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PAPER

The equation of state $PV = (s/d)U$ from semiclassical kinetic theory, the virial theorem, and the stress tensor

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Abstract

We present a pedagogical and self-contained derivation of the generalized equation of state

$$PV = \frac{s}{d}U$$

for a non-interacting quantum gas in d spatial dimensions with power-law dispersion $\epsilon(p) \propto p^s$, where $s > 0$. We generalize the kinetic derivation to a quantum gas in d dimensions by treating the particles as semiclassical wave packets and demonstrate that the pressure relation arises naturally from momentum transfer when localized wave packets collide with the container walls. We also provide a direct derivation of the equation of state using both the classical and quantum virial theorem. We then present a quantum-mechanical derivation based on the stress tensor, showing that the same equation of state follows directly from the homogeneity and isotropy of a bulk system in thermal equilibrium. Finally, we derive the generalized relation by an operator-based method grounded in the Heisenberg equation of motion.

1. Introduction

The equation of state, relating pressure P to volume V and internal energy U is a fundamental result in statistical mechanics. Consider a non-interacting quantum gas confined to d spatial dimensions whose single-particle energy depends on the momentum through the power-law dispersion

$$\epsilon(p) = Ap^s, \tag{1}$$

for any $s > 0$ and $A > 0$[3]. Here $p = |\mathbf{p}|$. For the familiar three-dimensional nonrelativistic ideal gas (corresponding to $s = 2$ and $d = 3$), the standard equation of state implies the relation $PV = \frac{2}{3}U$. In contrast, for a photon gas the dispersion is linear ($s = 1$), and in three dimensions this gives $PV = \frac{1}{3}U$. A quantum gas whose single-particle spectrum follows the power-law dispersion in (1), and which obeys either Bose–Einstein (BE) or Fermi–Dirac (FD) statistics, provides a standard textbook setting (see, for example, Pathria and Beale [1] or Huang [2]). In this case, the equation of state takes the form:

$$PV = \frac{s}{d}U. \tag{2}$$

[3] While $s = 1$ (e.g. photons, acoustic phonons, or Dirac electrons in graphene) and $s = 2$ (non-relativistic massive particles) are the most familiar cases, other values can arise in condensed matter physics. For example, low-energy excitations in ABC-stacked trilayer graphene exhibit a cubic dispersion, corresponding to $s = 3$. In such cases, as long as the system can be effectively described as a non-interacting quasiparticle gas, the generalized equation of state $PV = (s/d)U$ strictly holds (e.g. yielding $PV = \frac{3}{2}U$ for a 2D system with cubic dispersion).

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Standard textbook derivations of this relation often proceed by manipulating the grand partition function through mathematical identities, a route that can sometimes mask the underlying physical origin of the result [3]. Recall that in the grand-canonical ensemble one has $PV = -k_B T \ln \mathcal{Z}$, where $\mathcal{Z}$ refers to the grand-canonical partition function. For a generalized gas, the desired equation of state can then be obtained by performing an integration by parts in the expression for $\ln \mathcal{Z}$ in thermodynamic limit. While efficient, this technique is largely formal and does not, by itself, clarify the physical mechanism responsible for the relation.

However, many standard presentations do not show in a direct and transparent way how the above equation of state (2) follows from a kinetic-theory-type viewpoint in *quantum* statistical mechanics. In this paper, we first develop a semi-classical kinetic-theory derivation of the equation of state for quantum gases (both Bose and Fermi) by modeling the constituents as localized *wave packets*. This perspective makes the dynamical origin of pressure explicit and, at the same time, brings out a transparent link to the underlying quantum-mechanical description. Then, we derive the equation of state from the above general dispersion relation using both the classical and quantum virial theorem.

A more advanced-but ultimately more satisfying-derivation of the pressure for gases obeying BE or FD statistics is provided by Durand [4], who formulates pressure in terms of the quantum-mechanical stress tensor. In Durand's approach, pressure acquires a direct mechanical interpretation as the momentum flux encoded in the stress tensor. This makes the meaning of 'quantum pressure' especially transparent, because it is defined through the same physical idea used in kinetic theory-momentum flow-rather than through purely thermodynamic identities. He then shows that this momentum-flux definition reproduces precisely the result obtained from the standard thermodynamic expression, thereby clarifying what that formula is effectively computing. This result is obtained in a particularly general setting: the pressure acting on the boundary is related to how the energy eigenvalues vary under deformations of the confining surface. In the present work, we further show that such generality is not required in equilibrium, and we derive the equation of state of a quantum gas in equilibrium for $s = 2$ from the quantum mechanical stress tensor by using the homogeneity and isotropy of such a gas, and then develop a purely operator-based quantum derivation tailored to equilibrium systems for the general dispersion relation with $s > 0$.

The main pedagogical purpose of this article is to help students understand pressure microscopically as a consequence of momentum transfer and momentum flux, to identify the group velocity as the natural dynamical quantity in the kinetic description, and to see clearly how the factor s/d follows from the dispersion relation and the spatial dimensionality. In this way, the paper brings together topics that are often taught separately in the upper-level undergraduate curriculum, including kinetic theory, wave mechanics, operator methods, and equilibrium quantum statistical mechanics.

The structure of the paper is as follows. Section 2 is devoted to construct a semi-classical kinetic derivation of the equation of state for quantum gases (both Bose and Fermi) by treating particles as wave packets. In section 3, we derive the equation of the state by using virial theorem. Finally, in section 4, we derive the same pressure relation from the properties of the quantum stress tensor in a homogeneous and isotropic bulk system in thermal equilibrium for $s = 2$, without relying on boundary deformations, highlighting the conceptual unity between macroscopic thermodynamic relations and microscopic operators. We also give a direct derivation that yields the generalized equation of state using operator identities and Heisenberg picture version of the stress tensor. Appendix A is devoted to the derivation of the equation of state for a photon gas using the moving piston scenario, where a concrete and physically transparent explanation for the origin of radiation pressure is given in terms of wave-particle duality. We give the derivation of quantum stress tensor, as presented in Durand's work, in appendix B to make the manuscript self-contained. In appendix C, we justify the Heisenberg version of the stress tensor, deriving a conservation law for the momentum density operator in the Heisenberg picture.

2. Semi-classical kinetic theory derivation via wave packets

We now generalize the derivation of equation of state to a quantum gas in d dimensions with an arbitrary power-law dispersion relation (1) by treating the particles as quantum wave packets. Although the final calculation has the form of a kinetic-theory argument, the wave-packet picture plays an important conceptual role: it explains why a quantum state can be treated semiclassically as a localized carrier of momentum, why the relevant transport velocity is the group velocity, and why, near a wall, one may isolate an incoming component with positive normal momentum whose elastic reflection gives the usual momentum transfer. If one started directly from a particle picture with dispersion $\epsilon = Ap^s$, the same

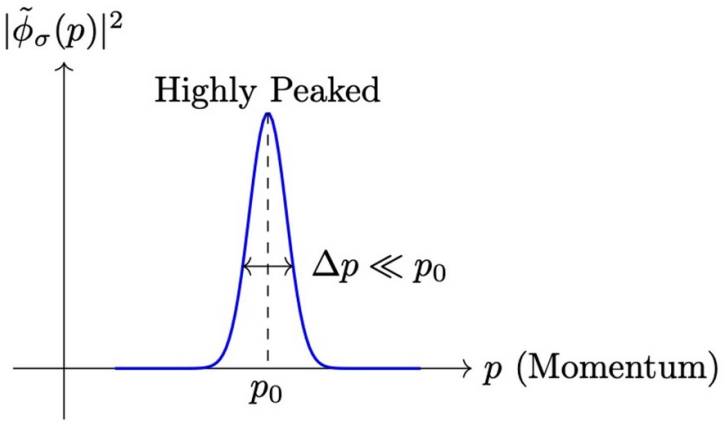

Figure 1. Momentum space representation of a 'highly peaked' wave packet. The particle has almost a well-defined momentum $\mathbf{p}_0$, allowing us to associate a specific group velocity $\mathbf{v}_g$ with its motion.

final formula would be obtained, but these specifically quantum-mechanical ingredients would remain ambigious or at best implicit.

For this, consider a system confined in a d-dimensional box of volume V. To define 'pressure' kinetically (via collisions with a wall), we need localized particles rather than the delocalized standing waves typical of a box of some shape. However, in a large box of ordinary shape, all but a small number of low-lying states can be understood semiclassically: for excited states with relatively large momentum, the de Broglie wavelength is small compared with the size of the box, and the states may therefore be approximated by localized wave packets. We construct *these wave packets* to represent particles in excited states in a special way to mimic well-defined momentum states of rectangular boxes. A particle in state σ is described by a wave function $\psi_\sigma(\mathbf{x})$, which is a superposition of momentum eigenstates [5–7] (we assume that there is a larger rectangular box containing our original box, so that in the thermodynamic limit the volume difference is negligible): $\psi_\sigma(\mathbf{x}) = \frac{1}{\sqrt{V}} \int e^{i\mathbf{p}\cdot\mathbf{x}/\hbar}\, \tilde{\phi}_\sigma(\mathbf{p})\, \mathrm{d}^d p$. We assume that the wave functions are *'highly peaked'*, where the probability amplitude in momentum space $|\tilde{\phi}_\sigma(\mathbf{p})|^2$ is sharply concentrated around a mean momentum $\mathbf{p}_0$ (see figure 1). Note that we use a rectangular box of very large volume (thermodynamic limit), and treat momentum states as if they are continuous as usual.

This assumption implies:

- The uncertainty in momentum is small ($\Delta p \ll p_0$), so the particle acts like a classical projectile with momentum $\mathbf{p}_0$.
- Inside the volume V, the spatial probability density is approximately uniform: $|\psi_\sigma(\mathbf{x})|^2 \simeq 1/V$ (as can be seen from our plane wave-decomposition).

Remark. The approximation $|\psi_\sigma(\mathbf{x})|^2 \simeq 1/V$ is not strictly true for a general container. It becomes exact only in geometries that admit periodic boundary conditions, where the eigenstates are plane waves with uniform spatial density and a clear separation into incoming and outgoing momentum states. For a box of general shape, however, the physically natural boundary conditions are Dirichlet or Neumann, and the corresponding eigenstates are standing-wave-like rather than pure plane waves. In that case, for a chosen wall normal, an eigenstate may be written as a superposition of two components with opposite normal momenta, one component moves toward the wall, while the other represents the reflected wave:

$$\psi_\sigma(\mathbf{x}) = \int_0^\infty \mathrm{d}p_1 \int \mathrm{d}^{d-1}p^\perp \left(\tilde{\phi}_\sigma\left(p_1, \mathbf{p}^\perp\right) \frac{e^{\left(ip_1 x^1 + i\mathbf{p}^\perp\cdot\mathbf{x}^\perp\right)/\hbar}}{\sqrt{V}} - \tilde{\phi}^*_\sigma\left(p_1, \mathbf{p}^\perp\right) \frac{e^{\left(-ip_1 x^1 + i\mathbf{p}^\perp\cdot\mathbf{x}^\perp\right)/\hbar}}{\sqrt{V}} \right). \quad (3)$$

We note that part of the wave function that 'collides' with the wall is the forward moving part, only the piece we use in the calculation above, and then as a result of an elastic collision we generate the back moving part, which interferes with the incoming part to create a standing wave pattern. In this sense, this explains why we only focus on the $p_1 > 0$ part in equations below. This separation of the momentum transfer piece (of the energy eigenstate) is essential to apply a kinetic approach.

The velocity relevant for transport and collisions in quantum mechanics is the group velocity. In quantum mechanics, group velocity is the velocity at which the envelope of a wave packet (and typically the peak of the probability density or energy density) propagates. For a wave packet built from energy-momentum (or frequency-wave-vector) eigenmodes, the velocity at which the packet's envelope propagates is given by the gradient of dispersion relation.

$$\mathbf{v}_{\mathrm{g}} = \nabla_{\mathbf{k}}\omega = \nabla_{\mathbf{p}}\,\epsilon\,(p)\,. \tag{4}$$

For our dispersion relation $\epsilon = Ap^s$, the gradient yields:

$$\mathbf{v}_{\mathrm{g}} = sAp^{s-2}\mathbf{p}\,, \tag{5}$$

Throughout our discussion, we assume that the particle energy distribution function $n(\epsilon(p))$ is rotationally invariant, that is, it is an isotropic distribution in momentum space. Note that as it stands, our kinetic arguments are valid for classical particles obeying Maxwell-Boltzmann distribution as well. However, in that case, we do not need wave packets unless we talk about the low density approximation of a quantum gas. Therefore, this distribution is explicitly assumed to be the quantum BE or FD distribution. Therefore, the average number of particles in an energy interval $[\epsilon, \epsilon + \delta\epsilon]$ is given by

$$n\,(\epsilon)\;\delta\epsilon = \frac{\rho\,(\epsilon)}{e^{\beta(\epsilon-\mu)} \pm 1}\,\delta\epsilon\,, \tag{6}$$

where $\rho(\epsilon)$ refers to the density of states around the energy ϵ. For excited states, using the fact that momentum states are good approximations, we express the distribution in terms of momentum and replace $\rho(\epsilon)\delta\epsilon$ (the number of available states in that energy interval) in the thermodynamic limit by $V\frac{\mathrm{d}p_1\cdots\mathrm{d}p_d}{h^d}$. Note that, strictly speaking, the dispersion relation $\epsilon = Ap^s$ must refer to the eigenvalues of the operator $A(-\hbar^2\nabla^2)^{s/2}$, which is indeed an isotropic and homogeneous operator. When we use the above mentioned highly peaked (around a definite momentum) wave functions, it is meaningful to represent the energy as $\epsilon = Ap^s$ in terms of the corresponding momentum eigenvalue p. So we are often referring to this choice for large systems, instead of the precise eigenvectors of the Hamiltonian operator. Therefore, we continue with this momentum state representation in the following calculations. The passage from the energy-space description to a momentum-space description is therefore approximate: the symbol $\approx$ below indicates the semiclassical replacement of the exact density of states in energy by the continuum density of momentum states in the thermodynamic limit. Hence,

$$n\,(\epsilon)\,\delta\epsilon \approx \mathrm{d}n\,(p) = f(p)\,V\frac{\mathrm{d}^d p}{h^d}\,, \tag{7}$$

where we have introduced the FD/BE distribution function $f(p) = \frac{1}{e^{\beta(Ap^s-\mu)}\pm 1}$ to simplify our writing. Alternatively, we can say that the quasi-classical approximation is valid in the thermodynamic limit, hence we use definitive momentum states to represent (excited) quantum states. Note that the degeneracy grows very rapidly with energy, therefore statistical considerations are dominated by the quasi-classical approximation except around very low temperatures. Even around very low temperatures, in typical examples, only the ground state must be treated separately as observed for example in the BEC condensation.

Pressure is the rate of momentum transfer per unit area. Consider a wall perpendicular to the x_1-axis.

i) Momentum Transfer: In the quasi-classical approximation, a particle with momentum component $p_1 > 0$ hits the wall and reflects elastically ($p_1 \to -p_1$). The momentum transferred to the wall is $\Delta p = 2p_1$. This makes sense again due to the fact that we have energy eigenstates of the system which are well approximated by states with definite momentum.

ii) Flux (The Swept Volume): In time $\mathrm{d}t$, particles with velocity $\mathbf{v}_g$ that are within a distance $\mathrm{d}x_1 = v_{g,1}\mathrm{d}t$ will hit the wall (figure 2). We observe that the probability of finding the particle within this region is approximately $\frac{1}{V}$. The number of such particles hitting area $\mathrm{d}A$ is:

$$\mathrm{d}N = \underbrace{f(p)\,V\frac{\mathrm{d}^d p}{h^d}}_{\text{Density}}\;\underbrace{(\mathbf{v}_g\cdot\boldsymbol{e_1})\;\mathrm{d}t\;\mathrm{d}A}_{\text{Volume}}\;\underbrace{\left(\frac{1}{V}\right)}_{\text{probability}}\,. \tag{8}$$

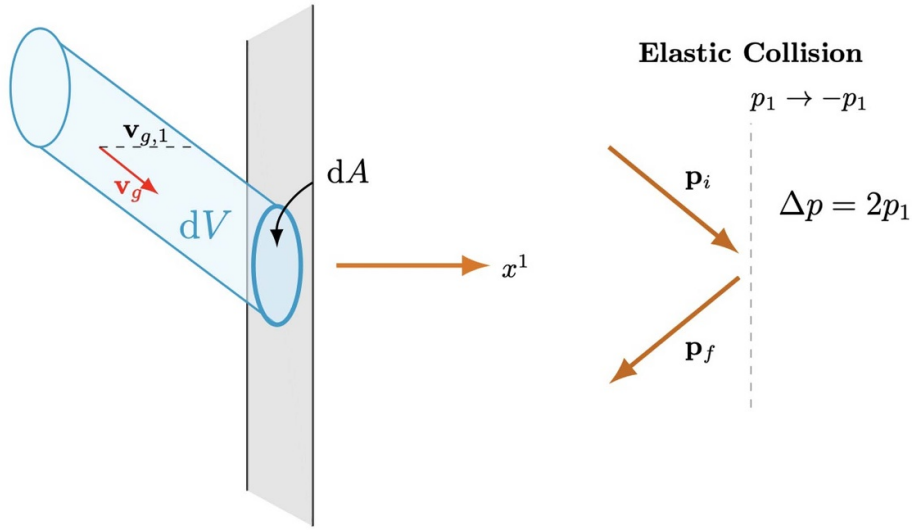

Figure 2. The flux calculation. A wave packet with group velocity $\mathbf{v}_g$ sweeps out a volume cylinder. Only particles within the volume $dV = (\mathbf{v}_g \cdot \boldsymbol{e}_1)\, dt\, dA$ will hit the area dA in time dt, where $\boldsymbol{e}_1$ is the unit vector along x_1 axis.

iii) Pressure Integral: The total momentum transferred to the wall in time dt is the integral of the single-particle transfer (Δp) weighted by the number of collisions (dN). The pressure P is defined as the force per unit area, which is the rate of momentum transfer per unit area:

$$P = \frac{1}{dA\ dt} \int_{p_1>0} \underbrace{(2p_1)}_{\Delta p} \underbrace{dN}_{\text{Collisions}} . \tag{9}$$

Note that the integral must be restricted to the incoming particles with positive p_1 components for momentum transfer to occur. Substituting the expression for dN, the factors dA and dt cancel out, leaving:

$$P = \int_{p_1>0} (2p_1)\,(v_{g,1}) f(p)\, \frac{d^d p}{h^d} = \int (p_1)\,(v_{g,1}) f(p)\, \frac{d^d p}{h^d}, \tag{10}$$

where in the last step, we removed the restriction of positive p_1 by removing the factor 2 and using the isotropy in the total integration. For an isotropic distribution function, average of the square of any component of momentum is identical in all directions ($\langle p_1^2 \rangle = \langle p^2 \rangle / d$), note that the momentum dependent part of the full expression is given by

$$f(p) p^{s-2} d^d p, \tag{11}$$

which is completely isotropic. Thus, within the integration, the substitution $p_1^2 \to p^2/d$ is allowed;

$$P = \frac{1}{d} \int p^2 \left(sAp^{s-2}\right) f(p)\, \frac{d^d p}{h^d} = \frac{s}{d} \int (Ap^s) f(p)\, \frac{d^d p}{h^d} = \frac{s}{d} \frac{U}{V}. \tag{12}$$

Thus, we obtain the equation of state (2).

At this point, it is also useful to mention an alternative derivation based on a slowly moving piston, in which the pressure is identified through the mechanical work done by the gas. In the semiclassical wave-packet picture, the packet is centered around a nearly definite momentum, so its reflected frequency undergoes the usual Doppler shift when the wall moves with speed u. As measured in the rest frame of the container, the reflected wave therefore acquires the shifted energy $\epsilon(p) + 2up_1$. A careful analysis, which is worked out in detail in a similar way for the photon gas in appendix A, shows that the rate of energy transfer to the gas is equal to $-dW = \frac{s}{d} \frac{U}{V} (u\, dt A)$, where A is the cross sectional area of the moving piston. As a result, by identifying the mechanical work with $-P dV$, one recovers the same equation of state. We note, however, that this argument again relies on an adiabatic approximation: the boundary must move sufficiently slowly that transitions between different energy levels may be neglected, at least apart from possible complications involving very low-lying states. See [8] for a recent

pedagogical discussion of the moving-piston method in the nonrelativistic $s = 2$ case. The same derivation using moving piston argument for the relativistic case for photon gas $s = 1$ has been discussed in appendix A.

3. Derivation through the virial theorem

Consider non-interacting particles in a body of volume V bounded by a surface S in a d dimensional space. The virial theorem states that

$$2\langle T\rangle = -\left\langle \sum_i \mathbf{r}_i \cdot \mathbf{F}_i \right\rangle , \tag{13}$$

where T is the total kinetic energy of the particles and $\mathbf{r}_i$ is the position vector of the ith particle and $\mathbf{F}_i$ is the force acting on the ith particle, and the braket $\langle\cdot\rangle$ denotes the time average, that is, time average of a function $G(t)$ is given by $\langle G\rangle = \frac{1}{\tau}\int_0^\tau G(t)\mathrm{d}t$. This is the standard result in textbooks of classical and statistical mechanics and can be found in [3, 9], and also see the recent article [10] which provides an extensive list of references containing the applications and derivations of it.

Suppose that Hamiltonian of the system is given by $H = \sum_i Ap_i^s + V^{\text{wall}}$ (here we assume that there is a highly repulsive potential concentrated at the boundary that reflects the particles). Let $\mathbf{r}_i$ and $\mathbf{p}_i$ be the position vectors and momenta of the ith particle. Then, the equations of motion are

$$\dot{\mathbf{r}}_i = \frac{\partial H}{\partial \mathbf{p}_i}, \qquad \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{r}_i}, \tag{14}$$

Hence, $\dot{\mathbf{r}}_i = sAp_i^{s-2}\mathbf{p}_i$. Consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\sum_i \mathbf{r}_i \cdot \mathbf{p}_i = \sum_i (\dot{\mathbf{r}}_i \cdot \mathbf{p}_i + \mathbf{r}_i \cdot \dot{\mathbf{p}}_i) \tag{15}$$

$$= \sum_i \left(sAp_i^{s-2}\mathbf{p}_i \cdot \mathbf{p}_i + \mathbf{r}_i \cdot \dot{\mathbf{p}}_i\right) \tag{16}$$

$$= \sum_i (sAp_i^s + \mathbf{r}_i \cdot \dot{\mathbf{p}}_i) \tag{17}$$

$$= sH + \sum_i \mathbf{r}_i \cdot \dot{\mathbf{p}}_i, \tag{18}$$

where the factor s is due to the homogenity of the kinetic energy (Hamiltonian) of degree s. Since the quantity $\sum_i \mathbf{r}_i \cdot \mathbf{p}_i$ is a bounded function, the mean value of its time derivative vanishes for large values of time. Then, it follows that

$$0 = sU + \left\langle \sum_i \mathbf{r}_i \cdot \dot{\mathbf{p}}_i \right\rangle , \tag{19}$$

where $U = \langle H\rangle$. The contribution coming from the surface potential can be neglected in the thermodynamic limit, hence the internal energy is simply $U = \langle\sum_i Ap_i^s\rangle$. The quantity $\dot{\mathbf{p}}_i$ is the force $\mathbf{F}_i$ acting on the individual particles. For a noninteracting gas, the only forces acting on the particles are those exerted by the confining wall. Thus, the quantity

$$\sum_i \mathbf{r}_i \cdot \mathbf{F}_i \tag{20}$$

receives contributions only at wall collisions. Upon taking a long-time average, the sum over these large number of discrete collision impulses may be replaced by the continuous boundary force exerted by the wall on the gas (which is interpreted as the macroscopic pressure), it follows that

$$\left\langle \sum_i \mathbf{r}_i \cdot \mathbf{F}_i \right\rangle = -P \oint_S \mathbf{r} \cdot \mathbf{n}\,\mathrm{d}S\,. \tag{21}$$

Then, using the divergence theorem in three dimensions we obtain

$$\left\langle \sum_i \mathbf{r}_i \cdot \dot{\mathbf{p}}_i \right\rangle = \left\langle \sum_i \mathbf{r}_i \cdot \mathbf{F}_i \right\rangle = -P \oint_S \mathbf{r} \cdot \mathrm{d}\mathbf{S} = -P \int_V (\nabla \cdot \mathbf{r})\, \mathrm{d}V = -dPV. \tag{22}$$

Therefore we obtain $PV = \frac{s}{d}U$. Similarly, in two dimensions $\mathrm{d}\mathbf{F} = -P\mathbf{n}\,\mathrm{d}s$,, where d$s$ is the arc-length element of the boundary curve for the two dimensional box. Then, using the two dimensional version of divergence theorem $-P\oint_S \mathbf{r}\cdot \mathrm{d}\mathbf{s} = -P\int_S (\nabla\cdot\mathbf{r})\,\mathrm{d}A$. Since $\nabla\cdot\mathbf{r} = 2$ in two dimensions, we obtain the relation $PV = \frac{s}{2}U$, where V refers to the surface area of the two dimensional box. One dimensional case is much simpler to see (the boundary consists of the two endpoints of the interval in this case). For the noninteracting case with $d = s = 2$, the equation of state was obtained in [11] using the same virial-theorem approach.

Remark. The generalized equation of state can also be derived from the quantum virial theorem [7], where the expectation value of an operator $\hat{A}$ is given by $\langle \hat{A} \rangle = Tr(\hat{A}\hat{\rho})$ and $\hat{\rho} = e^{-\beta \hat{H}}/Z$ is the density operator with $Z = Tr(e^{-\beta \hat{H}})$. In the Heisenberg picture, we consider the equilibrium expectation value of the virial operator $\sum_i \hat{\mathbf{x}}_i \cdot \hat{\mathbf{p}}_i$, (for simplicity we ignore the symmetrization) for which

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\langle \sum_i \hat{\mathbf{x}}_i \cdot \hat{\mathbf{p}}_i \right\rangle = s\left\langle \sum_i A|\hat{\mathbf{p}}_i|^s \right\rangle - \left\langle \sum_i \hat{\mathbf{x}}_i \cdot \left(\nabla_i V^{\mathrm{wall}}\right)(\hat{\mathbf{x}}_i) \right\rangle. \tag{23}$$

Because the system is in equilibrium, the density operator commutes with the Hamiltonian. Using the cyclicity of the trace, the expectation value of the virial is constant in time, therefore the left-hand side vanishes.

The remaining term involving V^{wall} has a simple physical meaning: it is the force exerted by the confining wall on the particles. In equilibrium, its expectation value may be identified with the macroscopic pressure force distributed over the boundary,

$$\left\langle \sum_i \hat{\mathbf{x}}_i \cdot \left(\nabla_i V^{\mathrm{wall}}\right)(\hat{\mathbf{x}}_i) \right\rangle = -P \oint \mathbf{x} \cdot \mathrm{d}\mathbf{S}. \tag{24}$$

Thus, the quantum-mechanical average of the wall force reproduces the same boundary term that appears in the classical virial argument. The rest of the proof is therefore unchanged, and one again obtains the generalized equation of state.

4. Purely quantum derivation via stress tensor

As noted in the introduction, our wave-packet approach is a hybrid between quasi-classical arguments and quantum mechanical reasoning. We now show that these relations can be derived purely in the realm of quantum mechanics using the quantum stress tensor, without relying on semi-classical approximations.

It is worth emphasizing that the scalar pressure appearing in the homogeneous equation of state is sufficient only when the system is spatially uniform and isotropic. For an inhomogeneous gas, the local mechanical response cannot in general be described by a single pressure variable. In such cases, the stress tensor provides the more fundamental formulation of the virial relation. Bobrov *et al* [12] derived a virial theorem for an inhomogeneous quantum medium in an external field within the canonical ensemble and showed that the precise form of the virial theorem depends on the boundary conditions imposed on the wave functions, in particular on the behavior of their spatial derivatives at the boundary of the volume enclosing the system. Their analysis also clarifies that the quantum stress tensor is naturally expressed in terms of averaged spatial derivatives of the many-particle wave functions. This observation places the homogeneous pressure-based relation used in the present work in a broader setting: it may be regarded as the isotropic and spatially uniform limit of a more general stress-tensor virial formulation.

Here, we derive the equation of state $PV = \frac{2}{3}U$ (usual kinetic energy in 3 dimensions) by following Durand's elegant quantum mechanical stress tensor approach [4] in a simpler and shorter way. In the original paper by Durand, a stress tensor calculation in the grand-canonical approach using the BE or FD statistics shows that the thermodynamic pressure is the same as the kinetic pressure. However, this is achieved by taking a more general step, where resulting pressure on the boundary is linked to the variation of the eigenvalues as one deforms the boundary surface [4]. We will show that this generality is not

needed for an equilibrium system. Indeed, one can find a connection to the thermodynamic pressure via the equation of state relating energy density to the pressure, as we will see now.

4.1. Equilibrium stress tensor for $s = 2$

We use the quantum stress tensor for a single particle case in the state ψ_σ, whose general form is given in [4] for N particles. Here we write it in a proper way for our purposes:

$$T_{ij}^{\sigma} = -\frac{\hbar^2}{2m}\left[\nabla_i\psi_\sigma^*\nabla_j\psi_\sigma + \nabla_j\psi_\sigma^*\nabla_i\psi_\sigma - \frac{1}{2}\delta_{ij}\nabla^2\left(\psi_\sigma^*\psi_\sigma\right)\right], \tag{25}$$

where ∇^2 denotes the Laplacian, which in index notation can be written as $\nabla_l\nabla_l$ (equivalently ∇_l^2). Throughout this paper, we adopt the Einstein summation convention, i.e. repeated indices are implicitly summed over. To make the exposition self-contained and pedagogically accessible, a simple derivation of this formula is included in appendix B.

For simplicity, we suppress the coordinate variable $\mathbf{x}$. Durand explicitly shows that in the grand-canonical approach, when we consider a multi-particle version of the stress tensor, we can find the desired expression in terms of single particle energy eigen-states by averaging over the corresponding distribution functions. Then, the average of the stress tensor with respect to BE or FD distributions is obtained,

$$\langle T_{ij}\rangle = \sum_\sigma T_{ij}^{\sigma} n(\epsilon_\sigma), \tag{26}$$

where $n(\epsilon_\sigma) = (e^{\beta(\epsilon_\sigma-\mu)} \pm 1)^{-1}$ is the equilibrium occupation number at inverse temperature $\beta = 1/k_BT$ and chemical potential μ, with the sign choice distinguishing FD $(+)$ from BE $(-)$ statistics, and ϵ_σ denotes the energy of the single-particle eigenstate labeled by σ. Here the bra-ket $\langle\cdot\rangle$ denotes the statistical average.

This is the quantum mechanical stress tensor of a gas in equilibrium, that means *it must obey the homogeneity and isotropy* as such a gas would be (away from the boundary) isotropic and homogeneous in the bulk. This implies that we must have only the pressure components (no shearing) and a constant pressure in the bulk. Thus, it must be of the following form

$$\langle T_{ij}\rangle = -P\delta_{ij}. \tag{27}$$

Note that it has been shown in [4] that the pressure acting on the boundary surface can be expressed as $P = -n_i\langle T_{ij}\rangle n_j$, where n_i's are the components of the outward unit normal vector.

The homogeneity means we can integrate it over the volume

$$\frac{1}{V}\int \mathrm{d}^dx\langle T_{ij}\rangle = -P\delta_{ij}\int\frac{\mathrm{d}^dx}{V}. \tag{28}$$

If we take the trace, we obtain

$$P = -\frac{1}{Vd}\int \mathrm{d}^dx\langle T_{ii}\rangle, \tag{29}$$

where $\delta_{ii} = d$ in d dimensions. Note that the Laplacian part $\nabla^2\left(\psi_\sigma^*\psi_\sigma\right)$ indeed is a pure divergence, and can be turned into a surface term (thanks to divergence theorem) giving us zero under Dirichlet or Neumann boundary conditions. The first symmetric part instead can be integrated using the identity $\nabla_i(\psi_\sigma^*\nabla_i\psi_\sigma) = (\nabla_i\psi_\sigma^*)(\nabla_i\psi_\sigma) + \psi_\sigma^*\nabla^2\psi_\sigma$ and the first term is again total divergence which vanishes. The rest of the term gives us the factor which is just the kinetic energy acting on the eigenfunctions of the Hamiltonian, so that

$$P = -\frac{1}{Vd}\int \mathrm{d}^dx\langle T_{ii}\rangle = \frac{2}{Vd}\sum_\sigma\left(\int \mathrm{d}^dx\psi_\sigma^*(\mathbf{x})\psi_\sigma(\mathbf{x})\right)\epsilon_\sigma n(\epsilon_\sigma), \tag{30}$$

where the factor 2 is due to the same contribution to the integral from the first two terms in (25). Since the eigenfunctions are assumed to be normalized, we finally obtain

$$PV = \frac{2}{d}\sum_\sigma \epsilon_\sigma n(\epsilon_\sigma) = \frac{2}{d}U, \tag{31}$$

as expected. Note that this relation holds in the thermodynamic formalism linking this kinetic derivation of pressure to the thermodynamic one. This gives a purely quantum mechanical derivation of the energy-pressure relation without relying on any semiclassical approximation.

4.2. Operator-based approach for general dispersions

Here we discuss how the stress tensor idea can be extended to more general dispersion relations. Note that this section is targeting more advanced readers, typically graduate students who are taking a statistical mechanics course, so it uses more advanced concepts. The following operator-based approach to the stress tensor for a non-interacting quantum gas, avoids explicit reference to momentum eigenstates and it is a natural choice.

For a single particle, we define the spatial stress operator (in the Heisenberg picture) to be the symmetrized momentum current:

$$\hat{T}_{ij} = -\frac{1}{2}\left(\hat{p}_i \hat{\dot{x}}_j + \hat{\dot{x}}_j \hat{p}_i\right), \tag{32}$$

where the symmetrization ensures Hermiticity and negative sign is our convention. Henceforth, the time dependence of operators is left implicit for ease of notation; all operators should be understood as time-dependent unless otherwise specified. Here the velocity operator is defined through the Heisenberg equation $\hat{\dot{x}}_i = \frac{i}{\hbar}[\hat{H}, \hat{x}_i]$. Note that to see how it is connected with the conservation of momentum density, we need to localize this operator. This means we must enforce position operator to be equal to a specific value $\mathbf{x}$, let us then write the momentum density:

$$\hat{\pi}_i(\mathbf{x}) = \frac{1}{2}\left[\hat{p}_i\, \delta(\hat{\mathbf{x}} - \mathbf{x}) + \delta(\hat{\mathbf{x}} - \mathbf{x})\, \hat{p}_i\right]. \tag{33}$$

Here $\hat{p}_i$ is the ith component of the momentum operator and $\mathbf{x}$ is assumed to be multiplied by the identity operator in Hilbert space. We provide a demonstration of equality of this expression to its version in the Schrödinger picture in appendix C for the convenience of the reader. Moreover we derive the stress tensor in the same appendix by relating it to the usual conservation law for momentum density. The time derivative of momentum density then yields

$$\partial_t \hat{\pi}_i(\mathbf{x}) = \partial_j \frac{1}{2}\left[\hat{p}_i \hat{\dot{x}}_j \delta(\hat{\mathbf{x}} - \mathbf{x}) + \delta(\hat{\mathbf{x}} - \mathbf{x})\, \hat{\dot{x}}_j \hat{p}_i + \partial_l B_{lji}\right]. \tag{34}$$

The last term is a total divergence and can be dropped as it does not lead to any change in the conservation law [13].

For Hamiltonians depending only on momentum, this symmetrization is in fact immaterial, but we keep it for definiteness. Consider a homogeneous kinetic Hamiltonian in a d dimensional space

$$\hat{H} = A|\hat{\mathbf{p}}|^s, \tag{35}$$

where $|\hat{\mathbf{p}}| := (\sum_{i=1}^{d} \hat{p}_i^2)^{1/2}$ and $s > 0$. The analysis of Bose and Fermi gases with a Hamiltonian of this form is presented as a set of exercises in [1]. Using the Heisenberg equation of motion [6], which holds as an operator equality and without approximation, the velocity operator is given by

$$\hat{\dot{x}}_j = \frac{i}{\hbar}\left[\hat{H}, \hat{x}_j\right] = sA\,|\hat{\mathbf{p}}|^{s-2}\hat{p}_j, \tag{36}$$

where we have formally used $[f(\hat{\mathbf{p}}), \hat{x}_j] = -i\hbar \frac{\partial f}{\partial \hat{p}_j}$. Substituting this into the definition of the stress tensor yields

$$\hat{T}_{ij} = -sA|\hat{\mathbf{p}}|^{s-2}\hat{p}_i\hat{p}_j. \tag{37}$$

It is explicitly shown in appendix C that the stress tensor (localized version) satisfies the usual conservation law,

$$\partial_t \hat{\pi}_i(\mathbf{x}) + \partial_j \hat{T}_{ji}(\mathbf{x}) = 0, \tag{38}$$

and we can express it in any basis we prefer.

We now turn to the many-particle system using this single particle version. Note that for a non-interacting system, single particle operators can be *lifted* to the many-particle space simply as sums over individual particle labels. Due to indistinguishability of identical particles, these are only labels, not actual particle tags, then the Hilbert space is constructed either by *symmetric* or *anti-symmetric* wave functions, depending on particles being Bosons or Fermions. It is often most convenient to work in the grand-canonical ensemble, where the particle number is not conserved, but the chemical potential of the reservoir, with which our system can exchange energy and particles, is kept fixed while the

average number of particles emerges as a result. This effectively means, we have a new multi-particle Hamiltonian $\hat{H}-\mu\hat{N}$ operating on the direct sum of multiple particle Hilbert spaces. The (equilibrium) density matrix for this system becomes simply,

$$\hat{\rho}_{\mathcal{F}}=\frac{1}{\mathcal{Z}}e^{-\beta\left(\hat{H}-\mu\hat{N}\right)}\,. \tag{39}$$

By evaluating the expectation value of our localized stress tensor in the thermal equilibrium state, we can, in principle, express everything in terms of single particle energy eigenstates. For general boundaries, this is a non-trivial issue, that we set aside for the time being. We evaluate the lifted stress tensor for the equilibrium density matrix and denote this expectation value as

$$\langle\hat{T}_{ij}^{\mathcal{F}}\left(\mathbf{x}\right)\rangle=\mathrm{Tr}_{\mathcal{F}}\hat{\rho}_{\mathcal{F}}\hat{T}_{ij}^{\mathcal{F}}\left(\mathbf{x}\right),\tag{40}$$

where $\mathcal{F}$ reminds us that we take the direct sum of all multi-particle Hilbert spaces (thus forming what is called a Fock space realization). In thermal equilibrium (assuming the thermodynamic limit), expectation values in the bulk become *translationally invariant and isotropic*. Consequently, the stress tensor must take the form

$$\langle\hat{T}_{ij}^{\mathcal{F}}\left(\mathbf{x}\right)\rangle=-P\delta_{ij},\tag{41}$$

where P is the pressure. We use this idea to simplify the left-hand side as well, since due to isotropy, we can sum the diagonal components and find as a result,

$$\frac{1}{d}\langle\sum_{i}\hat{T}_{ii}^{\mathcal{F}}\left(\mathbf{x}\right)\rangle=-P.\tag{42}$$

Now we observe that $\sum_i\hat{p}_i^2=(-\nabla^2)=|\hat{p}|^2$, which then turns one particle operator, to be lifted up the Fock space, into

$$\sum_{i}\hat{T}_{ii}\left(\mathbf{x}\right)=-\frac{1}{2}sA\left[|\hat{\mathbf{p}}|^{s}\delta\left(\hat{\mathbf{x}}-\mathbf{x}\right)+\delta\left(\hat{\mathbf{x}}-\mathbf{x}\right)|\hat{\mathbf{p}}|^{s}\right].\tag{43}$$

We can then evaluate the expectation value for this combination explicitly, and express the result in terms of single particle energy eignestates and their occupation numbers,

$$\langle\sum_{i}\hat{T}_{ii}^{\mathcal{F}}\left(\mathbf{x}\right)\rangle=-s\sum_{\sigma}n\left(\epsilon_{\sigma}\right)\psi_{\sigma}^{*}\left(\mathbf{x}\right)\left(A|\hat{\mathbf{p}}|^{s}\right)\psi_{\sigma}\left(\mathbf{x}\right),\tag{44}$$

where $n(\epsilon_\sigma)$ is either the FD or BE distribution function (at temperature T and chemical potential μ). Note that the apparent spatial dependence of the stress tensor naturally emerges through the position representation of the energy eigenstates, $\psi_\sigma(\mathbf{x})=\langle\mathbf{x}|\psi_\sigma\rangle$.

We make the following observation, there is no dependence on $\mathbf{x}$, on the right side of equation (42), thus integrating over space on both sides, we obtain,

$$P=\frac{s}{d}\frac{1}{V}\sum_{\sigma}n\left(\epsilon_{\sigma}\right)\int\mathrm{d}^{d}x\psi_{\sigma}^{*}\left(\mathbf{x}\right)\left(A|\hat{\mathbf{p}}|^{s}\right)\psi_{\sigma}\left(\mathbf{x}\right)=\frac{s}{d}\frac{1}{V}\sum_{\sigma}n\left(\epsilon_{\sigma}\right)\langle\psi_{\sigma}|\left(A|\hat{\mathbf{p}}|^{s}\right)|\psi_{\sigma}\rangle.\tag{45}$$

Recognizing the right-hand side as the total energy U divided by the volume V, we arrive at the equation of state (2).

Remark. It is possible to generalize this construction by taking even a more general dispersion relation. Suppose $A(s)$ is a positive function supported on the positive real axis. Then we can introduce non-interacting particles with a Hamiltonian,

$$\hat{H}=\int_{0}^{\infty}\mathrm{d}sA\left(s\right)|\hat{p}|^{s}.\tag{46}$$

This system is best understood as an effective dispersion model rather than a fundamental microscopic Hamiltonian. It may describe noninteracting quasiparticles whose energy admits an expansion or superposition in powers of momentum, as occurs in low-energy band expansions, relativistic corrections, and fractional or nonlocal kinetic theories. Then, the equation of state for P becomes

$$PV=\frac{1}{\mathrm{d}}\int_{0}^{\infty}\mathrm{d}ssA\left(s\right)\mathrm{Tr}\left[\hat{\rho}\left(T\right)|\hat{p}|^{s}\right]=\frac{1}{\mathrm{d}}\frac{\int_{0}^{\infty}\mathrm{d}ssA\left(s\right)\mathrm{Tr}\left[\hat{\rho}\left(T\right)|\hat{p}|^{s}\right]}{\int_{0}^{\infty}\mathrm{d}sA\left(s\right)\mathrm{Tr}\left[\hat{\rho}\left(T\right)|\hat{p}|^{s}\right]}U\tag{47}$$

where $\hat{\rho} = \sum_\sigma n(\epsilon_\sigma)|\psi_\sigma\rangle\langle\psi_\sigma|$ refers to thermal mixed state. Note that this is an interesting extension, in some sense we take a thermal-weighted average of s. This may be interpreted as an effective equation of state for a noninteracting thermal gas whose dispersion is a positive superposition of homogeneous momentum powers. In the pure power-law case $\epsilon(p) \propto p^s$, the ratio PV/U is fixed by the single scaling exponent s. By contrast, in the generalized case this ratio is determined by a thermal weighted average of the exponents contributing to the dispersion. Thus the above equation describes a crossover relation: if one term dominates, the usual form $PV = (s/d)U$ is recovered, whereas if several terms contribute appreciably, the system behaves as if it had an effective temperature-dependent exponent.

Conclusion

In this article, we have explored the generalized equation of state $PV = (s/d)U$ through multiple complementary theoretical lenses. We have demonstrated that this fundamental relation can be established using a semi-classical kinetic approach utilizing wave packets, the classical and quantum virial theorems, as well as a rigorous, purely quantum mechanical framework based on the stress tensor and Heisenberg equations of motion. These parallel derivations clearly elucidate the physical roles of the parameters s and d: the factor s enters naturally through the quantum definition of group velocity ($\mathbf{v}_g \propto \nabla_p p^s$) and the degree of homogeneity in the Hamiltonian, while the factor $1/d$ emerges from the geometric averaging of momentum components or coordinates in d-dimensional space.

In principle, a purely classical version of this model with the same dispersion relation and using the Gibbs distribution is possible from a statistical mechanics point of view. In that case, however, it is not possible to give a kinetic description, as we have no way of defining a transport velocity of particles to calculate the resulting momentum flux.

We make the following final remark, which itself can be instructive for students, as it provides an interesting connection between our approach and the modern concept of symmetry in quantum field theory: This identification of T_{ij} as the momentum current is precisely the content of the translational Ward identity, $\partial_\mu T^{\mu i} = 0$, whose spatial integral equates the rate of change of total momentum in a region to the momentum flux through its boundary. Recent pedagogical work has similarly applied kinetic theory to calculate forces on moving objects in quantum gases [14], highlighting the continued relevance of these fundamental approaches. The usefulness of the virial theorem in deriving equations of state is not restricted to ideal gases or to the simple power-law dispersions considered here. For example, Vasil'ev and Lyuboshits applied virial-theorem arguments to the electron gas in metals and discussed how the relation between kinetic and interaction energies constrains thermodynamic and electronic properties of metallic systems [15]. In the context of Bose systems, Bobrov, Schram, and Trigger used the virial theorem together with dimensional analysis to analyze a weakly non-ideal Bose gas at low temperature; their work emphasizes that interaction effects require a self-consistent determination of the chemical potential and other thermodynamic quantities, rather than a purely ideal-gas treatment [16]. More recently, Tanalikhit and Kanchanapusakit studied an ideal Fermi gas in a uniform gravitational field, deriving the density profile, chemical potential, and energy contributions in different gravitational regimes within a semiclassical framework [17]. These examples show that virial-type reasoning provides a unifying route to equations of state not only for ideal gases, but also for interacting Fermi and Bose gases and for quantum gases subject to external potentials.

From a pedagogical standpoint, this unified presentation provides a concrete setting for upper-level undergraduate and beginning graduate students to connect concepts that are often taught separately: kinetic theory, wave mechanics, and equilibrium quantum statistical physics. By treating pressure consistently as mechanical work, momentum transfer, or momentum flux across complementary theoretical frameworks, we hope this approach helps students bridge the gap between microscopic dynamics and macroscopic equations of state. Furthermore, the article provides instructors with a self-contained example that can serve as a reading assignment, a discussion topic, or a bridge between different parts of the physics curriculum.

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Data availability statement

No new data were created or analyzed in this study.

Appendix A. Insight from the photon gas ($s = 1$)

We consider the instructive example of a photon gas in three spatial dimensions ($d = 3$). An elegant kinetic-theory derivation of photon pressure, along with a broader discussion of photon-gas thermodynamics, may be found in [18]. Here we present the derivation showing how the equation of state for a photon gas emerges naturally when one combines the particle and wave aspects of photons in calculating work done by a moving piston. This treatment highlights how wave–particle duality can be used to give a concrete and physically transparent account of the origin of radiation pressure and naturally connects it to the thermodynamic concept of mechanical work. For readers of *European Journal of Physics*, this example also serves as a pedagogically familiar entry point, showing in a particularly transparent setting how microscopic momentum transfer, wave propagation, and thermodynamic work combine to produce an equation of state.

Consider a cylinder filled with photons in equilibrium at temperature T, closed by a perfectly reflecting mirror (piston) moving inward with velocity $\mathbf{u}$ along the z-axis as shown in figure 3. Here, we derive the pressure by considering the mechanical work done by a moving piston. For this, we need to take Doppler shift into account because the moving piston defines a natural rest frame where reflection is simpler and transformations between frames changes frequency of photon.

Lorentz Transformations and Doppler Shift

Let K be the laboratory frame (where the gas is initially at rest) and K' be the rest frame of the piston. A photon travels towards the piston with angular frequency ω and wave vector component k_z. A perfect mirror reflects photons of frequency ω and wave vector $\mathbf{k}$ according to the usual rule of incident angle being equal to the reflection angle while preserving the frequency in *its rest frame* K'. We can then calculate the change between the incident and reflected frequency by transforming first to the rest frame of the mirror and then back:

i) Transformation to Piston Frame ($K \to K'$): We apply the Lorentz transformation to the energy-momentum 4-vector $(\hbar\omega/c, \hbar\mathbf{k})$ [19]. The frequency ω' observed in the piston frame is given by:

$$\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \frac{\mathbf{u}}{c}\cdot\mathbf{k}\right). \tag{48}$$

Here, $\gamma = (1 - u^2/c^2)^{-1/2}$. Since the piston moves towards the photon (compression), the relative velocity leads to a blue shift. For a photon with $k_z > 0$ and piston velocity $-u\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector along the z axis, we find

$$\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} + \frac{u}{c}k_z\right). \tag{49}$$

ii) Reflection in Piston Frame: In the frame K', the piston acts as a stationary, perfect mirror. The reflection is elastic, so the frequency remains the same, but the normal component k_z of the wave

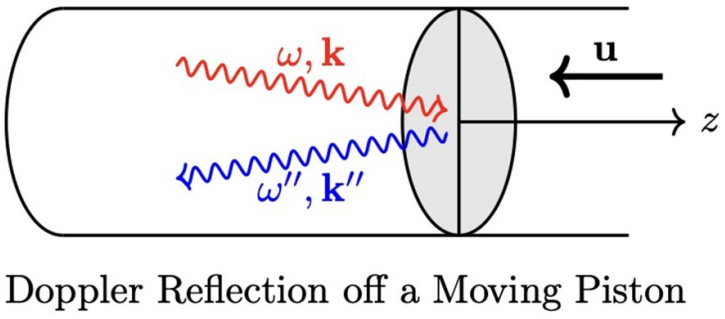

Figure 3. A photon reflecting off a piston moving inward with velocity $\mathbf{u}$. The reflected photon is Doppler shifted, gaining energy. The laboratory frame is K, and the piston rest frame is K'.

number (equally momentum) is reversed:

$$\omega'_{\text{ref}} = \omega' \quad \text{and} \quad k'_{z,\text{ref}} = -k'_{z,\text{inc}}\,. \tag{50}$$

iii) Transformation back to Laboratory Frame ($K' \to K$):
We now transform the reflected photon back to the lab frame K to find the final frequency ω''. Using the inverse Lorentz transformation for the frequency, we get

$$\frac{\omega''}{c} = \gamma\left(\frac{\omega'_{\text{ref}}}{c} + \frac{u}{c}k'_{z,\text{ref}}\right). \tag{51}$$

Since the reflected photon is moving away from the piston surface in the K' frame, and the piston frame is moving towards the observer, we get a second blue shift factor.

$$\frac{\omega''}{c} = \gamma\left(\frac{\omega'}{c} + \frac{u}{c}k'_z\right). \tag{52}$$

In the non-relativistic approximation $u \ll c$, we expand ω' and ω'' to first order in u/c and take $\gamma \approx 1$ and neglect terms of order $(u/c)^2$, to get

$$\omega' \approx \omega + uk_z\,, \tag{53}$$
$$\omega'' \approx \omega' + uk'_z \approx (\omega + uk_z) + uk_z = \omega + 2uk_z\,. \tag{54}$$

Then, the change in frequency is $\Delta\omega = \omega'' - \omega \approx 2uk_z$. If we now use *the quantum mechanical interpretation* of individual photons gaining energy due to collisions with the piston, the energy gained by the photon per collision is:

$$\Delta\epsilon = \hbar\,(\Delta\omega) \approx 2\hbar uk_z = 2up_z\,. \tag{55}$$

This energy gain $\Delta\epsilon$ represents the mechanical work performed by the piston on a single photon with a definite wave number. Note that we keep here the spirit of a typical derivation of black-body radiation where the wave nature of the system is often emphasized, while the particle aspect is not really used in the computation of the frequency increase due to the Doppler shift.

Calculating Pressure from Work

In a cavity of volume V, the number of available electromagnetic modes with wave vector in d^3k about $\mathbf{k}$ is $2\frac{V}{(2\pi)^3}\mathrm{d}^3k$ (the factor 2 is due to two independent transverse polarizations). In thermal equilibrium at temperature T, each mode has mean occupation $n_{\mathbf{k}} = (e^{\beta\hbar\omega} - 1)^{-1}$ (which is a BE distribution for the chemical potential $\mu = 0$). Here $\beta = 1/k_BT$. Hence the expected number of photons in that k-space element is their product:

$$\mathrm{d}n_{\mathbf{k}} = 2\underbrace{\left(\frac{1}{e^{\beta\hbar\omega} - 1}\right)}_{\text{BE Distribution}}\underbrace{\frac{V}{(2\pi)^3}\mathrm{d}^3k}_{\text{Density of States}}\,. \tag{56}$$

In spherical coordinates in $\mathbf{k}$ space, and using the dispersion relation $k = \omega/c$ (which implies then $\mathrm{d}^3k = k^2\,\mathrm{d}k\mathrm{d}\Omega = (\omega^2/c^3)\,\mathrm{d}\omega\mathrm{d}\Omega$), we obtain

$$\mathrm{d}n_{\mathbf{k}} = \frac{2V}{(2\pi)^3 c^3}\left(\frac{\omega^2}{e^{\beta\hbar\omega} - 1}\right)\mathrm{d}\omega\mathrm{d}\Omega\,, \tag{57}$$

where $\mathrm{d}\Omega$ denotes the differential solid angle. The total rate of work done ($\mathrm{d}W/\mathrm{d}t$) is found by integrating the energy transfer per collision ($\Delta\epsilon = 2up_z$) over the flux of photons hitting the piston. The volume swept out by the flux in a time interval $\mathrm{d}t$ is $\mathrm{d}V = A(c\cos\theta)\,\mathrm{d}t$. Neglecting contributions from the lateral surface of the piston, which is justified when the piston area A is large, this volume may be viewed geometrically as a slanted cylindrical element of base area A and effective height $(c\cos\theta)\,\mathrm{d}t$. Here θ is the angle between the photon wave vector and the outward normal to the piston surface. Thus, we have

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_0^\infty \int_{\Omega_+} \underbrace{\left(2u\frac{\hbar\omega}{c}\cos\theta\right)}_{\Delta\epsilon}\underbrace{(Ac\cos\theta)}_{\text{Flux}}\,\mathrm{d}n_{\mathbf{k}}\,, \tag{58}$$

where $\Omega_+ := \{(\theta,\phi) : 0 \leqslant \theta \leqslant \frac{\pi}{2}, 0 \leqslant \phi < 2\pi\}$ denotes the forward hemisphere, since we are concerned only with incoming photons. Substituting the explicit form of $\mathrm{d}n_{\mathbf{k}}$ and grouping terms, we get

$$\frac{\mathrm{d}W}{\mathrm{d}t} = 4uA\int_0^\infty \frac{V}{(2\pi)^3 c^3}\frac{\hbar\omega^3}{e^{\beta\hbar\omega}-1}\left(\int_{\Omega_+}\cos^2\theta\,\mathrm{d}\Omega\right)\mathrm{d}\omega\,. \tag{59}$$

Integrating the angular part yields

$$\int_{\Omega_+}\cos^2\theta\,\mathrm{d}\Omega = \int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi/2}\cos^2\theta\sin\theta\,\mathrm{d}\theta = 2\pi\left[-\frac{\cos^3\theta}{3}\right]_0^{\pi/2} = \frac{2\pi}{3}\,. \tag{60}$$

The remaining frequency integral is related to the total internal energy density (U/V). To see this, consider

$$\frac{U}{V} = \int_0^\infty\int_\Omega \hbar\omega\;\mathrm{d}n_{\mathbf{k}} = \int_0^\infty \frac{V\hbar\omega^3}{4\pi^2c^3\left(e^{\beta\hbar\omega}-1\right)}\,\mathrm{d}\omega\,. \tag{61}$$

Thus, the work equation simplifies to:

$$\frac{\mathrm{d}W}{\mathrm{d}t} = uA\left(\frac{1}{3}\frac{U}{V}\right). \tag{62}$$

Identifying the thermodynamic work rate as $\mathrm{d}W/\mathrm{d}t = P(\mathrm{d}V/\mathrm{d}t) = P(uA)$, we recover:

$$P = \frac{1}{3}\frac{U}{V}. \tag{63}$$

Incidentally, we see the importance of 'slow' motion of the piston ($u \ll c$) to claim that the process is quasi-static, thus the net work done is to be equated to $P\mathrm{d}V$ for the equilibrium pressure P of the photon gas. Practically it is very hard to violate this quasi-static condition for typical macroscopic objects, unless one is talking about an astronomical object like a star. On the other hand, one must remember that in a fully quantum mechanical treatment, a time dependent perturbation can also lead to transitions between different states, so the process should be slow enough to be treated as an adiabatic change.

Appendix B. Derivation of the quantum stress tensor

To derive the compact form of the quantum stress tensor for a specific stationary single-particle state σ, we start by expressing the standard Lagrangian density in terms of this state:

$$\mathcal{L} = \frac{i\hbar}{2}\left(\psi_\sigma^*\partial_t\psi_\sigma - \partial_t\psi_\sigma^*\psi_\sigma\right) - \frac{\hbar^2}{2m}\left(\nabla\psi_\sigma^*\right)\cdot\left(\nabla\psi_\sigma\right) - V\psi_\sigma^*\psi_\sigma\,. \tag{64}$$

By Noether's theorem, the invariance of the action under spatial translations yields the canonical energy-momentum tensor (stress tensor) for this state:

$$T_{ik}^\sigma = \frac{\partial\mathcal{L}}{\partial\left(\nabla_k\psi_\sigma\right)}\nabla_i\psi_\sigma + \frac{\partial\mathcal{L}}{\partial\left(\nabla_k\psi_\sigma^*\right)}\nabla_i\psi_\sigma^* - \delta_{ik}\mathcal{L}\,. \tag{65}$$

Evaluating the derivatives with respect to the field gradients, we obtain the kinetic terms:

$$\frac{\partial\mathcal{L}}{\partial\left(\nabla_k\psi_\sigma\right)} = -\frac{\hbar^2}{2m}\nabla_k\psi_\sigma^*\,, \tag{66}$$

$$\frac{\partial\mathcal{L}}{\partial\left(\nabla_k\psi_\sigma^*\right)} = -\frac{\hbar^2}{2m}\nabla_k\psi_\sigma\,. \tag{67}$$

Substituting these into the tensor definition gives:

$$T_{ik}^\sigma = -\frac{\hbar^2}{2m}\left(\nabla_i\psi_\sigma^*\nabla_k\psi_\sigma + \nabla_k\psi_\sigma^*\nabla_i\psi_\sigma\right) - \delta_{ik}\mathcal{L}\,. \tag{68}$$

To simplify the diagonal term $-\delta_{ik}\mathcal{L}$, we evaluate the Lagrangian 'on-shell'. Namely, we impose the equations of motion (the Schrödinger equation), which are formally derived by varying the action with respect to the fields ψ_σ^* and ψ_σ:

$$i\hbar\partial_t\psi_\sigma = -\frac{\hbar^2}{2m}\nabla^2\psi_\sigma + V\psi_\sigma \quad \text{and} \quad -i\hbar\partial_t\psi_\sigma^* = -\frac{\hbar^2}{2m}\nabla^2\psi_\sigma^* + V\psi_\sigma^* . \tag{69}$$

Inserting these on-shell conditions back into the Lagrangian density, the potential energy terms exactly cancel out:

$$\begin{aligned}\mathcal{L}_{\text{on-shell}} &= \frac{1}{2}\psi_\sigma^*\left(-\frac{\hbar^2}{2m}\nabla^2\psi_\sigma + V\psi_\sigma\right) + \frac{1}{2}\left(-\frac{\hbar^2}{2m}\nabla^2\psi_\sigma^* + V\psi_\sigma^*\right)\psi_\sigma \\ &\quad - \frac{\hbar^2}{2m}\left(\nabla\psi_\sigma^*\right)\cdot\left(\nabla\psi_\sigma\right) - V\psi_\sigma^*\psi_\sigma \\ &= -\frac{\hbar^2}{4m}\left(\psi_\sigma^*\nabla^2\psi_\sigma + \psi_\sigma\nabla^2\psi_\sigma^* + 2\nabla\psi_\sigma^*\cdot\nabla\psi_\sigma\right) .\end{aligned} \tag{70}$$

We recognize the term in the parentheses as the Laplacian of the probability density, $\nabla^2(\psi_\sigma^*\psi_\sigma)$. Thus, the on-shell Lagrangian simplifies beautifully to:

$$\mathcal{L}_{\text{on-shell}} = -\frac{\hbar^2}{4m}\nabla^2\left(\psi_\sigma^*\psi_\sigma\right) . \tag{71}$$

Substituting this result back into equation (68), we obtain the final, compact expression for the stress tensor of a single-particle state σ, equation (25) in the text:

$$T_{ik}^\sigma = -\frac{\hbar^2}{2m}\left[\nabla_i\psi_\sigma^*\nabla_k\psi_\sigma + \nabla_k\psi_\sigma^*\nabla_i\psi_\sigma - \frac{1}{2}\delta_{ik}\nabla^2\left(\psi_\sigma^*\psi_\sigma\right)\right] . \tag{72}$$

Appendix C. Momentum density operator in heisenberg picture

It is somewhat tricky to define a momentum density in Heisenberg picture. A possible alternative could be

$$\hat{\pi}_k(\mathbf{y}) = \frac{1}{2}\left(\hat{p}_k\delta(\hat{\mathbf{x}}-\mathbf{y}) + \delta(\hat{\mathbf{x}}-\mathbf{y})\hat{p}_k\right) . \tag{73}$$

The Dirac delta function above is defined by the operator form of the integral representation [7]:

$$\delta(\hat{\mathbf{x}}-\mathbf{y}) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{i\mathbf{k}\cdot(\hat{\mathbf{x}}-\mathbf{y})} . \tag{74}$$

One can see that this Heisenberg operator is simply related to the Schrödinger picture by [6, 7]

$$\delta(\hat{\mathbf{x}}-\mathbf{y}) = e^{i\hat{H}t/\hbar}\delta(\hat{\mathbf{x}}_S-\mathbf{y})\, e^{-i\hat{H}t/\hbar} , \tag{75}$$

which can be verified from a formal power series expansion–other representations of the Dirac delta function also leads to the same result. The states $|\psi_S\rangle$ refers to the Schrödinger representation, and the two representations $|\psi\rangle$ are related by $|\psi_S\rangle = e^{-\frac{i}{\hbar}\hat{H}t}|\psi\rangle$. Using the resolution of identity $\int|\mathbf{z}\rangle\langle\mathbf{z}|\mathrm{d}^3z = 1$ and the fact that $f(\hat{\mathbf{x}}_S)|\mathbf{z}\rangle = f(\mathbf{z})|\mathbf{z}\rangle$, we note that

$$\delta(\hat{\mathbf{x}}_S-\mathbf{y})|\psi_S\rangle = \int \mathrm{d}^3z\,\delta(\mathbf{z}-\mathbf{y})|\mathbf{z}\rangle\psi_S(\mathbf{z}) = |\mathbf{y}\rangle\psi_S(\mathbf{y}) . \tag{76}$$

Let us take the expectation value of momentum density and express it in typical Schrödinger picture:

$$\langle\psi_S|\left(\hat{\pi}_S\right)_k(\mathbf{x})|\psi_S\rangle = \langle\psi|\hat{\pi}_k(\mathbf{y})|\psi\rangle = \frac{1}{2}\langle\psi|\left(\hat{p}_k\delta(\hat{\mathbf{x}}-\mathbf{y}) + \delta(\hat{\mathbf{x}}-\mathbf{y})\hat{p}_k\right)|\psi\rangle , \tag{77}$$

which gives us

$$\left(\pi_S\right)_k(\mathbf{x}) = -\frac{i\hbar}{2}\left(\left(\psi_S^*\nabla_k\psi_S\right)(\mathbf{x}) - \left(\left(\nabla_k\psi_S^*\right)\psi_S\right)(\mathbf{x})\right) , \tag{78}$$

where $\langle\psi_S|(\hat{\pi}_S)_k(\mathbf{x})|\psi_S\rangle$ is denoted by $(\pi_S)_k(\mathbf{x})$ on the left hand side. If we have no external potential but only a momentum dependent Hamiltonian, which can be written as $A|\hat{\mathbf{p}}|^s$, the time evolution of momentum density operator (73) can be worked out as follows: Since the time derivative of momentum operator vanishes,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{p}} = \frac{i}{\hbar}\left[\hat{H},\hat{\mathbf{p}}\right] = \frac{i}{\hbar}\left[A|\hat{\mathbf{p}}|^s,\hat{\mathbf{p}}\right] = 0\,, \tag{79}$$

we only need to find the time derivative of the delta function. This can be found from the above integral representation of delta function:

$$\frac{\mathrm{d}}{\mathrm{d}t}\delta\left(\hat{\mathbf{x}}-\mathbf{y}\right) = \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, e^{-i\mathbf{k}\cdot\mathbf{y}}\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{i\mathbf{k}\cdot\hat{\mathbf{x}}}\right). \tag{80}$$

Since the derivative of an operator does not in general commute with itself, we need to use the so-called Duhamel's formula, given by

$$\frac{\mathrm{d}}{\mathrm{d}t}e^{\hat{A}(t)} = \int_0^1 \mathrm{d}\xi\; e^{\xi\hat{A}(t)}\dot{\hat{A}}(t)\,e^{(1-\xi)\hat{A}(t)}\,. \tag{81}$$

This formula can be proved from the operator identity $e^{\hat{X}}-e^{\hat{Y}} = \int_0^1 \mathrm{d}\xi\; e^{\xi\hat{Y}}(\hat{X}-\hat{Y})e^{(1-\xi)\hat{X}}$ (define $f(\xi) = e^{\xi\hat{X}}e^{(1-\xi)\hat{B}}$ and differentiate with respect to ξ, and then integrare from 0 to 1. The Duhamel's formula then follows by choosing $\hat{X} = \hat{A}(t+h)$ and $\hat{Y} = \hat{A}(t)$ and then taking the limit of $(e^{\hat{A}(t+h)} - e^{\hat{A}(t)})/h$ as $h\to 0$). Choosing $\hat{A}(t) = i\mathbf{k}\cdot\hat{\mathbf{x}}$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\delta\left(\hat{\mathbf{x}}-\mathbf{y}\right) &= \int \frac{\mathrm{d}^d k}{(2\pi)^d}\, i\mathbf{k}\cdot\left[\int_0^1 \mathrm{d}\xi\; e^{-i\mathbf{k}\cdot\mathbf{y}}e^{i\xi\,\mathbf{k}\cdot\hat{\mathbf{x}}}\left(\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{x}}\right)e^{i(1-\xi)\mathbf{k}\cdot\hat{\mathbf{x}}}\right]\\ &= -sA\frac{\partial}{\partial y_j}\int \frac{\mathrm{d}^d k}{(2\pi)^d}\int_0^1 \mathrm{d}\xi\; e^{-i\mathbf{k}\cdot\mathbf{y}}e^{i\xi\,\mathbf{k}\cdot\hat{\mathbf{x}}}|\hat{\mathbf{p}}|^{s-2}\hat{p}_j\,e^{i(1-\xi)\mathbf{k}\cdot\hat{\mathbf{x}}},\end{aligned} \tag{82}$$

where we have used the Heisenberg's equations of motion for $\hat{\mathbf{x}}$, given by equation (36). Thanks to the Baker-Campbell-Haussdorff formula [20] $e^{t\hat{A}}\hat{B}e^{-t\hat{A}} = B + t[A,B] + \frac{t^2}{2!}[A,[A,B]] + \cdots$, we obtain

$$e^{i\xi\,\mathbf{k}\cdot\hat{\mathbf{x}}}|\hat{\mathbf{p}}|^{s-2}\hat{p}_j e^{-i\xi\,\mathbf{k}\cdot\hat{\mathbf{x}}} = |\hat{\mathbf{p}}|^{s-2}\hat{p}_j + i\xi\;\mathbf{k}\cdot\left[\hat{\mathbf{x}},|\hat{\mathbf{p}}|^{s-2}\hat{p}_j\right] + \frac{(i\xi)^2}{2!}\,\mathbf{k}\cdot\left[\hat{\mathbf{x}},\left[\mathbf{k}\cdot\hat{\mathbf{x}},|\hat{\mathbf{p}}|^{s-2}\hat{p}_j\right]\right] + \cdots. \tag{83}$$

Then, the first term in the time derivative of momentum density operator can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\hat{p}_k\delta\left(\hat{\mathbf{x}}-\mathbf{y}\right)\right) = -sA\,\partial_j\left(|\hat{\mathbf{p}}|^{s-2}\hat{p}_k\hat{p}_j\delta\left(\hat{\mathbf{x}}-\mathbf{y}\right) + \partial_m\int \frac{\mathrm{d}^d k}{(2\pi)^d}\int_0^1 \mathrm{d}\xi\;\hat{p}_k\left[\hat{x}_m,F_j\left(\hat{\mathbf{p}},\xi\right)\right]e^{i\mathbf{k}\cdot(\hat{\mathbf{x}}-\mathbf{y})}\right), \tag{84}$$

where $\partial_j = \frac{\partial}{\partial y_j}$ and F_j is a function of ξ and involves nested commutators of $\hat{x}_j$ and $|\hat{\mathbf{p}}|^{s-2}\hat{p}_j$. Similarly, the other derivative term of the density $\frac{\mathrm{d}}{\mathrm{d}t}\delta(\hat{\mathbf{x}}-\mathbf{y})\hat{p}_k$, can be expressed in a similar manner, which shows that stress tensor up to a total divergence (ambiguity) is given by the expression claimed in the main text. The second term is a total spatial divergence and corresponds to the standard stress–tensor ambiguity of the replacement $T_{ij}\to T_{ij}+\partial_k B_{kij}$, which does not affect the momentum Ward identity or the bulk thermodynamics.

Remark. Equivalently, as is often done in practice, defining the local stress tensor using Weyl (symmetric) ordering from the outset often removes these improvement terms identically.

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