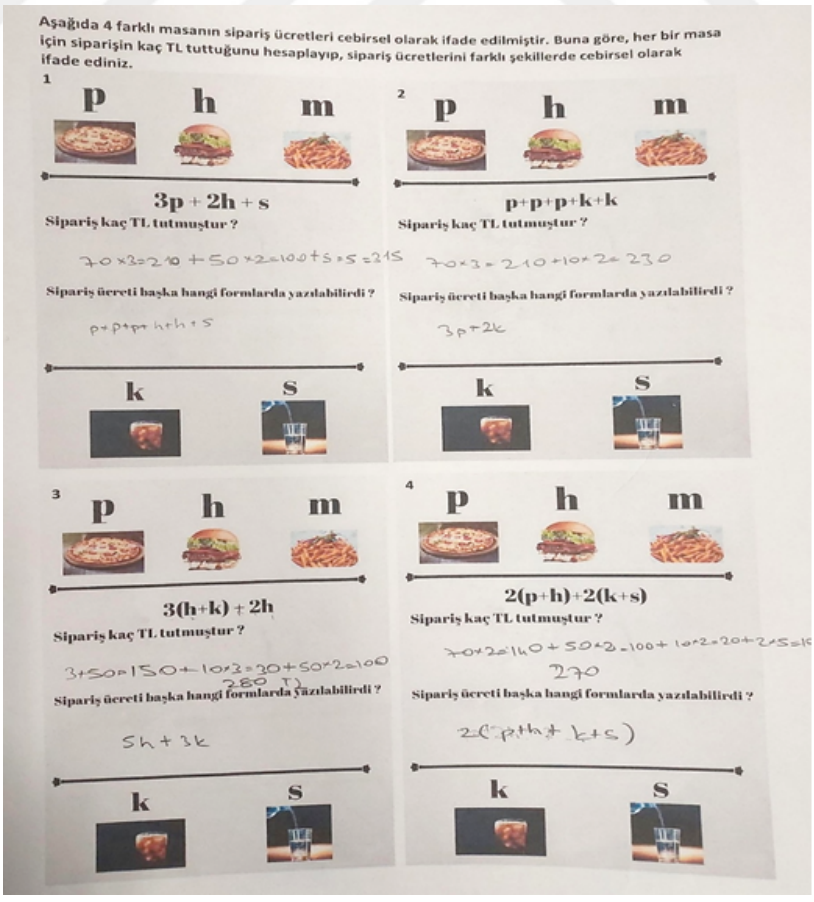


Figure 6.24. The worksheet of G1 in the second part of the third lesson.

G2 calculated the cost of the order correctly for three questions. In the first question, they formed the equivalent algebraic expression using the commutative property of addition since they changed the order of the collected terms. Their transformation in the second, third and fourth questions shows that they established a relationship between multiplication and addition in algebraic expressions since they transformed repeated addition into multiplication and vice versa. Their ability to translate algebraic expressions containing the distributive properties into equivalent forms shows that they can make sense of the distributive property of multiplication over the addition in algebraic expressions. Their worksheet is presented in Figure 6.25 below.

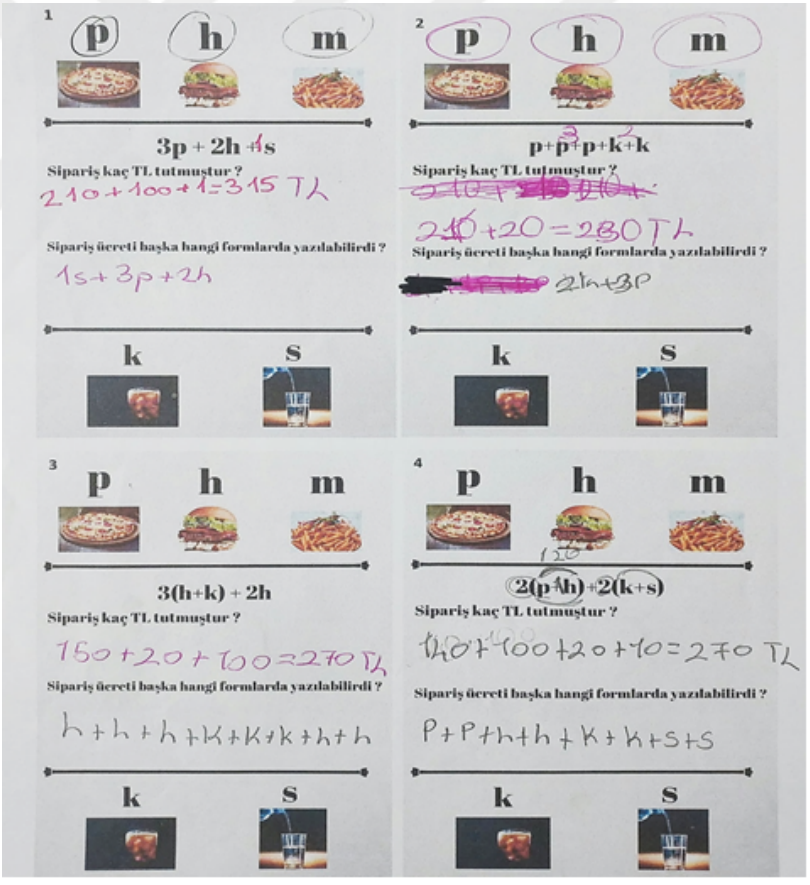


Figure 6.25. The worksheet of G2 in the second part of the third lesson.

When the worksheet of G3 was examined, it was seen that they calculated the cost of the order correctly for the first, third, and fourth questions. For the second question, they added one more variable, which represented the cost of the cola, and then

calculated the order cost. Therefore, the cost for the new order they wrote is correct, but the cost for the original form is incorrect. Besides, their equivalent expressions for the second question show that they could add this variable to use the distributive property of multiplication over addition by using the parenthesis for common factor three. Not only their expressions in the second question but also their equivalent algebraic expressions for the third and fourth questions show that they could make sense of the distributive property of multiplication over addition in algebraic expressions. They also used the commutative property of addition in the first question by changing the order of the collected terms. Along with this, since they wrote the equivalent algebraic expression by repeated addition format in the third and fourth questions, they established a relationship between multiplication and addition. Figure 6.26 shows their worksheet.

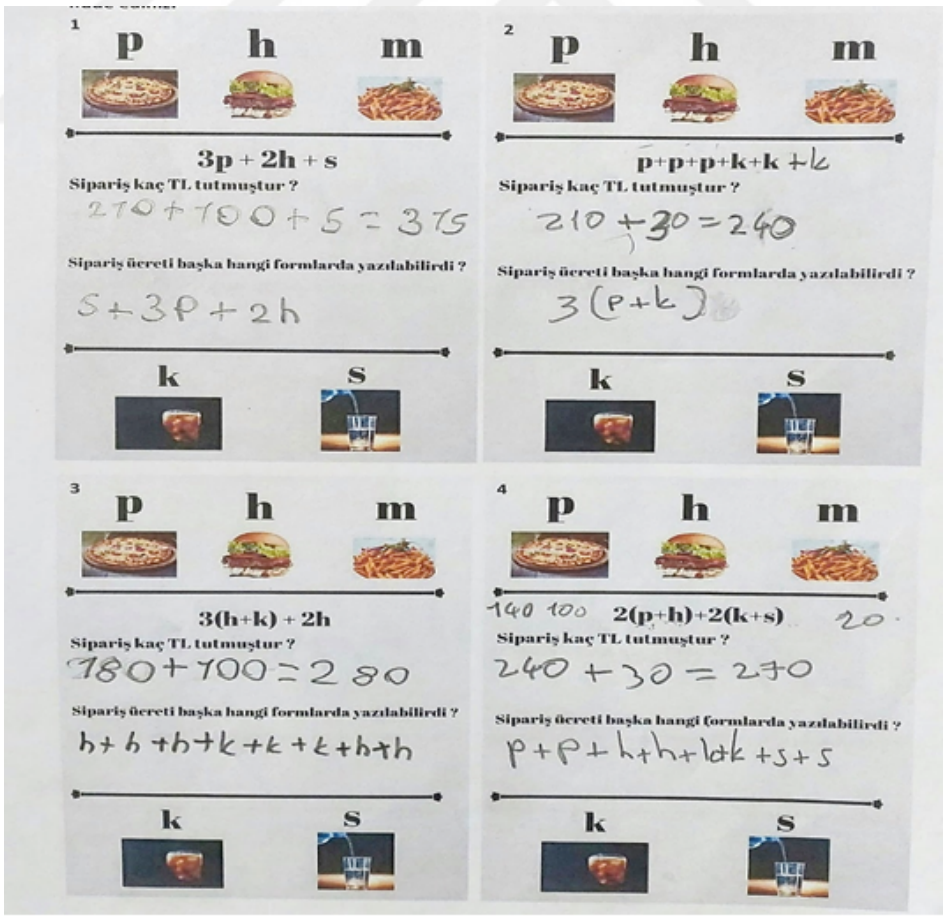


Figure 6.26. The worksheet of G3 in the second part of the third lesson.

The worksheet of G4 is shown in Figure 6.27 below. Their answers to the cost of the order in all four questions are incorrect, and no solution way is seen in the paper. On the other hand, they could establish a relationship between multiplication and addition in algebraic expressions since they transformed the multiplication to repeated addition format in the first, third, and fourth questions, and they transformed the repeated addition to the multiplicative format in the second question. Besides, since they could form the equivalent algebraic expressions to the expressions that include the distributive property in the third and fourth questions, they could make sense of the distributive property of multiplication over the addition in algebraic expressions.

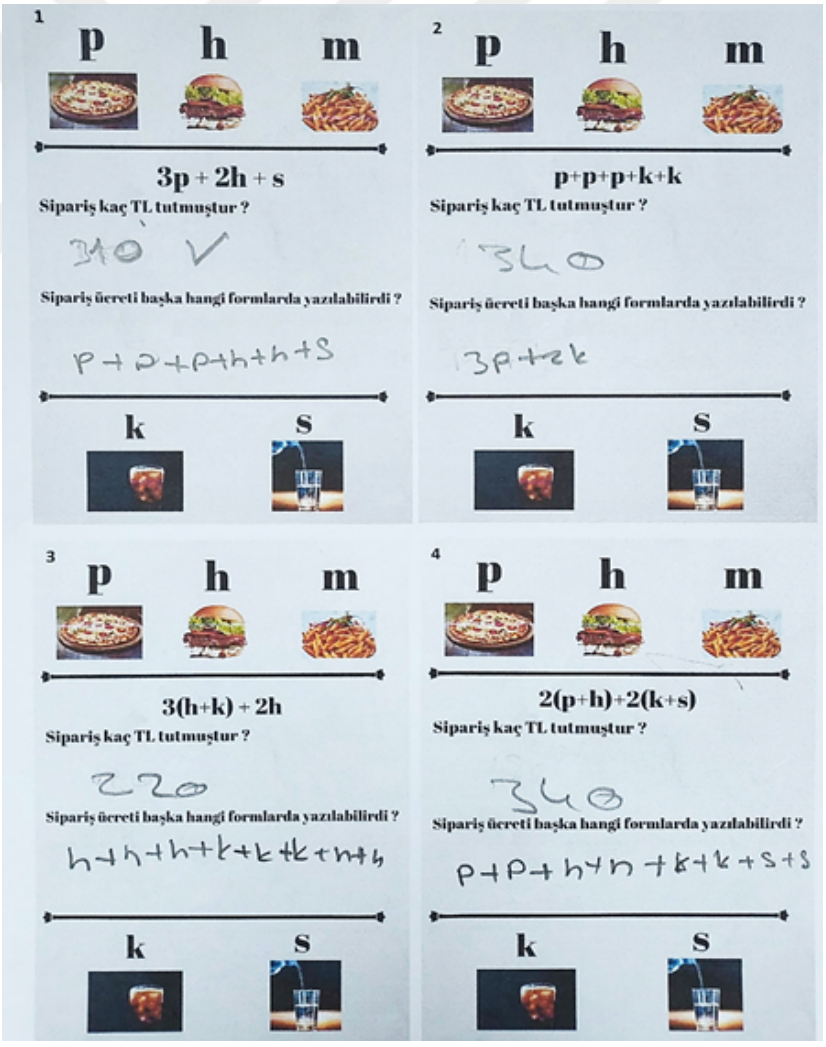


Figure 6.27. The worksheet of G4 in second part of the third lesson.

G5 correctly calculated the order cost for all four questions, as seen in Figure 6.28. The algebraic expressions they wrote show that they established a relationship between multiplication and addition. They wrote equivalent expressions in a multiplication or repeated addition format in all four questions to form equivalent expressions. On the other hand, they did not sum between expressions that relate to different food costs. Besides, since they used the distributive property of multiplication over the addition in the third and fourth questions, they could make sense of the distributive property of multiplication in algebraic expressions. In addition, they used equal signs between the expressions they wrote. This can show their relational understanding of the equal sign.

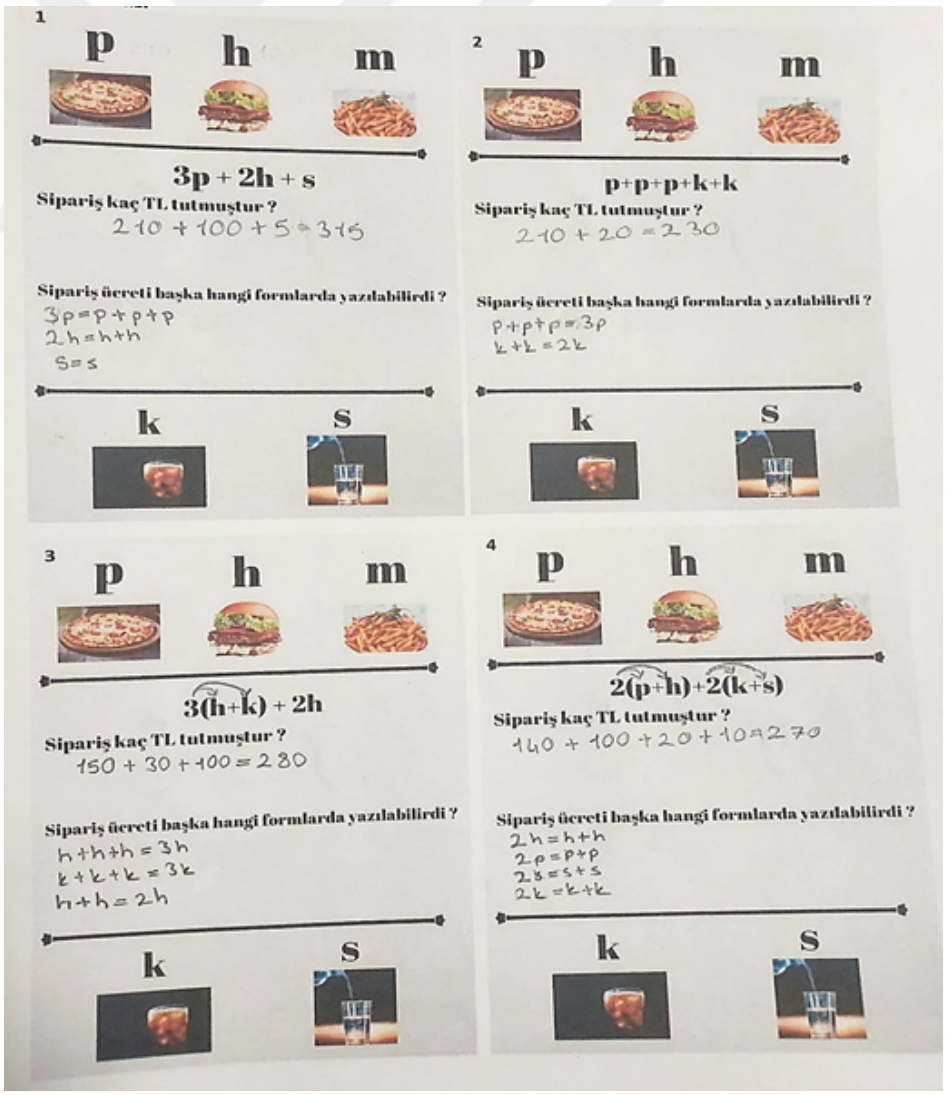


Figure 6.28. The worksheet of G5 in second part of the third lesson.

G6 correctly calculated the cost of the orders in the first three questions but incorrectly in the fourth question. As seen in Figure 6.29 below, they used both multiplication and repeated addition during their calculation. Their operation format was the same as the form of the algebraic expressions. For instance, the algebraic expression in the second question has the repeated addition format, and they calculated the cost in this way. Besides, algebraic expressions in the third and fourth questions include the distributive property, and it seems that they calculated the cost by using the distributive property. In the first question, their equivalent expressions show their understanding of the commutative property of addition in the algebraic expression since they changed the order of the collected terms. In the second and third questions, they also used the commutative property by changing the place of the collected terms. In the second question, they changed the repeated addition format to the multiplication; in the third and fourth questions, they changed the multiplication to the repeated addition. Therefore, their works show their understanding of the relationship between multiplication and addition in algebraic expressions. Besides, they used the distributive property correctly in the last two transformations. Thus, their understanding of the distributive property of multiplication over addition in algebraic expressions can be inferred.

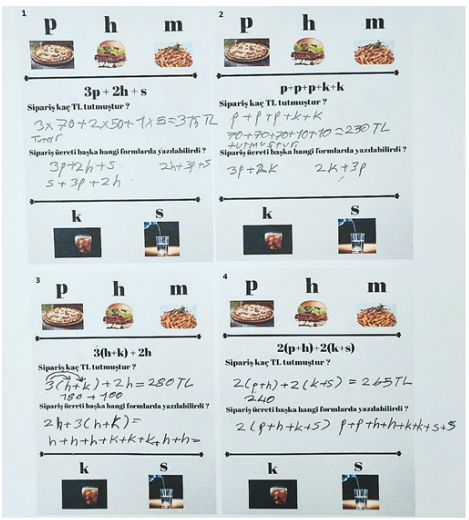


Figure 6.29. The worksheet of G6 in the second part of the third lesson.

The worksheet of G7 shows that they correctly calculated the cost of the orders in the second, third, and fourth questions but incorrectly in the first, as seen in Figure 6.30 below. Even though their calculation way was correct in the first question, their error was caused by the fact that they evaluated the price of water as 3 TL rather than 5 TL. They did not use multiplication during their calculation; they used repeated addition in all their calculations. Their equivalent algebraic expressions in the first and second questions show they can establish a relationship between multiplication and addition since they transformed multiplication into repeated addition and repeated addition into multiplication. Their transformations in the third and fourth questions show their understanding of the distributive property of multiplication over addition.

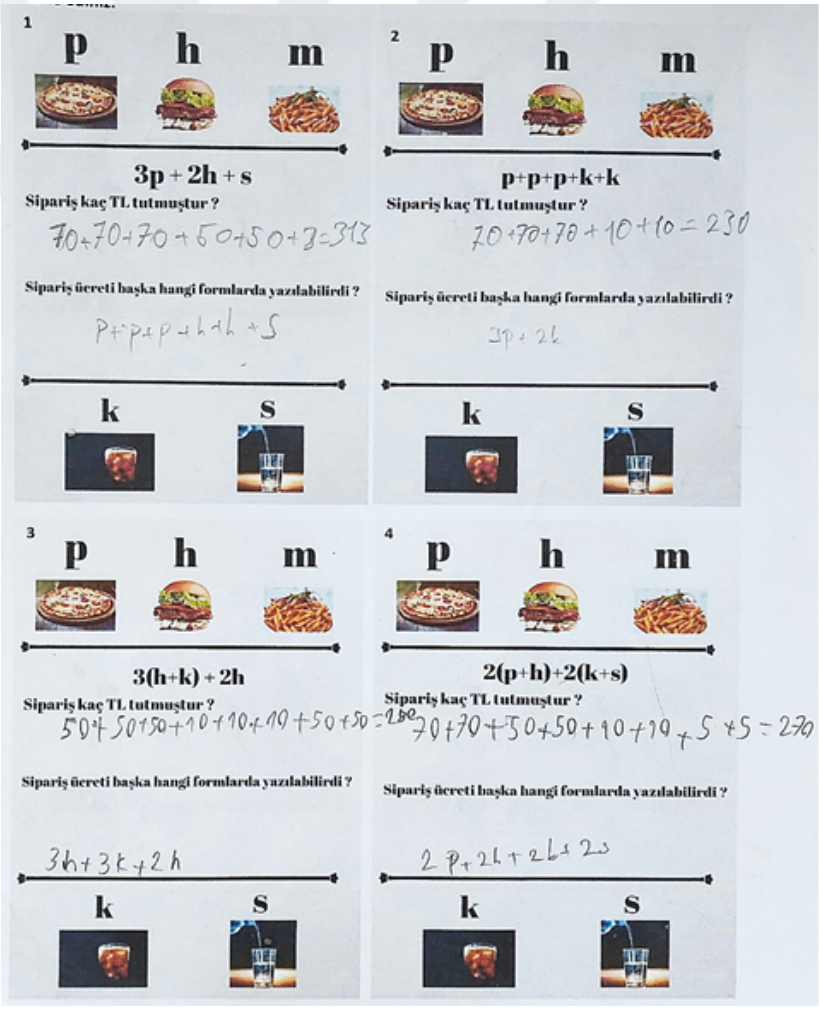


Figure 6.30. The worksheet of G7 in the second part of the third lesson.

G8 calculated the cost of the orders in the first, second, and fourth questions correctly but incorrectly in the second question, as seen in Figure 6.31 below. They transformed the multiplication to repeated addition in the first, third, and fourth questions and repeated addition to multiplication in the second questions. This shows that they can establish a relationship between multiplication and addition in algebraic expressions. In the third question, even if they incorrectly wrote the equivalent form of the algebraic expression, they were able to write correct repeated addition during their calculation of the cost. They also used the distributive property in the fourth question. They changed the expression that included the distributive property to repeated addition format during their calculation and also changed the form of distribution when they wrote the equivalent form. Therefore, it can be inferred that this group made sense of the distributive property and used it. Besides, they used commutative propery by changing the places of collected terms in the first and the fourth questions explicitly. This shows their understanding of commutative property in algebraic expressions.

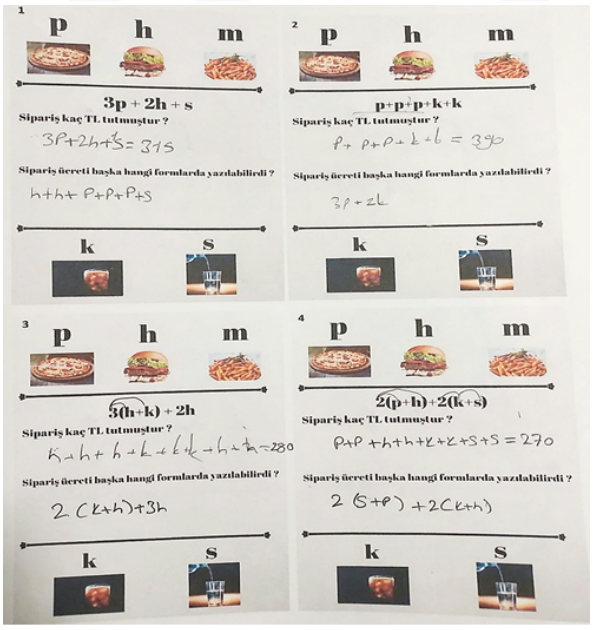


Figure 6.31. The worksheet of G8 in the second part of the third lesson.

Table 6.12 below summarizes the groups' works regarding the generalized arithmetic strand. In summary, all groups used the relationship between multiplication and addition in the equivalent forms they wrote. In addition to this property, one group

(G2) used the commutative property, one group (G1) used the distributive property, and three groups (G3, G6, and G8) used both the distributive and commutative properties. There is one group (G8) that writes incorrect equivalent expressions. Although this group wrote correct statements in three questions, the equivalent statement they wrote in one question was incorrect.

Table 6.12. Algebraic thinking indicators regarding generalized arithmetic.

Using the relationship between operation properties to form equivalent algebraic expressions.			
Codes	**Description**	**Example**	**Groups**
Multiplication as repeated addition	Students use the relationship between multiplication and addition.	Writing p+p+p+k+k as 3p+2k Writing 2h+3p+s as h+h+p+p+p+s	G4, G5, G7
Multiplication as repeated addition and commutative property	Students use the relationship between multiplication and addition use commutative property, and write equivalent forms.	Writing 2h+3p+s as h+h+p+p+p+s and s+3p+h Writing p+p+p+k+k as 3p+2k	G2
Multiplication as repeated addition and distributive property	Students use the relationship between multiplication and addition use distributive property, and write equivalent forms.	Writing 2h+3p+s as h+h+p+p+p+s, Writing p+p+p+k+k as 3p+2k Writing 2(p+k) as 2p+2k	G1

Table 6.12 Algebraic thinking indicators regarding generalized arithmetic. (cont.)

Codes	**Description**	**Example**	**Groups**
Multiplication as repeated addition, distributive and commutative properties	Students use the relationship between multiplication and addition, use distributive and commutative properties and write equivalent forms.	Writing 2h+3p+s as h+h+p+p+p+s, Writing p+p+p+k+k as 3p+2k Writing 2(p+k) as 2p+2k Writing 2h+3p + s as 3p+2h+s	G3, G6, G8
Incorrect equivalent form	Students attempt to write an equivalent form of an algebraic expression but incorrectly write.	The student wrote at least one example incorrectly. Writing 3(h+k)+2h as 2(k+h) +3h	G8
Using the equal sign relationally.			
Codes	**Description**	**Example**	**Groups**
Using	Students use equal sign relationally.	h+h=2h 3p+2h+s=315	G5, G8, G6
No Equal Sign	There is no equal sign.	Only expressions without equal sign	G1, G2, G3, G4, G7

6.3.1.4. Fourth Lesson. According to the MoNE (2018), the fourth lesson objective is the same as the third lesson's objective, which is calculating the value of the algebraic expression for different natural number values the variable can take. Thus, the three-act math task and activities provide students with the application of this objective. This objective was combined with the strand of Functional Thinking, which is a content strand of algebra, according to Kaput (2008).

In Act 1 of the three-act math task, a video that includes a number robot is shown to the student. The numbers 1, 2, 3, 4, 5, 6, and 7 are entered into this robot in the video, and these numbers come out as 14, 24, 34, 44, 54, 64, and 74 respectively. After the students watched this video, the teacher asked the students to estimate which number would come out of the machine if the number 9 entered into the machine. The whole class answered as 94. This shows that students noticed that 4 always appeared next to the numbers they entered, but there is no evidence yet whether the students found a formal rule. Then, the teacher and the students had the following dialogue. According to the dialogue, students can identify the relationship in a pattern through the recursive strategy, and no answers describe the relationship in the pattern by covariational strategy or correspondence rule.

> T: What do you notice?
> S5: The numbers increase by 10 by 10
> T: Other answers?
> S2: The "4" comes next to the numbers.
> T: Any other answer?
> T: What do you wonder?
> S1: Why do the numbers enter and exit the robot?
> S3: Why do numbers enter the robot and exit the robot with the number four?
> T: There can be a relationship between the numbers that enter to the robot and exist from the robot, is there any rule?
> S5: It increases by 10 by 10.
> T: As you see in the video, the number "1" enters the robot and exits from the robot as "14"; the number "2" enters the robot and exits as "24". You should find and describe a rule between the numbers that enter the robot and exist from the robot. This is our focus question.

During the second act, the teacher distributed a worksheet to the students, and the students worked on it. This worksheet includes two additional questions that ask the students to draw a table or a graph to show this relationship, find a rule, and write this rule algebraically. Students worked on solving these questions. As a clue, the teacher showed a few more steps of the pattern via video to encourage students to think better while working on the question.

G1 was able to identify correspondence relationships in the pattern and describe it with words. They drew a table to show input and output number values. Besides, they

found a general rule and wrote their rule algebraically. Their worksheet is presented in Figure 6.32.

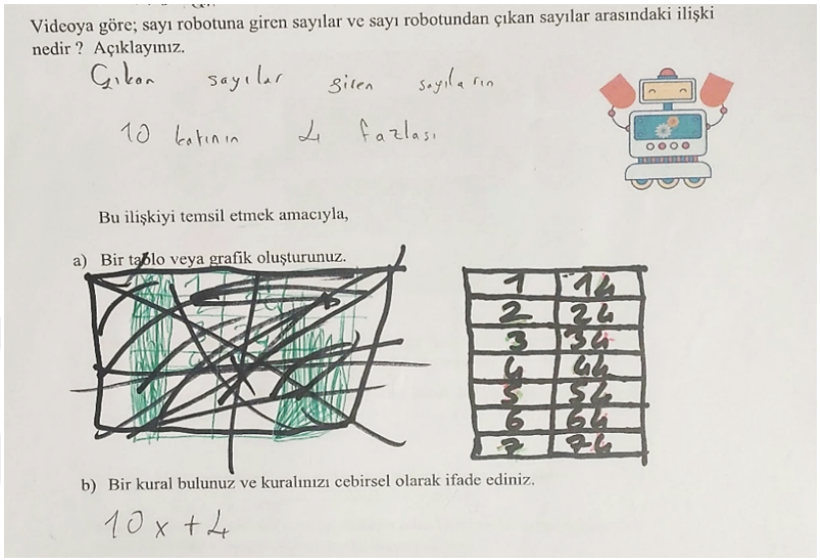


Figure 6.32. The worksheet of G1 in ACT 2 in the fourth lesson.

G2 identified the relationship in the pattern recursively and described it with words as "increasing by 10" as seen in Figure 6.33. They were able to draw a table to describe the input and output number values. On the other hand, their general rule was incorrect because it did not represent the relationship in the pattern.

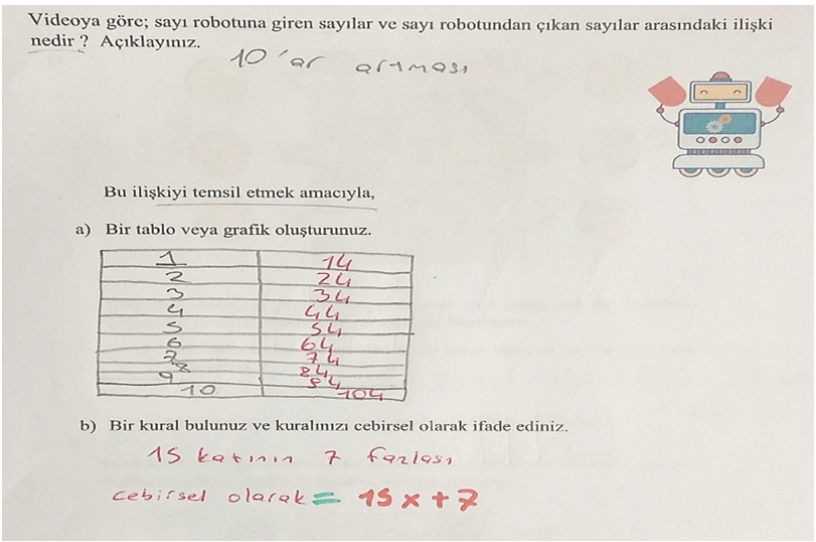


Figure 6.33. The worksheet of G2 in ACT 2 in the fourth lesson.

Similar to G2, G3 identified the relationship in the pattern via recursive strategy and described it with words as "increasing by 10 by 10". They also wrote a table to show input and output number values. On the other hand, they did not find a general rule; instead, they described the recursive relationship in the pattern with words again, as seen in Figure 6.34.

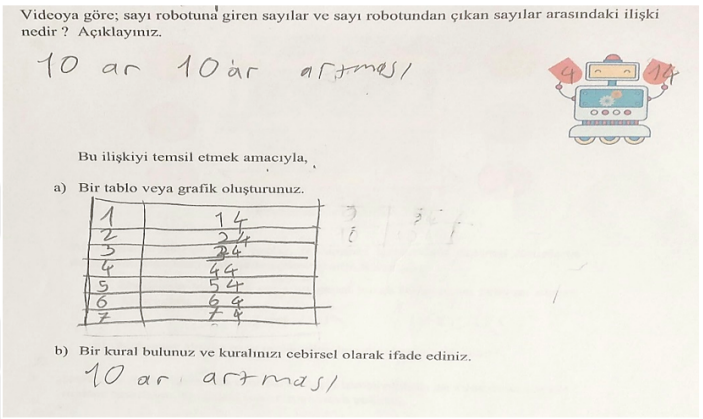


Figure 6.34. The worksheet of G3 in ACT 2 in the fourth lesson.

G4 found a recursive relationship in the pattern like G2 and G3 and described it with words as "increase by 10". They also drew a table to show the input and output number values. Besides, they were able to find the general rule and describe it by writing an algebraic expression and using words, as seen in Figure 6.35.

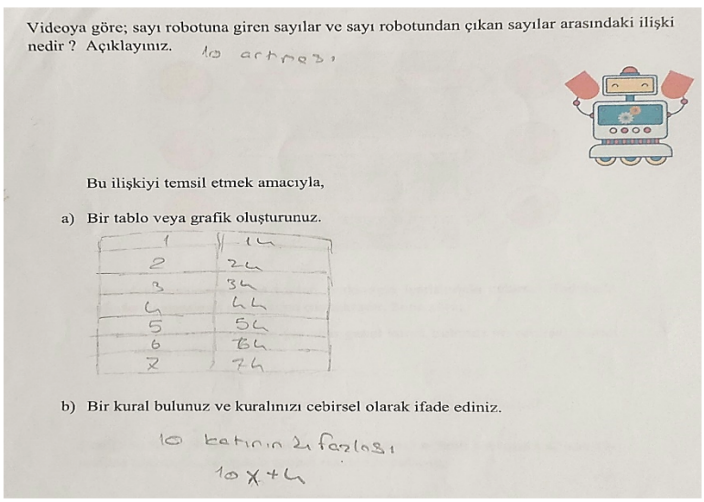
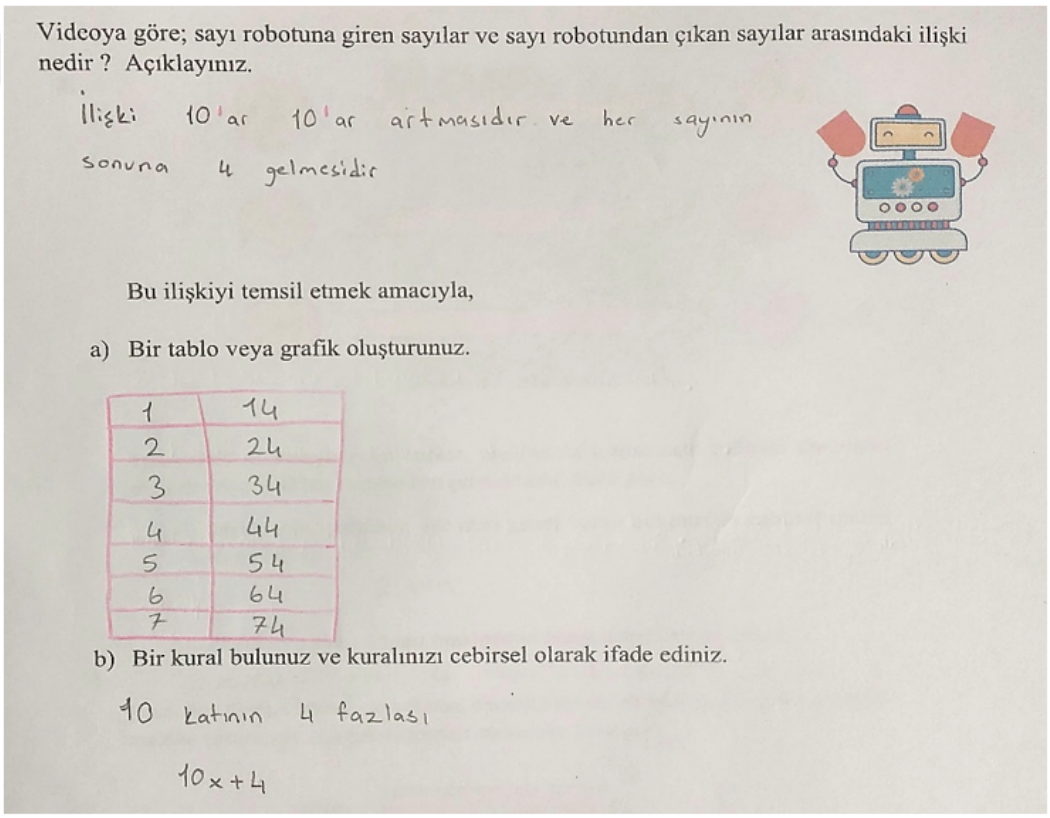


Figure 6.35. The worksheet of G4 in ACT 2 in the fourth lesson.

Like G2, G3, and G4, G5 found a recursive relationship in the pattern and described it as "increasing by 10 by 10" with words. They also wrote that 4 comes next to each number. Having four next to each number can be considered a different pattern rule, unlike recursive, covariational, or correspondence. Besides, they were able to draw a table to describe this relationship and found a general rule to show the relationship between the input and output number values. They described their rule via algebraic expression and using words. Figure 6.36 below presents their worksheet.


Videoya göre; sayı robotuna giren sayılar ve sayı robotundan çıkan sayılar arasındaki ilişki nedir ? Açıklayınız.

İlişki 10'ar 10'ar artmasıdır ve her sayının sonuna 4 gelmesidir

Bu ilişkiyi temsil etmek amacıyla,

a) Bir tablo veya grafik oluşturunuz.

1	14
2	24
3	34
4	44
5	54
6	64
7	74

b) Bir kural bulunuz ve kuralınızı cebirsel olarak ifade ediniz.

10 katının 4 fazlası

$10x+4$

Figure 6.36. The worksheet of G5 in ACT 2 in the fourth lesson.

G6 identified a correspondence relationship between the input and output number values similar to G1. They described this relationship with words and gave an example, as seen in Figure 6.37 below. They were able to draw a table to show this relationship and find a general rule by describing it algebraically and by words.

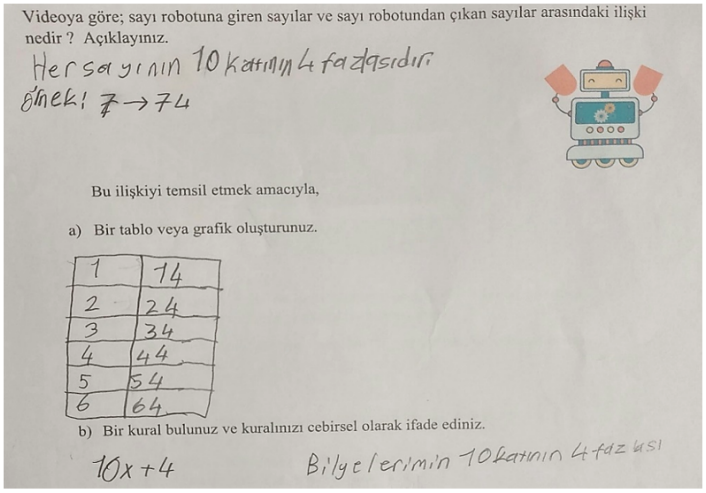


Figure 6.37. The worksheet of G6 in ACT 2 in the fourth lesson.

When the worksheet of G7 was examined, it was seen that a different rule of a pattern like G5 was described, which is that 4 comes next to each number. On the other hand, they could not describe recursive, covariational, or correspondence relationships. Besides, they could not formulate a general rule to express the relationship, but they were able to draw a table that shows the input and output number values. On the other hand, unlike the other groups, the input numbers 1 and 2 and corresponding output numbers 14 and 24 were not included in the table, and input 4 and output 44 were written twice. Figure 6.38 below shows their worksheet.

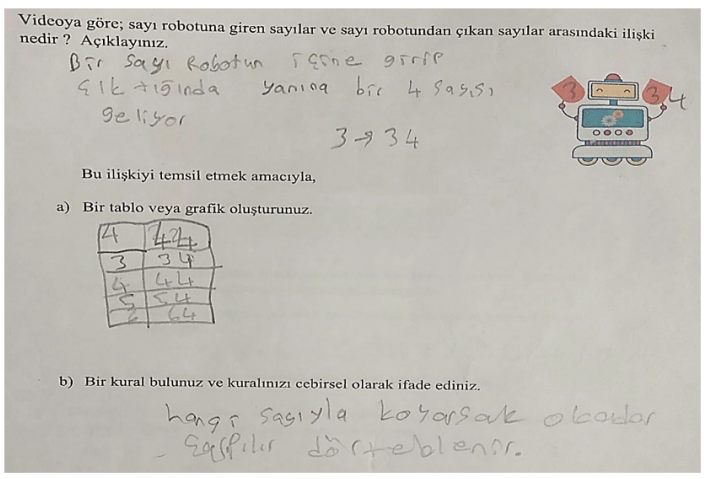


Figure 6.38. The worksheet of G7 in ACT 2 in the fourth lesson.

The worksheet of G8 shows that they identified the correspondence relationship in the pattern and described it with words. On the other hand, they could not formulate a correct general rule algebraically; they wrote more than one algebraic expression as a rule, which is incorrect. Like other groups, they were able to draw a table to show input and output number values. Figure 6.39 below shows their worksheet.

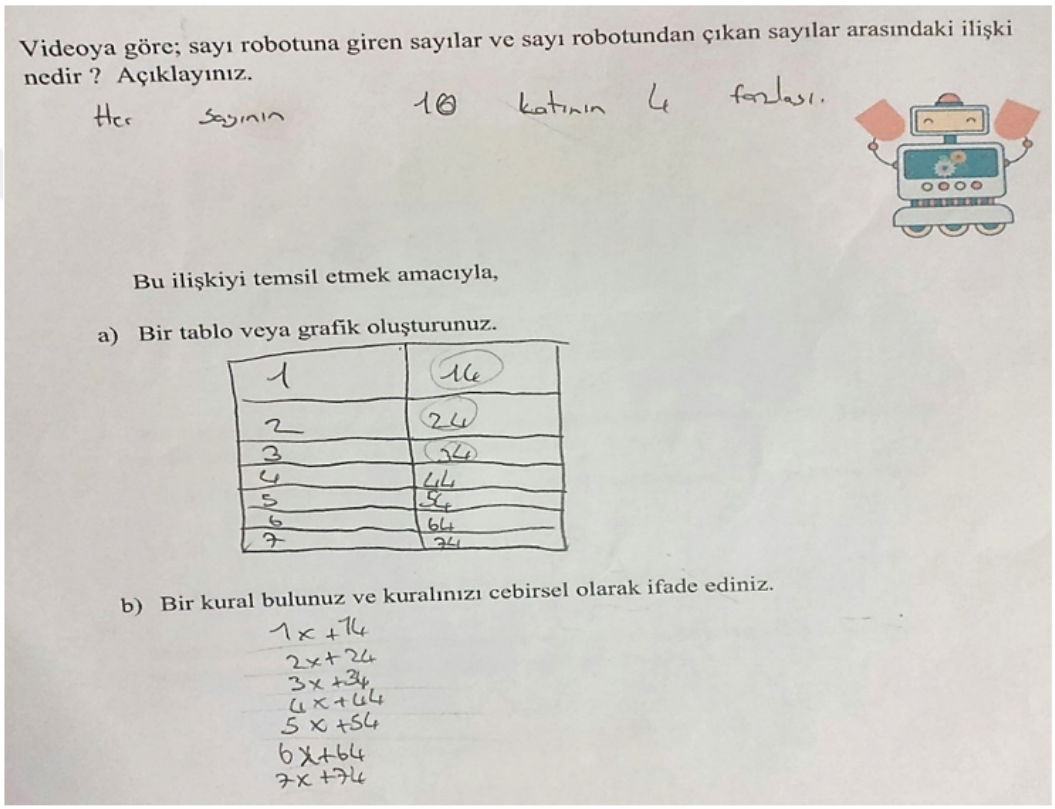


Figure 6.39. The worksheet of G8 in ACT 2 in the fourth lesson.

The groups' algebraic thinking indicators in terms of the functional thinking strand in ACT 2 is summarized in Table 6.13 below. The focus question asked groups to find the relationship between the numbers that enter the robot and exit the robot. On the other hand, three groups (G1, G6, and G8) could find and describe the correspondence relationship between input and output number values. Four groups (G2, G3, G4, and G5) focused on the difference between consecutive terms and described a recursive relationship. G7 described the relationship by explaining a number (four) that comes next to each number. Although this number (four) forms a pattern since it comes next to each number, this description is neither recursive, correspondence, or

covariational and therefore coded as "other". G5 also wrote this relationship; therefore, the description of G5 also has this code.

Besides, all groups except G7 were able to draw a complete table to represent the relationship in the pattern. The table drawn by G7 includes more than one missing term and repeated term. Therefore, it was considered an incomplete table.

Four groups (G1, G4, G5, and G6) formulated a general rule of a pattern that represents the correspondence relationship. Among these four groups, G1 formed a general rule by writing an algebraic expression. G4, G5, and G6 used both algebraic expressions and words to represent the general rule. Another four groups (G2, G3, G7, and G8) incorrectly formulated general rules. Specifically, G2 and G8 wrote incorrect algebraic expressions, and G3 and G7 wrote incorrect verbal expressions that do not represent the general correspondence rule. Table 6.13 shows students' algebraic thinking indicators in ACT 2.

Table 6.13. Algebraic thinking indicators regarding functional thinking.

Identifying the relationship in a pattern and describing it with words.			
Codes	**Description**	**Example**	**Groups**
Recursive	Students focus on the difference between consecutive terms and describe a recursive relationship.	increasing by 10 by 10	G2, G3 G4, G5
Correspondence	Students describe the correspondence relationship between input and output number values of the table.	4 more than 10 times of a number.	G1, G6, G8
Other	Students' descriptions are neither recursive, correspondence, or covariational.	4 comes to next to the each number	G5, G7

Table 6.13 Algebraic thinking indicators regarding functional thinking. (cont.)

Formulating a general rule of a pattern algebraically			
Codes	**Description**	**Example**	**Groups**
Formulated a general rule algebraically	Students form a correspondence rule algebraically.	10x+4	G1
Formulated a general rule algebraically and in words	Students form a correspondence rule algebraically and words.	10x + 4 and 4 more than 10 times of a number	G4, G5 and G6
Incorrectly Formulated algebraically	Students attempt to form a rule algebraically, but their rule is not a correct correspondence rule.	1x+14 15x+4 2x+24 3x+34 etc.	G2, G8
Formulated recursive rule with words	Students form a rule in words recursively rather than a correspondence rule.	Increasing by 10 by 10	G3
Incorrectly formulated with words	Students form a rule in words that do not explain the correct correspondence rule.	Whatever number we put in, it is multiplied by that number and 4 is added.	G7
Drawing a graph or a table to represent relationships in a pattern.			
Codes	**Description**	**Example**	**Groups**
Complete Table	Students draw a table that includes the corresponding values in the pattern.	1 14 2 24 3 34 Continues...	G1, G2, G3, G4, G5, G6, G8
Incomplete Table	Students attempt to draw a table, but the table is incomplete.	4 44 3 34 4 44	G7

In the third act, students shared their answers, and the solution was revealed with a photograph showing the rule in the robot algebraically and words. Examples

were given if different numbers were entered into the robot and how these numbers would exist from the robot by highlighting the correspondence relationship between the numbers. Thus, the calculation of the value of algebraic expression for different natural number values was emphasized within the framework of functional thinking.

After the three-act math task, students worked on a different worksheet with four questions. The first question of the worksheet is presented in Figure 6.40.

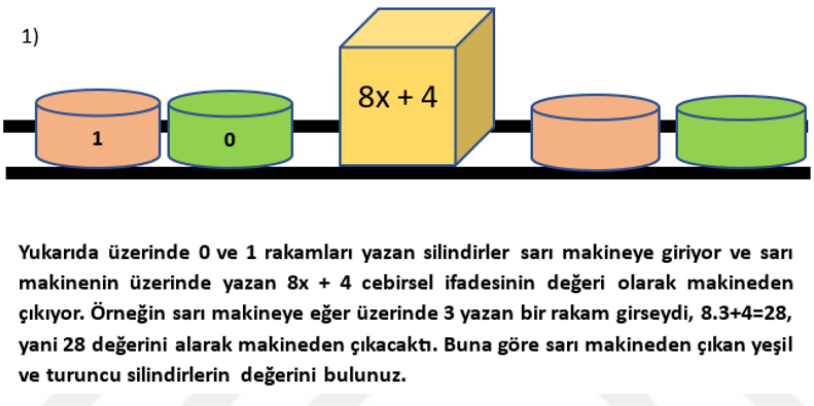


Figure 6.40. The first question of the second worksheet in the fourth lesson.

When the solutions of the groups were examined, it was seen that five groups (G1, G2, G4, G5, and G6) correctly found the corresponding terms. They found corresponding terms in the same way that is presented in Figure 6.41 in the worksheet of G1. One group (G3) attempted to find it but miscalculated it, as seen in Figure 6.42. There was no solution in the worksheet of two groups (G7 and G8).

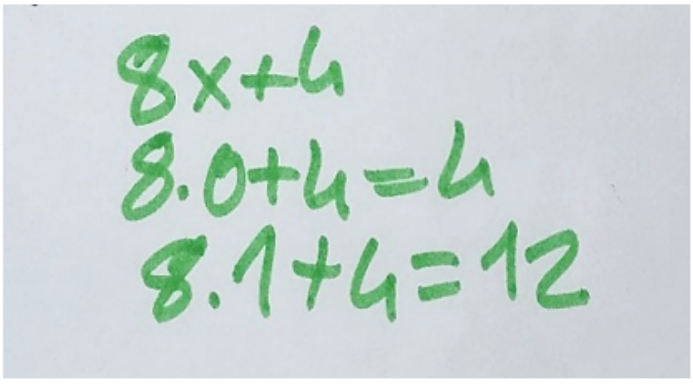


Figure 6.41. The answer of G1 for the first question in the worksheet.

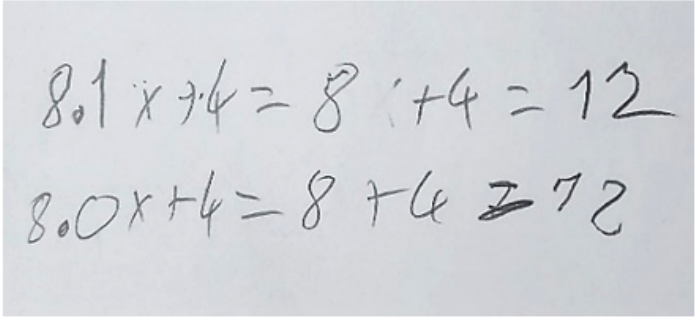


Figure 6.42. The answer of G3 for the first question in the worksheet.

The second question of the worksheet is presented in Figure 6.43 below. There are two items in the question; the first asked students to form a general correspondence rule, and the second asked students to find the corresponding term of the number four.

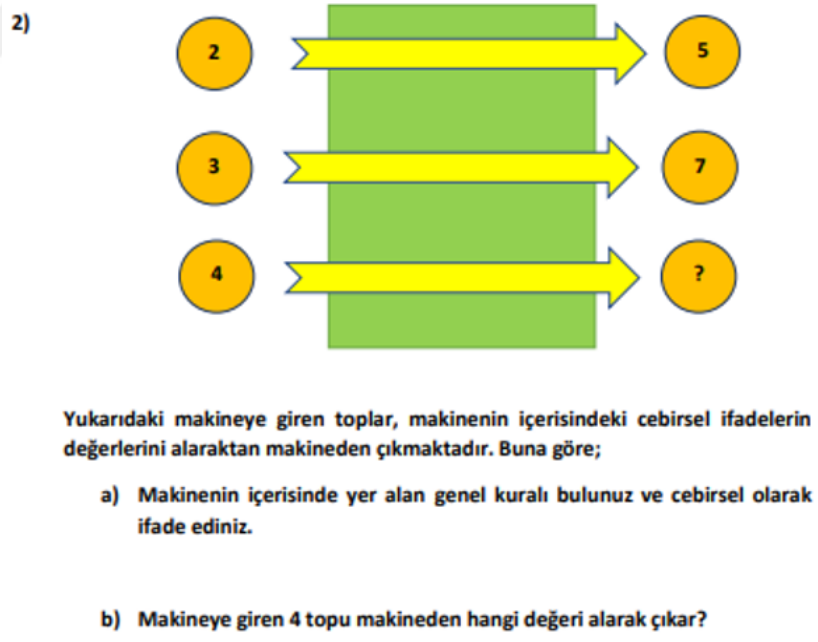


Figure 6.43. The second question of the second worksheet in the fourth lesson.

After analyzing the eight groups' worksheets, it was seen that five groups (G1, G2, G4, G5, and G6) correctly formulated an algebraic general rule that represents a correspondence relationship between input and output number values. Among these five groups, three groups (G1, G5, and G6) formed the rule by using only algebraic

expression, as seen in the worksheet of G5 (Figure 6.44); two groups (G2 and G4) used both algebraic expression and words to form the general rule as seen in the worksheet of G2 (Figure 6.45)

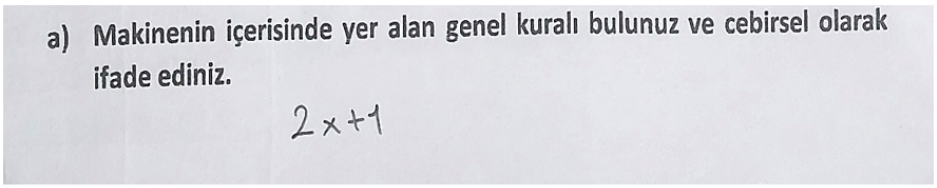


Figure 6.44. The general rule of G5 in the second question.

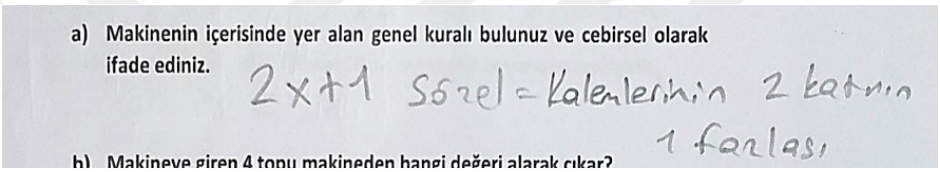


Figure 6.45. General rules of G2 in the second question.

Two groups (G3 and G8) among all groups formulated a general by using words; they did not write an algebraic expression to represent the rule as seen in the worksheet of G8 (Figure 6.46). One group (G7) formed more than one incorrect rule algebraically; this group's worksheet is shown in Figure 6.47.

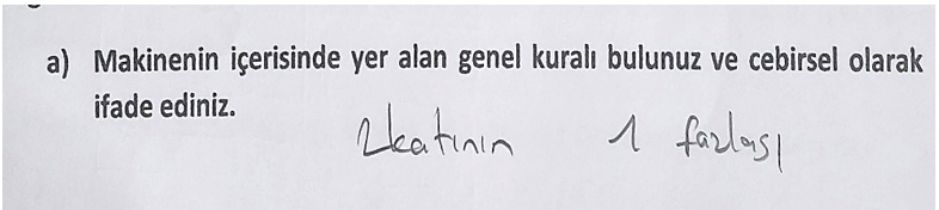


Figure 6.46. The general rule of G8 in the second question.

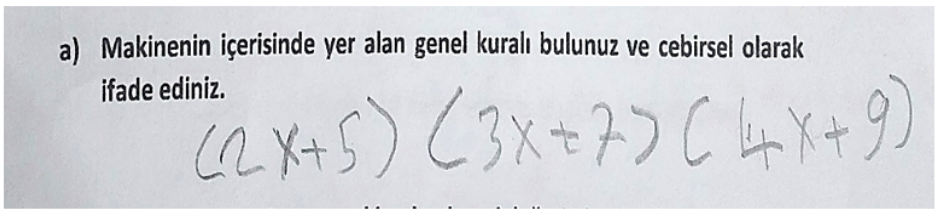


Figure 6.47. General rules of G7 in the second question.

When groups' answers were examined for the second item, seven groups (G1, G3, G4, G5, G6, G7, and G8) correctly found the corresponding terms of the number "four" that entered the machine. Conversely, one group (G2) found incorrectly ,and wrote only the answer (Figure 6.48) Among the groups who found the corresponding term correctly, six groups (G1, G3, G4, G5, G7, and G8) wrote only the answer, and there was no written solution way, as seen in the worksheet of G5 (Figure 6.49). One group (G6) used the correspondence rule they wrote to find a solution; their worksheet is shown in Figure 6.50 below.



Figure 6.48. Answer of G2 in the second item of the second question.

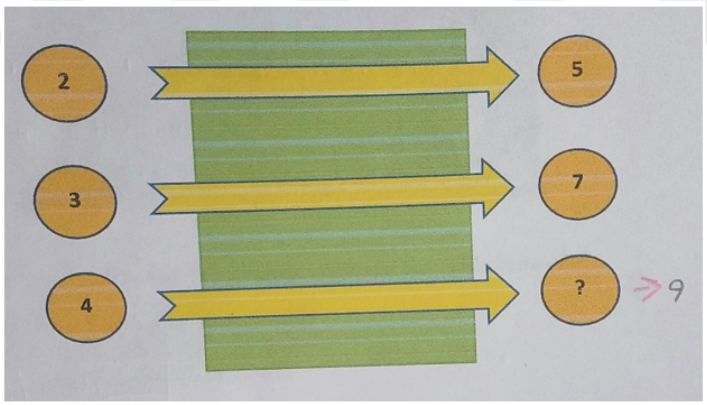


Figure 6.49. Answer of G5 in the second item of the second question.

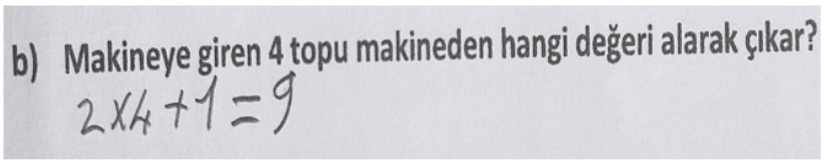


Figure 6.50. Answer of G6 in the second item of the second question.

The analysis of the groups' worksheet for the second question was summarized in Table 6.14 below based on related categories of functional thinking.

Table 6.14. Algebraic thinking indicators in the second question.

Finding the corresponding terms of the pattern			
Codes	**Description**	**Example**	**Groups**
Found the corresponding term with no written calculation	Students find the corresponding terms correctly without written calculations.	Writing only "9"	G1, G3, G4, G5, G7, G8
Found the corresponding term by using the correspondence rule	Students find the corresponding terms correctly by using the correspondence rule.	2x+1=2.4+1=9	G6
Incorrectly found the corresponding term with no written calculation	Students find the corresponding term incorrectly without written calculations.	Writing only "5"	G2
Formulating a general rule of a pattern algebraically			
Codes	**Description**	**Example**	**Groups**
Formulated a general rule algebraically	Students form a correspondence rule algebraically.	2x+1	G1, G5, G6
Formulated a general rule algebraically and in words	Students form a correspondence rule algebraically and words.	2x+1, 1 more than twice a number	G2, G4
Formulated a general rule in words	Students form a correspondence rule using words.	1 more than twice a number	G3, G8
Incorrectly Formulated algebraically	Students attempt to form a rule algebraically, but their rule is not a correct correspondence rule.	(2x+5) (3x+7) (4x+9)	G7

The third question of the worksheet is shown in Figure 6.51 below. Two items were in the question, and the groups' answers were analyzed according to the same themes and categories as those in the second question. Unlike the second question, the third question asked students to form their own rules and, thus, create their own machines.

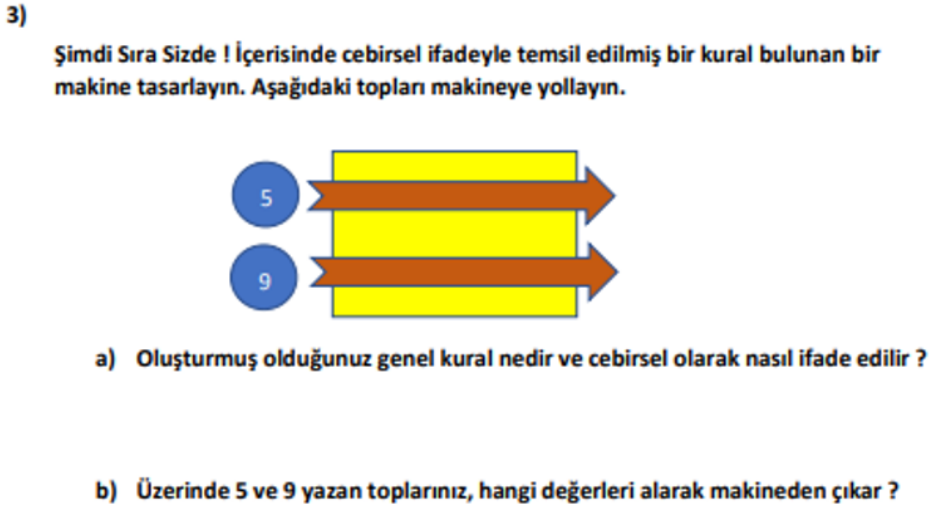


Figure 6.51. The third question of the second worksheet in the fourth lesson.

When the worksheets were examined for the first item, it was seen that six groups (G1, G2, G3, G4, G5, and G6) formulated a general rule algebraically, and two groups (G7 and G8) formulated a general rule by words. Among the six groups who could write general algebraic expressions, four groups (G1, G2, G4, and G5) formed the general rule by using only algebraic expressions, and two groups (G3 and G6) formed the rule by using both words and algebraic expressions. Table 6.15 shows the general rules formed by the groups.

Table 6.15. General rules formed by groups.

Group Name	**General Rules**
G1	3x+7
G2	3x+5
G3	500x-300, 300 less than 500 times of my books
G4	5x+1
G5	4x+4
G6	2x + 3, 3 more than 2 times a number
G7	Multiplying by 5 and add 9
G8	Multiplying by 3 and add 2

When the groups' worksheet was examined for the second item, two groups (G1 and G8) correctly found the corresponding terms without a written solution as seen

in Figure 6.52 in the worksheet of G8, four groups (G2, G3, G4, and G6) found the corresponding terms by using their rule as seen in the worksheet of the G6 in Figure 6.53 below. One group (G5) incorrectly found the corresponding terms without a solution way as seen in Figure 6.54, and one group (G7) had no answer.

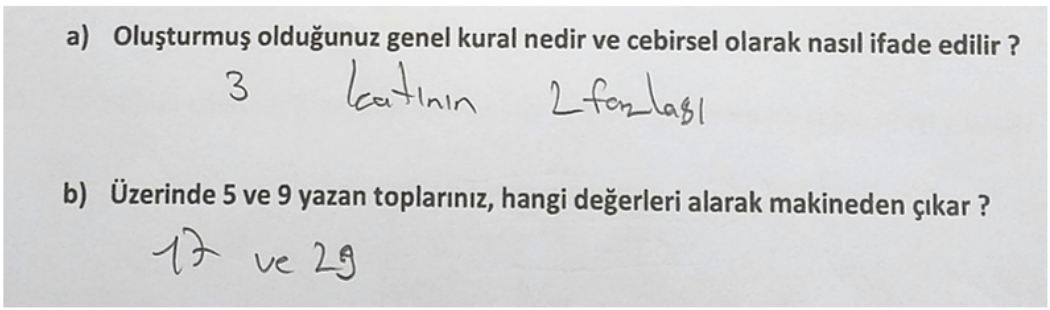


Figure 6.52. Answers of G8 in the third question.

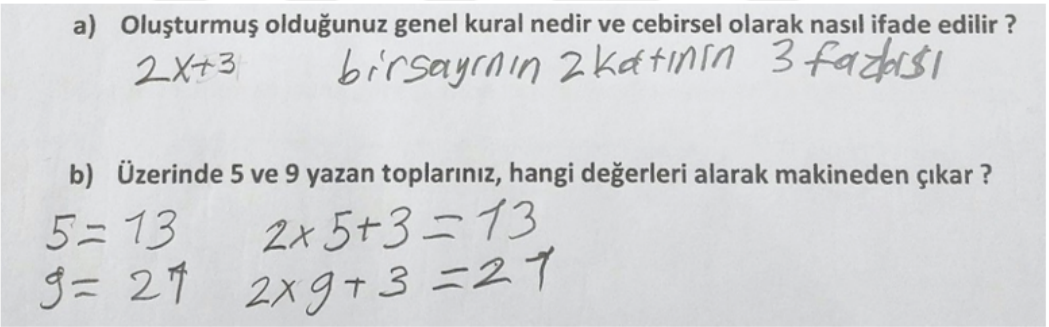


Figure 6.53. Answers of G6 in the third question.

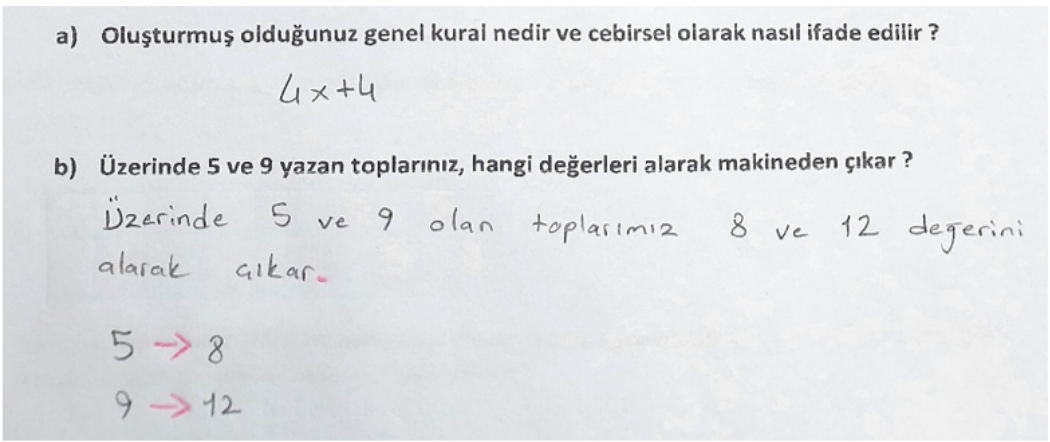


Figure 6.54. Answer of G5 in the third question.

Table 6.16 below summarizes the groups' works regarding functional thinking for the third question. For the first item, G1, G2, G4, and G5 formulated their general rule using only algebraic expressions, and G7 and G8 formulated their rule using only words. G3 and G6 used both words and algebraic expressions.

For the second item, G2, G3, G4, and G6 found the corresponding term using the correspondence rule they wrote. G1 and G8 also found the corresponding term, but there is no solution way in their worksheet. G5 also had no solution way and incorrectly found the answer. Besides, G7 had no answer.

Table 6.16. Algebraic thinking indicators in the third question.

Finding the corresponding terms of the pattern			
Codes	**Description**	**Example**	**Groups**
Found the corresponding term with no written calculation	Students find the corresponding terms correctly without written calculations.	Writing only "17 and 29"	G1, G8
Found the corresponding term by using the correspondence rule	Students find the corresponding terms correctly by using the correspondence rule.	2 x 5 + 3 =13	G2, G3, G4, G6
Incorrectly found the corresponding term with no written calculation	Students find the corresponding term incorrectly without written calculations.	Writing only answers as "8 and 12"	G5
No answer	There is no answer, blank.		G7

Table 6.16 Algebraic thinking indicators in the third question. (cont.)

Formulating a general rule			
Codes	**Description**	**Example**	**Groups**
Formulated a general rule algebraically	Students form a correspondence rule algebraically.	4x+4	G1, G2, G4, G5
Formulated a general rule algebraically and in words	Students form a correspondence rule algebraically and words.	2x + 3, 3 more than 2 times a number	G3, G6
Formulated a general rule in words	Students form a correspondence rule using words.	Multiplying by 5 and add 9	G7, G8

The fourth question in the worksheet is presented in Figure 6.55 below. There are four items in the question, and groups' answers to these items were analyzed according to related categories in functional thinking.

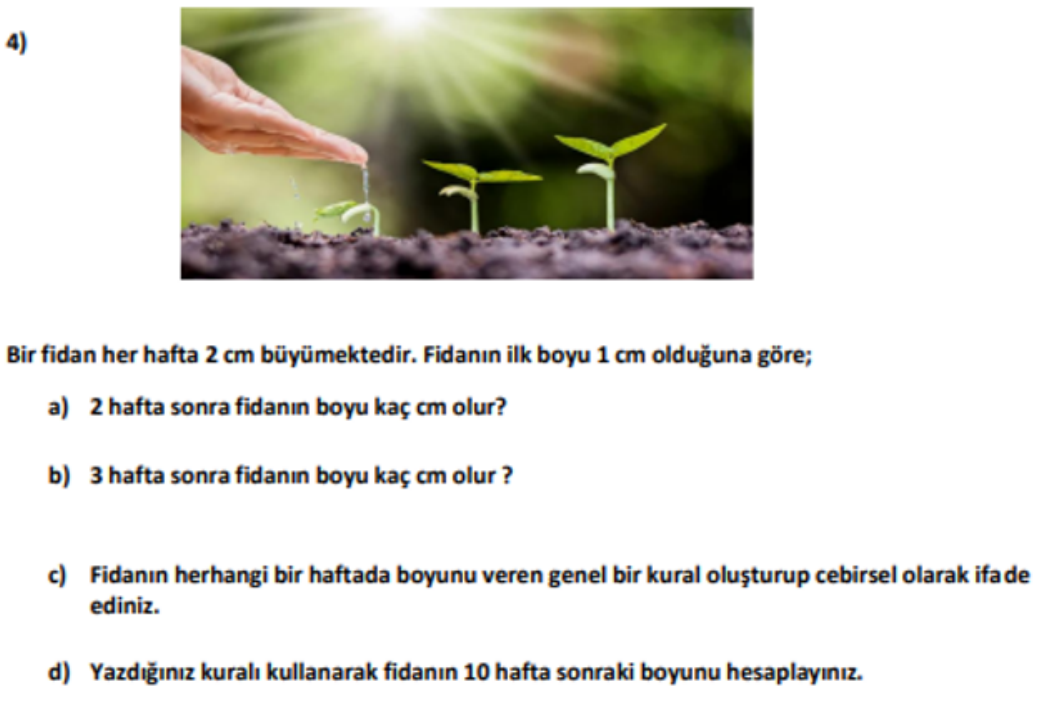


Figure 6.55. The fourth question of the second worksheet in the fourth lesson.

When the groups' worksheet was examined for the first two items, seven groups (G1, G2, G3, G4, G5, G6, and G7) could find the close terms of the pattern correctly. Among these seven groups, five groups (G1, G2, G4, G5, and G7) found close terms without written calculation, as seen in the worksheet of G4 in Figure 6.56, and two groups (G3 and G6) found close terms with written calculations, as seen in the worksheet of G3 in Figure 6.57 below. In addition, one group (G8) found the close terms incorrect, as seen in Figure 6.58 below.

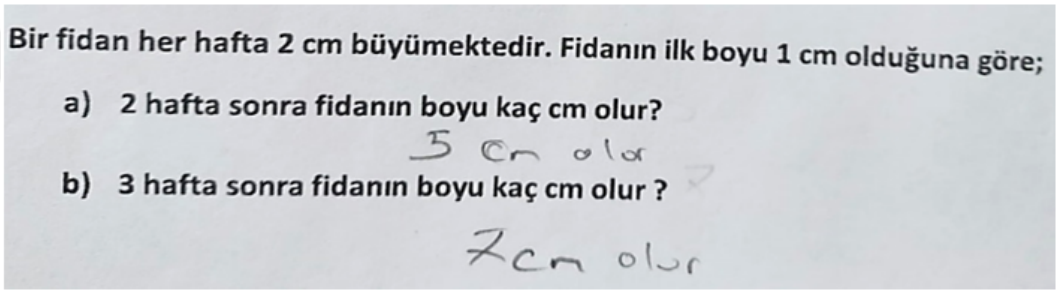


Figure 6.56. Answer of G4 in the first two items of the fourth question.

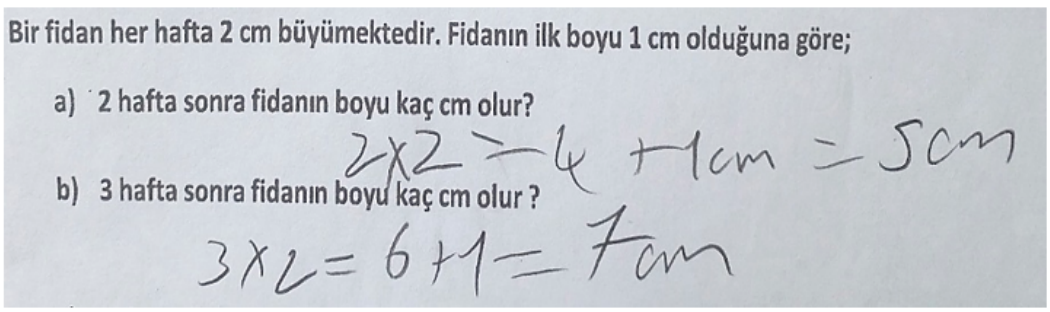


Figure 6.57. Answer of G3 in the first two items of the fourth question.

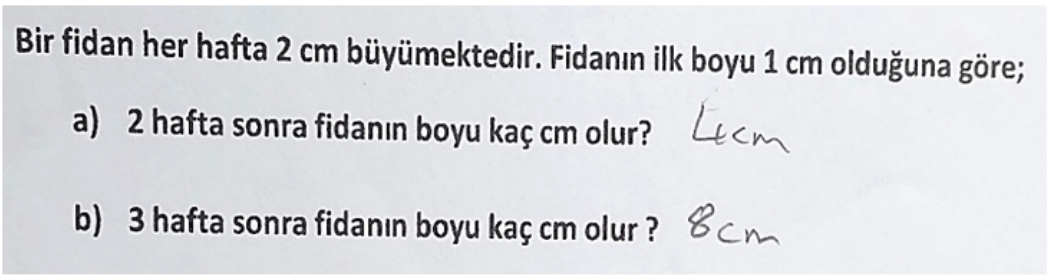


Figure 6.58. Answer of G8 in the first two items of the fourth question.

When groups' worksheets for the third item were analyzed, it was seen that three groups (G2, G4, and G6) could formulate a general rule of a pattern that represents the correspondence relationship. Two groups among these groups (G2 and G4) wrote algebraic expressions as "2x+1", as seen in Figure 6.59 in the worksheet of G2, and one group (G6) wrote both algebraic expressions and verbal expressions, as seen in Figure 6.60 below. One group (G8) did not write algebraic expressions to represent the general rule. Instead, this group wrote only the verbal expression, as seen in Figure 6.61 below. Three groups among all groups (G3, G5, and G7) wrote incorrect verbal expressions that did not represent the correspondence relationship, as seen in the worksheet of G3 in Figure 6.62 below. Besides, one group (G1) has no answer for this item.

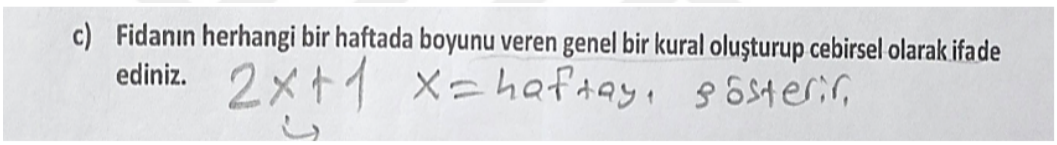


Figure 6.59. Answer of G2 in the third item of the fourth question.

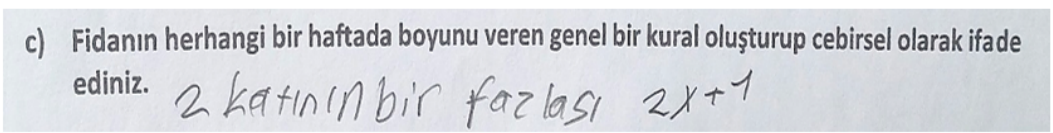


Figure 6.60. Answer of G6 in the third item of the fourth question.

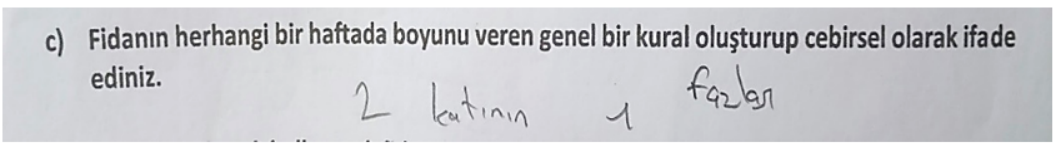


Figure 6.61. Answer of G8 in the third item of the fourth question.

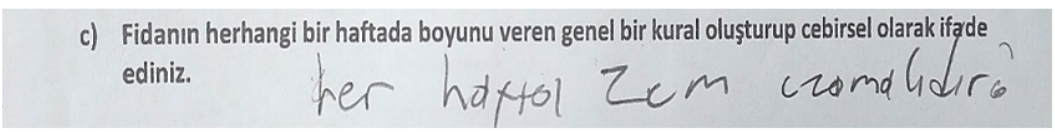


Figure 6.62. Answer of G3 in the third item of the fourth question.

When the groups' worksheets were examined for the fourth item, there were two groups (G2 and G4) who used the general rule explicitly to find the distant term of

the pattern, as seen in the worksheet of group G2 in Figure 6.63. Four groups (G3, G5, G6, and G7) found the distant term correctly but did not use the general rule explicitly, as seen in the worksheet of group G6 in Figure 6.64. In addition, two groups (G1 and G8) among all groups could not find the distant term correctly, and there was no evidence of using the general rule for their calculation, as shown in the worksheet of group G8 in Figure 6.65.

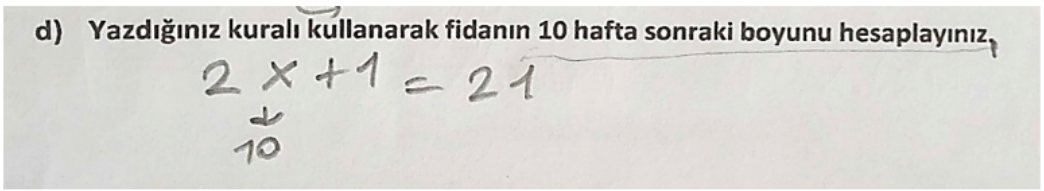


Figure 6.63. Answer of G2 in the fourth item of the fourth question.

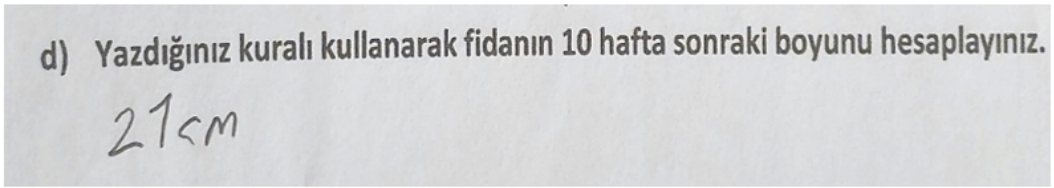


Figure 6.64. Answer of G6 in the fourth item of the fourth question.

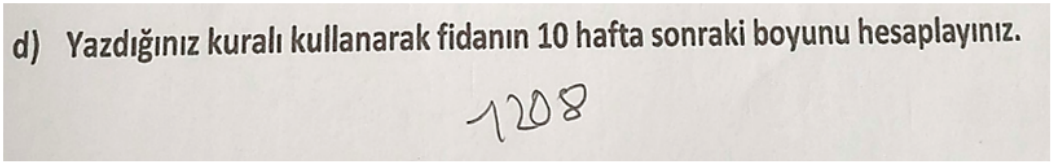


Figure 6.65. Answer of G8 in the fourth item of the fourth question.

Groups' algebraic thinking indicators regarding functional thinking for the fourth question were summarized in Table 6.17 below.

Table 6.17. Algebraic thinking indicators in the fourth question.

Finding close term of the pattern			
Codes	**Descripiton**	**Example**	**Groups**
Found close term without written calculation	Students find the close terms correctly without written calculations.	Writing only 5 and 7	G1, G2, G4, G5, G7
Found close term using correspondence relationship	Students find the close term correctly using correspondence relationships.	2x2= 4+1= 5 2x3= 6+1= 7	G3, G6
Incorrectly found close term without written calculation.	Students incorrectly find the close term and there is no solution way.	Writing only 4 and 8	G8
Formulating a general rule of a pattern algebraically			
Codes	**Descripiton**	**Example**	**Groups**
Formulated a general rule algebraically.	Students form a correspondence rule algebraically.	2x+1	G2, G4
Formulated a general rule algebraically and in words	Students form a correspondence rule algebraically and words.	2x+1, 1 more than 2 times a number	G6
Formulated a general rule in words	Students form a correspondence rule using words.	1 more than 2 times a number	G8
Incorrectly formulated general in words.	Students incorrectly form a rule in words.	It should grow 2cm every week.	G3, G5, G7
No Answer	There is no answer in the worksheet.	-	G1

Table 6.17 Algebraic thinking indicators in the fourth question. (cont.)

Using the general rule of a pattern to find the distant term.			
Codes	**Description**	**Example**	**Groups**
Explicitly using the general rule and found distant term correctly.	Students explicitly use the general rule and find the distant term correctly.	2x10+1 = 21	G2, G4
Not explicitly using the general rule but found the distant term correctly.	Students find the distant term correctly without using the general rule explicitly.	Writing only "21"	G3, G5, G6, G7
Not explicitly using the general rule and found the distant term incorrectly.	Students find the distant term incorrectly without using the general rule explicitly.	Writing only 1208	G1, G8

6.3.1.5. Fifth Lesson. According to MoNE (2018), the fifth lesson objective is to explain the meaning of simple algebraic expressions together with modeling studies. With three-act math tasks and the following activities, students' algebraic thinking regarding modeling was analyzed.

In the first act, the students were shown a video, which included a game interface and an algebraic expression that appeared in it. Once the algebraic expression appears as a code in the game format, the modeling of this algebraic expression emerges with algebra tiles. Thus, one example related to modeling with algebra tiles was shown to the students. However, this video had no definitions or explanations for algebra tiles; modeling was given without referencing them and their meanings. Figure 6.66 shows a section in this video.

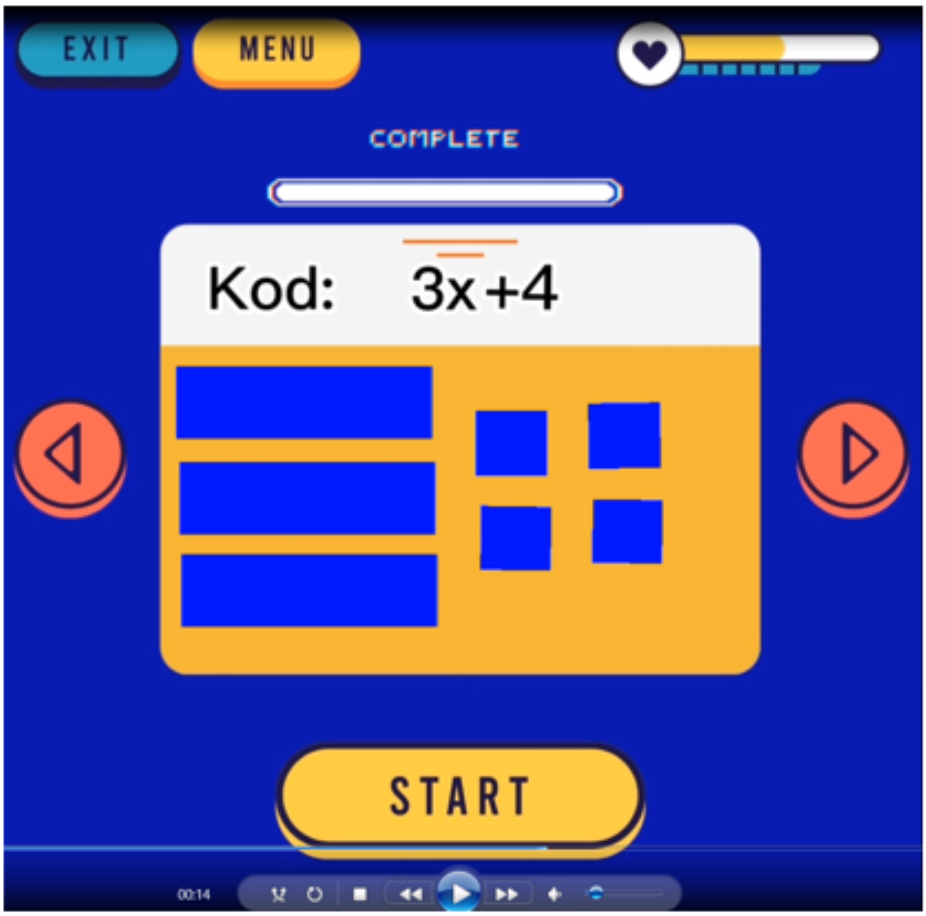


Figure 6.66. A section from the video in ACT 1 of the fifth lesson.

In continuation of the video, the algebraic expression 4x+5 appears, but no tiles appears on the screen. After the video was shown, the dialogue between the teacher and students was as follows.

T: What do you notice?
S1: Teacher, 3x
S3: Three rectangles and four squares
S7: Five boxes.
S8: Game
S12: The value of each square is +1, the value of each rectangle is 1x.
T: Yes, rectangles represent 1x
Class: Squares represent +1.
T: Yes, squares represent +1.

As the dialogue showed, students noticed the meaning of algebra tiles. In other words, they recognized that the rectangles represent "x" and squares represent one.

Besides, when the teacher asked them what they wondered about, their answers were game-related, like the game's name and the continuation of the game. Then the teacher wrote the focus question as "What will be the code for 4x+5 appearing on the screen?". At first, students wrote their estimation of how many squares and rectangles there would be; then, in the second act, they started to work on this question by drawing the model as a group. Meanwhile, a photo that includes the meaning of algebra tiles was shown to the students as a clue.

When the answers were analyzed, all the groups could draw the appropriate models with an expression of 4x+5. Among these groups, two groups (G1 and G5) wrote what each shape represented in the algebraic expression (x or 1), as seen in the worksheet of G5 in Figure 6.67, while six groups (G2, G3, G4, G6, G7, G8) just drew the shapes as seen in the worksheet of G4 in Figure 6.68.

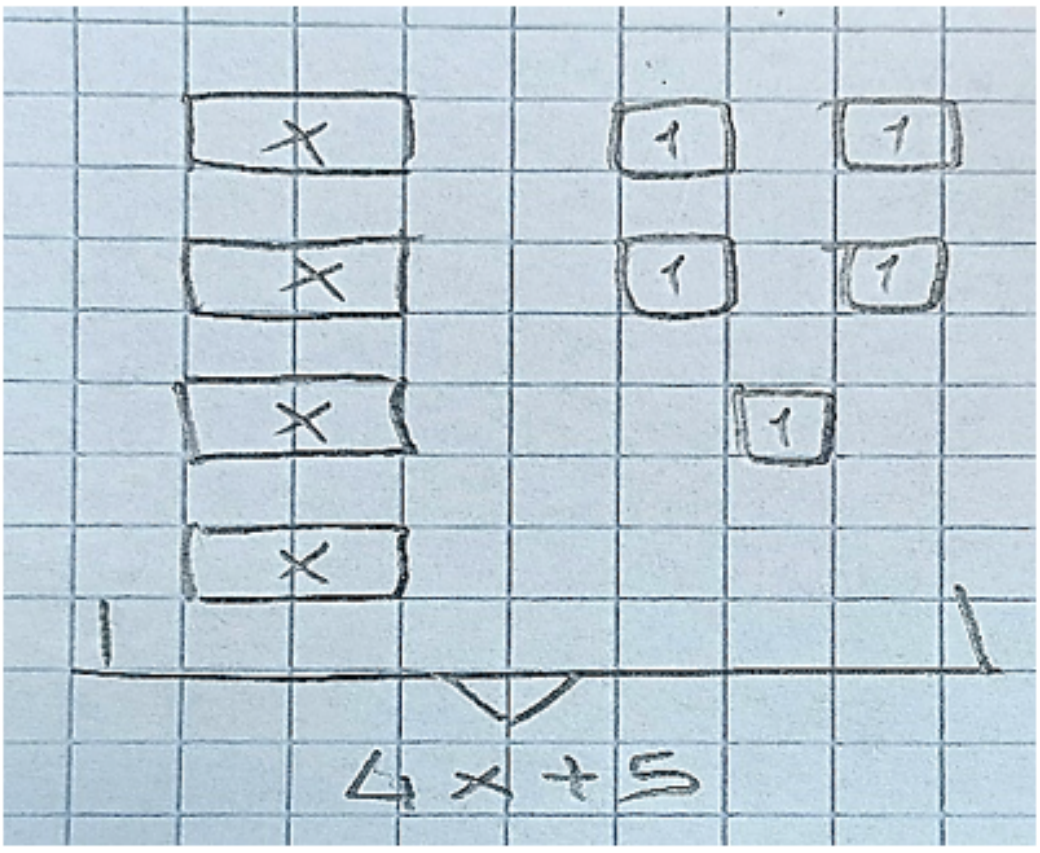


Figure 6.67. The worksheet of G5 in ACT 2 of the fifth lesson.

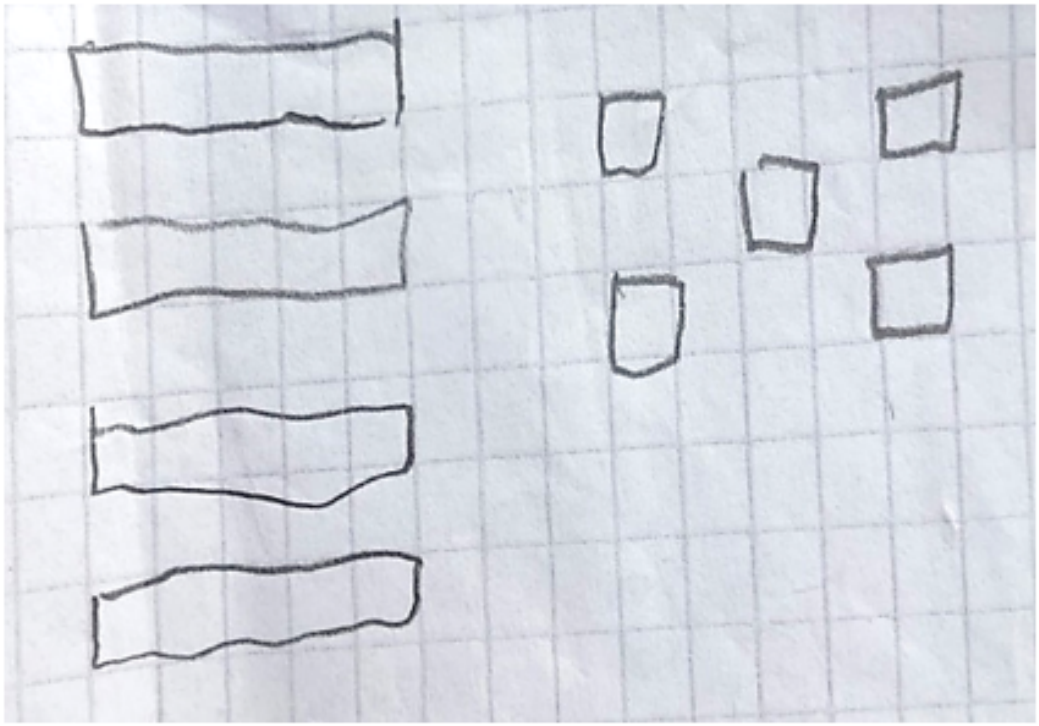


Figure 6.68. The worksheet of G4 in ACT 2 of the fifth lesson.

During the third act, the continuation of the video, which shows the modeling of 4x+5, was shown to the students. Then, negative algebra tiles (-x) and (-1) were introduced to the students, and some examples were given of how to write algebraic expressions related to algebra tiles.

Afterward, the teacher distributed a new worksheet for writing algebraic expressions suitable for the given algebra tiles. Examination of the groups' worksheet shows that four groups (G2, G3, G6, and G8) completed the entire worksheet correctly.

In order to illustrate, the worksheet of the G8 is presented in Figure 6.69 and Figure 6.70 below.

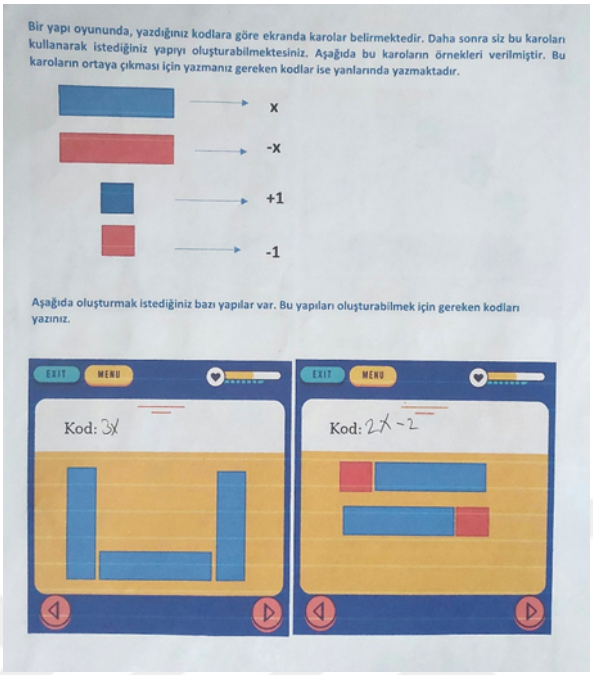


Figure 6.69. The first page of the worksheet of G8 in the fifth lesson.

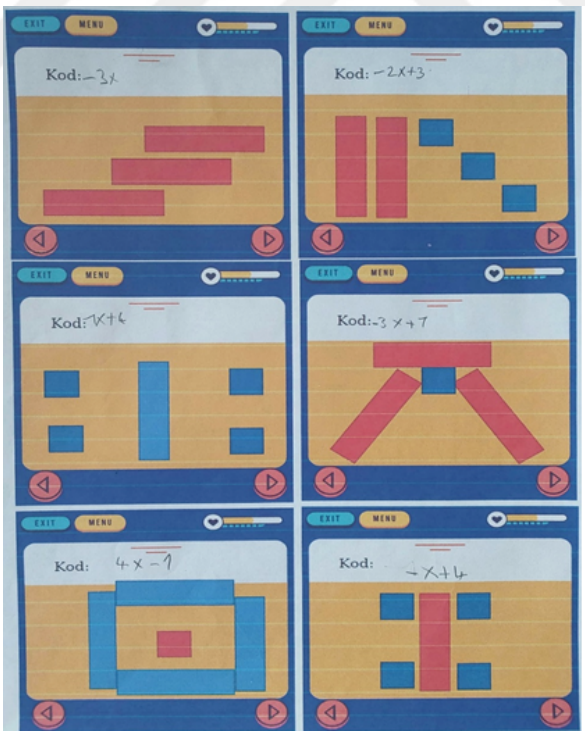


Figure 6.70. The second page of the worksheet of G8 in the fifth lesson.

One group (G5) wrote correct algebraic expressions for all models except the last. Figure 6.71 shows the answer of G5 for the last question. Besides, one group (G1) wrote incorrect expressions for the last two models. Figure 6.72 shows their worksheet.

In the worksheet of G1, even though the written format is correct, the variables are unsuitable for the algebra tiles.

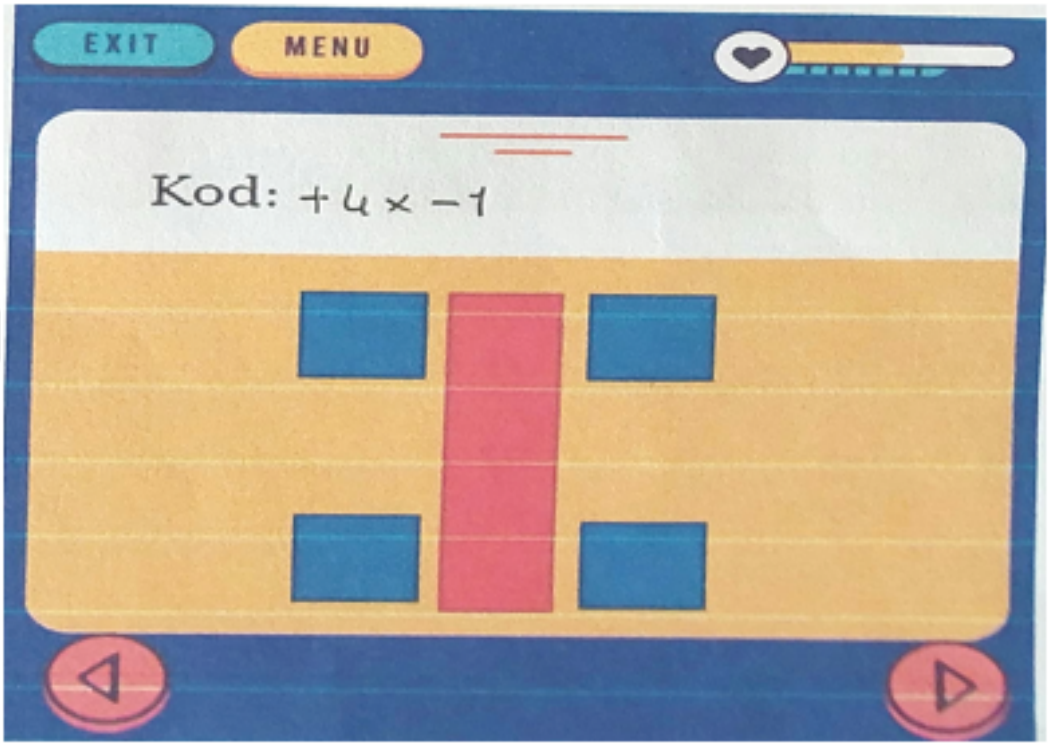


Figure 6.71. The last expression of G5.

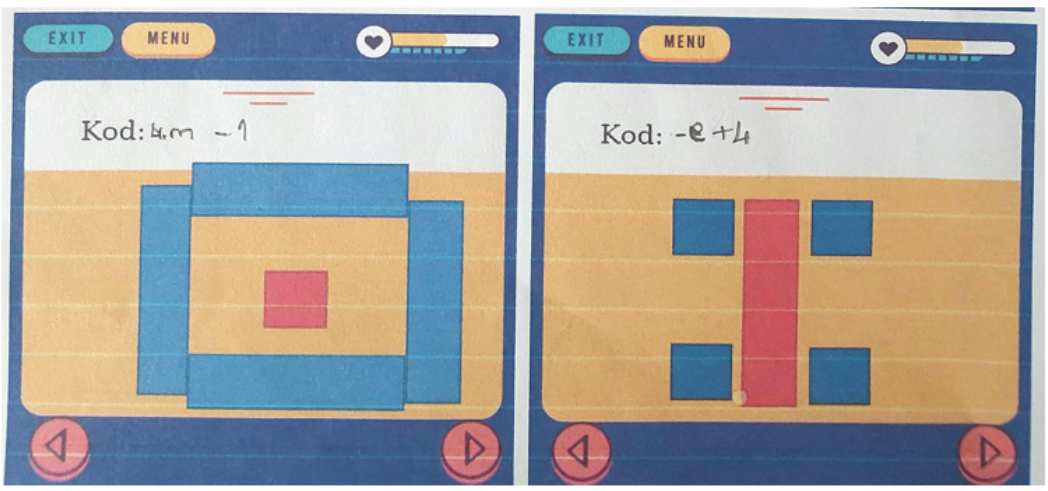


Figure 6.72. The last two expression of G1.

One group (G4) wrote incorrect algebraic expressions for the two models in the last four models (Figure 6.73). Besides, one group (G7) wrote no expression for the last four models on the second page, although algebraic expressions for other models were correct.

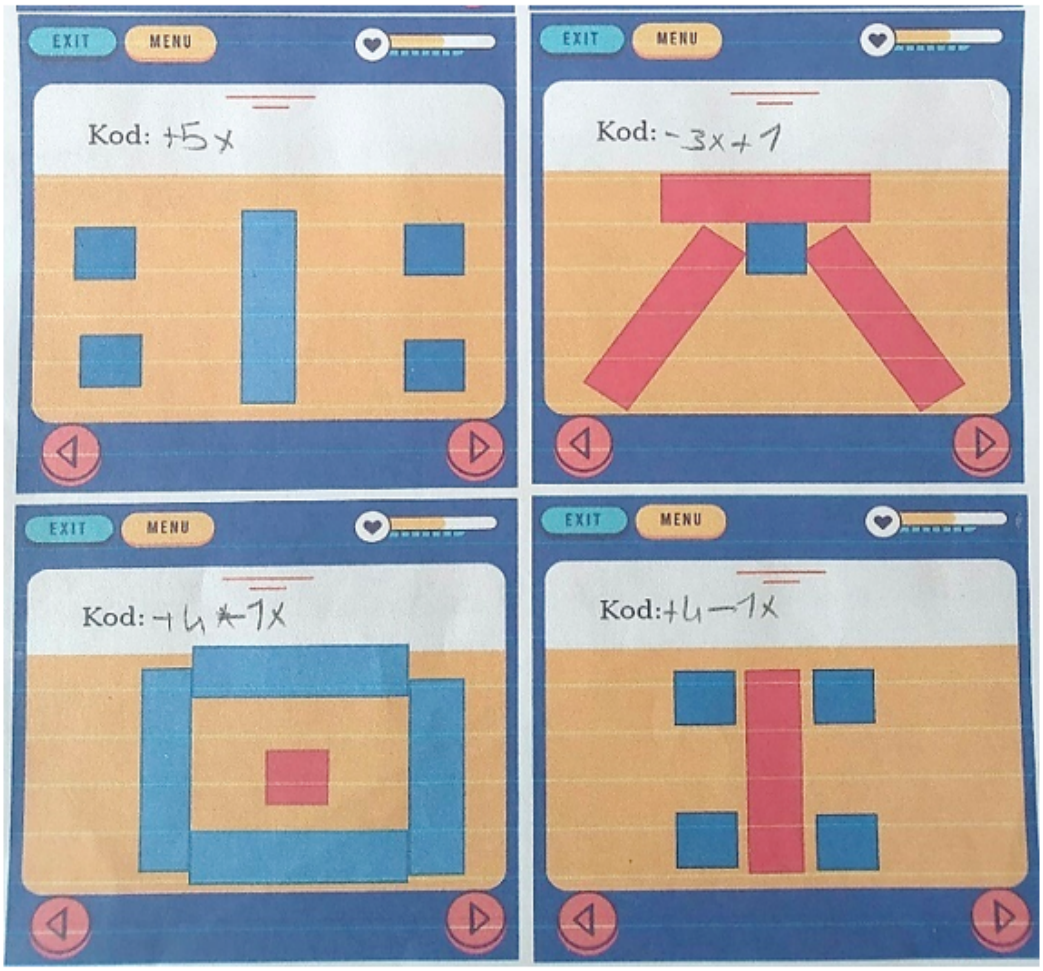


Figure 6.73. Two wrong expressions of G4.

Figure 6.74 below shows the emergent codes and the related category of the modeling. As summarized in the table, even though all groups could write correct expressions, three of them (G1, G4, and G5) also wrote incorrect expressions, and one of them (G7) had no answer in the last four questions as well.

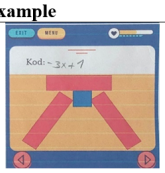
Transforming a real-life (or-context based) situation into a mathematical model			
Codes	**Description**	**Example**	**Groups**
Appropriate algebraic expression for all eight models.	Students wrote algebraic expressions that were suitable to the model for all questions. There is no incorrect answer.		G2, G3, G6, G8
Appropriate algebraic expressions for seven models.	There is one incorrect and seven correct expressions.	Writing 4x-1 instead of 4-x.	G5
Appropriate algebraic expressions for six models.	There are two incorrect and six correct expressions.	Writing 4m-1 instead of 4x-1. Writing -e+4 instead of -x+4. Writing 5x instead of x+4	G1, G4
No answer for last four models.	There are four correct answers, but there is no answer for the last four models.		G7

Figure 6.74. Algebraic thinking indicators of groups regarding modeling in the first activity in the fifth lesson.

After this worksheet, the teacher distributed another worksheet related to modeling algebraic expressions with appropriate algebra tiles. When groups' works were examined, it was seen that six groups (G1, G2, G3, G5, G6, and G8) correctly made models that were appropriate to the algebraic expressions in all examples, as seen in the worksheet of G5 in Figure 6.75. One group (G4) made one incorrect model, as seen in Figure 6.76. Besides, one group (G7) modeled algebraic expressions with inappropriate tiles, as shown in Figure 6.77.

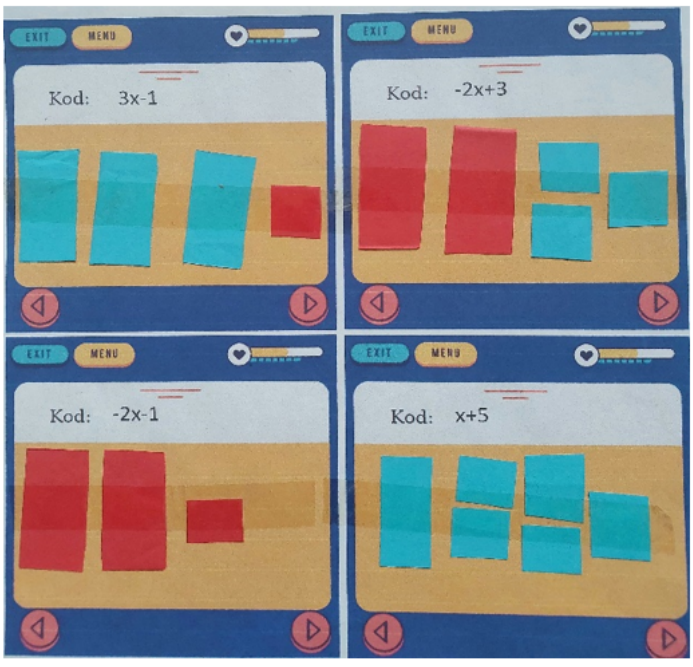


Figure 6.75. Appropriate models of G5.

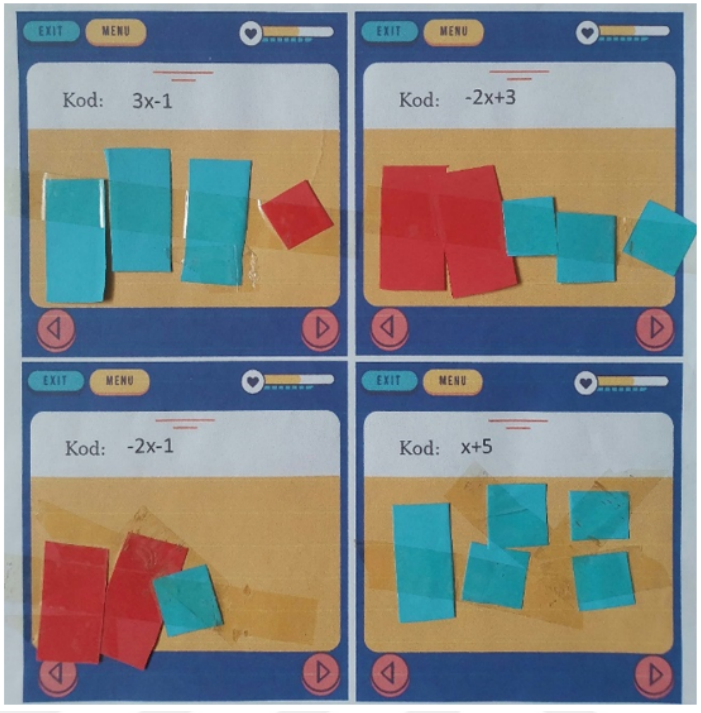


Figure 6.76. Models of G4.



Figure 6.77. Inappropriate models of G7.

Groups' algebraic thinking indicators under the modeling theme are presented in Table 6.18 below with emerged codes. From here, it can be understood that most groups could model the algebraic expressions correctly.

Table 6.18. Algebraic thinking indicators of groups regarding modeling in the second activity in the fifth lesson.

Transforming a real-life (or-context based) situation into a mathematical model		
Codes	**Description**	**Groups**
Forming appropriate models that fit the given algebraic expression	All four models are appropriate for the given algebraic expressions.	G1, G2, G3, G5, G6, G8
Forming one inappropriate model.	There is one inappropriate and three appropriate models.	G4
Forming inappropriate models.	All four models are inappropriate, which means they do not fit the given algebraic expression.	G7

6.3.2. Students' Algebraic Thinking After the Instruction

In order to examine the algebraic thinking of the students at the end of the lessons, six students were selected based on their achievement test results, communication skills, and willingness. According to the achievement test, high-level students were labeled as H1 and H2, medium-level students as M1 and M2, and low-level students as L1 and L2 for the analysis.

6.3.2.1. Analysis of Student Answers for the First Question. The first question in the interview was analyzed under the themes of generalized arithmetic and modeling. There

are three items (i, ii, and iii) in the question, and students' answers for the first item are given below (Table 6.19).

Table 6.19. Students' answers in the first item of the first question.

A customer asked the waiter for 4 hamburgers and 2 ayran. a) How can the waiter algebraically write the total order price in the order book? Find more than one way.	
Students	**Dialogues**
H1	H1: He/She can write 4x + 2x. T: How else can he/she write? So, are there any other ways? H1: It could be 2x + 4x. T: Why are these equal to each other? H1: Change of the places. They are equal even if they change places since we make additions. The result does not change. T: Okay, are there any other methods that came to your mind? H1: No.
H2	H2: By using letters (writes h+h+h+h+a+a) T: Okay, you can say your thoughts at the same time. Why did you write like this? H2: Because there are 4 hamburgers and we write them using letters. Since there are 4 of them, we show them with letters. I wrote 4 "h" plus "h". 2 ayran also "a"+"a" T: So how else can you write this? For example, this is the right way, could there be other different ways to write? H2: Yes, 4h + 2a T: How could you write it that way? And, what kind of relationship is there between the two expressions? H2: I wrote it because 4 hamburgers were requested. There is such a relationship between them. T: Are there any other methods that came to your mind? H2: No.

Table 6.19 Students' answers in the first item of the first question. (cont.)

Students	**Dialogues**
M1	M1: 4x + 2a (writes 4x+2a) T: So how else can you write this? M1: 4 pieces of x , (writes x+x+x+x), 2 pieces of ayran, (writes a+a) T: What is the relationship between them? M1: 4x+2a is short way. T: Is there another way that came to your mind? M1: No
M2	M2: I represented 4 hamburgers also 2 ayran. Then I made algebraic expressions, as algebraic expression 4 "x" plus 2 "a" (writes 4x + 2a). T: Hmm, so how would you write these differently? M2: Oh, can it be 4 x plus and 2 a. Now this represents 4 hamburgers x+x+x+x, plus a+a. (writes x+x+x+x+a+a) T: So, how do you relate to these? Why could you write it like this? M2: We could do it this way because he/she wanted 4 hamburgers, and I showed them with 4 x's. T: : Is there another way that came to your mind? M2: No
L1	L1: x + 4 and x + 2 T: Why did you write in this way? L2: Because there are 4 hamburgers and 2 ayran T: Is there another way to represent this? L1: No
L2	L2: Since there are 4 hamburgers, 4x, that is, 4 times plus 2 because he also wants 2 ayran (writes 4x + 2) T: Why did you write in this way? L2: I showed 4 hamburgers as 4x and 2 ayran as 2. T: How else could you have written this? Could it have been written in different ways? L2: Yes.

Table 6.19 Students' answers in the first item of the first question. (cont.)

Students	**Dialogues**
	T: How?
	L2: Is it like this (writes 1h+1h+1h+1h+1a+1a)
	T: What is the relationship between them ?
	L2: Same things.
	T: Are there any other methods that came to your mind?
	L2: No.

According to the students' answers, H1 refers to the commutative property of addition by explaining the equality of 4x+2x and 2x+4x. This shows that he used the properties of operations to form equivalent expressions. However, he failed to recognize that different variables should represent different quantities since he used the same variable to represent the price of the ayran and hamburger.

The first algebraic expression of H2 was in repeated addition format, and she transformed it to a multiplicative format that is equivalent. Although she did not explicitly explain the relationship between addition and multiplication when the teacher asked, these two forms of algebraic expression she wrote show that she established a relationship between multiplication and addition and used it to form an equivalent expression.

M1 transformed his first algebraic expression, which is a multiplicative format, to the additive format by using properties of operations. His equivalent algebraic expression and explanation show that he could establish a relationship between multiplication and addition and use it.

Similarly, M2 wrote two equivalent algebraic expressions by using the relationship between multiplication and addition. Her first expression was in the form of multiplication, and she changed it to the addition format. Therefore, M2 used the relationship between properties of operations to form equivalent expressions.

L1 could not write algebraic expressions correctly. For four hamburgers, he wrote x+4 instead of 4x to represent the total price of hamburgers. For two ayrans, he wrote x+2 rather than 2x to represent the total price of ayran. Even though he used variables to express an arithmetic operation algebraically, his algebraic expression was incorrect, and no other form was written. Therefore, there is no evidence of using the relationship between operation properties to form equivalent algebraic expressions. He also failed to recognize that different variables should represent different quantities since he used the same variable for the prices of the ayran and hamburgers. L2 also could not write algebraic expressions correctly. Although she showed the price of four hamburgers as 4x, she represented the price of two ayrans as two and wrote 4x+2. When the teacher asked for the equivalent form of the expression, she used different variables, h, and a, and wrote in repeated addition format. Since the variables are not identical to the first expression's variable and the expressions are not equivalent, there is no evidence of using the relationship between operation properties to write equivalent algebraic expressions. Students' answers were summarized with the related codes and categories regarding generalized arithmetic in Table 6.20 and modeling in Table 6.21.

Table 6.20. Algebraic thinking indicators of students regarding generalized arithmetic in the first item of the first question in the interview.

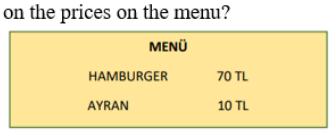
Using the relationship between operation properties to form equivalent algebraic expressions.			
Codes	**Description**	**Example**	**Students**
Commutative Property	Students write an equivalent form of an algebraic expression using commutative property.	Writing 4x+2x as 2x+4x	H1
Multiplication as repeated addition	Students use the relationship between multiplication and addition, thus write equivalent forms.	Writing 4x+2a as x+x+x+x+a+a	H2, M1, M2
No equivalent form	Students do not write an equivalent form of an algebraic expression.	-	L1
Incorrect equivalent form	Students attempt to write an equivalent form of an algebraic expression but incorrectly write.	Writing 4x+2 as 1h+1h+1h+1h+1a+1a	L2

Table 6.21. Algebraic thinking indicators of students regarding modeling in the first item of the first question in the interview.

Transforming a real-life (or-context based) situation into a mathematical model			
Codes	**Description**	**Example**	**Students**
Appropriate algebraic expression	Students correctly transform real-life (or context-based) situations into an algebraic expression.	4x+2a, x+x+x+x+a+a, 4h+2a, h+h+h+h+a+a	H2, M1, M2, L2
Inappropriate algebraic expression	Students incorrectly transform real-life (or context-based) situations into an algebraic expression.	4x+2 x+4 x+2	L1, L2
Partially appropriate algebraic expression	Students correctly transform real-life (or context-based) situations into an algebraic expression but use the same letters for different variables.	4x+2x, 2x+4x	H1

Through the second item, it was examined whether students found the value of order cost and whether they stated the variables during their calculation. Students' answers are presented in Figure 6.78 below.

b) How much TL will the order cost based on the prices on the menu?



Students	**Dialogues**
H1	H1: Since one hamburger is x TL, 4 hamburgers are 4x TL. 4 times 70 equals 280 TL. Since he ordered 2 ayran and ayran is 10 TL, 2 times 10 equals 20 TL. And the total order price is 300 TL.
H2	H2: Here, it is said that the hamburger is 70 TL, I multiplied 70 by 4, it became 280 TL. Since ayran was 10 TL, It is said that 2 ayran. And It made 20 TL. Since I calculated 280 and 20, we will add them. Then, 300.
M1	M1: 280, 300 T: How did you find 300 directly? M1: The hamburger is 70, I multiplied 70 by 4; I multiplied 10 by 2 and then added them.
M2	M2: Here it is asked how much TL it costs. So I immediately multiplied 70 by four. Because there are 4 hamburgers, it made 280. I made ayran the same way. I multiplied 10 by 2 and it became 20. I added 20 with 280 and it made 300.
L1	L1: 230 TL. T: How did you find 230? L1: I multiplied 4 by 70 to get 210. I multiplied 2 by 10 to get 20 and added.
L2	L2: Since the hamburger costs 70 TL and he wants 4 pieces, we multiply it by 4. We find 280, then since he wants 2 ayran, and the price of ayran is 10 TL, we multiply it by two and get 20, and when we add them together, it equals 300.

Figure 6.78. Students' answers in the second item of the first question in the interview.

When students' answers were analyzed, it was seen that all students except L1 correctly calculated the cost of the order and found correct results. L1's error was a multiplication error, but his way was right. Even though students found the correct results, only H1 clearly stated the variables during the calculation. For the other students (H2, M1, M2, L1, and L2), students calculated the cost order by a simple arithmetic operation.

The third item in the first question aims to examine students' explanations about the meanings of the variables they used. Students' answers are shown in Table 6.22 below.

Table 6.22. Students' answers in the third item of the first question in the interview.

c) What do the letters represent in the algebraic expression you wrote?	
Students	**Dialogues**
H1	H1: Letters represent price.
H2	H2: It represents how much money hamburgers and ayran.
M1	M1: Hamburger and ayran. No, the price of hamburger ayran.
M2	M2: Variables represent letters, letters represent things. The variables represent hamburger and ayran
L1	L1: Hamburger and ayran.
L2	L2: Represents the variable. T: Why does it represent the variable? L2: Because we cannot write hamburger and ayran longly. T: What does it represent regarding the order? L2: Represents hamburger and ayran regarding the order.

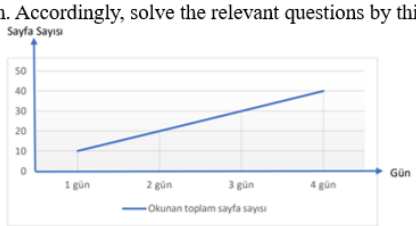
When students' answers were examined, half of the students students viewed variables as the price of hamburgers and ayran (H1, H2, and M1), and the other half viewed the variables as an object, which means hamburger and ayran (M2, L1, and L2). Therefore, some students (M2, L1, and L2) could not attach the correct meaning to the variable they used in their algebraic expressions. Table 6.23 below summarizes the students' answers regarding the meaning of the variable.

Table 6.23. Algebraic thinking indicators of students regarding generalized arithmetic in the third item of the first question in the interview.

Explaining the meaning of variable			
Codes	**Description**	**Example**	**Students**
Quantity	Students express that variables stand as a quantity of an object.	"h" represents the "price of hamburger" "a" represent the " price of ayran".	H1, H2, M1
Object	Students express that the letters represents a label for an object.	"h" represents the "hamburger" "a" represents the "ayran".	M2, L1, L2

6.3.2.2. Analysis of Student Answers for the Second Question. The second question in the interview is under the theme of functional thinking. There are five items in the question. Students' answers were analyzed based on the related categories of functional thinking. Their answers to the first item are presented in Figure 6.79.

Aslı reads a book regularly every day. Aslı represented the number of days and the total number of pages she had read so far in a graph. Accordingly, solve the relevant questions by thinking aloud.



a) What is the relationship between the number of days and the total number of pages read?

Students	**Dialogues**
H1	H1: She says how many pages of books she read in 4 days. These are increasing by 10 by 10. There is such a rule.
H2	H2: Here, on the first day, she read 10 pages; on the second day, she read 20 pages; on the 3rd day, she read 30 pages; and on the 4th day, she read 40 pages. Aslı reads 10 pages in one day and 40 pages in 4 days. So, in one day, she reads 10 pages; on the second day, it becomes 20; on the third day, it becomes 30; and on the fourth day, it becomes 40.
M1	M1: Number of days, number of days, reading 10 pages in 1 day. On the 2nd day, it same and, it goes up to 4 days.
M2	M2: What kind of relationship? Well, there is a relationship like this, it increases by 10 by 10.
L1	L1: The number of pages read increases day by day.
L2	L2: The number of pages is up to 50. What is the relationship between the number of days and the total pages she had read so far? 10 10 goes because 0, 10, 20, 30, 40, 50.

Figure 6.79. Students' answers in the first item of the second question in the interview.

As Figure 6.79 above shows, three students (H1, M2, and L2) described a recursive relationship in the pattern represented in the graph. The other three students (H2, M1, and L1) described a covariational relationship.

In other words, H1, M2, and L2 only focused on the difference between consecutive terms, and H2, M1, and L1 focused on two quantities that vary with respect to each other. Their descriptions are summarized in Table 6.24 below with the related category and associated codes.

Table 6.24. Algebraic thinking indicators of students regarding functional thinking in the first item of the second question in the interview.

Identifying the relationship in a pattern and describing it with words.			
Codes	**Description**	**Example**	**Students**
Recursive	Students focus on the difference between consecutive terms and describe a recursive relationship.	It increases by 10 by 10.	H1, L2, M2
Covariational	Students describe the covariational relationship that explains how two quantities vary with respect to each other.	The number of pages read increases day by day.	H2, M1, L1

With the second item of the second question in the interview, it was analyzed whether the students could find the close term of the pattern and in which way they tried to find it. Students' answers to the second item are presented in Figure 6.80 below.

Aslı reads a book regularly every day. Aslı represented the number of days and the total number of pages she had read so far in a graph. Accordingly, solve the relevant questions by thinking aloud.



b) How many pages of a book does Aslı read in 7 days?

Students	Dialogues
H1	H1: Since there are 40 pages in 4 days, there will be 70 pages in 7 days. T: How did you calculate it in your head? H1: Since she reads 10 pages each day, if there were 7 days, 7 times 10 equals 70.
H2	H2: If she reads 40 now, she will read 50 pages of the book on the fifth day, 60 pages on the 6th day, and 70 pages on the 7th day.
M1	M1: How many pages of a book does Aslı read in 7 days, Aslı, in 7 days.. T: What are you thinking right now? M1: Adding 30 to 40. 70 T: 70, how did you find 70? M1: If she reads 40 pages in 4 days, I add 3 more days until it is 7 days. I add up both of them, I add how many pages she reads in 3 days. T: Okay, how many pages does she read in 3 days? M1: 30 T: How many pages does she read in 4 days? M1: 40. I add 40 and 30.
M2	M2: If she reads 10 pages on the first day; 2nd day is 20, 3rd day is 30, 4th day is 40, 5th day is 50, 6th day is 60, 7th day is 70.
L1	L1: If she reads 10 pages day by day, we multiply 7 by 10 and get 70
L2	L2: How many pages does Aslı read in 7 days? She reads 70 pages in 7 days. T: How did you find 70? L2: I found it in my head. For example, since this 10 increased by 10, I multiplied 7 and 10 and got 70.

Figure 6.80. Students' answers in the second item of the second question in the interview.

When the students' answers above were analyzed, it was seen that two students (H2 and M2) found close terms by extending the pattern, and four students (H1, M1, L1, and L2) found close terms using the correspondence relationship which means by multiplying the number of days with a number of pages in per day.

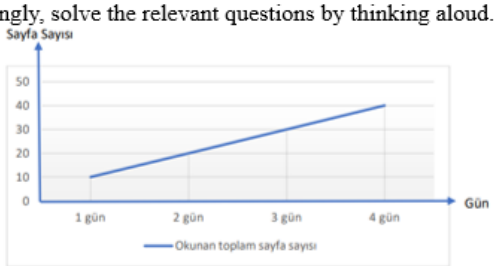
The related category and associated codes for their answers are depicted in Table 6.25 below.

Table 6.25. Algebraic thinking indicators of students regarding functional thinking in the second item of the second question in the interview.

Finding close term of the pattern			
Codes	**Descripiton**	**Example**	**Students**
Found close term by extending the pattern	Students find the close term correctly by extending the pattern.	2nd day is 20, 3rd day is 30, 4th day is 40, 5th day is 50, 6th day is 60, 7th day is 70	H2, M2
Found close term using correspondence relationship	Students find the close term correctly using correspondence relationships.	7 times 10 equals 70	H1, M1, L1, L2

With the third item of the second question, students' answers were analyzed according to how they found the distant term and by which method. Their answers are shown in Figure 6.81 below.

Aslı reads a book regularly every day. Aslı represented the number of days and the total number of pages she had read so far in a graph. Accordingly, solve the relevant questions by thinking aloud.



c) How many pages of a book does Aslı read in 50 days?

Students	**Dialogues**
H1	H1: Since there are 10 pages in a day, we find 500 by multiplying 50 times 10
H2	H2: Let's multiply 50 by 10, 50 times 10 equals 500. T: Well, why did you multiply 50 by 10? H2: She reads 10 pages every day. Then it is asked how many pages she will read in 50 days. So I multiplied 50 by 10 and got 500.
M1	M1: 500 T: How did you find 500? M1: If she reads 100 pages in 10 days, 10-20-30-40-50, 50 days equals 500.
M2	M2: Every day it goes ten, ten, it is said 50 days, here I multiplied 50 days by 10. 500
L1	L1: 500 T: How did you find it? L1: She reads 10 pages of a book a day. We multiply 10 by 50 and get 500.
L2	L2: How many pages of a book does Aslı read in 50 days? When we multiply 50 by 10, we get 500.

Figure 6.81. Students' answers in the third item of the second question in the interview.

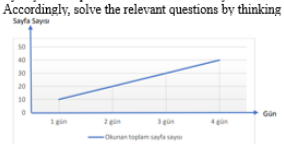
Students' answers show that all students (H1, H2, M1, M2, L1, and L2) found distant terms by using the correspondence relationship between the number of days and the number of pages per day (Table 6.26).

Table 6.26. Algebraic thinking indicators of students regarding functional thinking in the third item of the second question in the interview.

Finding distant term of the pattern			
Codes	**Descripiton**	**Example**	**Students**
Found distant term using correspondence relationship	Students find the distant term correctly using correspondence relationships.	We multiply 10 by 50 and get 500.	H1, H2, M1,M2, L1, L2

The fourth item of the second question in the interview asked students to find a general rule that reflects the number of pages on any given day. In other words, students are expected to form a general correspondence rule and write it algebraically. Students' answers to this item are depicted in Figure 6.82 below.

Aslı reads a book regularly every day. Aslı represented the number of days and the total number of pages she had read so far in a graph. Accordingly, solve the relevant questions by thinking aloud.



d) Form a general rule that will allow us to find the total number of pages that Aslı has read on any given day, based on the number of days, and express your rule algebraically.

Students	**Dialogues**
H1	H1: If we do not know the number of days she read, we can write x times 10. I wrote x times 10. (writes 10x) T: So what does ``x'' mean? H1: Number of days she read. T: So what does 10 mean? H1: 10 indicates the number of pages she read in one day.
H2	H2: We express it as 10x. What never changes here is the number of pages, but since the number of days changes, I made 10 times ``x''. T: Exactly why, what does the 10 here mean? H2: 10 here indicates 10 pages of a book a day. It represents 10. "x" represents how many days, the number of days.
M1	M1: Is x – 10? T: How did you find x – 10? M1: Sir, it is not clear how many days she read. So x – 10. Or x + 10? T: Why x + 10? M1: Sir, because it has increased. It increased day by day.
M2	M2: 10 times ``k''. T: What does the "k" here mean? M2: The ``k'' here stands for something. Each day of Aslı, any day. T: Any day is okay. What does 10 represent? M2: Number of pages read per day.
L1	L1: "x" times 10. T: What does "x" mean? L1: "x" represents any day and 10 represents the number of pages read in any day.
L2	L2: I don't know the day, so let me give it "f". It could be like this: 10 + f T: What does 10 mean? L2: It refers to the number of pages, and ``f'' refers to day. It increases 10 10.

Figure 6.82. Students' answers in the fourth item of the second question in the interview.

As seen in Figure 6.82 above, four students (H1, H2, M2, and L1) could write a correct general rule that represents the correspondence relationship between two variables.

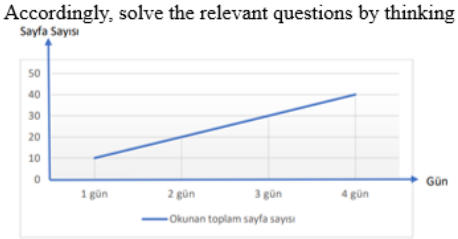
Specifically, H1, H2, and L1 wrote 10x, and M2 wrote 10f as a rule. On the other hand, two students (M1 and L2) wrote incorrect rules that do not represent the correspondence relationship. M1 first wrote x-10, then changed it to x+10, and L2 wrote 10+f as a rule. Their answers were coded as in Table 6.27 in the related category.

Table 6.27. Algebraic thinking indicators of students regarding functional thinking in the fourth item of the second question in the interview.

Formulating a general rule of a pattern algebraically			
Codes	**Description**	**Example**	**Students**
Formulated a general rule algebraically	Students form a correspondence rule algebraically.	10x	H1, H2, M2, L1
Incorrectly formulated algebraically	Students attempt to form a rule algebraically, but their rule is not a correct correspondence rule.	10+f x + 10	M1, L2

The fifth item of the second question aimed for students to find the pattern's distant term using the correspondence rule they formed. Students' answers to the item are depicted in Figure 6.83 below.

Aslı reads a book regularly every day. Aslı represented the number of days and the total number of pages she had read so far in a graph. Accordingly, solve the relevant questions by thinking aloud.



e) How many pages of a book does Aslı read in 90 days? Find this by using the general rule you formed.

Students	Dialogues
H1	H1: Based on the rule, since she reads 10 pages in one day and here is x times 10, we can find this as 900 by multiplying 90 and 10.
H2	H2: I multiply 90 by 10. In 90 days, it writes 90 because the day has already changed. I multiply 90 by 10, and 10 means something that how many pages she read in a day. 90 times 10 equals 900.
M1	M1: In 90 days, If this is the number of days, 900. T: How did you think of the 900? M1: It said 90 days, she reads 100 pages in 10 days. 200 in 20 days, it goes up to 900 in 90 days.
M2	M2: We multiply 90 by 10, 900. 10 is the number of pages read and "k" is the number of days.
L1	L1: x times 10 is 900
L2	L2: We will multiply 90 by 10, 900. T: Did you find this using this rule? L2: Yes, because it increases by 10 by 10.

Figure 6.83. Students' answers in the fifth item of the second question in the interview.

As seen above, all students could correctly find the distant term of the pattern. On the other hand, while four students (H1, H2, M2, and L1) explicitly used the general rule they formed in order to find the distant term of the pattern, two students (M1 and L2) did not use the general rule since the general statements they formed already were wrong. Their answers were coded, as shown in Table 6.28 below.

Table 6.28. Algebraic thinking indicators of students regarding functional thinking in the fifth item of the second question in the interview.

Using the general rule of a pattern to find the distant term.		
Codes	**Description**	**Students**
Explicitly using the general rule and found distant term.	Students explicitly use the general rule and find the distant term correctly.	H1, H2, M2, L1
Not explicitly using the general rule but found the distant term correctly.	Students find the distant term correctly without not using the general rule explicitly.	M1, L2

6.3.2.3. Analysis of Student Answers for the Third Question. The third question in the interview is based on the modeling theme and consists of three items. Through these three items, students are expected to model context-based situations by writing algebraic expressions and to find the value of modeled situations in context according to the given values of variables. Students' answers to the first item were given in Figure 6.84 below.

Mr. Mehmet started to build a wall from identical bricks. Below is a model of the wall he built. Accordingly, answer the following questions by thinking aloud.



a) Write an algebraic expression to express the horizontal length of the wall in cm, built by Mr. Mehmet

Students	Dialogues
H1	H1: (Writes 6x+2y), since it gives two y centimeters, I wrote them as 2y centimeters, and because it gives 6 x centimeters, I wrote them as 6x centimeters.
H2	H2: This is x. Let's write y here, this is x, x, x, this is y. Here 2y, 2y+6x. (Writes 2y+6x)
M1	M1: He used 2 pieces of ``y" cm, and 6 pieces of ``x" cm. (writes 2y+6x)
M2	M2: Here, it is asked us for an algebraic expression. So I said, there are 2 "y" cm, 2y, and there are 6 of them + 6x. (writes 2y+6x)
L1	L1: Here y + 8. Because there are 8 horizontal bricks, y + 8.
L2	L2: Teacher, for example, if this is y cm, here is 3x cm, 6x cm, then 2 times y cm. The horizontal length came from 6x. If each section is x cm, if there are 6 of them, we call it 6x, then plus 2. Since each section is a horizontal length, we call it 6x + 2y, since each section is y. Since there are 2 of them, we say 6x + 2y.

Figure 6.84. Students' answers in the first item of the third question in the interview.

As shown above, while five students (H1, H2, M1, M2, and L2) could write correct algebraic expressions that represent the horizontal length of the wall, one student (L1) wrote an incorrect expression. Students' answers were coded according to the appropriateness of the given context. Emerged codes associated with the category of modeling are presented in Table 6.29 below.

Table 6.29. Algebraic thinking indicators of students regarding modeling in the first item of the third question.

Transforming a real-life (or-context based) situation into a mathematical model			
Codes	**Description**	**Example**	**Students**
Appropriate algebraic expression	Students correctly transform real-life (or context-based) situations into an algebraic expression.	2y+6x	H1, H2, M1, M2, L2
Inappropriate algebraic expression	Students incorrectly transform real-life (or context-based) situations into an algebraic expression.	y+8	L1

The second item of the third question expected students to find the horizontal length of the wall according to the given values of the variables. Students' answers to the second item are presented in Figure 6.85 below.

Mr. Mehmet started to build a wall from identical bricks. Below is a model of the wall he built. Accordingly, answer the following questions by thinking aloud.



The dimensions of identical bricks are y = 50 cm and x = 10 cm. Based on this information, answer b) and c) of the question.

b) How many centimeters of horizontal length according to the bricks' dimensions?

Students	**Dialogues**
H1	H1: Since there are two 50s, the sum of these two is 100 centimeters, and since this is 10, this is 30, and this is 60. 6 times 10 equals 60. It means 160 cm and the horizontal length can be 160 centimeters.
H2	H2: We will put 50 instead of y, and we will add 10 instead of "x". 50 and 50 more equal 100. 50+50=100. If there are 6 of them, let's multiply 6 and 10 to get 60. Let's add 100 and 60 equals 160.
M1	M1: If he sews two horizontal lengths, the sum of the two is 100. He also sewed six vertical lengths, each of which makes 10, 60. We add 100 and 60 to get 160.
M2	M2: I multiplied 50 by 2 because y is 50 cm. I multiplied 10 by 6 because x is also equal to 10. I added 100 and 60 and got 160.
L1	L1: 50-60-70-80-130-140-150-160, horizontal length 160 cm.
L2	L2: It is 160 centimeters. T: How did you find 160? L2: Since there were 2 "y" and "y" was 50, I multiplied 50 by 2 and got 100. Since there were 6 "x" and they gave 10 to "x", I multiplied 6 and 10 to get 60, added the two, and got 160.

Figure 6.85. Students' answers in the second item of the third question in the interview.

When students' answers were analyzed, it was seen that all students correctly found horizontal length. Even though students found the correct results, three students (H2, M2, and L2) stated that variables stand for numbers during their calculations. On the other hand, the other three students (H1, M1, L1) did simple arithmetic calculations without referencing the variables.

With the last item of the third question, students are asked to model 180 cm length by combining variables. Thus, students are expected to write algebraic expressions that are appropriate to the 180 cm length of the wall. Students' answers to the third item are presented in Figure 6.86 and Figure 6.87 below.

Mr. Mehmet started to build a wall from identical bricks. Below is a model of the wall he built. Accordingly, answer the following questions by thinking aloud.



The dimensions of identical bricks are y = 50 cm and x = 10 cm. Based on this information, answer b) and c) of the question.

c) Mr. Mehmet wants to build a wall with a horizontal length of 180 cm. How many bricks can he use, and how can he place the bricks? Give more than one example. Write algebraic expressions that show the horizontal length in cm in each of your examples.

Students	Dialogues
H1	H1: By using 2 extra vertical bricks, we can make 180 cm. Since this horizontal length is 160, by using 2 more vertical bricks, one for 10 and two for 20 (writes 2y+8x). T: What else can you do to make 180 cm? H1: We can subtract 1y cm from here (2y+8x) and replace it with 5x cm (writes y+13x). T: What else? H1: We can place 3y cm and 3x cm (writes 3y+3x). T: Can you explain by thinking aloud how you write your algebraic expressions? H1: Since one y cm is 50 cm and we will make 180 cm, 13 pieces of 10 cm make 130 cm and when we add 50 cm, 180 cm is completed. Here(3y+3x), since 3x cm is 30 cm and 3 of y are 150 cm, then it becomes 180cm.
H2	It could be something. We can add twenty to 160. 2x can be come. T: How else can he do it? H2: He can do it like this, I think he can bring 8 x's side by side and 2 (y) side by side. (writes 2y+8x) T: What else can he do to make it 180? H2: Well, we can bring 4 x's here. Let these stay here and we can bring y here and add them here again. T: Do we change their locations? H2: Yes. Let's start with 4x; Let's bring y here and then put 4 x in the middle of them. We can write 4x + 1y + 4x + y. T: What else? H2: We can also write 3y + 3x. We can also write this with the distributive property. T: Hmmm! Then show it with the distributive property. H2: 3 times y + x in parentheses. [writes 3(y+x)] T: What else? H2: I can't think of anything else.

Figure 6.86. Two students' answers in the third item of the third question.

M1	M1: If he sews two more vertical lengths, it becomes 180 cm. If the total here is 160, adding 2 more vertical cm makes it 180. He can sew 18 vertical lengths. If vertical lengths are 10 each, 18x10. It can sew 3 horizontal lengths and 3 vertical lengths. If each horizontal is 50, if I sew 3 horizontals, it is 150. If each vertical is 10, I sew 3 verticals, that's 30. If we add 30 to 150, it's 180 (writes 2y+8x, 18x, and 3y+3x) T: How did you write these algebraically? M1:2y+8x T: Yes. M1:18x T: Yes. Why 2y+8x? M1: Because he sews 2 horizontal and 8 vertical lengths. If we sum these, we get 180. T: Why is this 18x? Because there are 18 vertical lengths, it becomes 180. T: Okay. Why is this 3y+3x? M1: In here, there are 3 vertical and 3 horizontal lengths.
M2	M2: This was 10 cm, 2 pieces of 10. T: How did you do it? So explain that too. M2: This is counted as 10. I added 2x. T: How do you write algebraically? M2: 2y + 8x T: What else? M2: Something like this came to my mind; this is 50 cm, this is 10 cm. Like this, 10, 20, 30, 40, 50. 3y + 3x. T: So how did this 180 happen? M2: How did 180 happen? Heh, I found it sir, this is 50, this is 50, this is 50, 150. Here is 30 too. T: Now can you summarize it all again? M2: This is the same shape; When I added only 2, it became 8, so 2y plus 8x. In the second one, I added 2 and made it horizontal, that is, 50 centimeters. It became 3 y plus 3 x. Then, by showing it here, first 50 cm, then 10, 10, 10 = 30 cm, 50, 50 cm.
L1	L1: Here we have 160cm, we can use 2 more x cm bricks. T: How else can he do it? L1: I remove one y cm. 3x = 30, +50 = 80. If we add 3 more it becomes 110. If we add 2 more x cm it becomes 130. If we add one horizontal length(y), it becomes 180. T: Can you write these algebraically ? L1: (Writes, x + 8 + y + 2) T: Is there any other way? L1: -
L2	L2: If this is 160, I add two more "x" and it becomes 180. T: How do you write this algebraically? L2: Since 6 ``x", then I put this on these ``x" and it becomes 8. 8x+2y. T: What else can you do? L2: There are 8 vertical lengths. There are 2 horizontal lengths and if we add them together, we get 180 T: T: How do you express it algebraically? L2: Oh, it's exactly the same thing, but it's different, but it's the same, okay.

Figure 6.87. Four students' answers in the third item of the third question.

When students' answers were analyzed, all students could correctly find 180 cm horizontal lengths by combining the variables. On the other hand, one student (L1) could not write the correct algebraic expression representing the horizontal length that derives his combination of the variables. The other five students (H1, H2, M1, M2, and L2) could write the correct algebraic expressions related to the 180 cm length they mentioned by combining variables. Table 6.30 shows students' different algebraic expression types.

Table 6.30. Students' expression types that model 180 cm horizontal length.

Students	**Algebraic expressions that model 180 cm horizontal length**
H1	2y+8x, 3y+3x, y+13x
H2	2y+8x, 3y+3x, 3(y+x), 4x+1y+4x+1y
M1	2y+8x, 3y+3x, 18x
M2	2y+8x, 3y+3x
L1	x+8+y+2
L2	8x+2y

H1 mentioned three different combinations of variables that equal 180 cm, which are 2y+8x, y+13x, and 3y+3x. Similarly, H2 also found 3y+3x and 2y+8x to get 180 cm. Unlike H1, H2 also mentioned another form of 2y+8x, 4x + 1y + 4x + 1y, by changing the places of bricks. Besides, H2 also mentioned the distributive property of addition over multiplication and wrote 3(x+y) as a different form of the 3y+3x. Similar to the H1 and H2, M1 also wrote 2y+8x and 3y+3x. Furthermore, unlike these students (H1 and H2), M1 wrote 18x by combining 10 cm bricks to get 180 cm. Like the first three students (H1, H2, and M1), M2 wrote 2y+8x and 3y+3x to get a 180 cm horizontal length. Like the other students (H1, H2, M1, and M2), L2 combined eight x and two y and wrote the expression 8x+2y. In addition to students (H1, H2, M1, M2, and L2) who could write correct algebraic expressions related to the 180 cm horizontal length that was formed, there was one student (L1) who could not write the correct algebraic expression even though he mentioned in the correct form by combining the length of the bricks. In other words, even though he could form 180 cm lengths using numbers associated with horizontal and vertical lengths of the brick, his expression shows he could not model the length with correct algebraic expressions. Table 6.31 shows the classifications of their algebraic expressions in terms of modeling theme according to the appropriateness to the 180 cm length that is in the context.

Table 6.31. Algebraic thinking indicators of students regarding modeling in the third item of the third question in the interview.

Transforming a real-life (or-context based) situation into a mathematical model			
Codes	**Description**	**Example**	**Students**
Appropriate algebraic expression	Students correctly model 180 cm length by using algebraic expressions.	3y+3x	H1, H2, M1, M2, L2
Inappropriate algebraic expression	The student wrote incorrect algebraic expression to model 180 cm length.	x+8+y+2	L1

7. DISCUSSION AND CONCLUSION

In this study, how students think algebraically during and at the end of the RME-based lessons enriched with three-act math tasks and whether these lessons change the sixth-grade students' algebra achievement and attitudes toward mathematics were investigated. In the first part of this section, findings related to the instruction phase were summarized and discussed in terms of the RME approach and three-act math tasks. Then, the results of each research question were discussed in light of the literature with some recommendations. In the end, the limitation of this study was indicated, and recommendations for future research and practice were presented.

7.1. RME-Based Lessons Enriched With Three Act Math Tasks

The lessons were evaluated in terms of the RME approach; some conclusions were drawn about the implication of this approach with three-act math tasks. Therefore, in this section, the lessons based on the RME approach with three-act math tasks were discussed by highlighting some specific aspects.

In accordance with the reality principle of the RME approach, all lessons started with a real-life (or context-based) problem revealed through a three-act math task. In other words, students started the lessons with a problem that needed a solution that formed after the classroom discussion based on their noticings and wonderings about the real-life scenario presented in video or photograph format. This led to students building their learning. For example, in a three-act math task in the first lesson, the total number of tangerines in the basket was unknown, and in the second lesson, the amount of bulgur in one glass was unknown. Instead of being directly taught by the teacher as "We use variables to represent unknown quantities", students were first made to wonder about unknown quantities. After arousing curiosity, they searched for a solution to represent unknown quantities and worked on writing a general expression of an operation that includes an unknown. Thus, students mathematized a real-life

problem that was presented through three-act math tasks, and this enabled learning new concepts: variable and algebraic expression.

Three-act math tasks also allowed students to notice and think about which information they needed to know to solve the focus question. In this way, students could think about important concepts about the subject. For example, in the first and second lessons, the students told the teacher that they needed to know the value of quantities to mathematically represent a situation. After the necessary clue in ACT 2 under the guidance principle, they understood that quantities are unknown, and without knowing the value of quantities, they should write an expression that represents an operation that includes an unknown. Thus, they found a way to represent unknown quantities with the help of the lessons based on RME and three-act tasks. Throughout the lesson, in accordance with activity and interactivity principles of RME, they actively participated in the learning process and interacted with each other; they asked questions and discussed what they wondered and noticed in ACT 1, worked on solving the focus question on ACT 2, and shared their solutions in ACT 3 and reach a conclusion about the content like what is variable, what is algebraic expression. They were in a similar process to how mathematics was invented.

The third lesson aimed to make the students explore how to calculate the value of algebraic expression for different natural number values that variables can take. Therefore, with the three-act math task, a problem situation was presented in accordance with the reality principle of RME, and the students were able to mathematize the real-life situation. By looking at the order note, they realized that the letters written on the order meant the prices of the food. When the teacher asked the meaning of the variable in the class, the students stated that the variable was “price”. Therefore, with the problem situation represented with a three-act math task, they had a chance to think about the meaning of the variables and infer about them. Therefore, three-act math tasks provide students with an opportunity to think about critical points of the subject to be taught.

At the same time, students wondered about the different aspects of this situation, such as the prices of food and the cost of the order note. Based on what they wondered, the focus question was determined as, "How much will the order cost?". After that, they considered which information was required to solve this question. This was critical because it led students to think about how they find the order cost and use the variables' values in an expression. Based on their wonderings and thoughts, they had a chance to explore the lesson's main point. This information they thought about (in this case, the numerical value of the variables) was actually one of the main points of the topic. Their wondering about the order cost, which will be revealed by calculating the value of algebraic expression, was the second main point of the lesson. Thus, with the help of the three-act math tasks, students' attention can be drawn to these critical points of the lesson.

Similarly, in the fourth and fifth lessons, the context-based situation provided through three-act math tasks also enabled students to notice and wonder about critical aspects of the point. For instance, in the fourth lesson, students noticed a pattern and wondered the underlying reason for the occurrence of this pattern. This then led to thinking functionally and writing an algebraic general rule by considering the values of variables that will take within the expression. Similarly, in the fifth lesson, they noticed the algebra tiles within a game inference and had a chance to think about why the tiles emerged in that way. This provided students with a way to establish a relationship between an algebraic expression and its model with tiles. Thus, students noticed the meaning of algebra tiles as "1" and "x". Through this, rather than learning from the teacher, they noticed and explored critical points of the lesson on their own.

Before teaching a subject, students' noticing and wondering about some aspects of the concepts in the context of real-life can point out critical points of the subject, spark curiosity and make students more willing to learn. Solving a problem in line with their curiosity can ensure exploring a concept and an effective understanding of it. They actively used mathematics to solve main questions derived from their noticing and wonderings, and in this way, meaningful learning can take place. While they are

solving the problem, their thinking about what information they should use to solve a problem also supports their comprehension of important aspects of the topic in a meaningful manner.

Throughout the RME-based lessons enriched with three-act maths tasks, students learn a subject through active participation by doing mathematics. For instance, while they were working on the question and calculating how much the order cost, they discovered on their own how the value of the algebraic expression could be calculated. Thus, they actively participated in their learning. In line with the interactivity principle, the instructions supported a communicative and collaborative environment. By means of this, they shared what they wondered about and noticed in Act 1, worked with groups in Act 2, and shared and discussed their solutions in Act 3. The clues in Act 2 align with RME's guidance principle and enable the "guided re-invention" process.

Furthermore, the instructions also comprise the intertwinement principle because various mathematical areas are connected, like properties of operations, algebraic expressions, functional relationships, and modeling. For example, students transformed algebraic expressions into different equivalent forms by using their knowledge of the operation properties in arithmetic. This situation also created the environment for horizontal mathematization followed by vertical mathematization. A strong bond was established between these areas; they understood the relationships. In this way, it can be said that they are also in a gradual thinking process by the level principle.

Lessons based on the RME approach enriched with three-act math tasks can be a powerful instructional technique that provides students with meaningful learning, fosters curiosity, highlights essential points of the subject and, leads to deep mathematical understanding.

7.2. Students' Algebra Achievement

In this study, sixth-grade students' achievement on the algebra topic was investigated before and after RME-based lessons enriched with three-act math tasks. According to the results, students' achievement scores on the post-test were significantly higher than the students' achievement scores on the pre-test. Therefore, the result of this study can imply that the RME approach with three-act math tasks is an effective instructional approach to teaching algebra in sixth grade. The effectiveness of the RME approach on students' achievement was investigated in many of the studies (Altaylı, 2012; Çakır, 2011; Dönmez, 2018; Kesmezoğlu, 2023; Özkan, 2019; Sevim, 2019; Ünal and İpek, 2010; Yonucuoğlu and Bindak, 2021; Zakaria and Syamaun, 2017).

When the studies on the effectiveness of the RME approach on sixth-grade students' algebra achievement were evaluated, most of these studies had experimental and control groups (Çakır, 2011; Özkan, 2019). Studies by Çakır (2011) and Özkan (2019) show that the achievement of students who took the lesson based on RME was significantly higher than the control group's achievement score who took the traditional instruction. Another study examining the effect of RME-based instruction on seventh-grade students' achievement in algebra shows that the experimental group's achievement score was significantly higher than the control group's (Kesmezoğlu, 2023).

Since the current study focuses on students' algebraic thinking in RME-based lessons besides the achievement, there was no control group, and students' achievement was evaluated in pre and post-tests. On the other hand, this study also showed the effectiveness of the RME-based lessons because students' achievement in the post-test was significantly higher than in the pre-test, and the effect size was very large. Besides, unlike the other research, the current study used three-act math tasks in RME lessons as a material. Since the current study shows the students' achievements in RME-based lessons with three-act math tasks, it differs from other research that focuses on only the RME approach.

Along with this, the results of this study supported the results of the other studies that investigate the RME approach on students' achievement scores (Altaylı, 2012; Çakır, 2011; Kesmezoğlu, 2023; Özkan, 2019; Yonucuoğlu and Bindak, 2021; Zakaria and Syamaun, 2017). Therefore, it can be concluded that this study not only demonstrates the effectiveness of realistic mathematics education but also reveals its efficiency with a specific task type. Therefore, it supports the current literature and brings a new perspective to the RME approach.

7.3. Students' Attitudes Toward Mathematics

The second research question of this study was to investigate students' attitudes toward mathematics before and at the end of the RME-based lessons enriched with three-act math tasks. The results showed that after students attended the RME-based lessons enriched with three-act math tasks, their attitude towards mathematics scores significantly increased. In the literature, different researchers examined the difference in attitude scores towards mathematics between those who participated in RME-based education and those who participated in traditional education (Arseven, 2010; Bıldırcın, 2012; Dönmez, 2018; Işık and Altay, 2019; Sevim, 2019; Üzel and Uyangör, 2006; Zakaria and Syamaun, 2017). The studies of Arseven (2010), Işık and Altay (2019), Sevim (2019), Üzel and Uyangör (2006) and Zakaria and Syamaun (2017) showed that the attitude towards the mathematics of students who attended the RME-based instruction was higher than the attitude toward mathematics of students who attended the traditional instruction. In line with these, the current study demonstrated a significant improvement in mathematics attitudes after RME-based lessons with three act math tasks, as evidenced by the significantly different post-test scores. However, some studies contradict these results.

In the studies of Bıldırcın (2012) and Dönmez (2018), students' attitudes toward mathematics scores did not differ between the experimental group and the control group. Therefore, the literature has inconsistent results on the efficiency of RME on students' mathematics attitudes. On the other hand, this study contributed to the

literature by showing the role of RME-based lessons enriched with three-act math tasks on students' attitudes toward mathematics and brings a different lens to the current literature.

7.4. Students' Algebraic Thinking

Algebraic thinking is necessary to understand patterns, relations, and functions, represent mathematical situations and analyze them via algebraic symbols, and make mathematical modeling to investigate various contexts and quantitative relationships (NCTM, 2000). Therefore, it comes to be an essential place for mathematics education and mathematics curriculum.

In the literature, numerous research focuses on important focal themes for algebraic thinking. These were summarized by Kieran *et al.* (2016) as "i) generalizing related to patterning activity, (ii) generalizing related to properties of operations and numerical structure, (iii) representing relationships among quantities, and (iv) introducing alphanumeric notation". (p.5). In that sense, Kieran *et al.* (2016) provided us with a good summary of the main studies in the literature related to algebraic thinking.

In this research, by taking MoNE's (2018) objectives and Kaput's (2008) algebra content strands that are generalized arithmetic, functional thinking, and modeling into consideration, different three-act math tasks and worksheets were prepared to examine students' algebraic thinking.

In the first lesson, a real-life problem was presented with a three-act math task, and students answered the focus question of how to mathematically express a situation in which three tangerines are taken from the basket. In other words, they work on writing a general expression of an operation that includes an unknown quantity. When the students' expressions were examined, it was seen that they used shapes, a word (unknown), letters, and arbitrary numbers to write a general expression. Groups using numbers were unable to generalize, but groups using letters and shapes were able to

generalize by writing a general representation of an operation that include an unknown. Although algebraic expression topics were not taught to the students before, some groups were able to use letters.

Regarding this, the study of Blanton and Kaput (2004) showed that students could use letters to represent variables at a very early age, starting from the third-grade level. From another point of view, one possible reason for this can be that in the earlier grades, there are activities in the textbooks that include letters or shapes instead of the unknown. In addition, some teachers can also use shapes and letters instead of unknowns in early grade-levels' lessons. These aspects might be another reason for this situation. Future research can focus on students' generalization types to express an operation that includes an unknown quantity and the underlying reasons for these generalization types.

Along with this, in the three-act math task of the second lesson, it was observed that the general expressions written by the students consisted of symbols, letters, and shapes. When the expressions they wrote were examined in detail, it was seen that G2 and G3, who wrote expressions using letters and arbitrary numbers instead of variables in the first lesson, used only letters in the second lesson; G1 and G5, who used only letters in the first lesson, also used only letters in the second lesson, and G4, who used arbitrary numbers instead of variables in the first lesson, used letters in this lesson; G7 who used both letters and shapes in the first lesson, switched to using letters in the second lesson but could also create wrong algebraic expressions; G6 who also used letters and shapes in the first lesson, used only shapes in the second lesson; G8 who used a word (unknown) in the first lesson, used symbols and letters in the second lesson but wrote wrong algebraic expressions. In other words, in the three-act task of the second lesson, all groups except G6 were able to create algebraic expressions using letters. In relation to that, in the study of Wilkie and Clarke (2016), some students used symbols or shapes to represent a variable; after learning about variables, they transformed their expressions into more formal representations. However, it was observed that there were problems in the algebraic expressions created by the groups that used shapes (G6 and

G7) and words (G8) in the first lesson. No problems were encountered in the algebraic expressions of groups that used arbitrary numbers/letters and only letters instead of variables in the first lesson. Table 7.1 summarizes students' expression types in both of the lessons' three-act math tasks.

Table 7.1. Comparison between Lesson 1 and Lesson 2 regarding students' expression types in three-act math tasks.

Groups	**3 Act Task in Lesson 1**	**3 Act Task in Lesson 2**
G1 and G5	Only letters	Letters
G2 and G3	Letters and Numbers	
G4	Numbers	
G6	Letters and Shapes	Shapes
G7	Letters and Shapes	Letters but there is one incorrect form.
G8	Word (Unknown)	Symbols and letters but incorrect

Interestingly, the groups who wrote expressions using shapes in the first lesson experienced difficulties using letters in the second lesson. On the other hand, Nayıroğlu (2022) stated that using shapes first supports the transition to using letters later instead of unknowns. Therefore, future research can focus on the effectiveness of using shapes before using letters instead of unknown quantities. Future studies can also examine students' progress in algebraic expressions according to the different variable notations, such as shapes, letters, and symbols.

In the second activity of the second lesson, all students could use letters. However, some mistakes were noticed in some groups. Some of them chose to use capital letters as variables and represent different quantities with the same variable. This type of error is also found in the study of Molina *et al.* (2017). In addition, one group used the multiplication symbol (G7) in their expressions, which we do not encounter in algebraic expressions. When this group's (G7) expression was examined in the three-act math task, it was seen that they made the same error.

The second point examined in algebraic thinking was writing equivalent algebraic expressions using operation properties. Concerning this, Stacey and MacGregor (1997a) pointed out that "One of the most frequent algebraic acts is manipulation – changing an expression into an equivalent expression that has the same value, for example, replacing 3y + 6 by 3(y + 2)" (p. 256). Examining writing equivalent algebraic expressions was significant because this algebraic activity relies on students' understanding of the use of properties of numbers (Bush and Karp, 2013) and operation properties (Kaput, 2008) and using these properties to manipulate the symbols. It was examined through the second lesson, the third lesson, and the interviews.

In the three-act task in the second lesson, unlike the first lesson, students tried to create general expressions with both addition and multiplication. In this way, it was examined whether they could use the operation property, the relationship between addition and multiplication, for algebraic expressions. Six groups used this property correctly in algebraic expressions. However, one group (G6) could not write a multiplicative expression equivalent to the additive expression, and another group (G8) wrote wrong equivalent expressions. One group (G7) also wrote one incorrect expression in addition to their correct expressions. G6's failure to write equivalent algebraic expressions might be due to the fact that they only used shapes, and G8's incorrect multiplicative expressions may be due to his incorrect construction of expressions. In other words, expressions written by G8 could not be written multiplicatively.

Four of these six groups who wrote equivalent expressions in three-act math tasks (G1, G2, G3, G7) did not write equivalent algebraic expressions in the following activity. Although they could write equivalent expressions in the three-act math task, the fact that they did not write equivalent algebraic expressions in the following activity may stem from the second activity being more complex than the three-act math task. Although one group (G8) formed incorrect algebraic expressions in the three-act math task, they formed correct algebraic expressions in the second activity. However, this group did not write an equivalent algebraic expression. Two groups (G4 and G5) were able to write equivalent algebraic expressions in both the three-act task and the second

activity. G4 only established a relationship between addition and multiplication, while G5 used the distributive property and the relationship between addition and multiplication. Therefore, G5 differs from other groups because they also use distributive property.

Although G1, G2, and G3 did not write equivalent algebraic expressions in the second activity of the second lesson and (They wrote equivalent algebraic expressions in the three-act math task only) when we look at the third lesson's three-act math task, they (G1, G2, and G3) were able to write equivalent expressions. In this process, G1 and G2 established a relationship between addition and multiplication, while G3 used the commutative property of addition. On the second activity page of the third lesson, these groups again established a relationship between addition and multiplication while writing equivalent expressions. In addition, G1 used the distributive property, G2 used the commutative property, and G3 used both the distributive and commutative properties. In this regard, as lessons progress, they (G1, G2, and G3) could use multiple operation properties.

In the second lesson's activities, G4 established a relationship between addition and multiplication and wrote equivalent algebraic expressions. In the three-act task of the third lesson, they were able to write equivalent algebraic expressions using two types of operation properties (relationship between addition and multiplication, commutative property). In the second activity of the third lesson, they only used the relationship between addition and multiplication. Therefore, G4 used mainly the relationship between addition and multiplication.

Similar to G4, G5 established a relationship between addition and multiplication in both activities in the second lesson. This group, unlike other groups, was able to use the distributive property in the second worksheet of the second lesson. In the three-act math task of the third lesson, similar to G4, they established a relationship between addition and multiplication and also used the commutative property. However, in the second activity of the third lesson, they wrote an equivalent expression using only

the relationship between addition and multiplication. Therefore, it can be said that although they used one property in some activities, in general, G5 could use various operation properties together.

G6, on the other hand, was able to write equivalent algebraic expressions by establishing a relationship between addition and multiplication only in the second activity of the second lesson. They used the commutative property in the third lesson's three-act math task. In the second activity of the third lesson, they formed equivalent algebraic expressions using the relationship between addition and multiplication, the distributive property, and the commutative property. Thus, as lessons progress, they could use various operation properties to get equivalent algebraic expressions.

G7 could only write equivalent algebraic expressions by establishing a relationship between addition and multiplication in the three-act task of the second lesson but did not write equivalent algebraic expressions in the second activity of the second lesson. In the third lesson's three-act math task, they could write equivalent algebraic expressions and only used the relationship between addition and multiplication. Again, in the second activity of the third lesson, they created equivalent algebraic expressions using only the relationship between addition and multiplication. Therefore, this group could only use one operation property to form equivalent algebraic expressions.

G8, on the other hand, could not write an equivalent algebraic expression in the second lesson. However, in the third lesson's three-act math task, they could write equivalent algebraic expressions using the relationship between addition and multiplication. In the second activity of the third lesson, they wrote equivalent algebraic expressions using the commutative property, distributive property, and the relations between addition and multiplication. Therefore, although they did not write equivalent expressions in the second lesson, they could use various operation properties in the third lesson.

When we evaluate everything together, the most common operation property used to transform algebraic expressions into equivalent forms is the relationship between multiplication and addition. In the interview, students also used this property the most. Students who use commutative and distributive properties are fewer. This can stem from students' familiarity with this operation property because students use this property in almost every arithmetic situation. For instance, many mathematics problems require understanding this relationship. Without understanding this relationship, even simple math problems can not be solved. On the other hand, problems that require understanding commutative and distributive properties are fewer than problems understanding the relationship between multiplication and addition. Therefore, it can also be said that this stems from students' past experiences in arithmetic. If we evaluate all groups, it can be seen that some groups perform better in forming equivalent algebraic expressions and using more than one arithmetic property together. Therefore, the reason for this can be students' development in arithmetic properties. Concerning this, the study of Britt and Irwin (2011) showed that students who developed in arithmetic in terms of the additive, multiplicative, and proportional operations were also able to use these symbolically, and they stated that students who recognize the underlying structure of operational properties were able to represent numerical generalizations by using symbols. Another study by Pitta-Pantazi *et al.* (2022) showed that students who understand arithmetic well and can perform arithmetic problems can also solve algebraic problems. They used two types of tasks: arithmetic and algebraic. One of the areas they examined is the properties of operations. They found that students who could perform arithmetic tasks in this area could also perform algebraic tasks in the same area. They stated that students' abilities in arithmetic support the manipulation of algebraic expressions syntactically. Similarly, Warren (2003) mentioned that students' understanding of operations properties, such as associative and commutative properties, supports their understanding of algebra.

However, the current study did not investigate students' past arithmetic experiences and their level of development in arithmetic. In the discussion above, only some assumptions related to possible underlying reasons for students' performance in

transforming algebraic expressions into equivalent ones were made according to the findings. Based on all these, future research can examine students' manipulation of algebraic expressions in a broader context, establish connections with their past arithmetic experiences and their development levels in arithmetic, and investigate the role of similar or different learning environments in working with symbols in this way.

Concerning the commutative and distributive properties, Rumsey (2012) found that students tend to observe commutative property rather than distributive property in arithmetic operations. Similarly, the study of Squire *et al.* (2004) showed that students' comprehension of commutative property is better than comprehension of distributive property. On the other hand, the findings of this study did not reveal a strong difference in students' use of these properties to form equivalent expressions. The difference between the current study and other studies can stem from the fact that this study's aim is not students' comprehension or observation of the properties like other studies but how these properties were used in the algebraic manipulation.

The meaning of variable is one of the fundamental understandings of algebraic thinking (Knuth *et al.*, 2005). The current study also examines students' interpretation of this essential concept. This point was examined mainly in the interview. Variables have different meanings (Lloyd and Herbel-Eisenmann, 2011). Some students view variables as objects rather than measures or amounts of the objects (Blanton *et al.*, 2011) and see them as labels (Stacey and MacGregor, 1997b; Usiskin, 1988). Along with this, this is a gradual process; they gradually gain an understanding of the correct meaning of the variable (Blanton *et al.*, 2017). When the interview was analyzed, it was observed half of the students understood the variables as objects, and the other half viewed them as the quantities of an object. Therefore, some students had a misunderstanding about the meaning of the variable. In the literature, a lot of research investigated students' understanding of the variables. For instance, the study of Kuchemann (1978) showed that students can understand the variable as an object or the label of the object. Another study, conducted by Weinberg *et al.* (2004) examined students' understanding of variables in the case of the problem: "Cakes cost c dollars each and

brownies cost b dollars each. Suppose I buy four cakes and three brownies. What does 4c + 3b stand for?" (Weinberg *et al.*, 2004, p. 4). According to the findings, most students mistakenly interpreted "4c + 3b" as "four cakes and three brownies", failing to recognize that the variables represented "cost of a cake" and "cost of brownies", respectively. Therefore, the finding of this study is parallel with other studies on students' understanding of the meaning of variables (Kuchemann, 1978; Weinberg *et al.*, 2004). There can be various reasons for students misunderstanding variables. One of them is the using the initial letters of the object to represent an unknown quantity (MacGregor and Stacey, 1997). In this study, this can be a reason for students' misunderstanding of variables. On the other hand, this can be prevented with teachers emphasizing that the variables are quantity, not the object or label (Stacey and MacGregor, 1997b).

The second content strand of algebra examined in this study was functional thinking. This strand was combined with the objective of calculating the value of algebraic expressions for different natural number values that variables take because students were asked to find corresponding terms through a functional rule. In the first act of the three-act math task, students were shown a context-based situation, and curiosity was aroused. Students were asked what they noticed and what they wondered about. Based on these, students were asked to explain the relationship between the numbers entering and exiting in the number robot, to find an algebraic rule to represent this relationship, and to draw a table or graph to represent the relationship. As Kalchman and Koedinger (2005) stated, different types of representations, such as verbal, algebraic, and table, are important for students' understanding of functional relationships.

When the groups' algebraic indicators for functional thinking were examined, it was observed that every group, except one group (G7), could create a table to represent corresponding terms correctly. On the other hand, groups generally found a recursive relationship rather than a correspondence relationship in their explanations. This can be because describing a recursive relationship requires less sophisticated thinking (Blanton *et al.*, 2015). A few groups also recognized a different relationship, neither recursive nor correspondence or covariational. In addition, it was seen that instead of

writing a general correspondence rule, some groups wrote a rule that was neither recursive nor correspondence or covariational. Some groups' rules are based on a recursive strategy. These rules were not evaluated as a general rule. Besides, half of the groups could write a general rule of the pattern algebraically.

After three-act math tasks, in the worksheet, a question related to finding the corresponding terms was asked students. Students were expected to calculate the value of algebraic expressions to find the corresponding term of a pattern. It was seen that groups are more successful in finding the corresponding terms of the pattern than describing correspondence relationships and forming a general rule. Similar to this result, the study of McEldoon and Rittle-Johnson (2010) showed that students are more successful in applying the rule to determine corresponding values. The reason for this can be using a rule to find corresponding values requires arithmetic strategies and does not require deep comprehension of functional relationships (McEldoon and Rittle-Johnson, 2010).

Specifically, in the three-act math task, the second and third worksheet questions, G1 found a correspondence relationship and formulated a general algebraic rule. In the first question of the worksheet, they also correctly found corresponding terms of a pattern. On the other hand, in the last question, they did not write a general algebraic rule or find distant terms of the pattern. They only found the close term of the pattern. Although G1 is successful in the three-act math task and the other questions in the worksheet, the reason for their performance in the last question can be related to the context of the problem (McEldoon *et al.*, 2010). Students may have difficulty finding the general rule and distant terms when a different context is included. Another reason can be other questions in the worksheet focused on the relationship between input and output numbers explicitly, which may affect students' strategies (Lannin *et al.*, 2006); on the other hand, the last question focused on the relationship in an implicit manner.

When we look at the second group, they found a recursive relationship in the three-act math task and could not write a correct algebraic rule to represent the rela-

tionship. The reason can be interpreting the pattern as recursive can limit students' functional thinking (Tanışlı, 2011). On the other hand, they found corresponding terms in the first and the third questions of the worksheet by using a pre-determined general rule. This can be interpreted as, even though students could not formulate a general rule of a pattern, they could use a determined rule to find the corresponding terms (McEldoon and Rittle-Johnson, 2010). The reason for this can be forming general rules requires more sophisticated thinking and can be seen at a higher level (Blanton *et al.*, 2015). On the other hand, even if they wrote the general rule incorrectly in the three-act math task, they wrote the general rule for other questions correctly. Regarding this, different studies stated that students gradually develop in finding correspondence relationships (Stephens *et al.*, 2017; Syawahid *et al.*, 2020; Tanışlı, 2011). In the last question, they also used the rule they found to calculate the distant term. As Blanton *et al.* (2011) stated, finding the correspondence rule supports students in finding the distant terms of the pattern.

Similar to the second group, the third group also found a recursive relationship in the three-act math task and did not find the general algebraic rule. Instead, they defined a recursive relationship with words. In the last question of the worksheet, they also defined a recursive rule in words. Although they defined the recursive rule, they correctly found close and distant terms in the last question. Their way of doing this was by making arithmetic calculations. This might imply that students could use correspondence relationships in the form of arithmetic structure without transforming it into algebraic structure. Interviews with some students also support this because some students find close and distant terms of the pattern using correspondence relationships without algebraic rules. This can be viewed as an arithmetic generalization of the pattern (Radford, 2008). The second and third questions also supported students' ability to use correspondence relationships since they could find corresponding terms of the pattern.

Examination of the fourth group shows that although they first described the recursive relationships in the three-act math task, they could form general algebraic

rules in the task and other questions. In addition, they correctly found corresponding terms and close and distant terms. Related to this, Blanton *et al.* (2015) stated that understanding functional relationships can co-appear with recursive understanding. Along with this, when students first encounter the pattern, they can describe recursive relationships naturally (Lannin, 2004). Besides, in the three-act math task, the independent variable was increasing one, and the dependent variable was ordered ten by ten; students can tend to observe recursive relationships rather than covariational or correspondence relationships for this reason (Lannin *et al.*, 2006; Stacey and Macgregor, 2001).

Like G4, G5 also described a recursive relationship in the three-act math task. On the other hand, they formulated general rules in other questions except the fourth question in the worksheet. As mentioned before, the reason for this in the last question can be related to the context of the problem. Although they could not formulate a general rule in the last question, they found close and distant terms correctly. In this question, this can also depend on arithmetic generalization (Radford, 2008). Since they formulated a general rule in other questions and found distant terms in the last question, they have an understanding of the correspondence relationships. However, they did not notate an algebraic general rule when the context changes. Therefore, future research can examine students in terms of writing general rules in different contextual situations.

When the sixth group's (G6) works were analyzed as a whole, it could be seen that they were successful in describing correspondence relationships, finding general rules and representing them algebraically and words, finding corresponding terms of the pattern, and finding close and distant terms of the pattern. Therefore, it can be said that this group has the capability of functional thinking in terms of different aspects defined by different researchers (Blanton *et al.*, 2011; Chimoni *et al.*, 2018; Confrey and Smith, 1995; Smith, 2008).

On the other hand, G7 could not find recursive or correspondence relationships in the three-act math task and could not formulate a general rule and draw a table correctly. In the first question of the worksheet, there was no finding of corresponding terms of the pattern. In the second question of the worksheet, they wrote an incorrect general rule algebraically but found corresponding terms correctly. The reason can be students might suppose a recursive rule and make additions between consecutive terms. In the third question, they could formulate their general rule to form a pattern by using words, but they could not find corresponding terms. In the last question, although they found close and distant terms of the given pattern, they could not find a general rule algebraically. Therefore, it can be said that this group had difficulty finding a general rule of a pattern algebraically. On the other hand, they could find corresponding terms of the pattern in one question and close and distant terms in the last question. The reason for this can be that students made an arithmetic calculation or extended the pattern. Therefore, future research can examine arithmetic calculations or the thinking of students who do not algebraically formulate a general rule of a pattern.

Lastly, although G8 found the correspondence relationship in the three-act task, they could not write a correct general rule related to this relationship. In the first question of the worksheet, there was no answer to the question regarding finding the corresponding term. In the second and third questions, they verbally formulated a general rule and found the corresponding terms correctly. In the last question, even though they verbally formulated a general rule of the pattern, they could not find the close and the distant terms correctly. Therefore, it can be said that even though this group found the correspondence relationship, they had difficulty representing this algebraically and using it in different contexts.

Therefore, considering everything, some groups had difficulties forming algebraic rules even though they found general rules verbally. This finding coincides with the study of Swafford and Langrall (2000); 10 sixth-grade students participated in their study, and students were asked to describe the problem both symbolically by variables and verbally and then solve it. It was found that students tend to form verbal de-

scriptions rather than variable notation. Similarly, the study of McEldoon and Rittle-Johnson (2010) showed that even though some students generate a rule, they cannot form a symbolic rule algebraically.

When students' functional thinking in the interview was analyzed, although three students first observed a recursive relationship in the context (H1, L2, and M2), two of them could find the distant term by using a correspondence relationship (H1, L2) rather than extending the pattern. On the other hand, when the distant term was asked, all students could use correspondence relationships to find distant terms (H1, H2, M1, M2, L1, L2). In the lessons, some students had difficulties finding the distant term of the pattern; on the other hand, in the interviews, all students, even low achievers, could use correspondence relationships to find the distant terms. This can be due to the fact that the relationship is a graphical representation. Regarding this, Markovits *et al.* (1988) discovered that students often perform better with functions when they are presented graphically. When students' performances in forming algebraic general rules and using this rule to find a distant term were examined in the interview, four students (H1, H2, M2, L1) correctly found a general algebraic rule of the pattern and used this rule to find the distant term. As in this case and in the lessons, forming a general rule supports finding distant terms of the pattern (Blanton *et al.*, 2011). Students who found distant terms but did not find correct general rules made arithmetic calculations by using correspondence relationships. Therefore, even though some students recognized the generalization, they incorrectly transformed it into an algebraic structure. As in the case of other questions, this is because forming an algebraic rule requires higher thinking (Blanton *et al.*, 2015).

The third content strand of algebra examined in this study was mathematical modeling. Modeling was examined in terms of transforming real-life (or context-based situations) into mathematical models. In the first two lessons, while students were writing general expressions, they modeled the real-life situation with different methods by using letters, symbols, words, or shapes. The fifth lesson was also based on modeling. Students wrote algebraic expressions that fit the given models and formed models that

fit the given algebraic expressions. When we compared these two situations, it was seen that groups were more capable of forming models that fit the given algebraic expressions than writing an algebraic expression that fit the given models. In addition, in the first and last questions of the interview, they wrote algebraic expressions to model context-based situations. During this process, some students wrote incorrect expressions like x+8+y+2 rather than 8x+2y. This type of misunderstanding was found in different studies in the literature (Akyüz and Hangül, 2014; Sleeman, 1984; Steinberg *et al.*, 1990). For instance, the study of Steinberg *et al.* (1990) shows that students can understand 3x as x+3. Similarly, Akyüz and Hangül (2014) pointed out students' understanding of 4x as 4+x. This study also shows that students write y+8 rather than 2y+6x, which is another mistake in writing algebraic expressions. Moreover, students' answers for the last item of the third question show that while some students write more than one expression to model a situation by transforming a context-based situation into an algebraic expression, others write more than one expression to model a situation by using operation properties. In other words, some students created different expressions based directly on real life, while others did this by manipulating symbols. Another point observed in this question was that students with higher achievement scores could write more equivalent algebraic expressions and manipulate the symbols better.

7.5. Limitations

As the nature of the research, this study has limitations. Firstly, the study has some issues regarding generalizability because it was limited to one public school in the Akdeniz district of Mersin in the spring term of the 2022/2023 year.

The second limitation is regarding the time factor. Because of the MoNE (2018) annual plan, lessons lasted only two weeks and 10 lesson hours, which means a short period. The length of the implementations is important to get deeper and more accurate results.

The third limitation was regarding the scales. It is possible that participants may not respond to the "Mathematics Attitude Scale" honestly. Along with this, student's achievement is limited to only one instrument, the "Algebra Achievement Test". In addition, some extraneous variables can affect the changes in students' scores.

Another limitation is related to the qualitative part of the study. Since the MoNE (2018) objectives include limited objectives regarding the algebraic thinking around Kaput's (2008) strands, some algebraic thinking components could not be included in the lessons, such as properties of even and odd numbers, computational strategies, working with the number system, equations, and inequalities.

7.6. Recommendations for Further Research and Practices

In this section, some recommendations were made based on the current study and related literature to use RME-based lessons enriched with three-act math tasks in the lessons efficiently and to lead in conducting various research that can develop mathematics education in different aspects.

7.6.1. Recommendations for Further Research

By considering different aspects of RME-based lessons enriched with three-act math tasks and considering the current study and its results, suggestions for future studies can be made.

This research briefly describes the lessons in the qualitative analysis section, but the purpose of the current study was not to examine its implementation procedure deeply. Therefore, qualitative research can be conducted to investigate the implementation of RME-based lessons enriched with three-act math tasks and to investigate the experiences of both students and teachers. During this, it can be suggested that what students notice and wonder about in three-act math tasks can be examined. Based on these, students' thinking styles can be revealed in various aspects.

In addition, teachers' experience of preparing RME-based lessons enriched with three-act math tasks and the quality and applicability of their lessons can be examined. Besides, how teachers perceive the mathematization processes of real-life situations can be investigated in detail based on the instructions they prepared.

The findings of this study revealed how students work with letters in RME-based lessons enriched with three-act math tasks. Future studies can investigate the role of similar or different learning environments on students working with letters. The findings of this study also showed that students' performance in transforming algebraic expressions into equivalent ones differs, and literature showed that this could stem from the students' past arithmetic experiences and developmental levels in arithmetic. However, this study does not focus on these possible reasons. Therefore, future studies can examine students' algebraic manipulation by making connections with their past arithmetic experiences and their development levels in arithmetic.

This study was limited in terms of algebraic thinking components around Kaput's (2008) content strands due to the MoNE (2018) objectives. RME-based lessons enriched with three-act math tasks can be designed and implemented in earlier grades; in this way, students' algebraic thinking can be examined in broader aspects by considering algebraic thinking components in earlier grades.

Furthermore, RME-based lessons enriched with three-act math tasks can be used in different mathematical subject areas, and its role can be examined on different variables like achievement, attitude, and different mathematical competencies such as problem-solving, problem posing, critical thinking, analytical thinking, creative thinking, and mathematical modeling.

Examining and comparing the effectiveness of RME-based lessons enriched with three-act math tasks on teaching and learning different mathematical subjects and concepts might be important. In this way, the effect of RME-based lessons enriched with three-act math tasks on students' reasoning about different topics like proportional

reasoning, probabilistic, and statistical reasoning can also be investigated. During this time, the number of participants can be smaller for in-depth analysis.

Further research can also aim to develop students' mathematical reasoning on various subject areas and mathematical competencies by using RME-based lessons enriched with three-act math tasks. Therefore, instructions can be designed according to this aim, and students' development levels can be examined. Moreover, studies can be conducted to examine how these lessons connect different mathematical competencies and develop and combine these skills.

7.6.2. Recommendations for Practice

When the results of the current study were evaluated, it can be concluded that RME-based lessons enriched with three-act math tasks are an efficient approach to increase students' achievement and attitude towards mathematics and can be combined with algebraic thinking content strands to examine students' algebraic thinking.

Besides, this study can provide educators and practitioners with a new viewpoint on designing and using the RME approach with a specific material type and combining these with algebraic thinking content strands. Therefore, this study has the potential to significantly contribute to the literature as it is one of the first studies to emerge in the field of mathematics education in that way. The profundity of these contributions can be enhanced with some suggestions; therefore, some recommendations will be made to shed light on implementing this new instructional design in the teaching-learning process. Educators can design three-act math tasks by considering the principles of the RME approach and can use these in the lessons. At the beginning of the lessons, essential concepts related to the subject can become intriguing and a situation that students notice and wonder about by using three-act math tasks. Thus, students' attention can be drawn to essential concepts in the subject. In that way, the process of re-invention of mathematics, an important view in RME, can be started with curiosity.

In the three-act math task, the focus question is created from in-class discussions guided by the teacher and has to be a question that leads to learning an essential element of the topic. Therefore, lessons should be well designed in advance. The possible wondering and noticing of the students and how the teacher will direct these should be determined in advance. In this way, the subject to be taught can be effectively introduced. The clue provided in ACT 2 of the three-act task should enable students to solve the focus question and have the potential to lead to learning of the topic through problem-solving. Therefore, it should be well thought out and designed in advance. It should be the kind of clue that strengthens students' thinking processes. In ACT 3 of the three-act math task, all students should be involved in class discussion, and the teacher should guide this discussion to highlight significant aspects of the topic.

If teachers examine the components that students noticed and wondered about in a three-act task, inferences can be made about students' possible understandings of a subject or misconceptions. For example, observations such as students' recognition of a recursive pattern rather than a covariational relationship can provide educators with insight into students' understanding in later stages of the course. Anticipating these scenarios can create an opportunity for educators to improve and enrich their lessons.

Finally, these lessons require students' active participation in the learning process and have the potential to enable them to re-invent mathematics. In this regard, teachers should provide necessary guidance to students and the teaching and learning process.

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APPENDIX A: INTERVIEW QUESTIONS

In this section, three interview questions were presented to examine students' algebraic thinking at the end of the lessons.

1) Aşağıdaki soruları düşüncelerinizi sesli biçimde ifade ederek çözünüz.

a) Bir müşteri garsondan 4 hamburger ve 2 ayran istemiştir. Garson sipariş ücretini, sipariş defterine cebirsel olarak nasıl yazabilir ? Birden fazla yol bulunuz.

b) Menüdeki fiyatlara göre sipariş kaç TL tutacaktır?

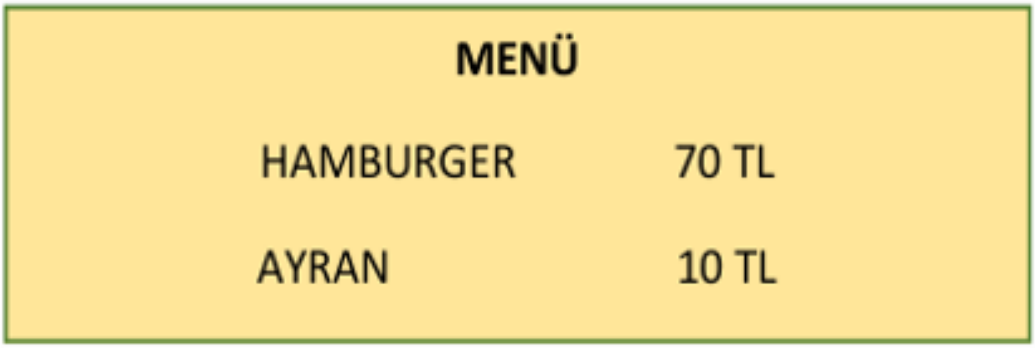

Figure A.1. Menü.

c) Yazdığınız cebirsel ifade de harfler neyi temsil etmektedir ?

2) Aslı her gün düzenli olarak kitap okumaktadır. Aslı, gün sayısı ile o güne kadar okumuş olduğu toplam sayfa sayısını bir grafikte temsil etmiştir. Buna göre ilgili soruları düşüncelerinizi sesli bir şekilde ifade ederek çözünüz.



Figure A.2. Grafik.

a) Gün sayısı ile o güne kadar okunan toplam sayfa sayısı arasında nasıl bir ilişki vardır?

b) Aslı 7 günde kaç sayfa kitap okur?

c) Aslı 50 günde kaç sayfa kitap okur?

d) Gün sayısına göre Aslı'nın herhangi bir günde, o zamana kadar okumuş olduğu toplam sayfa sayısını bulmamızı sağlayacak genel bir kural oluşturunuz ve kuralınızı cebirsel olarak ifade ediniz.

e) Aslı 90 günde kaç sayfa kitap okur ? Oluşturduğunuz kuraldan yola çıkarak bulunuz.

3) Mehmet Bey özdeş tuğlalardan bir duvar örmeye başlamıştır. Aşağıda ördüğü duvarın bir modeli verilmiştir. Buna göre aşağıdaki soruları düşüncelerinizi sesli olarak ifade ederek cevaplayınız.

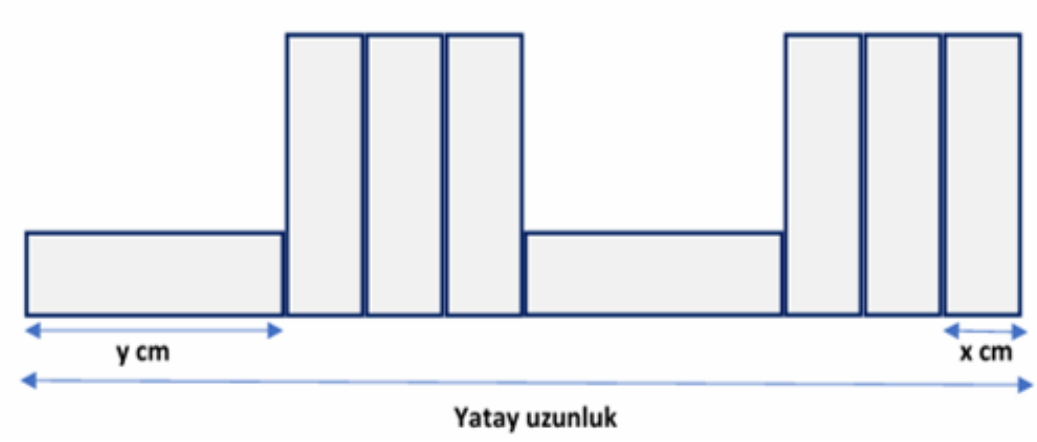

Figure A.3. Duvar.

a) Mehmet Bey'in ördüğü duvarın yatay uzunluğunu cm cinsinden ifade etmek için bir cebirsel ifade yazınız.

Özdeş tuğlaların boyutları y = 50 cm ve x = 10 cm' dir. Bu bilgiye göre sorunun b) ve c) şıklarını cevaplayınız. b) Tuğlaların verilen boyutlarına göre ördüğü yatay uzunluk kaç cm'dir ?

c) Mehmet Bey yatay uzunluğu 180 cm olan bir duvar örmek istemektedir. Kaç tane tuğla kullabilir ve tuğlaları ne şekilde yerleştirebilir? Birden fazla örnek veriniz. Her bir örneğinizdeki yatay uzunluğu cm cinsinden gösteren cebirsel ifadeleri yazınız.

APPENDIX B: SAMPLE LESSON PLAN

In this section, a sample lesson plan based on RME enriched with three-act math tasks was presented.

Grade: Sixth grade

Time: 2 x 40 minutes

Objective: Students will be able to calculate the value of the algebraic expression for different natural number values that the variable will take.

Materials: Three Act Math Task, Worksheets

In the introduction part of the lesson, a three-act math task was used as material for students to do horizontal mathematization. Students are divided into groups of four in order to provide a communicative and collaborative environment (Interactivity Principle). The three-act math task attracts students' attention and provides a meaningful introduction to the course. They mathematize a real-life situation in informal ways.

Three Act Math Task

ACT 1:



Figure B.1. ACT 1.

Ask Students: What do you notice?

What do you wonder? Focus Question: How much will the order cost? (Reality Principle)

ACT 2:

Students try to understand the letters. They are expected to conclude that the letters are the prices of the foods on the menu. In order to solve the focus question, students are expected to ask about the prices of the foods. The teacher shares the photo below as a clue (Guidance Principle) and distributes a worksheet for their work on the focus question (Activity Principle). Clue:



Figure B.2. Clue.

Worksheet:



Figure B.3. ACT 2.

ACT 3: Students share their solutions and thinking styles as a group (Interactivity Principle). Then, the teacher shows the photo below as the solution.



Figure B.4. ACT 3.

Along with this, the teacher asks some questions to enrich the discussion environment and to support the emergence of important details of the subject (Guidance Principle).

- What do the letters represent in the order ? (Meaning of variable)
- What do the numbers represent in the order? (Concept of coefficient)
- How did you calculate the order cost? (Generalization of Arithmetic: Properties of Operations, variable as unknown quantity)
- In what other ways can the waiter write the order note? (Generalization of Arithmetic: Properties of Operations: multiplication as repeated addition, distributive property of addition, commutative property of addition, associative property of addition)

Different equivalent forms of 4h+2m+6k were discussed in the classroom around the properties of operations (Intertwinement Principle). With the three-act math task and the discussion environment in the classroom, students reach knowledge. They discover that variables represent unknown numbers in an algebraic expression and how to calculate the value of the algebraic expression by giving values to variables under the guidance of the teacher (Guided Reinvention). Later, in the continuing part of the lesson, the teacher distributes another worksheet. Students calculate the values of different order notes (algebraic expressions) and write different equivalent forms. This worksheet also provides an opportunity for formative assessment.

At the end of the lesson, groups share their solutions and strategies in class and comment on other groups' solution strategies (Interactivity Principle). During this time, students are focused on how they write algebraic expressions and equivalent forms, calculate expressions' values, and what similar terms, terms, and coefficients are. Learning becomes formal, and important concepts and information are illuminated. Thus, students pass through vertical mathematization. The teacher gives homework for the assessment of the lesson. In the homework, they are asked to write order notes (algebraically) using the menu they used in the lesson and to calculate the value of the orders.