HEISENBERG DOUBLE, SUPERYMMETRIC DUALITIES AND PENTAGON RELATION

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B.S., Physics, Boğaziçi University, 2021

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Physics

Boğaziçi University

2024

ACKNOWLEDGEMENTS

My fascination with the dynamics of the world around us has always been a guiding light in my life. This curiosity ignited during my high school years, leading me to study physics. Pursuing knowledge in this field is far from a solitary endeavor; it thrives on careful guidance, friendship, and a supportive environment that fosters active engagement and thoughtful reflection. In this spirit, I reserve my most profound appreciation for my thesis advisor, Ilmar Gahramanov. His exceptional expertise, steadfast guidance, and insightful critiques have been the beacon that illuminated my path in research. I sincerely thank Can Kozçaz and Hakan Erkol for their intellectual friendship and mentorship. Their contributions have profoundly influenced my academic and personal growth. To my friends— Ege, Elif, Mustafa, Erdal, and Berke—your unwavering support has been a wellspring of strength and happiness. Their mentorship has been an invaluable pillar throughout this journey. A special thanks to Tolga and his mother, Gülperi and Özge, whose kindness and encouragement have provided comfort throughout this journey. I am grateful to my collaborator, H. Kübra, for her insights and partnership. Above all, my deepest gratitude goes to my family. To my mother, whose unwavering faith in me has been the foundation of my success, and to my sister, whose continuous support has been an invaluable source of reassurance.

ABSTRACT

HEISENBERG DOUBLE, SUPERYMMETRIC DUALITIES AND PENTAGON RELATION

In this thesis, we explore the pentagon identity as it appears in Quantum Groups within the Heisenberg double algebra and non-perturbative results derived using the superconformal index technique in three-dimensional supersymmetric gauge theories with four supercharges. The pentagon identity provides insights into the interplay between geometry, topology, and quantum field theory, note that it appears particularly through the 3d-3d correspondence. In the first part, we discuss the operator form of the pentagon identity, which is related to quantum groups. In the second part, we show interesting integral identities that appear in index calculations through mirror symmetry, satisfied quantum dilogarithm functions or variants of it, and their relation to supersymmetric dualities in three dimensions. We examine known pentagon identities on $S^2 \times S, S^3, S^3_b$, S^3/Z_r, and S^3_b/Z_r, and present a new pentagon identity on the orbifold S^3_b/Z_r.

ÖZET

HEISENBERG DOUBLE, SÜPERSİMETRİK DUALİTELER VE PENTAGON DENKLEMİ

Bu tezde, Heisenberg çift cebiri içindeki Kuantum Gruplarında ve dört süper yük içeren üç boyutlu süpersimetrik ölçü teorilerinde süperkonformal indeks tekniği kullanılarak elde edilen pertürbatif olmayan sonuçlarda ortaya çıkan pentagon ilişkisini inceliyoruz. Pentagon ilişkisi, geometri, topoloji ve kuantum alan teorisi arasındaki etkileşimlere dair içgörüler sağlar, Öyle görülüyor ki özellikle 3d-3d karşılıklığı aracılığıyla ortaya çıkıyor. Ilk bölümde, kuantum gruplarla ilgili olan Pentagon ilişkisinin operatör formunu tartışıyoruz. İkinci bölümde, indeks hesaplamalarında ayna simetrisi aracılığıyla ortaya çıkan, kuantum dilogaritma fonksiyonları veya onun varyantları tarafından sağlaan ilginç integral kimliklerini ve bunların üç boyutlu süpersimetrik dualitelerle ilişkisini inceliyoruz. $S^2 \times S, S^3, S^3_b, S^3/Z_r$ ve S^3_b/Z_r gibi geometrilerde bilinen Pentagon ilişkisini inceliyoruz ve orbifold S^3_b/Z_r üzerinde yeni bir integral pentagon ilişkisi sunuyoruz.

TABLE OF CONTENTS

ACKNOWLEDGEMENTS iii
ABSTRACT iv
ÖZET v
LIST OF FIGURES viii
LIST OF TABLES ix
LIST OF SYMBOLS x
LIST OF ACRONYMS/ABBREVIATIONS xi
1. INTRODUCTION 1
2. THE REPRESENTATION OF QUANTUM GROUPS 4
2.1. Introduction 4
2.2. Hopf Algebra 5
2.3. Universal Enveloping Lie Algebra 9
2.4. q-Deformed Universal Enveloping Lie Algebra 10
3. OPERATOR PENTAGON IDENTITY 12
3.1. Heisenberg Double 12
3.2. Heisenberg Double Arising From The Borel Half $U_q(sl(2,\mathbb{R}))$ 14
4. INTEGRAL PENTAGON IDENTITY 18
4.1. 3d $\mathcal{N}=2$ Supersymmetric Algebra 18
4.2. Supersymmetric Partition Function In $3d$ $\mathcal{N}=2$ Theories 19
4.2.1. Witten Index 19
4.2.2. Superconformal Index 20
4.2.3. Lens Index 23
4.3. Integral Identities $3d$ $\mathcal{N}=2$ Theories 25
4.3.1. Pentagon Arising From Dimensional Reduction 25
4.3.2. Pentagon Arising From Mirror Symmetry 27
4.3.3. Integral Pentagon Identity With $\mathcal{N}=2$ On S^3_b 30
4.3.4. Integral Pentagon Identity with $\mathcal{N}=2$ on $S^2\times S^1$ 31
4.3.5. Flavour Group Symmetry Reduction 32

4.3.6. Integral Pentagon Identity With $\mathcal{N} = 2$ On $S_b^3/\mathbb{Z}_r$ 33
5. CONCLUSION 37
REFERENCES 39
APPENDIX A: LIE ALGEBRA 45
APPENDIX B: HYPERGEOMETRIC FUNCTIONS AND THEIR IDENTITIES 49
B.1. Elliptic Gamma 49
B.2. Barnes Double Gamma 49
B.3. Hyperbolic Hypergoemetric Gamma 50
B.4. Double Sine Function 51
APPENDIX C: USEFUL RELATIONS 53
APPENDIX D: PROOF OF INTEGRAL PENTAGON IDENTITY 54

LIST OF FIGURES

Figure 4.1. Star Triangle relation. 26

LIST OF TABLES

Table 4.1. SQED charges. 29

Table 4.2. Charges in the free Wess-Zumino theory. 29

Table 5.1. Summary of vector multiplets on various manifolds 37

Table 5.2. Characteristics of manifolds and their corresponding chiral multiplets 38

LIST OF SYMBOLS

A	Description of Hopf algebra
A^*	Description of dual Hopf algebra
$\mathcal{A}$	Description of algebra
$\mathcal{A}^*$	Description of dual algebra
$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$	Description of direct sum
id	Identity element
S	Canonical element of the mentioned space
U_q	q-deformed algebra
α	Rapidity parameters
β	Rapidity parameters
$\gamma^{(2)}$	Hyperbolic gamma function
Δa_{ij}	Description of coproduct of element a_{ij}
ω	Quasi-periodic parameters of hypergeometric functions

LIST OF ACRONYMS/ABBREVIATIONS

2D	Two Dimensional
3D	Three Dimensional
LI	Lens Index
SI	Superconformal Index
UELA	Universal Enveloping Lie Algebra

1. INTRODUCTION

The pentagon identity is a fascinating area of research that provides deep insights into the interplay between geometry and topology. This thesis aims to construct the Pentagon identity from gauge theory and algebraic point of view and emphasis the various topics related to the Pentagon identity, including its theoretical foundations, mathematical frameworks, and applications.

The historical background of the Pentagon identity traces back to algebraic structures associated with associativity. In the 1960s, S. Mac Lane introduced the Pentagon condition in the monoidal category theory, ensuring the tensor product's associativity up to natural isomorphism.

The algebraic structure was established after the discovery of quantum groups. Quantum groups, introduced independently by Drinfeld and Jimbo in the 1980s, are deformations of classical Lie groups and algebras. These structures have deep connections to integrable systems, knot theory, and representation theory. While quantum groups do not inherently require an inverse map, they include a unique map called the antipodal map, which offers a rich framework for describing symmetries and transformations in various contexts. The Pentagon operator is a canonical element of the tensor product of Heisenberg double algebras, constructed via quantum groups.

For instance, after introducing Topological Quantum Field Theory (TQFT), Reshetikhin and Turaev generalized the Jones polynomial to a family of three-dimensional manifolds known as RT invariants. These invariants have been used to distinguish between three-dimensional manifolds and can be expressed using quantum groups.

The integral form of the pentagon identity emerges in the study of superconformal and lens index computation. These tools are designed to check the consistency of dualities, providing a method to verify that two dual theories yield the same physical

predictions encoded in their partition functions. Specifically, these integral identities are used to confirm Seiberg-like dualities in three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories. The equality of partition functions, calculated differently, provides strong evidence for the proposed dualities.

In the computation of partition functions in 3D, the process often begins with calculating the four-dimensional index before reducing the theory to a 3D partition function. It has been observed that in 4D, the index is described by elliptic hypergeometric functions. Upon reduction to 3D, these functions transform into quantum dilogarithm functions, reflecting the underlying dimensional reduction and the corresponding change in the mathematical structure.

Certain theories have the distinct characteristic of being expressible via the star-triangle relation in four dimensions. These theories can be described using the pentagon identity upon reduction to three dimensions. Unlike their four-dimensional counterparts, three-dimensional gauge theories do not have chiral anomalies. This lack of chiral anomalies permits the consideration of theories in non-trivial background gauge fields coupled with global symmetries. These couplings introduce new discrete parameters for global symmetries in the superconformal index expression. This modified index is known as the generalized superconformal index. Utilizing the aforementioned techniques on the generalized superconformal index can yield more comprehensive integral pentagon identities.

The pentagon identity also arises in tetrahedron index [1,2] When computing the partition function of a 3D $\mathcal{N} = 2$ supersymmetric theory on a triangulated manifold, each tetrahedron contributes a specific factor to the overall partition function. The 2-3 Pachner move, which replaces two tetrahedra sharing a common face with three tetrahedra that fill the same volume, alters the local triangulation. However, the overall partition function remains unchanged due to the specific form of these contributions, designed to cancel out appropriately.

Consider M as a notable example of a nontrivial three-manifold, specifically a bipyramid that can be divided into two or three tetrahedra. These partitions result in two distinct UV descriptions of the theory T_M. When two tetrahedra are joined with suitable polarization, the resulting theory is $N_f = 1$ SQED. Alternatively, joining three tetrahedra yields the XYZ model, which features three chiral multiplets interconnected through a cubic superpotential. The mirror symmetry observed in these theories demonstrates that constructing T_M for any three-manifold is not dependent on the specific triangulation used.

Thus, a significant relationship exists between 3D$\mathcal{N} = 2$ supersymmetric gauge theories and three-manifolds. This relationship is called 'class $\mathcal{R}$ theories. The pentagon identity encapsulates information about the geometry of the associated three-manifolds.

2. THE REPRESENTATION OF QUANTUM GROUPS

Groups describe symmetries through transformations. Quantum groups extend this concept by incorporating transformations with a 'linearized inverse' known as the antipode. Despite this more relaxed structure, quantum groups preserve essential properties for describing symmetries. For instance, they play a crucial role in various structures, such as quantum inverse scattering, exactly solvable lattice models, and low-dimensional topology governed by quantum groups.

2.1. Introduction

Quantum groups were founded in the 1970s by the Leningrad group led by L. Faddeev. The quantum group $U_q(sl_2)$ firstly was discovered by Piotr P. Kulish and N. Yu. Rashetikhin [3]. Higher spin quantum groups extend quantum integrable models such as the sine-Gordon model and the Heisenberg XXZ spin chain, ultimately revealing $U_q(sl_2)$. In the early 1980s, Vladimir Sklyanin introduced the Hopf algebra structure to $U_q(sl_2)$ [4]. Michio Jimbo and Vladimir Drinfeld then generalized this to arbitrary Kac-Moody algebras, establishing the theory of quantum groups [5] [6]. Drinfeld introduced the term "quantum groups" at the International Mathematicians Congress held in Berkeley in 1986 [7]. Quantum groups are a class of Hopf algebras characterized by nontrivial deformations of either the enveloping algebras of semisimple Lie algebras or the algebra of regular functions on related algebraic groups. They are deeply connected with various branches of mathematics and physics. The Hopf algebra structure is pivotal, allowing for the definition of algebraic elements such as comultiplication and the antipode, which relate to the deformation quantization of algebraic structures.

2.2. Hopf Algebra

A Hopf algebra is characterized as an unital and counital algebra that incorporates an antipodal map. An algebra is defined as a unital associative algebra, represented as $(\mathcal{A}, m, \eta)$, where $\mathcal{A}$ denotes a vector space over a field k. The multiplication is given by the map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, and the unit map is denoted by $\eta : k \to \mathcal{A}$. This structure satisfies

$$\begin{aligned} m(m \otimes \mathrm{id}) &= m(\mathrm{id} \otimes m), \\ m(\mathrm{id} \otimes \eta) &= m(\eta \otimes \mathrm{id}). \end{aligned} \tag{2.1}$$

The other component, a counital coassociative coalgebra, is defined as $(\mathcal{A}, \Delta, \epsilon)$. The comultiplication map is $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, and the counit map is $\epsilon : \mathcal{A} \to k$. This structure must satisfy the following axioms

$$\begin{aligned} (\mathrm{id} \otimes \Delta) \circ \Delta &= (\Delta \otimes \mathrm{id}) \circ \Delta, \\ (\mathrm{id} \otimes \epsilon) \circ \Delta &= (\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}. \end{aligned} \tag{2.2}$$

A Hopf algebra integrates the structures of unital associative and counital associative coalgebras with $(\mathcal{A}, m, \eta, \Delta, \epsilon, S)$. The antipode S, a pivotal linear map within this framework, satisfies the relation $S : \mathcal{A} \to \mathcal{A}$ and the equations $m(S \otimes \mathrm{id}) \circ \Delta = m(\mathrm{id} \otimes S) \circ \Delta = \eta \circ \epsilon$. The antipode plays a crucial role in the Hopf algebra structure, satisfying the specific properties.

(i) S acts as an involution, ensuring that $S(S(a)) = a$ for every $a \in \mathcal{A}$.

(ii) As an algebraic homomorphism, S satisfies $S(a)S(b) = S(ab)$ for all elements $a, b \in \mathcal{A}$.

(iii) S functions as a coalgebra homomorphism, indicated by $S(\Delta(a)) = \Delta(S(a))$ for each $a \in \mathcal{A}$.

(iv) S preserves the counit, maintaining $S(1) = 1$, where 1 represents the unit element in $\mathcal{A}$.

Dual Hopf algebra is defined as follows: if there exists Hopf algebra A, then $(\mathcal{A}^*, m^*, \eta^*, \Delta^*, \epsilon^*, S^*)$ is its dual A^*. The original Hopf algebra A uses its multiplication to combine algebra elements. In contrast, A^* combines linear functionals on A through

its unique multiplication. We utilize the standard algebraic bracket $\langle\rangle$: A $\otimes\mathcal{A}^* \to k$. In this framework, we identify the operator of algebra and its dual as

$$\Delta = m^* \quad m \equiv \Delta^* \quad \epsilon \equiv= \eta^* \quad \eta \equiv \epsilon^*. \tag{2.3}$$

Duality bracket satisfies

$$\begin{aligned} &\langle m(1 \otimes m)(a \otimes b \otimes c), d\rangle = \langle m(m \otimes id)(a \otimes b \otimes c), d\rangle, \\ &\langle (1 \otimes m)(a \otimes b \otimes c), m^*(d)\rangle = \langle (m \otimes id)(a \otimes b \otimes c), m^*(d)\rangle, \\ &\langle a \otimes b \otimes c, (1 \otimes m^*)\, m^*(d)\rangle = \langle a \otimes b \otimes c, (m^* \otimes id)\, m^*(d)\rangle. \end{aligned} \tag{2.4}$$

For $a, b, c \in \mathcal{A}$ and $d \in \mathcal{A}^*$, similar correspondences hold for Δ. The unital property of η is mirrored by the counital property of η^*, and the counital property of ϵ is mirrored by the unital property of ϵ^*.

For example, consider a free-associative Hopf algebra. Let A be a finite set of letters from an alphabet, $A = \{1, a_1, a_2, a_3, \ldots, a_n\}$, and let A^* be the set of all possible words formed by these letters, $A^* = \{1, a_1a_1, a_1a_2, \ldots\}$. The comultiplication map is defined as

$$\Delta a_n = \sum_{i=0}^{n} a_i \otimes a_{n-i}, \tag{2.5}$$

a_0 is unit ea element, which is 1

$$\begin{aligned} \Delta a_2 &= a_0 \otimes a_2 + a_1 \otimes a_1 + a_2 \otimes a_0, \\ \Delta a_3 &= a_0 \otimes a_3 + a_1 \otimes a_2 + a_2 \otimes a_1 + a_3 \otimes a_0. \end{aligned} \tag{2.6}$$

We have the property that

$$\Delta(a_i a_j) = \Delta a_i \Delta a_j. \tag{2.7}$$

It is easy to calculate the combination of the words. It is important to note that if algebra A would have one variable, then it would be one variable polynomial algebra with one variable. Take an element $a \in A$ then the count satisfies the following

$$\begin{aligned} &(id \otimes \epsilon)\Delta a = a \otimes 1, \quad (\epsilon \otimes 1)\Delta a = 1 \otimes a, \\ &\epsilon(a) = \begin{cases} 1, & \text{if } a = 1 \\ 0, & \text{if } a \neq 1. \end{cases} \end{aligned} \tag{2.8}$$

Before finding the antipode relation, we need to investigate the algebraic structure of the free associative algebra. Free associative algebra is a graded bialgebra, and thus, its underlying vector space can be decomposed into a direct sum of subspaces

$$\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n, \tag{2.9}$$

where $\mathcal{H}_n$ is the subspace of $\mathcal{H}$ consisting of all elements of degree n. In other words, every element $h \in \mathcal{H}$ can be uniquely written as a finite sum of elements of the form $h = \sum_{n\geq 0} h_n$, where $h_n \in \mathcal{H}_n$.

- $\mathcal{H}$ is a graded algebra, i.e., $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ and $\mathcal{H}_m\mathcal{H}_n \subseteq \mathcal{H}_{m+n}$ for all non-negative integers m and n.
- $\mathcal{H}$ is a graded coalgebra, $\Delta(\mathcal{H}_n) \subseteq \bigoplus_{m=0}^{n} \mathcal{H}_m \otimes \mathcal{H}_{n-m}$ and $\epsilon(\mathcal{H}_n) = 0$ for all $n \geq 1$.
- The antipode map preserves the grading, $S(\mathcal{H}_n) \subseteq \mathcal{H}_n$ for all $n \geq 0$.

A graded bialgebra can have an antipode defined recursively. Lets take $x \in \mathcal{H}_n \quad n \geq 1$, we can decompose Δx as

$$\Delta x = x \otimes 1 + \sum_{i=1}^{n} y_i \otimes z_i. \tag{2.10}$$

The counit relations are given by

$$(id \otimes \epsilon)\Delta x = x \otimes \epsilon(1) + \sum_{i=1}^{m} y_i \otimes \epsilon(z_i) = x \otimes 1. \tag{2.11}$$

Thus, we have $\epsilon(1) = 1, \quad \epsilon(z) = 0, \quad z_i \in \mathcal{H}_n,$

$$\begin{aligned} m(S \otimes id)\Delta(x) &= \eta\epsilon(x), \\ m(S \otimes id)\left(x \otimes 1 + \sum_{i=1}^{n} y_i \otimes z_i\right) &= 0. \end{aligned} \tag{2.12}$$

Thus we have

$$S(x) = -\sum_{i=1}^{n} S\left(y_i\right) z_i, \tag{2.13}$$

furthermore

$$\Delta(x) = x \otimes 1 + \sum_{i=1}^{m} y_i \otimes z_i, \tag{2.14}$$

we obtain

$$\begin{aligned}
\Delta a_1 &= a_1 \otimes 1 + 1 \otimes a_1, \\
\Delta a_2 &= a_2 \otimes 1 + a_1 \otimes a_1 + 1 \otimes a_2, \\
\Delta a_3 &= a_3 \otimes 1 + a_2 \otimes a_1 + a_1 \otimes a_2 + 1 \otimes a_3, \\
S(a_2) &= a_1 a_1 - a_2, \\
S(a_1) &= -a_1, \\
S(a_3) &= -a_1 a_1 a_1 + a_2 a_1 + a_1 a_2 - a_3.
\end{aligned} \tag{2.15}$$

Another example is the Universal Enveloping Hopf Algebra of a Lie algebra $\mathfrak{g}$, which captures the full algebraic structure of $\mathfrak{g}$.

- Comultiplication Δ: For any generator x in $\mathfrak{g}$, with 1 being the identity element of $U(\mathfrak{g})$, the comultiplication is defined as
$$\Delta(x) = x \otimes 1 + 1 \otimes x, \tag{2.16}$$
extending linearly to the entirety of $U(\mathfrak{g})$. This map signifies the distributive property of x over tensor products, reflecting the algebra's inherent structure.
- Counit ϵ: The counit map extracts scalar values from $U(\mathfrak{g})$, nullifying Lie algebra elements while preserving the identity
$$\epsilon(x) = 0 \quad \forall x \in \mathfrak{g}, \quad \epsilon(1) = 1. \tag{2.17}$$
- Antipode S: Serving as an algebraic inverse, the antipode for any $x \in \mathfrak{g}$ is given by $S(x) = -x$, ensuring that $m(S \otimes \mathrm{id})\Delta(x) = \eta\epsilon(x)$, which simplifies to $S(x) + x = 0$.

Extending linearly to the entirety of $U(\mathfrak{g})$. This map signifies the distributive property of x over tensor products, reflecting the algebra's inherent structure.

The algebra $U(\mathfrak{g})$ is infinite-dimensional, possessing an unbounded basis. This aspect underscores the algebra's comprehensive nature, which can encapsulate complex Lie algebraic structures.

In the above generalization, let's consider $\mathfrak{g}$ as $\mathfrak{sl}(2,\mathbb{C})$. The commutator relation reads

$$[x,y]=2y,\quad [x,z]=-2z,\quad [y,z]=x, \tag{2.18}$$

consisting of elements x, y and z. The coproduct map, which defines how elements split into tensor products

$$\Delta(x)=x\otimes 1+1\otimes x,\quad \Delta(y)=y\otimes 1+1\otimes y,\quad \Delta(z)=z\otimes 1+1\otimes z. \tag{2.19}$$

The counit map acts as a sort of "trace" on the algebra and is defined by

$$\epsilon(x)=0,\quad \epsilon(y)=0,\quad \epsilon(z)=0. \tag{2.20}$$

The antipode map, which acts as a sort of "inverse" on the algebra, is given by

$$S(x)=-x,\quad S(y)=-y,\quad S(z)=-z. \tag{2.21}$$

2.3. Universal Enveloping Lie Algebra

The enveloping Lie algebra $U(\mathfrak{g})$ extends and preserves the structure of the original Lie algebra, thus generalizing it. Consider a basis $\{x_i\}_{i=1}^n$ for $\mathfrak{g}$. The algebra $\mathcal{U}(\mathfrak{g})$ is infinite-dimensional, and a basis for $\mathcal{U}(\mathfrak{g})$ consists of the set $\left\{\prod_{i=1}^k \rho\left(x_i\right)^{a_i}\right\}_{k=1}^n$, where $a_i\in\mathbb{N}$.

Take $U(sl(2))$ as an example. This algebra is generated by the elements $\rho(X)$, $\rho(Y)$, and $\rho(H)$. The basis elements X, Y, and H of $\mathfrak{sl}(2)$ correspondences to $sl(2)$

$$X=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad Y=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{2.22}$$

The commutators

$$[X,Y]=H,\quad [H,X]=2X,\quad [H,Y]=-2Y. \tag{2.23}$$

As a vector space, $\mathcal{U}(\mathfrak{sl}(2))$ can be described using the monomials of these generators. The basis vectors of this space are given by

$$\rho(X)^k\rho(H)^l\rho(Y)^m, \tag{2.24}$$

where k, l, and m are non-negative integers. This indicates that any element of $\mathcal{U}(\mathfrak{sl}(2))$ can be expressed as a linear combination of these monomials.

$$\begin{aligned}\rho(Y)\rho(X) &= \rho(X)\rho(Y) - \rho(H),\\ \rho(H)\rho(X) &= \rho(X)\rho(H) + 2\rho(X),\\ \rho(Y)\rho(H) &= \rho(H)) + 2\rho(Y),\end{aligned} \tag{2.25}$$

thus

$$\begin{aligned}X^a H^b &= (H-2a)^b X^a,\\ Y^a H^b &= (H+2a)^b Y^a,\\ [X^a, Y] &= aX^{p-1}(H+a-1) = a(H-a+1)X^{a-1}.\\ [X, Y^a] &= aY^{p-1}(H-a+1) = a(H+a-1)Y^{a-1},\end{aligned} \tag{2.26}$$

2.4. q-Deformed Universal Enveloping Lie Algebra

It is a deformation of $U(\mathfrak{g})$, denoted as $U_q(\mathfrak{g})$, where q is the deformation parameter. This emergent algebra includes additional generators x_i, y_i, K_i, and K_i^{-1}. The defining relations are as follows

$$\begin{aligned}x_i y_j - y_j x_i &= \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}}, \quad q_i = q^d, \quad [k]_{q^d} = \frac{q^{dk} - q^{-dk}}{q^d - q^{-d}},\\ K_i K_j - K_j K_i &= 0, \quad K_i x_j = q^{\frac{A_{ij}}{2}} x_j K_i, \quad K_i y_j = q^{-\frac{A_{ij}}{2}} y_j K_i,\\ &\sum_{k=0}^{1-A_{ij}} (-1)^k \begin{pmatrix}1-A_{ij}\\ k\end{pmatrix}_q x_i^{1-A_{ij}-k} x_j x_i^k = 0,\\ &\sum_{k=0}^{1-A_{ij}} (-1)^k \begin{pmatrix}1-A_{ij}\\ k\end{pmatrix}_q y_i^{1-A_{ij}-k} y_j y_i^k = 0,\end{aligned} \tag{2.27}$$

where $\binom{n}{j} = \frac{[n]!_q}{[j]_q![n-j]_q!}$ and we get $d \in \mathbb{N} \setminus \{0\}$. The Hopf algebra structure of $U_q(\mathfrak{g})$ reads

$$\begin{aligned}\Delta K_i &= K_i \otimes K_i,\\ \Delta x_i &= K_i^{-1} \otimes x_i + x_i \otimes K_i,\\ \Delta y_i &= K_i^{-1} \otimes y_i + y_i \otimes K_i\\ \epsilon(K_i) &= 1, \quad \epsilon(x_i) = \epsilon(y_i) = 0,\\ S(K_i) &= K_i^{-1}, \quad S(x_i) = -qx_i, \quad S(y_i) = -qy_i.\end{aligned} \tag{2.28}$$

These relations define the structure of the q-deformed $U(\mathfrak{g})$ Hopf algebra. To demonstrate the non-commutativity

$$\begin{aligned}\Delta(K_i)\Delta(x_j) &= (K_i \otimes K_i)(x_j \otimes K_j + K_j^{-1} \otimes x_j),\\ &= K_i x_j \otimes K_i K_j + K_i K_j^{-1} \otimes K_i x_j,\\ &= q^{\frac{A_{ij}}{2}}(x_j K_i \otimes K_j K_i + K_j^{-1} K_i \otimes x_j K_i),\\ &= q^{\frac{A_{ij}}{2}}\Delta(x_j)\Delta(K_i).\end{aligned} \tag{2.29}$$

The Lie bracket yields

$$\begin{aligned}[\Delta(x_i), \Delta(y_j)] &= -y_j x_i \otimes K_j K_i + y_j K_i^{-1} \otimes K_j x_i +\\ & K_j^{-1} x_i \otimes y_j K_i + K_j^{-1} K_i^{-1} \otimes y_j x_i\\ &= [x_i, y_j] \otimes K_i K_j + K_i^{-1} K_j^{-1} \otimes [x_i, y_i]\\ &= \frac{\delta_{ij}}{q - q^{-1}}\left(K_i^2 \otimes K_i^2 - K_i^{-2} \otimes K_i^{-2}\right)\\ &= \delta_{ij}\frac{\Delta(K_i)^2 - \Delta(K_i)^{-2}}{q - q^{-1}}.\end{aligned} \tag{2.30}$$

These derivations show that the emergent structure is consistent.

Let's consider $U_q(\mathfrak{sl}(2))$ as an example, with generators K, x, K^{-1}, y

$$\begin{aligned}KK^{-1} &= K^{-1}K = 1,\\ \Delta x &= x \otimes K + K^{-1} \otimes x, \quad \Delta y = y \otimes K + K^{-1} \otimes y,\\ \epsilon(x) &= \epsilon(y) = 0, \quad \epsilon(K) = 1,\\ S(K) &= K^{-1}, \quad S(x) = -qx, \quad S(y) = -qy.\end{aligned} \tag{2.31}$$

The commutation relations are

$$[x, y] = \frac{K^2 - K^{-2}}{q - q^{-1}}, \quad Kx = qxK, \quad K^{-1}x = q^{-1}xK^{-1}, \quad Ky = q^{-1}yK. \tag{2.32}$$

Taking the limit $q \longrightarrow 1$ will end up $U(sl_2)$.

3. OPERATOR PENTAGON IDENTITY

3.1. Heisenberg Double

The Heisenberg double algebra arises from pairing a Hopf algebra with its dual. It represents a deformation of the classical universal enveloping algebra of a Lie algebra and is used to describe the algebraic structure of certain quantum systems. Although it does not follow the Hopf algebra structure, it stands as an algebra in its own right.

The Heisenberg double appears naturally within the context of Teichmüller space. Kashaev's research emphasized the canonical element of the Heisenberg double, using Faddeev's quantum dilogarithm, highlighting its importance in the quantization of Teichmüller's theory.

The Heisenberg double is defined as the tensor product of a Hopf algebra $\mathcal{A}$ and its dual $\mathcal{A}$ on the vector space $H(\mathcal{A}) \cong \mathcal{A} \otimes \mathcal{A}$ with a duality bracket $(,) : \mathcal{A} \times \mathcal{A}^* \to \mathbb{C}$. This duality bracket satisfies certain properties

$$\begin{aligned} &\left(E_\alpha, E^\beta\right) = \delta_\alpha^\beta, \\ &\left(E_\alpha, E^\rho E^\sigma\right) = \left(\Delta\left(E_\alpha\right), E^\rho \otimes E^\sigma\right), \\ &\left(E_\alpha E_\beta, E^\rho\right) = \left(e_\alpha \otimes E_\beta, \Delta\left(E^\rho\right)\right). \end{aligned} \tag{3.1}$$

We have the following multiplication operations

$$\begin{aligned} &\left(E_\alpha \otimes 1\right)\left(E_\beta \otimes 1\right) = m_{\alpha\beta}^\gamma \left(E_\gamma \otimes 1\right), \\ &\left(1 \otimes E^\alpha\right)\left(1 \otimes E^\beta\right) = \mu_\gamma^{\alpha\beta} \left(1 \otimes E^\gamma\right), \\ &\left(E_\alpha \otimes 1\right)\left(1 \otimes E^\beta\right) = m_{\rho\gamma}^\beta \mu_\alpha^{\gamma\sigma} \left(1 \otimes E^\rho\right)\left(E_\sigma \otimes 1\right). \end{aligned} \tag{3.2}$$

The canonical element is particular to the tensor product of two instances of the Heisenberg double $H(\mathcal{A})$. The canonical element S, a key component of the pentagon relation, is characterized by $S \in \mathcal{H}(\mathcal{A}) \otimes \mathcal{H}(\mathcal{A})$. This element satisfies the pentagon relation, which is defined as follows

$$S = E_\alpha \otimes 1 \otimes 1 \otimes E^\alpha, \tag{3.3}$$

where it satisfies a fundamental identity called the pentagon equation given by

$$S_{12}S_{13}S_{23} = S_{23}S_{12}. \tag{3.4}$$

The basis is selected such that $E_\beta \otimes 1$ is denoted as E_β and $1 \otimes E^\gamma$ is denoted as E^γ. The proof as follows

$$\begin{aligned} S_{12}S_{13}S_{23} &= (E_\alpha \otimes E^\alpha \otimes 1)(E_\beta \otimes 1 \otimes E\beta)(1 \otimes E_\gamma \otimes E^\gamma) = E_\alpha E_\beta \otimes E^\alpha E_\gamma \otimes E^\beta E^\gamma, \\ &= m^\rho_{\alpha\beta} E_\rho \otimes E^\alpha E_\gamma \otimes \mu^{\beta\gamma}_\sigma E^\sigma = E_\rho \otimes m^\rho_{\alpha\beta}\mu^{\beta\gamma}_\sigma E^\alpha E_\gamma \otimes E^\sigma = E_\rho \otimes E_\sigma E^\rho \otimes E^\sigma, \\ &= (1 \otimes E_\rho \otimes E^\rho)(E_\sigma \otimes E^\sigma \otimes 1) = S_{23}S_{12}. \end{aligned} \tag{3.5}$$

Let's consider the following structure as an example

$$E_l = \frac{x_l}{l!}, \qquad E_l E_m = \binom{l+m}{l} E_{l+m}, \qquad \Delta(E_l) = \sum_{s=0}^{l} (E_{l-s} \otimes E_s). \tag{3.6}$$

The dual of the Hopf algebra has the following properties

$$E^l = y^l \quad E^l E^m = E^{l+m}, \quad \Delta\left(E^l\right) = \sum_{s=0}^{l} \binom{l}{s} \left(E^{l-s} \otimes E^s\right). \tag{3.7}$$

The Heisenberg double

$$\begin{aligned} (E_n \otimes 1)(E_m \otimes 1) &= \binom{n+m}{n} (E_{n+m} \otimes 1), \\ (1 \otimes E^n)(1 \otimes E^m) &= \left(1 \otimes E^{n+m}\right), \\ (E_n \otimes 1)(1 \otimes E^m) &= \sum_{s=0}^{n} \binom{m}{n-s} \left(1 \otimes E^{m-n+s}\right)(E_s \otimes 1), \end{aligned} \tag{3.8}$$

the canonical element S is

$$S = \sum_{k=0}^{\alpha} \frac{X_k}{k!} \otimes Y^k = \exp(X \otimes Y), \tag{3.9}$$

where X=x $\otimes$1 and $Y = 1 \otimes y$ and by construction $[x, y] = 1$.

Let's consider another example that is a special case of Volkov's Pentagon relation, the algebra with having generator G and H and commutation relation

$$[H, G] = G. \tag{3.10}$$

The comultiplication map reads

$$\begin{aligned} \Delta(G) &= G \otimes \exp(hH) + 1 \otimes G, \\ \Delta(H) &= H \otimes 1 + 1 \otimes H. \end{aligned} \tag{3.11}$$

The basis of the algebra $\mathcal{A}$ is defined as

$$E_{m,n} = \frac{H^m G^n}{m! q_n}. \tag{3.12}$$

The dual algebra $\mathcal{A}^*$ is generated by two elements, $\tilde{H}$ and F

$$\tilde{H}F - F\tilde{H} = -hF. \tag{3.13}$$

The comultiplication map reads

$$\Delta(\tilde{H}) = \tilde{H} \otimes 1 + 1 \otimes \tilde{H}, \quad \Delta(F) = F \otimes \exp(-\tilde{H}) + 1 \otimes F. \tag{3.14}$$

The basis is defined as

$$E^{m,n} = \tilde{H}^m F^n. \tag{3.15}$$

The permutation relations within the Heisenberg double $\mathcal{H}(\mathcal{A})$ can be expressed as follows

$$[H, \tilde{H}] = 1, \quad [G, \tilde{H}] = 0, \quad [G, F] = (1-q)q^{-H}, \quad [H, F] = -F. \tag{3.16}$$

The fundamental element can be expressed in a particular structure

$$S = \sum_{m,n=0}^{\infty} E_{m,n} \otimes E^{m,n} = \exp(H \otimes H)(G \otimes F; q)_{\infty}^{-1}. \tag{3.17}$$

We need to have a final ingredient to construct the Pentagon identity; thus, we define

$$U = 1 \otimes G \otimes F, \qquad V = G \otimes F \otimes 1. \tag{3.18}$$

The five terms relation reads

$$(U; q)_{\infty}([U, V]/(1-q); q)_{\infty}(V; q)_{\infty} = (V; q)_{\infty}(U; q)_{\infty}. \tag{3.19}$$

Let us define the Casimir element

$$C = UV - qVU, \qquad [U, C] = 0, \qquad [V, C] = 0, \tag{3.20}$$

where $C = 0$, the algebraic relation $UV = qVU$ is called Rogers' pentagon relation [8]. This relation is fundamental in quantum algebra and has implications for studying quantum groups and integrable systems. A further generalization of this system is presented in [9] where the relation is extended to $UV = q^2VU$. This generalization is known as Volkov's pentagon relation.

3.2. Heisenberg Double Arising From The Borel Half $U_q(sl(2, \mathbb{R}))$

The exploration of the canonical element of the Heisenberg double of the Borel half of $U_q(\mathfrak{sl}(2))$ and its self-dual representations was initially carried out in [10]. It was revealed that the Teichmüller theory of Riemann surfaces can be effectively described utilizing the canonical element of the Heisenberg double of the Borel half of $U_q(\mathfrak{sl}(2))$.

Kashaev demonstrated that the canonical element of the Heisenberg double, which adheres to the pentagon equation, can be interpreted as a quantized flip operator that connects various triangulations of a specific quadrilateral.

Consider the framework, the Borel half of $\mathcal{U}_q(\mathfrak{sl}(2))$ with generators X and Y, and its dual $\mathcal{U}_q(\mathfrak{sl}(2))^*$ with generators $\hat{X}$ and $\hat{Y}$, which satisfy the commutation relations

$$\begin{aligned}
[X,\hat{X}] &= \frac{1}{2\pi i},\\
[X,Y] &= -ibY,\\
[X,\hat{Y}] &= ib\hat{Y},\\
[\hat{X},Y] &= 0,\\
[\hat{X},\hat{Y}] &= +ib\hat{Y},\\
[Y,\hat{Y}] &= q\left(1-q^{-2}\right)e^{2\pi bX},
\end{aligned} \tag{3.21}$$

with coproduct

$$\begin{aligned}
\Delta(X) &= 1\otimes X + X\otimes 1,\\
\Delta(\hat{X}) &= 1\otimes \hat{X} + \hat{X}\otimes 1,\\
\Delta(Y) &= Y\otimes e^{2\pi bX} + 1\otimes Y,\\
\Delta(\hat{Y}) &= \hat{Y}\otimes e^{-2\pi b\hat{X}} + 1\otimes \hat{Y}.
\end{aligned} \tag{3.22}$$

Before proceeding further, to avoid confusion, it is important to clarify that the functions $G_b, g_b, \phi_b, S_b, s_b, e_b$ and $\gamma^{(2)}$ are all interrelated through shifts, inverse or multiplications by elementary functions. They are described by Barnes's Double gamma functions (described in Appendix). The structure of the Heisenberg double can be described as being generated by the set $e_{(s,t)}\otimes\hat{e}_{(s',t')}$, where $s,s',t,t'\in\mathbb{R}$, and this set is closed under the operations of the Heisenberg double

$$\begin{aligned}
E_{(s,t)} &= \frac{1}{2\pi}\Gamma(-is)G_{\mathrm{b}}(-it)q^{-\left(ib^{-1}t\right)^2}(2\pi iX)^{is}Y^{ib^{-1}t},\\
\hat{E}_{(s,t)} &= \hat{X}^{is}\hat{Y}^{ib^{-1}t}.
\end{aligned} \tag{3.23}$$

The canonical element emerges as

$$S = \exp(2\pi iX\otimes\hat{X})g_{\mathrm{b}}^{-1}(Y\otimes\hat{Y}). \tag{3.24}$$

The function g_{b} in the basis and canonical element representation satisfies the pentagon relation, which is a fundamental coherence law in monoidal categories

$$g_{\mathrm{b}}(U)g_{\mathrm{b}}(V) = g_{\mathrm{b}}(V)g_{\mathrm{b}}\left(q^{-1}UV\right)g_{\mathrm{b}}(U). \tag{3.25}$$

for non-cocommutative variables U and V, where U and V form a Weyl pair such that $UV = q^2VU$. Likewise we observe that the canonical element equation (3.24) leads to Faddeev's quantum dilogarithm $\Phi_{\mathrm{b}}(x)$ pentagon identity.

Let's consider this structure in Hilbert space. The self-dual representation $\pi : H(U_q(\mathcal{B})) \to L^2(\mathbb{R})$ maps the generators of the Heisenberg double to operators on the Hilbert space $L^2(\mathbb{R})$. This self-dual representation unifies the quantum group and its dual on the same Hilbert space, providing a unified framework for the representation

$$\begin{aligned} \pi(X) &= p, \\ \pi(Y) &= e^{2\pi bq}, \\ \pi\left(\hat{X}\right) &= q, \\ \pi\left(\hat{Y}\right) &= e^{2\pi b(p-q)}, \end{aligned} \tag{3.26}$$

where p and q are operators acting on the Hilbert space with $[p,q] = \frac{1}{2\pi i}$ on $L^2(\mathbb{R})$. The canonical element reads

$$S = e^{2\pi i p_1 q_2} g_{\mathrm{b}}^{-1}\left(e^{2\pi b(q_1+p_2-q_2)}\right). \tag{3.27}$$

The multiplication of the basis elements encodes the canonical element through the adjoint action

$$\mathrm{Ad}\left(S^{-1}\right)(1 \otimes E_{(s,t)}) = \Delta(E_{(s,t)}). \tag{3.28}$$

Thus we have

$$\begin{aligned} \Delta(X) = S^{-1}(1\otimes X)S = g_{\mathrm{b}}(Y\otimes\hat{Y})e^{-2\pi iX\otimes\hat{X}}(1\otimes X)e^{2\pi iX\otimes\hat{X}}g_{\mathrm{b}}^{-1}(Y\otimes\hat{Y}), \\ = g_{\mathrm{b}}(Y\otimes\hat{Y})(1\otimes X + X\otimes 1)g_{\mathrm{b}}^{-1}(Y\otimes\hat{Y}) = 1\otimes X + X\otimes 1, \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} \Delta(Y) &= S^{-1}(1\otimes Y)S = g_{\mathrm{b}}(Y\otimes\hat{Y})(1\otimes Y)g_{\mathrm{b}}^{-1}(Y\otimes\hat{Y}), \\ &= g_{\mathrm{b}}\left(e^{2\pi b(q_1+p_2-q_2)}\right)e^{2\pi bq_2}g_{\mathrm{b}}^{-1}\left(e^{2\pi b(q_1+p_2-q_2)}\right), \\ &= e^{\pi bq_2}g_{\mathrm{b}}\left(e^{-i\pi b^2}e^{2\pi b(q_1+p_2-q_2)}\right)g_{\mathrm{b}}^{-1}\left(e^{i\pi b^2}e^{2\pi b(q_1+p_2-q_2)}\right)e^{\pi bq_2}, \\ &= e^{\pi bq_2}\left(1+e^{2\pi b(q_1+p_2-q_2)}\right)e^{\pi bq_2}, \\ &= e^{2\pi bq_2} + e^{2\pi b(q_1+p_2)} = 1\otimes Y + Y\otimes e^{2\pi bX}. \end{aligned}$$

For the dual part we write

$$Ad(S)(\hat{E}_{(s,t)}\otimes 1) = \Delta(\hat{E}_{(s,t)}). \tag{3.30}$$

We get

$$\begin{aligned}\Delta(\hat{X}) &= S(\hat{X}\otimes 1)S^{-1} = e^{2\pi i X\otimes\hat{X}} g_{\mathrm{b}}^{-1}(Y\otimes\hat{Y})(\hat{X}\otimes 1)g_{\mathrm{b}}(Y\otimes\hat{Y})e^{-2\pi i X\otimes\hat{X}},\\ &= e^{2\pi i X\otimes\hat{X}}(\hat{X}\otimes 1)e^{-2\pi i X\otimes\hat{X}} = 1\otimes\hat{X}+\hat{X}\otimes 1,\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\Delta(\hat{Y}) &= S(\hat{Y}\otimes 1)S^{-1} = e^{2\pi i O\otimes\hat{O}} g_{\mathrm{b}}^{-1}(Y\otimes\hat{Y})(\hat{Y}\otimes 1)g_{\mathrm{b}}(Y\otimes\hat{Y})e^{-2\pi i X\otimes\hat{X}},\\ &= e^{2\pi i X\otimes\hat{X}} g_{\mathrm{b}}^{-1}\left(e^{2\pi b(q_1+p_2-q_2)}\right) e^{2\pi b(p_1-q_1)} g_{\mathrm{b}}\left(e^{2\pi b(q_1+p_2-q_2)}\right) e^{-2\pi i X\otimes\hat{X}},\\ &= e^{2\pi i X\otimes\hat{X}} e^{\pi b(p_1-q_1)} g_{\mathrm{b}}^{-1}\left(e^{+i\pi b^2} e^{2\pi b(q_1+p_2-q_2)}\right),\\ &\times g_{\mathrm{b}}\left(e^{-i\pi b^2} e^{2\pi b(q_1+p_2-q_2)}\right) e^{\pi b(p_1-q_1)} e^{-2\pi i X\otimes\hat{X}},\\ &= e^{2\pi i X\otimes\hat{X}} e^{\pi b(p_1-q_1)}\left(1+e^{2\pi b(q_1+p_2-q_2)}\right) e^{\pi b(p_1-q_1)} e^{-2\pi i X\otimes\hat{X}},\\ &= e^{2\pi i X\otimes\hat{X}}\left(e^{2\pi b(p_1-q_1)}+e^{2\pi b(p_1+p_2-q_2)}\right) e^{-2\pi i X\otimes\hat{X}},\\ &= e^{2\pi i X\otimes\hat{X}}\left(\hat{Y}\otimes 1+e^{2\pi b X}\otimes\hat{Y}\right) e^{-2\pi i X\otimes\hat{X}} = 1\otimes\hat{Y}+\hat{Y}\otimes e^{-2\pi b\hat{X}}.\end{aligned} \tag{3.32}$$

In conclusion, we have presented a detailed description of the Heisenberg double on a Hilbert space, including the self-dual representation π and the canonical element S. This representation provides a unified framework for understanding the behavior of quantum systems and their duals. The coproducts derived from the canonical element S satisfy the coassociativity condition, ensuring the consistency of the Heisenberg double structure.

4. INTEGRAL PENTAGON IDENTITY

4.1. 3d $\mathcal{N} = 2$ Supersymmetric Algebra

We briefly review supersymmetric algebra [11] [12]. Poincaré algebra consists of Lorentz algebra and translations with Lorentz transformations respectively

$$\begin{aligned}[M_{\mu\nu}, M_{\rho\sigma}] &= i\left(\eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\rho}M_{\mu\sigma} - \eta_{\mu\sigma}M_{\nu\rho} + \eta_{\nu\sigma}M_{\mu\rho}\right), \\ [M_{\mu\nu}, P_\rho] &= i\left(\eta_{\nu\rho}P_\mu - \eta_{\mu\rho}P_\nu\right).\end{aligned} \tag{4.1}$$

The $\mathcal{N} = 2$ supersymmetric algebra in three dimensions plays a crucial role in the analysis of supersymmetric gauge theories, particularly in areas such as Seiberg-like dualities in 3D and its relations to string theory and M theory through the AdS/CFT correspondence and the 3d-3d correspondence. The anticommutation of supercharges

$$\{Q_\alpha, Q_\beta\} = \left\{\bar{Q}_\alpha, \bar{Q}_\beta\right\} = 0. \tag{4.2}$$

Anticommutation of Conjugate Supercharges

$$\left\{Q_\alpha, \bar{Q}_\beta\right\} = 2\gamma^i_{\alpha\beta}P_i + 2i\epsilon_{\alpha\beta}Z, \tag{4.3}$$

where $\alpha = 1, 2$ and $\gamma^i_{\alpha\beta}$ are the gamma matrices. Z represents a central charge, a scalar operator that commutes with all other operators. $\epsilon_{\alpha\beta}$ is the antisymmetric tensor in two dimensions, crucial for defining spinor bilinears, and P_μ is the translation operator. Commutators with Poincaré generators

$$\begin{aligned}[P_\mu, Q_\alpha] = 0, \quad \left[P_\mu, \bar{Q}_\alpha\right] &= 0, \\ [M_{\mu\nu}, Q_\alpha] &= -\frac{1}{2}\left(\gamma_{\mu\nu}\right)^\beta_\alpha Q_\beta, \\ \left[M_{\mu\nu}, \bar{Q}_\alpha\right] &= -\frac{1}{2}\left(\gamma_{\mu\nu}\right)^\beta_\alpha \bar{Q}_\beta.\end{aligned} \tag{4.4}$$

An additional $U(1)$ R-symmetry is the algebra's automorphism group, rotating the supercharges accordingly

$$\begin{aligned}[R, Q_\alpha] &= Q_\alpha, \\ \left[R, \bar{Q}_\alpha\right] &= -\bar{Q}_\alpha.\end{aligned} \tag{4.5}$$

In instances involving superconformal symmetry, the framework includes two additional bosonic generators, specifically the special conformal transformations K_μ and dilatations D_μ, as well as two fermionic generators, S_α and $\bar{S}_\alpha$. This structure is the

foundation of the $\mathcal{N}=2$ superconformal algebra in three dimensions, which is structured as a supergroup with an isometry group

$$SO(3,2)_{\text{conf}} \times SO(2)_R. \tag{4.6}$$

$SO(3,2)_{\text{conf}}$ represents the conformal group in three dimensions, which includes rotations, translations, dilations, and special conformal transformations. the R-symmetry associated with $\mathcal{N}=2$ supersymmetry is typically $U(1)$, which is isomorphic to $SO(2)$. Altogether it is subgroup of $\text{Osp}(2 \mid 2,2)$. The key relation which is used in the calculation of superconformal index given by general anti-commutation relation

$$\left\{\bar{Q}_\alpha, \bar{S}_\beta\right\} = M_{\mu\nu}\left[\gamma^\mu, \gamma^\nu\right]_{\alpha\beta} + 2\varepsilon_{\alpha\beta}D - 2\varepsilon_{\alpha\beta}R. \tag{4.7}$$

Specific supercharge to calculate the superconformal index, Q, and its conjugate $Q^\dagger$ are chosen to satisfy the simplified relation

$$\left\{\bar{Q}, Q\right\} = \Delta - R - j_3, \tag{4.8}$$

This typically simplifies to focus on the anticommutator that isolates the protected BPS states, where the conformal dimension is Δ, the R-charge is R, and the spin component is j_3.

4.2. Supersymmetric Partition Function In $3d$ $\mathcal{N}=2$ Theories

4.2.1. Witten Index

We begin by briefly mentioning the Witten Index [13–15]. Witten Index is a tool to check whether supersymmetry is broken or unbroken in a given theory

$$\{Q,Q\} = 0, \quad \left\{Q,Q^\dagger\right\} = \Delta, \quad \left\{Q,(-1)^F\right\} = 0, \tag{4.9}$$

where the supercharge is Q and the fermionic number operator is $(-1)^F$. Typically, Δ represents the Hamiltonian, ensuring that supersymmetric transformations conserve energy and the Hamiltonian is bounded below. The expression symbolizes the Witten index

$$\mathcal{I}_W = Tr((-1)^F e^{-\beta\Delta}). \tag{4.10}$$

The Witten index distinguishes itself from the conventional partition function by incorporating the fermionic number operator $(-1)^F$, which introduces a spectral asymmetry

into the system. A non-zero Witten index ($\mathcal{I}_W \neq 0$) generally signifies the preservation of supersymmetry, reflecting an equilibrium between bosonic and fermionic states within the system [16]. Consequently, the index effectively measures the net number of zero-energy and fermionic vacuum states.

4.2.2. Superconformal Index

The Witten index in supersymmetric gauge theories can be generalized by incorporating the global symmetries of the theory that commute with the fermionic generators Q and $Q^\dagger$. This extended index was first formulated for four-dimensional theories [15,17]. Supersymmetric localization techniques have facilitated in-depth studies of precisely calculable quantities in supersymmetric gauge theories across multiple dimensions [18, 19]. In a d-dimensional supersymmetric theory, the superconformal index is defined as a partition function on the manifold $S^{d-1} \times S^1$ [15, 20, 21]

$$\mathrm{I}\left(\{t_i\}\right) = \mathrm{Tr}\left[(-1)^F e^{-\beta\left\{Q,Q^\dagger\right\}} \prod t_i^{F_i}\right]. \tag{4.11}$$

The trace is computed over the Hilbert space on S^{d-1}. Here, F_i represents the generators for global symmetries that commute with Q and $Q^\dagger$. At the same time, t_i are the additional regulators, or fugacities (chemical potentials), associated with these global symmetries.[1] . The index specifically targets short states that stay invariant under a certain subalgebra of the superconformal algebra. Moreover, it remains invariant under the renormalization group flow and is mainly designed to test dualities and anomalies and examine the spectrum of operators in SCFTs. Consequently, this potent analytical tool allows us to compute exact quantities, including superconformal indices, partition functions on compact manifolds, Wilson loops, 't Hooft loops, surface operators, and more. We focus on $\mathcal{N} = 2$ theories on $S^2 \times S^1$ as follows

$$I\left(q, \{t_i\}\right) = \mathrm{Tr}\left[(-1)^{\mathrm{F}} e^{-\beta\left\{Q,Q^\dagger\right\}} q^{\frac{1}{2}(\Delta + j_3)} \cdot \prod t_i^{F_i}\right], \tag{4.12}$$

Δ denotes the energy, j_3 is the third component of angular momentum on S^2, and F_i signifies the charge related to global symmetry, represented by the fugacity t_i. In the context of the three-dimensional $N = 2$ superconformal algebra, the supersymmetric

[1] Unlike the Witten Index, the Superconformal Index generally depends on parameters such as temperature or other external conditions, making it sensitive to the dynamical content of the theory beyond just the ground states

charge Q has quantum numbers $\Delta = \frac{1}{2}$, $j_3 = -\frac{1}{2}$, and an R-charge $R = 1$. The supercharges $Q^\dagger$ and Q obey the following anticommutation

$$\Delta - R - j_3 = \frac{1}{2}\{Q, S\}. \tag{4.13}$$

Only short states for which $\frac{1}{2}Q, S = \Delta - R - j_3 = 0$ contribute to the index. Hence, the index is independent of the temperature β. Still, it is sensitive to the t_i and q parameters[2] . The superconformal index can be calculated using a path integral on $S^2 \times S^1$ by employing the localization technique [19]. This method yields a matrix integral represented as follows

$$\begin{aligned} I\left(q, \{t_i\}\right) = \sum_{m \in \mathbb{Z}^{\text{rank } G}} \frac{1}{|W_m|} \int e^{-S_{CS}^{(0)}} e^{\mathrm{i}bb_0} q^{\frac{1}{2}\epsilon_0} \prod_{j=1}^{\text{rank } F} t_j^{q_{0j}} \\ \times \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} \text{ind}\left(z_i^n, t^n, q^n; m\right)\right] \prod_{i=1}^{\text{rank } G} \frac{dz_i}{2\pi \mathrm{i} z_i}. \end{aligned} \tag{4.14}$$

Summing over the magnetic fluxes $m = (m_1, \ldots, m_{\text{rank}\, G})$ on the 2-sphere, where

$$m_i = \frac{1}{2\pi} \int_{S^2} F_i. \tag{4.15}$$

The variable m_i denotes the GNO [3] charge of the monopole configuration, which, in the examples we analyze, takes integer values. The prefactor $|W_m|$ corresponds to the product of the factorials of the rank of each subgroup G_i within G, where the monopoles decompose G into the product $G_1 \times G_2 \times \cdots \times G_k$. For example, for the $U(N)$ gauge group, $|W_m|$ is the product of the N_k factorials. We are dealing with a Chern-Simons free index because our focus is primarily on the Pentagon identity, necessitating a $U(1)$ gauge group. The full superconformal index is presented here for completeness. The zero-point energy contribution is denoted by q_{0j}

$$q_{0j}(m) = -\sum_{\Phi} \sum_{\rho \in R_\Phi} \frac{|\rho(m)|}{2} f_j(\Phi). \tag{4.16}$$

The contribution from the Casimir energy associated with the vacuum state on the 2-sphere, which is affected by the magnetic m flux as follows

$$\epsilon_0(m) = \frac{1}{2} \sum_{\Phi} (1 - \Delta_\Phi) \sum_{\rho \in R_\Phi} |\rho(m)| - \sum_{\alpha \in G} \frac{|\alpha(m)|}{2}. \tag{4.17}$$

Inside the index, (4.14) single letter index is calculated as

[2]The superconformal index quantifies the number of BPS states, weighting them according to their quantum numbers.

[3]The GNO (Goddard, Nuyts, and Olive) charge, also referred to as the magnetic weight, is a method for characterizing the magnetic charge of monopoles in non-abelian gauge theories.

$$\begin{aligned}\text{ind}\left(e^{ig_j}=z_j,t_j,q;m_j\right)&=-\sum_{\alpha\in G}q^{\frac{|\alpha(m)|}{2}}e^{\mathrm{i}\alpha(g)}\\&+\sum_{\Phi}\sum_{\rho\in R_\Phi}\left[e^{\mathrm{i}\rho(g)}\prod_j t_j^{f_j}\frac{q^{\frac{1}{2}|\rho(m)|+\frac{1}{2}\Delta_\Phi}}{1-q}-e^{-\mathrm{i}\rho(g)}\prod_j t_j^{-f_j}\frac{q^{\frac{1}{2}|\rho(m)|+1-\frac{1}{2}\Delta_\Phi}}{1-q}\right].\end{aligned}\tag{4.18}$$

The initial term accounts for the contributions from the vector multiplets, whereas the following line specifies the contributions from the matter multiplets, denoted by Φ. Here, j ranges over the rank of the flavor symmetry group. Once the single-letter index is established, determining the full multi-letter index involves combinatorial calculations. The calculation culminates in what is referred to as the 'plethystic exponential [4] . The generalized superconformal index incorporates integer parameters corresponding to global symmetries. As noted by Kapustin and Willett [22], the generalized superconformal index includes integer parameters associated with global symmetries. As discussed by Kapustin and Willett [22], the superconformal index of a three-dimensional supersymmetric gauge theory can be extended by considering the theory in the presence of a non-trivial background gauge field that interacts with the global symmetries. This approach introduces additional discrete parameters related to global symmetries into the superconformal index, these parameters are not summed over

$$\begin{aligned}I\left(q,\{t_a\},\{n_a\}\right)=\sum_{m_1,\ldots,m_{\text{rank}(G)}}\frac{1}{|W_m|}\oint^{\text{rank}G}\prod_{j=1}^{2\pi\mathrm{i}z_j}dz_j Z_{gauge}\left(z_j,m_j;q\right)\\\times\prod_{\Phi}Z_\Phi\left(z_j,m_j;t_a,n_a;q\right),\end{aligned}\tag{4.19}$$

here n_α are new discrete variables. Gauge field contribution

$$Z_{gauge}\left(z_j,m_j;q\right)=\prod_{\alpha\in\text{ad}(G)}q^{-\frac{|\alpha(m)|}{2}}\left(1-e^{\alpha(g)}q^{\frac{|\alpha(m)|}{2}}\right).\tag{4.20}$$

Flavour group contribution yields

$$\begin{aligned}Z_\Phi=\prod_{\rho\in R_\Phi}\left(q^{\frac{1-r_\phi}{2}}\prod_j t(\Phi)^{-f(\Phi)}e^{-\mathrm{i}\rho(g)}\right)^{\frac{|\rho(m)+f(\Phi)n(\Phi)|}{2}}\\\times\frac{\left(q^{\frac{1}{2}|\rho(m)+f(\Phi)n(\Phi)|+\frac{1-r_\Phi}{2}}e^{-\mathrm{i}\rho(g)}t(\Phi)^{-f(\Phi)};q\right)_\infty}{\left(q^{\frac{1}{2}|\rho(m)+f(\Phi)n(\Phi)|+\frac{r_\Phi}{2}}e^{\mathrm{i}\rho(g)}t(\Phi)^{f(\Phi)};q\right)_\infty}.\end{aligned}\tag{4.21}$$

We have the Chern-Simon term free index, as mentioned.

[4] It is typically calculated using a plethystic exponential function of the single-letter index, which effectively sums over all possible symmetrized combinations of the single-field contributions. The plethystic exponential is used extensively in the enumeration of symmetric functions and is given by $PE[f(t)]=\exp\left(\sum_{n=1}^{\infty}\frac{1}{n}f\left(t^n\right)\right)$ where $f(t)$ is the single-letter index as a function of fugacities.

4.2.3. Lens Index

For our consideration of the $3d$ $\mathcal{N} = 2$ index, the Lens index generalizes the superconformal index for theories on a 3-dimensional manifold derived from quotient the 3-sphere S^3 by a $\mathbb{Z}_r$ action. Understanding that the global structures of related gauge groups are essential for supersymmetric dual theories is crucial.

A significant difference emerges when the theoretical framework shifts from a conventional manifold to an orbifold, transitioning from $\mathcal{M}_3$ to a lens space represented as $L(p, q)$. With its non-contractible cycles, the lens space presents a distinct topological context that impacts state behavior. These non-contractible cycles allow for non-trivial principal bundles that describe gauge fields in such spaces. The global structure of the gauge group, shaped by the topology of the underlying space, determines the types of permissible principal bundles and their contributions, influencing the theory's field configurations and physical phenomena.

In these theories, G is the gauge group, and F is the global symmetry group. The gauge group multiplets are in the adjoint representation of G, while the chiral multiplets are in the appropriate representations of both G and F, specifically in the fundamental representation. The supersymmetry algebra also includes an $SO(2)$ R-symmetry that rotates the supercharges. Supersymmetric localization allows the exact partition function calculation on the squashed lens space $S^3_b/\mathbb{Z}_r$. This partition function was initially derived in earlier works [23] and further investigated in subsequent studies. [24–26]. The squashed sphere

$$S^3_b = \left|b^2\right| x^{\,2} + b^{-2}|y|^2 = 1 \quad (x, y) \in \mathbb{C}^2, \tag{4.22}$$

with

$$(x, y) \sim \left(e^{\frac{2\pi i}{r}} x, e^{-\frac{2\pi i}{r}} y\right). \tag{4.23}$$

Thus, the partition function arises exactly

$$Z = \sum_m \int \frac{\prod^{\text{rank } G} dz_j}{2\pi i \prod_k |\mathcal{W}_k|} Z_{\text{cl}}[z, m] Z_{\text{vector}}\,[z, m] Z_{\text{matter}}\,[z, m]. \tag{4.24}$$

The sum here is taken over the holonomies.

$$m = \frac{r}{2\pi}\int_{\mathbb{C}} A_\mu dx^\mu, \tag{4.25}$$

where A_μ represents the gauge field

$$G \to \prod_{k=0}^{r-1} G_k. \tag{4.26}$$

The short review of the supersymmetric localization can be found in [27] and for $\mathcal{N}=2$ supersymmetric theories in [28]. Through dimensional reduction, the lens partition function can also be obtained from $S^1 \times S^3/\mathbb{Z}_r$. The classical term $Z_{\rm cl}$ arises, but we focus solely on theories without Chern-Simons and Fayet-Iliopoulosterms. The partition has $Z_{\rm vector}$ from the vector multiplets and $Z_{\rm matter}$ from the chiral multiples. These contributions are as follows

$$Z_{\rm matter} = \prod_i \prod_{\rho_i} \prod_{\phi_i} \hat{s}_{b,-\rho_i(m)-\phi_i(n)}\left(i\frac{Q}{2}\left(1-\Delta_i\right)-\rho_i(z)-\phi_i(\Phi)\right). \tag{4.27}$$

The index i labels the chiral multiplets, with ρ_i and ϕ_i representing the weights of the gauge and flavor group representations, respectively, by noting that ϕ_i corresponds to the real masses associated with the flavor group. Δ_i denotes the Weyl weight of the i-th chiral multiplet. Here, $Q = b + \frac{1}{b}$, where the squashing parameter b^2 is given by ω_2/ω_1 [29, 30][5] . The function $\tilde{s}_{b,-m}$ is the modified double sine function defined in the appendix. The one-loop contribution of the vector multiplet[6] for theory with non-abelian gauge group [7] , combined with the Vandermonde determinant, is expressed as

$$\begin{aligned} Z_{\rm vector} &= \prod_\alpha \frac{1}{\hat{s}_{b,\alpha(m)}\left(\mathrm{i}\frac{Q}{2}+\alpha(z)\right)}, \\ &= \prod_{\alpha>0} 4\sinh\frac{\pi}{r}\left(\frac{\alpha(z)}{\omega_1}+\mathrm{i}\alpha(m)\right)\sinh\frac{\pi}{r}\left(\frac{\alpha(z)}{\omega_2}-\mathrm{i}\alpha(m)\right), \end{aligned} \tag{4.28}$$

where the product is taken over the positive roots α of the gauge group G.

[5]The squashing parameter can be either a real number or a phase.

[6]The vector multiplet is composed of a gauge field, a complex Dirac fermion, a real scalar field, and an auxiliary scalar field.

[7]For Abelian vector fields, the one-loop contributes trivially

4.3. Integral Identities $3d$ $\mathcal{N}=2$ Theories

4.3.1. Pentagon Arising From Dimensional Reduction

Seiberg and Witten analyzed the dimensional reduction of four-dimensional supersymmetric field theories compactified on $\mathbb{R}^3\times S^1$ [31]. It is known that the superconformal index of four-dimensional SUSY Yang-Mills theories can be expressed in elliptic hypergeometric integrals because of their effectiveness of capturing the properties of the chiral and gauge contributions of the theories. Later, it was shown that [32] the superconformal index can be represented using elliptic hypergeometric integrals

$$\begin{aligned}\frac{(p;p)_\infty(q;q)_\infty}{4\pi\mathrm{i}}\int_{\mathbb{T}}\frac{\prod_{j=1}^{6}\Gamma\left(t_j z^{\pm1};p,q\right)}{\Gamma\left(z^{\pm2};p,q\right)}\frac{\mathrm{d}z}{z}&=\prod_{1\le j<k\le 6}\Gamma\left(t_j t_k;p,q\right),\\ \Gamma(z;p,q)=\prod_{j,k=0}^{\infty}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k},&\quad |p|,|q|<1,\quad z\in\mathbb{C}^*.\end{aligned}\tag{4.29}$$

This is known as the Star-Triangle formula. Spiridonov established the above integral identity with the balancing condition $\prod_{i=1}^{6}t_i=pq$. The integral identity represents an elliptic generalization of the Euler integral of the first kind, commonly known as the Beta integral. It can be thought of as a generalization of the Askey-Wilson integral. We have two theories

- Theory A: The left-hand side of the index represents chiral multiplets of flavor group $SU(6)$, and by changing the superpotential term, $SU(6)$ can be decomposed to such that $SU(2)\times SU(2)\times SU(2)$. Theory has the gauge group $SU(2)$,
- Theory B: The right-hand side represents the theory's superconformal index, which lacks gauge fields. Instead, the theory includes meson fields, part of the fifteen-dimensional antisymmetric tensor representation of the $SU(6)$ flavor group.

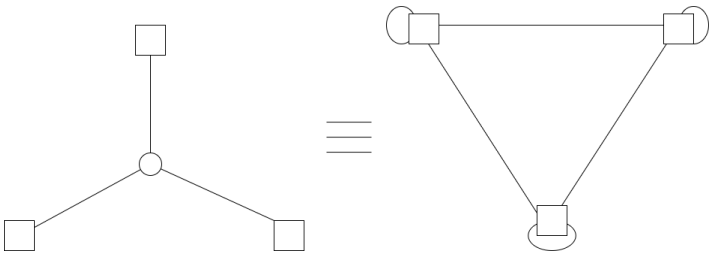


Figure 4.1. Star Triangle relation.

The above integral identity is known as a master solution, which describes a neighbor interaction model with both discrete and continuous spins on a 2D lattice. Note that this 2D lattice theory was recently discovered to be dual to quiver gauge theory on $S^3 \times S^1$. We reformulate the elliptic gamma function and take the necessary limit to suit our objectives

$$\begin{gathered} z = \mathrm{e}^{2\pi i r u}, \quad p = \mathrm{e}^{2\pi i r \omega_1}, \quad q = \mathrm{e}^{2\pi i r \omega_2}, \quad t_j = \mathrm{e}^{2\pi i r \alpha_j}, \\ \lim_{r\to 0} \exp\left(\pi i \frac{2z-\omega_1-\omega_2}{12r}\right) \Gamma\left(\mathrm{e}^{2\pi i r z}; \mathrm{e}^{2\pi i r \omega_1}, \mathrm{e}^{2\pi i r \omega_2}\right) = \gamma^{(2)}\left(z; \omega_1, \omega_2\right), \end{gathered} \tag{4.30}$$

applying the above limit to equation (4.29), we effectively reduce the theory to three dimensions, namely

$$\frac{1}{2}\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{6} \gamma^{(2)}\left(\alpha_j \pm u; \omega_1, \omega_2\right)}{\gamma^{(2)}\left(\pm 2u; \omega_1, \omega_2\right)} \frac{\mathrm{d}u}{i\sqrt{\omega_1\omega_2}} = \prod_{1\le k<l\le 6} \gamma^{(2)}\left(\alpha_k + \alpha_l; \omega_1, \omega_2\right). \tag{4.31}$$

$\sum_{j=1}^{6} \alpha_j = \omega_1 + \omega_2$ is satisfied. The Boltzmann weigh can be defined to rewrite the result as the form of Star-Triangle. However we reparametrize it again to reduce its gauge group $SU(2)$ to $U(1)$

$$\alpha_j = \mu + a_j, \quad \alpha_{j+3} = -\mu + b_j, \quad j \in \{1, 2, 3\}. \tag{4.32}$$

where the balancing condition happens to be $\sum_{j=1}^{3}\left(a_j + b_j\right) = Q = \omega_1 + \omega_2$ and shifting the integration parameter $u \to u + \mu$

$$\frac{1}{2}\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{6} \gamma^{(2)}\left(\alpha_j \pm (u+\mu); \omega_1, \omega_2\right)}{\gamma^{(2)}\left(\pm 2(u+\mu); \omega_1, \omega_2\right)} \frac{\mathrm{d}u}{i\sqrt{\omega_1\omega_2}} = \prod_{1\le k<l\le 6} \gamma^{(2)}\left(\alpha_k + \alpha_l; \omega_1, \omega_2\right), \tag{4.33}$$

and by taking the limit $\mu \longrightarrow \infty$ using the limits of

$$\begin{gathered} \lim_{|u|\to\infty} \mathrm{e}^{\frac{\pi i}{2} B_{2,2}(u;\omega_1,\omega_2)} \gamma^{(2)}\left(u; \omega_1, \omega_2\right) = 1, \text{ if } \arg\omega_1 < \arg u < \arg\omega_2 + \pi, \\ \lim_{|u|\to\infty} \mathrm{e}^{-\frac{\pi i}{2} B_{2,2}(u;\omega_1,\omega_2)} \gamma^{(2)}\left(u; \omega_1, \omega_2\right) = 1, \text{ if } \arg\omega_1 - \pi < \arg u < \arg\omega_2. \end{gathered} \tag{4.34}$$

We get the following nice integral identity

$$\int_{-i\infty}^{i\infty} \prod_{j=1}^{3} \gamma^{(2)}\left(a_j - u, b_j + u; \omega_1, \omega_2\right) \frac{\mathrm{d}u}{i\sqrt{\omega_1\omega_2}} = \prod_{k,l=1}^{3} \gamma^{(2)}\left(a_k + b_l; \omega_1, \omega_2\right). \tag{4.35}$$

It turns out to be that it can be written as five terms identity

$$\int_{-i\infty}^{i\infty} \prod_{j=1} \mathcal{B}\left(a_j - u, b_j + u\right) \frac{\mathrm{d}u}{i\sqrt{\omega_1\omega_2}} = \mathcal{B}\left(a_2 + b_1, a_3 + b_2\right) \mathcal{B}\left(a_1 + b_2, a_3 + b_1\right), \quad (4.36)$$

where

$$\mathcal{B}(x, y) = \frac{\gamma^{(2)}\left(x; \omega_1, \omega_2\right) \gamma^{(2)}\left(y; \omega_1, \omega_2\right)}{\gamma^{(2)}\left(x + y; \omega_1, \omega_2\right)}. \quad (4.37)$$

The integral identity (4.36) arises in the $U_q(sl_2)$ context of quantization of Teichmüller space noted by R. M. Kashaev[8] . It appears that the operator form of the integral pentagon identity (4.36) may be obtained as a canonical element of the Heisenberg double algebra stated before

$$S = \exp(2\pi i X \otimes \hat{X}) g_{\mathrm{b}}^{-1}(Y \otimes \hat{Y}), \quad (4.38)$$

with below identification

$$g_b^{-1}(e^{Ax}) = B\gamma^{(2)}\left(Cx; \omega_1, \omega_2\right) e^{Dx^2}, \quad (4.39)$$

A, B, C, and D are constants. One may derive the integral form of the above identity by using the canonical element (4.38) at the end. Both constructions are part of the same algebraic structure.

4.3.2. Pentagon Arising From Mirror Symmetry

The following formula characterizes the tetrahedron index

$$\mathcal{I}_q[m, z] = \prod_{i=0}^{\infty} \frac{1 - q^{i-\frac{1}{2}m+1} z^{-1}}{1 - q^{i-\frac{1}{2}m} z}, \quad m \in Z \text{ with } |q| < 1. \quad (4.40)$$

Here, z represents a continuous variable fugacity associated with the gauge group. The tetrahedron index is used to glue and combine essential pieces of triangulated manifolds to capture the partition function (or superconformal index), which is invariant under 2-3 Pachner moves, as proven in [33]. This invariance leads to the pentagon identity. The partition function receives contributions from various multiplets that reflect the underlying symmetries of the theory. Specifically, one can interpret the chiral multiplet in the partition function as the contribution from the flavor symmetry group, highlighting the role of flavor symmetries in the dynamics of the gauge theory. When constructing more complex 3d $\mathcal{N} = 2$ theories, these chiral multiplet contributions act

[8] One can derive the integral pentagon identity from the Saalschütz summation formula and prove the identity

as building blocks. A simple but crucial example demonstrated in [34] that combining two tetrahedra results in an $\mathcal{N}_f = 1$ SQED, whereas joining three tetrahedra leads to the XYZ model, which involves three chiral multiplets linked by a cubic superpotential. The mirror symmetry of these theories reinforces the idea that constructing the theory for any 3-manifold does not depend on the specific triangulation used.

- Theory A: The theory is called SQED $\mathcal{N} = 2$, having two chiral fields with opposite charge Q and $\tilde{Q}$, respectively $+1$ and -1 under the local symmetry group $U(1)$ responsible for the interaction mediated via the vector multiplet V.In addition to $U(1)$ gauge symmetry, the theory has an additional global symmetry group of $U(1)$ consisting of two parts $U(1)_A$ and $U(1)_J$. The former is often named axial symmetry, under which $\tilde{Q}$ and Q have the same charge $+1$. The latter is sometimes called topological $U(1)_J$, which is related to the conservation of certain topological quantities. These symmetries are global, affecting all points in space-time uniformly. The index we get
$$I_e = \sum_{m\in Z} q^{|m|/3} \int_{\mathbb{T}} \frac{dz}{2\pi i z} \frac{\left(q^{5/6+|m|/2}z;q\right)_\infty \left(q^{5/6+|m|/2}z^{-1};q\right)_\infty}{\left(q^{1/6+|m|/2}z;q\right)_\infty \left(q^{1/6+|m|/2}z^{-1};q\right)_\infty}. \tag{4.41}$$
- Theory B: The theory is known as the XYZ model or the free Wess-Zumino model, consisting of three chiral superfields: X, Y, and Z. However, we will refer to them as q, $\tilde{q}$, and S, which interact via a trilinear superpotential $W = XYZ$ with the following assigned charges. We get the index
$$I_m = \frac{\left(q^{2/3};q\right)_\infty^3}{\left(q^{1/3};q\right)_\infty^3}, \tag{4.42}$$

we redefine the tetrahedron index
$$\mathcal{I}_q[m,z] = (-q^{\frac{1}{2}})^{\frac{(m+|m|)}{2}} z^{-\frac{(m+|m|)}{2}} \prod_{i=0}^{\infty} \frac{1-q^{i+\frac{1}{2}|m|+1}z^{-1}}{1-q^{i+\frac{1}{2}|m|}z}. \tag{4.43}$$

We have $I_e = I_m$, thus pentagon identity
$$\sum_{m\in Z} \oint \frac{dz}{2\pi i z} z^{-m} \mathcal{I}_q\left[m; q^{1/6}z^{-1}\right] \mathcal{I}_q\left[-m; q^{1/6}z\right] = \mathcal{I}_q^3\left[0; q^{1/3}\right]. \tag{4.44}$$

We also have a different pentagon identity involving tetrahedron indices. In essence, the hyperbolic gamma functions can be in terms of tetrahedron $\gamma^{(2)} \to \mathcal{I}_q$ then apply that concept to equation 4.35
$$\sum_{m\in Z} \oint \frac{dz}{2\pi i z} (-z)^{-3m} \prod_{i=1}^{3} \mathcal{I}_q\left[-m, q^{\frac{1}{6}}\xi_i z\right] \mathcal{I}_q\left[m, q^{\frac{1}{6}}\eta_i z^{-1}\right] = \prod_{i,j=1}^{3} \mathcal{I}_q\left[0, q^{\frac{1}{3}}\xi_i\eta_j\right], \tag{4.45}$$

where $\prod_{i=1}^{3}\xi_i = \prod_{i=1}^{3}\eta_i = 1$ the balancing condition and define the following

$$\mathcal{B}[m;a,b] = \frac{\mathcal{I}_q[m,a]\mathcal{I}_q[-m,b]}{\mathcal{I}_q[0,ab]}, \tag{4.46}$$

by using the following property

$$\mathcal{I}_q[m,a]\mathcal{I}_q\left[m,qa^{-1}\right] = 1, \tag{4.47}$$

we get the Pentagon identity

$$\sum_{m\in Z}\oint \frac{dz}{2\pi i z}(-z)^{-3m}\prod_{i=1}^{3}\mathcal{B}\left[m;\xi_i z^{-1},\eta_i z\right] = \mathcal{B}\left[0;\xi_1\eta_2,\xi_3\eta_1\right]\mathcal{B}\left[0;\xi_2\eta_1,\xi_3\eta_2\right], \tag{4.48}$$

where we redefine the parameters $\xi_i \to q^{-1/6}\xi_i$ and $\eta_i \to q^{-1/6}\eta_i$ with balancing condition $\prod_{i=1}^{3}\xi_i = \prod_{i=1}^{3}\eta_i = q$ [2].

Table 4.1. SQED charges.

	U(1)	$\mathrm{U}(1)_J$	$\mathrm{U}(1)_A$
$\tilde{Q}$	-1	0	$+1$
Q	$+1$	0	$+1$

Table 4.2. Charges in the free Wess-Zumino theory.

	$\mathrm{U}(1)_V$	$\mathrm{U}(1)_A$
X	$+1$	$+1$
Y	-1	$+1$
Z	0	-2

4.3.3. Integral Pentagon Identity With $\mathcal{N}=2$ On S_b^3

The partition function of the ellipsoid typically takes the following form [34]

$$\begin{aligned} Z = & \frac{1}{|W|}\int_{i=1}^{\text{rank}G}\frac{dz_i}{i\sqrt{\omega_1\omega_2}}\exp\left(\frac{ik\pi z_i^2}{\omega_1\omega_2}+\frac{\pi i\xi z_i}{\omega_1\omega_2}\right) \\ & \times\frac{\prod_{j=1}^{\text{rank}F}\gamma^{(2)}\left(\omega\Delta_j+\rho_j(z)+\tilde{\rho}_j(\mu);\omega_1,\omega_2\right)}{\prod_{\alpha\in R_+}\gamma^{(2)}\left(\alpha(z);\omega_1,\omega_2\right)\gamma^{(2)}\left(-\alpha(z);\omega_1,\omega_2\right)}. \end{aligned} \tag{4.49}$$

The isometry group of a round 3-sphere is $SO(4)$, which is closely related to the double cover represented by $SU(2)\times SU(2)$. Squashing characterizes the SCI by breaking its isometries to $SU(2)\times U(1)$, resulting in a non-trivial, reduced symmetry group.

One example of pentagon identity for this case is widely recognized in the literature and governed by the quantum group $U_q(sl_2)$, especially in conformal field theories like Liouville and Toda theories, as the expression of correlation functions and conformal blocks with $b\neq 1$ [35]

$$\int dz s_b(x-z)s_b(x+z) = s_b\left(2x-\frac{iQ}{2}\right)s_b\left(\frac{iQ}{2}-x\right)s_b\left(\frac{iQ}{2}-x\right). \tag{4.50}$$

The function s_b represents the double sine function[9]

$$s_b(x) = e^{-\frac{i\pi}{2}x^2}\prod_{j=1}^{\infty}\frac{1+e^{2\pi bx+2\pi ib^2\left(j-\frac{1}{2}\right)}}{1+e^{2\pi b^{-1}x+2\pi ib^{-2}\left(\frac{1}{2}-j\right)}}, \tag{4.51}$$

or equal representation

$$= \exp\left(\int_{\mathbb{R}+i0}\frac{e^{-2ixz}}{(4\sinh(zb)\sinh(z/b))}\frac{dz}{z}\right). \tag{4.52}$$

Another example can be given: On one side, an 'electric' theory features a $U(1)$ gauge symmetry and six chiral multiplets. Conversely, there is a 'magnetic' theory with no gauge symmetry and nine chiral multiplets

$$\begin{aligned} \sum_{m=-\infty}^{\infty}\oint\frac{1}{z^{3m}}\frac{dz}{2\pi iz}\prod_{i=1}^{3}\mathcal{B}\left(a_iz,n_i+m;b_iz^{-1},m_i-m\right) & \\ =\mathcal{B}\left(a_1b_2,n_1+m_2;a_3b_1,n_3+m_1\right) & \\ \times\mathcal{B}\left(a_2b_1,n_2+m_1;a_3b_2,n_3+m_2\right), & \end{aligned} \tag{4.53}$$

where $\mathcal{B}(a,n;b,m)$ is

$$\mathcal{B}(a,n;b,m) := \frac{\left(q^{1+\frac{n}{2}}a^{-1};q\right)_\infty}{\left(q^{\frac{n}{2}}a;q\right)_\infty}\frac{\left(q^{1+\frac{m}{2}}b^{-1};q\right)_\infty}{\left(q^{\frac{m}{2}}b;q\right)_\infty}\frac{\left(q^{\frac{n+m}{2}}ab;q\right)_\infty a^{\frac{m}{2}}b^{\frac{n}{2}}}{\left(q^{1+\frac{n+m}{2}}(ab)^{-1};q\right)_\infty}, \tag{4.54}$$

[9]It is recognized as the non-compact variant of Faddeev's function in the literature.

two limit case $\omega_2 \to \infty$ and $q \to 1$ are examined for above pentagon identity in [36] and its links to pentagon identities.

4.3.4. Integral Pentagon Identity with $\mathcal{N} = 2$ on $S^2 \times S^1$

The following index identity is derived from mirror symmetry as validated in [22]

$$\begin{aligned}\sum_{s\in\mathbb{Z}/2}\int \frac{dz}{2\pi i z} z^{2n} w^{2s} \left(x^{1/2} z^{\pm 1}\alpha^{-1}\right)^{|s\mp m|} \frac{\left(z_q^I(\pm 1\alpha^{-1}x^{2|s\mp m|+3/2};x^2\right)_\infty}{\left(z^{\pm 1}\alpha x^{2|s\pm m|+1/2};x^2\right)_\infty} \\ = \left(x^{1/2}\alpha w^{\pm 1}\right)^{|m\pm n|} \frac{\left(\alpha w^{\pm 1}x^{2|m\pm n|+3/2};x^2\right)_\infty}{\left(\alpha^{-1}w^{\pm 1}x^{2|m\mp n|+1/2};x^2\right)_\infty}.\end{aligned} \tag{4.55}$$

Further exploration into different representations, as discussed by [37], reveals the following integral identity

$$\begin{aligned}&\sum_{s\in\mathbb{Z}}\oint \frac{dz}{2\pi i z}(-w)^s z^{n-s} \frac{\left(z\alpha^{-1}q^{\frac{m+s}{2}+\frac{3}{4}};q\right)_\infty}{\left(z^{-1}\alpha q^{\frac{m+s}{2}+\frac{1}{4}};q\right)_\infty} \frac{\left(z\alpha^{-1}q^{\frac{m-s}{2}+\frac{3}{4}};q\right)_\infty}{\left(z\alpha q^{\frac{m-s}{2}+\frac{1}{4}};q\right)_\infty} \\ &= (-w)^n \frac{\left(\alpha w q^{\frac{m+n}{2}+\frac{3}{4}};q\right)_\infty}{\left(\alpha^{-1}w^{-1}q^{\frac{m-n}{2}+\frac{1}{4}};q\right)_\infty} \frac{\left(\alpha w q^{\frac{m-n}{2}+\frac{3}{4}};q\right)_\infty \left(\alpha^{-2}q^{m+\frac{1}{2}};q\right)_\infty}{\left(\alpha^{-1}w q^{\frac{m+n}{2}+\frac{1}{4}};q\right)_\infty \left(\alpha^{2}q^{m+\frac{1}{2}};q\right)_\infty},\end{aligned} \tag{4.56}$$

by introducing the function $\mathcal{B}_q(m,z)$ simplifies the representation

$$\mathcal{B}_q(m,z) = \frac{\left(zq^{\frac{m}{2}+\frac{1}{2}};q\right)_\infty}{\left(z^{-1}q^{\frac{m}{2}};q\right)_\infty}. \tag{4.57}$$

This function allows us to rewrite the integral identity as a pentagon identity, offering a more structured expression

$$\begin{aligned}&\sum_{s\in\mathbb{Z}}\oint \frac{dz}{2\pi i z}(-w)^s z^{n-s}\mathcal{B}_q\left(m-s, z\alpha^{-1}q^{\frac{1}{4}}\right)\mathcal{B}_q\left(m+s, z^{-1}\alpha^{-1}q^{\frac{1}{4}}\right) \\ =&(-w)^n\mathcal{B}_q\left(m+n, w\alpha q^{\frac{1}{4}}\right)\mathcal{B}_q\left(m-n, w^{-1}\alpha q^{\frac{1}{4}}\right)\mathcal{B}_q\left(m+\frac{1}{2}, \alpha^{-2}q^{\frac{1}{4}}\right).\end{aligned} \tag{4.58}$$

This index aligns with the XYZ model for $m = n = 0$.

Another example occurring on $\mathbb{RP}^2 \times \mathbb{S}^1$ is discussed in [37]. Mirror symmetry leads to the identities presented in [38].

$$
\begin{aligned}
& q^{1/8}\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\oint_{C_0}\frac{dz}{2\pi i z}z^s\sum_{m=0}^{1}a^{-1/2+m}q^{-1/4+m/2} \\
& \quad\times\frac{(z^{-1}aq^{1+m};q^2)_\infty\,(zaq^{1+m};q^2)_\infty}{(za^{-1}q^m;q^2)_\infty\,(z^{-1}a^{-1}q^m;q^2)_\infty} \\
& = q^{-\frac{1}{8}}a^{-1/2-|s|}\frac{\left(a^{-1}q^{1/2+|s|};q^2\right)_\infty}{(aq^{1/2+|s|};q^2)_\infty}\frac{\left(a^{-1}q^{3/2+|\tilde{s}|};q^2\right)_\infty}{(aq^{3/2+|\tilde{s}|};q^2)_\infty}\frac{(a^2q;q^2)_\infty}{(a^{-2};q^2)_\infty}.
\end{aligned}
\tag{4.59}
$$

By defining the following function [37]

$$
\mathcal{B}\left(z,m,q^2\right)=z^{-\frac{1}{4}+\frac{m}{2}}q^{-\frac{1}{8}+\frac{m}{4}}\frac{(zq^{m+1};q^2)_\infty}{(z^{-1}q^m;q^2)_\infty},
\tag{4.60}
$$

one discovers a non-trivial Pentagon relation

$$
\begin{aligned}
& \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\oint_{C_0}\frac{dz}{2\pi i z}z^s\sum_{m=0}^{1}\mathcal{B}\left(z^{-1}a,m;q^2\right)\mathcal{B}\left(za,m;q^2\right) \\
& \qquad=\mathcal{B}\left(a^{-1}q^{-\frac{1}{2}},|s|;q^2\right)\mathcal{B}\left(a^{-1}q^{-\frac{1}{2}},|s|+1;q^2\right)\mathcal{B}\left(a^2,0;q^2\right).
\end{aligned}
\tag{4.61}
$$

..

4.3.5. Flavour Group Symmetry Reduction

We can apply dimension reduction techniques to lower the dimensionality of the flavor symmetry group. Let's give a quick example [39], consider the following integral expression given by

$$
\int_{-i\infty}^{i\infty}\frac{dx}{i\sqrt{\omega_1\omega_2}}\prod_{i=1}^{3}\gamma^{(2)}(x+f_i)\gamma^{(2)}(-x+g_i)=\prod_{i,j=1}^{3}\gamma^{(2)}(f_i+g_j),
\tag{4.62}
$$

with

$$
\sum_{i=1}^{3}(f_i+g_i)=Q.
\tag{4.63}
$$

Equation (4.63) is the balancing constraint, and Q controls the quasi-periodicity through its dependence on ω_1 and ω_2. In many applications of QFTs, with $Q=\omega_1+\omega_2=b+b^{-1}$, setting $\omega_1\omega_2=1$ allows us to set $b=\sqrt{\frac{\omega_1}{\omega_2}}$. We use the notation $\gamma^{(2)}(x,b,b^{-1})\equiv S_b(x)$

$$
\int_{-i\infty}^{i\infty}\frac{dx}{i}\prod_{i=1}^{3}S_b\left(x+f_i\right)S_b\left(-x+g_i\right)=\prod_{i,j=1}S_b\left(f_i+g_j\right),
\tag{4.64}
$$

This identity encodes the Heisenberg double algebra of $\mathcal{U}_q(sl(2))$. Fixing the flavour symmetry parameter $g_3\to-i\infty$, and applying the limit $f_3\to i\infty$, with $g_3+f_3=Q-f_1-f_2-g_1-g_2$, and setting $\omega_1=b$ and $\omega_2=b^{-1}$, we obtain

$$\int_{-i\infty}^{i\infty} e^{i\pi[(y(f_1+f_2+g_1+g_2)+f_1f_2-g_1g_2)]} S_b(y+f_1)S_b(y+f_2)S_b(-y+g_1)S_b(-y+g_2)\frac{dy}{i}$$
$$= S_b(Q-f_1-f_2-g_1-g_2)S_b(f_1+g_1)S_b(f_1+g_2)S_b(f_2+g_1)S_b(f_2+g_2).$$

Thus we reduce the flavour symmetry group $SU(3) \times SU(3)$ to $SU(2) \times SU(2)$. The expression $e^{i\pi[y(f_1+f_2+g_1+g_2)+f_1f_2-g_1g_2]}$ is intriguing from the standpoint of understanding this nontrivial contribution, particularly in the context of flavor symmetry reduction.

4.3.6. Integral Pentagon Identity With $\mathcal{N}=2$ On $S_b^3/\mathbb{Z}_r$

The Pentagon identity in this setting illustrates the application of flavor symmetry reduction and the utilization of group theoretical data extracted from supersymmetric theories to construct duality pairs

$$\begin{aligned} Z_{\text{chiral}} = &\prod_{\rho}\prod_{\phi} e^{\frac{i\pi}{2r}\left((\rho(m)+\phi(n))(r-\rho(m)-\phi(n))-(r-1)(\rho(m)+\phi(n))^2\right)} \\ &\times \gamma^2\left(i(\rho(z)+\phi(\Phi))+\omega_1(\rho(m)+\phi(n))+\frac{\omega_1+\omega_2}{2};\omega_1 r,\omega_1+\omega_2\right) \\ &\times \gamma^2\left(iz+\omega_2\left(r-(\rho(m)+\phi(n))\right)+\frac{\omega_1+\omega_2}{2};\omega_2 r,\omega_1+\omega_2\right). \end{aligned} \tag{4.65}$$

In this context, ρ and ϕ represent the weights of the gauge group's representation and the flavor group, respectively. It's noted that the chiral multiplet's Weyl weight is integrated into the flavor fugacity. The one-loop contribution from the vector multiplet is trivial as we consider $U(1)$ gauge theories.

- Theory A: This theory is characterized by a $U(1)$ gauge symmetry and an $SU(3)_L \times SU(3)_R$ flavor symmetry. It includes three fundamental chiral multiplets and three anti-fundamental chiral multiplets without any Chern-Simons term.
- Theory B: The dual part acks gauge degrees of freedom but retains the same global symmetries as Theory A. It features nine "mesons" that are part of the fundamental representation of the $SU(3)_L \times SU(3)_R$ flavor group

$$
\begin{aligned}
&\sum_{y=0}^{[r/2]} \epsilon(y) \int_{-i\infty}^{i\infty} \prod_{i=1}^{3} \gamma^{(2)}\left(a_i - z + \omega_1\left(u_i - y\right); \omega_1 r, \omega_1 + \omega_2\right) \\
&\quad \times \gamma^{(2)}\left(a_i - z + \omega_2\left(r - \left(u_i - y\right)\right); \omega_2 r, \omega_1 + \omega_2\right) \\
&\quad \times \gamma^{(2)}\left(b_i + z + \omega_1\left(v_i + y\right); \omega_1 r, \omega_1 + \omega_2\right) \\
&\quad \times \gamma^{(2)}\left(b_i + z + \omega_2\left(r - \left(v_i + y\right)\right); \omega_2 r, \omega_1 + \omega_2\right) \frac{dz}{ir\sqrt{\omega_1\omega_2}} \\
&= \prod_{i,j=1}^{3} \gamma^{(2)}\left(a_i + b_j + \omega_1\left(u_i + v_j\right); \omega_1 r, \omega_1 + \omega_2\right) \\
&\quad \times \gamma^{(2)}\left(a_i + b_j + \omega_2\left(r - \left(u_i + v_j\right)\right); \omega_2 r, \omega_1 + \omega_2\right).
\end{aligned}
\tag{4.66}
$$

The integral equation provided evaluates the sum over y from 0 to $[r/2]$, incorporating the function $\epsilon(y)$ and the $\gamma^{(2)}$ function, which is integral to the hyperbolic hypergeometric functions involved. The integral is over complex variables, extending from $-i\infty$ to $i\infty$, and includes terms that involve the flavor fugacities a_i for $SU(3)_L$ and b_i for $SU(3)_R$, alongside the parameters u_i, v_i, ω_1, and ω_2. The equation is subject to the constraints that the sums of $a_i + b_i$ and $u_i + v_i$ must equal $\omega_1 + \omega_2$ and r, respectively. The function $\epsilon(y)$ is specified such that it equals 1 at $y = 0$ and $y = \lfloor r/2 \rfloor$, and 2 otherwise. We first decouple the theory by sending $a_3 \longrightarrow \infty$ and $b_3 \longrightarrow -\infty$ and then we decouple again the flavour symmetries by doing for $a_2 \longrightarrow \infty$ and $b_2 \longrightarrow -\infty$ and define $a_1 + b_1 = g_1,\ a_2 + b_2 = g_2,\ u_1 + v_1 = t_1,\ u_2 + v_2 = t_2$ we have the following integral identity

$$
\begin{aligned}
&\sum_{y=0}^{\left\lfloor \frac{r}{2} \right\rfloor} \int_{-i\infty}^{i\infty} \frac{dz}{ir\sqrt{\omega_1\omega_2}} e^{\frac{2\pi i z}{\omega_1\omega_2 r}\left(\frac{2g_1+g_2-\omega}{2}\right) + \frac{i\pi y}{r}(t_1+2t_2-r)} \\
&\quad \times \gamma^{(2)}\left(z + \omega_1 y, \omega_1 r, \omega\right) \gamma^{(2)}\left(z + \omega_2(r-y), \omega_2 r, \omega\right) \\
&\quad \times \gamma^{(2)}\left(-z + g_1 + \omega_1(t_1 - y), \omega_1 r, \omega\right) \gamma^{(2)}\left(-z + g_1 + \omega_2(r - t_1 + y), \omega_2 r, \omega\right) \\
&= e^{\frac{2\pi i}{\omega_1\omega_2 r}\left(\frac{g_1}{2}\right)\left(\frac{2g_2+g_1-\omega}{2}\right) + \frac{i\pi}{r}\left(\frac{-t_1}{2}\right)(t_1+2t_2-r)} \\
&\quad \times \gamma^{(2)}\left(\omega - g_1 - g_2 + \omega_1(r - t_1 - t_2), \omega_1 r, \omega\right) \\
&\quad \times \gamma^{(2)}\left(\omega - g_1 - g_2 + \omega_2(t_1 + t_2), \omega_2 r, \omega\right) \\
&\quad \times \gamma^{(2)}\left(g_2 + \omega_1 t_2, \omega_1 r, \omega\right) \gamma^{(2)}\left(g_2 + \omega_2(r - t_2), \omega_2 r, \omega\right) \\
&\quad \times \gamma^{(2)}\left(g_1 + \omega_1 t_1, \omega_1 r, \omega\right) \gamma^{(2)}\left(g_1 + \omega_2(r - t_1), \omega_2 r, \omega\right).
\end{aligned}
\tag{4.67}
$$

By transforming the fugacities such that $\frac{g_1}{2}$ is redefined as α, z as $z+\alpha, g_2$ as $\beta-\alpha, t_1$ as $2n_\alpha, y$ as $y+n_\alpha$, and t_2 as $n_\beta - n_\alpha$, the following is obtained

$$\begin{aligned}
&\sum_{y=0}^{[r/2]} \epsilon(y) \int_{-i\infty}^{i\infty} e^{\frac{2\pi i}{\omega_1\omega_2 r}(z+\alpha)\left(\frac{3\alpha+\beta-\omega}{2}\right)+\frac{i\pi}{r}(y+n_\alpha)\left(2n_\beta-r\right)} \\
&\times \gamma^{(2)}\left(z+\alpha+\omega_1\left(y+n_\alpha\right);\omega_1 r,\omega\right) \\
&\times\gamma^{(2)}\left(z+\alpha+\omega_2\left(r-y-n_\alpha\right)\right);\omega_2 r,\omega\big)\,\gamma^{(2)}\left(-z+\alpha+\omega_1\left(-y+n_\alpha\right);\omega_1 r,\omega\right) \\
&\times\gamma^{(2)}\left(-z+\alpha+\omega_2\left(r+y-n_\alpha\right)\right);\omega_2 r,\omega\big)\,\frac{dz}{ir\sqrt{\omega_1\omega_2}} \\
&= e^{\frac{2\pi i}{\omega_1\omega_2 r}(\alpha)\left(\frac{2\beta-\omega}{2}\right)+\frac{i\pi}{r}(-n_\alpha)\left(2n_\beta-r\right)}\gamma^{(2)}\left(\omega-\alpha-\beta+\omega_1\left(r-n_\alpha-n_\beta\right);\omega_1 r,\omega\right) \\
&\times \gamma^{(2)}\left(\omega-\alpha-\beta+\omega_2\left(n_\alpha+n_\beta\right);\omega_2 r,\omega\right)\gamma^{(2)}\left(\beta-\alpha+\omega_1\left(n_\beta-n_\alpha\right);\omega_1 r,\omega\right) \\
&\times \gamma^{(2)}\left(\beta-\alpha+\omega_2\left(r-n_\beta+n_\alpha\right);\omega_2 r,\omega\right) \\
&\times \gamma^{(2)}\left(2\alpha+\omega_1\left(2n_\alpha\right);\omega_1 r,\omega\right)\gamma^{(2)}\left(2\alpha+\omega_2\left(r-2n_\alpha\right);\omega_2 r,\omega\right).
\end{aligned} \tag{4.68}$$

The form of the pentagon relation for mirror symmetry is presented in [24]. The specific case when $r=1$ corresponds to the identity established in [39, 40]. In conclusion, they are reducing the flavor symmetry of the pentagon relation from $SU(3)\times SU(3)$ to $U(1)\times U(1)$ yields another pentagon integral identity with block

$$\mathcal{B}\left(x,y\right)=\gamma^{(2)}\left(x+\omega_1 y;\omega_1 r,\omega\right)\gamma^{(2)}\left(x+\omega_2\left(r-y\right)\right);\omega_2 r,\omega\big). \tag{4.69}$$

We have the identity

$$\begin{aligned}
&\sum_{y=0}^{[r/2]} \epsilon(y) \int_{-i\infty}^{i\infty} e^{\frac{2\pi i}{\omega_1\omega_2 r}(z+\alpha)\left(\frac{3\alpha+\beta-\omega}{2}\right)+\frac{i\pi}{r}(y+n_\alpha)\left(2n_\beta-r\right)} \\
&\times \mathcal{B}\left(z+\alpha,y+n_\alpha\right)\mathcal{B}\left(-z+\alpha,-y+n_\alpha\right)\frac{dz}{ir\sqrt{\omega_1\omega_2}} \\
&= e^{\frac{2\pi i}{\omega_1\omega_2 r}(\alpha)\left(\frac{2\beta-\omega}{2}\right)+\frac{i\pi}{r}(-n_\alpha)\left(2n_\beta-r\right)}\mathcal{B}\left(\omega-\alpha-\beta,r-n_\alpha-n_\beta\right) \\
&\times \mathcal{B}\left(\beta-\alpha,n_\beta-n_\alpha\right)\mathcal{B}\left(2\alpha,2n_\alpha\right).
\end{aligned} \tag{4.70}$$

There is another pentagon identity on $S^3/\mathbb{Z}_r$, which interestingly satisfies both the pentagon and star-triangle identities [41]. The original theory, obtained in [42], satisfies the star-triangle relation with gauge group $G=SU(2)$. A pentagon identity can be derived on the same manifold using the gauge group symmetry breaking method described in [43]. Taking the quotient group to infinity means we arrive at the pentagon identity on the manifold $S^2\times S$ presented before.

- Theory A, the gauge group is $SU(2)$ and the flavor group is $SU(6)$. The chiral multiplets are in the fundamental representation of both the gauge group and the flavor group, while the vector multiplet is in the adjoint representation of the gauge group.
- Theory **B**, the dual theory **A** lacks gauge symmetry, and there are fifteen chiral multiplets in the antisymmetric tensor representation of the flavor group. Here, Theory B serves as the low-energy description of Theory **A**, characterized purely by composite gauge singlets

We have the following identity

$$\begin{aligned}
&\sum_{y=0}^{[r/2]}\epsilon(y)e^{\frac{\pi iC}{2}}\int_{-\infty}^{\infty}\frac{dz}{r\sqrt{-\omega_1\omega_2}}\\
&\times\prod_{i=1}^{3}\gamma^{(2)}\left(-i\left(a_i-z\right)-i\omega_1\left(u_i-y\right);-i\omega_1 r,-i\omega_1-i\omega_2\right)\\
&\times\gamma^{(2)}\left(-i\left(a_i-z\right)-i\omega_2\left(r-\left(u_i-y\right)\right);-i\omega_2 r,-i\omega_1-i\omega_2\right)\\
&\times\gamma^{(2)}\left(-i\left(b_i+z\right)-i\omega_1\left(v_i+y\right);-i\omega_1 r,-i\omega_1-i\omega_2\right)\\
&\times\gamma^{(2)}\left(-i\left(b_i+z\right)-i\omega_2\left(r-\left(v_i+y\right)\right);-i\omega_2 r,-i\omega_1-i\omega_2\right)\\
&=\prod_{i,j=1}^{3}\gamma^{(2)}\left(-i\left(a_i+b_j\right)-i\omega_1\left(u_i+v_j\right);-i\omega_1 r,-i\omega_1-i\omega_2\right)\\
&\times\gamma^{(2)}\left(-i\left(a_i+b_j\right)-i\omega_2\left(r-\left(u_i+v_j\right)\right);-i\omega_2 r,-i\omega_1-i\omega_2\right).
\end{aligned}\tag{4.71}$$

Here, $C=-2y+(u_1+u_2+u_3-v_1-v_2-v_3)$, constrained by $\sum_{i=1}^{3}(a_i+b_i)=\omega_1+\omega_2$ and $\sum_{i=1}^{3}(u_i+v_i)=0$

$$\begin{aligned}
\mathcal{B}\left(z_1,u_1;z_2,u_2\right)=&\frac{\gamma^{(2)}\left(-iz_1-i\omega_1u_1;-i\omega_1 r,-i\omega_1-i\omega_2\right)}{\gamma^{(2)}\left(-i\left(z_1+z_2\right)-i\omega_1\left(u_1+u_2\right);-i\omega_1 r,-i\omega_1-i\omega_2\right)}\\
&\times\frac{\gamma^{(2)}\left(-iz_2-i\omega_1u_2;-i\omega_1 r,-i\omega_1-i\omega_2\right)}{\gamma^{(2)}\left(-i\left(z_1+z_2\right)-i\omega_2\left(r-u_1-u_2\right);-i\omega_2 r,-i\omega_1-i\omega_2\right)}\\
&\times\gamma^{(2)}\left(-iz_1-i\omega_2\left(r-u_1\right);-i\omega_2 r,-i\omega_1-i\omega_2\right)\\
&\times\gamma^{(2)}\left(-iz_2-i\omega_2\left(r-u_2\right);-i\omega_2 r,-i\omega_1-i\omega_2\right),
\end{aligned}\tag{4.72}$$

we have the integral identity

$$\begin{aligned}
&\frac{1}{r\sqrt{-\omega_1\omega_2}}\sum_{y=-\lfloor r/2\rfloor}^{\lfloor r/2\rfloor}e^{\frac{\pi iC}{2}}\int_{-\infty}^{\infty}dz\prod_{i=1}^{3}\mathcal{B}\left(a_i-z,u_i-y;b_i+z,v_i+y\right)\\
&=\mathcal{B}\left(a_1+b_2,u_1+v_2;a_2+b_3,u_2+v_3\right)\mathcal{B}\left(a_1+b_3,u_1+v_3;a_2+b_1,u_2+v_1\right).
\end{aligned}\tag{4.73}$$

5. CONCLUSION

This work examines the Pentagon identity through algebraic methods and supersymmetric index calculations. It is believed that there is a map between these objects. One can systematically obtain the operator form of the Pentagon identity from the integral formalism [44] and vice versa.

For the integral side of this identity, the integral pentagon identities for partition functions are believed to be related to certain invariants of corresponding 3-manifolds through the 3d-3d correspondence. This correspondence connects three-dimensional $\mathcal{N} = 2$ supersymmetric theories with triangulated 3-manifolds, and the geometric aspects of the $\mathrm{SL}(2,\mathbb{Z})$ action are naturally encoded in the 3d-3d correspondence [1]. This arises from the compactification of the six-dimensional $\mathcal{N} = (2,0)$ theory on a 3-manifold, resulting in a 3d field theory. The partition function generally arises as follows

$$Z = \sum_{m} \int \frac{1}{|W|} \prod_{j}^{rankG} \frac{dz_j}{2\pi i r} Z_{\text{classical}}\,[z,m] Z_{\text{one-loop}}\,[z,m]. \tag{5.1}$$

One can construct the theory and its mirror dual using group theoretical data. The one-loop contributions from chiral and vector multiplets can be followed from the tables.

Table 5.1. Summary of vector multiplets on various manifolds

Manifold	Vector multiplet
$S^2 \times S^1$	$\prod_\alpha q^{-\frac{1}{2}\lvert\alpha(m)\rvert}\left(1-\alpha(z)^{\pm}q^{\frac{\alpha(m)}{2}}\right)$
$S^3_b/\mathbb{Z}_r$	$\prod_\alpha \left(\hat{s}_{b,\alpha(m)}\left(i\frac{Q}{2}\pm\alpha(z)\right)\right)^{-1}$
S^3_b	$\prod_\alpha \left(\gamma^{(2)}\left(\alpha(z)^{\pm};\omega_1,\omega_2\right)\right)^{-1}$

For the operator side of pentagon identity, using Teichmüller's topological field theory of 3-dimensional manifolds, topological invariants are constructed [45]. This theory associates each tetrahedron in a triangulated 3-manifold with an operator such that $\mathrm{T}_{vw} : \mathcal{H}_v \otimes \mathcal{H}_w \to \mathcal{H}_v \otimes \mathcal{H}_w$ satisfies the Pentagon identity [46] within the Heisenberg Double algebra ($\mathcal{U}_q(sl(2))$). The operator is given by

$$S = \Phi_{\mathrm{b}}^{-1}\left(q_1 - p_1 + p_2\right) e^{2\pi i p_1 q_2}. \tag{5.2}$$

This operator can be interpreted as a flip map between triangulations in Teichmüller's theory

$$S_{12}S_{13}S_{23} = S_{23}S_{12}. \tag{5.3}$$

The Pentagon identity emerges as a fundamental relation governing the consistency condition that encodes the quantization of Teichmüller space through the quantum dilogarithm function [47]. The same algebraic structure $U_q(sl_2)$ governs the symmetries of following integral pentagon identity

$$\int_{-i\infty}^{i\infty} \prod_{j=1} \mathcal{B}\left(a_j - u, b_j + u\right) \frac{\mathrm{d}u}{i\sqrt{\omega_1\omega_2}} = \mathcal{B}\left(a_2 + b_1, a_3 + b_2\right) \mathcal{B}\left(a_1 + b_2, a_3 + b_1\right), \tag{5.4}$$

where $\mathcal{B}(x,y) = \frac{\gamma^{(2)}(x;\omega_1,\omega_2)\gamma^{(2)}(y;\omega_1,\omega_2)}{\gamma^{(2)}(x+y;\omega_1,\omega_2)}$. As a future direction, it appears that for each integral pentagon identity, there is an associated operator pentagon identity related to the same quantum group that governs the Teichmüller space. Therefore, it would be interesting to obtain the map $S \leftrightarrow \mathcal{B}$. Additionally, constructing the invariant of the lens space and investigating the pentagon identity in theory space for quiver theories would be valuable. Lifting the pentagon identity to theory space for quiver theories would also be intriguing, exploring its relation with Seiberg mutation via cluster algebras.

Table 5.2. Characteristics of manifolds and their corresponding chiral multiplets

Manifold	Chiral multiplet
$S^2 \times S^1$	$\prod_j \frac{\left(q^{1-\frac{\Delta_j}{2}+\frac{\left\lvert\rho_j(m)+\phi_j(n)\right\rvert}{2}} \rho_j(z)^{-1}\phi_j(a)^{-1};q\right)}{\left(q^{\frac{\Delta_j}{2}+\frac{\left\lvert\rho_j(m)+\phi_j(n)\right\rvert}{2}} \rho_j(z)\phi_j(a);q\right)}\Bigg)_\infty$
$S_b^3/\mathbb{Z}_r$	$\prod_j \hat{s}_{b,-\rho_j(m)-\phi_j(n)}\left(i\frac{Q}{2}\left(1-\Delta_j\right) - \rho_j(z) - \phi_j(a)\right)$
S_b^3	$\prod_j \gamma^{(2)}\left(\frac{(\omega_1+\omega_2)}{2}\Delta_j + \rho_j(z) + \phi_j(a); \omega_1, \omega_2\right)$

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APPENDIX A: LIE ALGEBRA

A Lie algebra $\mathfrak{g}$ denotes a finite-dimensional vector space over a field F (the real or complex numbers) equipped with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket, which satisfies the following properties

Bilinearity

$$\begin{aligned} [ax + by, z] &= a[x, z] + b[y, z] \\ [z, ax + by] &= a[z, x] + b[z, y] \end{aligned} \tag{A.1}$$

for all $x, y, z \in \mathfrak{g}$ and $a, b \in F$.

Alternativity

$$[x, x] = 0 \tag{A.2}$$

for all $x \in \mathfrak{g}$. (This implies $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.)

Jacobi Identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{A.3}$$

for all $x, y, z \in \mathfrak{g}$. The Jacobi identity is a fundamental property ensuring the coherence of the algebraic structure of $\mathfrak{g}$.

Examples

1. Abelian Lie Algebra: Any vector space V with the Lie bracket defined as

$$[x, y] = 0 \text{ for all } x, y \in V. \tag{A.4}$$

2. Matrix Lie Algebra: The space of all $n \times n$ matrices over F with the Lie bracket defined as $[A, B] = AB - BA$, where A and B are matrices.

3. The Lie Algebra $\mathfrak{su}(2)$

$$\mathfrak{su}(2) = \left\{ A \in \mathfrak{sl}(2, \mathbb{C}) \mid A + A^\dagger = 0 \right\} \tag{A.5}$$

where $A^\dagger$ is the Hermitian conjugate of A.

Structure Constants A Lie algebra $\mathfrak{g}$ can be expressed in terms of a basis $\{e_i\}$ where the Lie bracket of the basis elements is given by

$$[e_i, e_j] = \sum_k c_{ij}^k e_k \tag{A.6}$$

The coefficients c_{ij}^k are called the structure constants of the Lie algebra.

Representations

A Lie algebra representation ρ of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear map $\rho : \mathfrak{g} \to \text{End}\ (V)$ that adheres to the following criteria: - Lie Bracket Preservation: For all $x, y \in \mathfrak{g}$, the representation ρ satisfies $\rho([x, y]) = [\rho(x), \rho(y)]$, where the brackets denote the Lie bracket in $\mathfrak{g}$ and the commutator in End (V), respectively. - Linearity: The map ρ is linear, meaning for any $x \in \mathfrak{g}$ and $v, w \in V$, and for any scalar a, the following properties hold

$$\begin{aligned} \rho(x)(v + w) &= \rho(x)v + \rho(x)w \\ \rho(x)(av) &= a\rho(x)v \end{aligned} \tag{A.7}$$

The first criterion ensures the representation mirrors the algebraic structure of $\mathfrak{g}$. In contrast, the second criterion establishes that the action of $\mathfrak{g}$ on V through ρ is linear, resembling the behavior of a linear operator.

Examples

1. The representation of Angular Momentum Operators in Quantum. Mechanics

The angular momentum operators, denoted as L_x, L_y, and L_z, represent the components of angular momentum along the three Cartesian axes

$$[L_x, L_y] = i\hbar L_z, \quad [L_y, L_z] = i\hbar L_x, \quad [L_x, L_x] = i\hbar L_y, \tag{A.8}$$

To analyze these operators in quantum mechanics, we use a representation ρ that maps each angular momentum operator to a specific linear operator on the Hilbert space of quantum states, where $\hbar$ is the reduced Planck constant. This complex vector space

represents possible quantum states. For instance, $\rho(L_x)$ corresponds to the operator for L_x, affecting the state as if measuring or rotating around the x-axis. The representation ρ preserves the commutation relations of the angular momentum operators

$$\rho([L_z, L_y]) = [\rho(L_x), \rho(L_y)], \tag{A.9}$$

2. The representation of $\mathfrak{sl}(2,\mathbb{C})$ is the spin- $\frac{1}{2}$ representation, which acts on a 2-dimensional complex vector space $V = \mathbb{C}^2$. In this representation, the basis elements H, E, and F are represented by the following matrices

$$\rho(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \rho(E) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \rho(F) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{A.10}$$

These matrices act on vectors in $\mathbb{C}^2$ as follows. For a vector $v = \binom{v_1}{v_2} \in \mathbb{C}^2$

$$\begin{aligned} \rho(H)v &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ -v_2 \end{pmatrix}, \\ \rho(E)v &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_2 \\ 0 \end{pmatrix} \\ \rho(F)v &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ v_1 \end{pmatrix} \end{aligned} \tag{A.11}$$

These matrices preserve the commutation relations of the Lie algebra

$$[\rho(H), \rho(E)] = 2\rho(E), \quad [\rho(H), \rho(F)] = -2\rho(F), \quad [\rho(E), \rho(F)] = \rho(H) \tag{A.12}$$

The spin- $\frac{1}{2}$ representation is particularly important in quantum mechanics, describing the intrinsic angular momentum of particles like electrons.

The adjoint representation

It is a way to represent the Lie algebra as matrices acting on itself via the Lie bracket. The basis of the Lie algebra g is $\{X_i\}$ where i runs over some index set. The Lie bracket of two elements $\boldsymbol{X}_i$ and $\boldsymbol{X}_j$ can be written as

$$[\boldsymbol{X}_i, \boldsymbol{X}_j] = \sum_k c_{ij}^k \boldsymbol{X}_k \tag{A.13}$$

where c_{ij}^k are the structure constants of the Lie algebra. The adjoint representation ad is a map from g to the space of linear transformations on $\mathfrak{g}$ (i.e., to the general

linear Lie algebra $\mathfrak{gl}((g))$. Specifically, for an element $X \in \mathfrak{g}$, the adjoint action of X on another element $Y \in \mathfrak{g}$ is defined as

$$\mathrm{ad}(X)(Y) = [X, Y] \tag{A.14}$$

This can be represented in matrix form. If X is represented as $X = \sum_i x_i X_i$, then the adjoint action ad (X) is a linear transformation given by a matrix whose entries are determined by the structure constants

$$(\mathrm{ad}(X))_{ij} = \sum_k x_k c^i_{kj} \tag{A.15}$$

This matrix acts on the basis vectors X_i via matrix multiplication. In other words, the (i, j)-th entry of the matrix representing $\mathrm{ad}(X)$ is the coefficient of X_i in the expression for $[\boldsymbol{X}, X_j]$

For example, if g is the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ with basis elements H, E, F and Lie brackets

$$[H, E] = 2E, \quad [H, F] = -2F, \quad [E, F] = H \tag{A.16}$$

then, the adjoint representation matrices are

$$\mathrm{ad}(\boldsymbol{H}) = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad \mathrm{ad}(E) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathrm{ad}(F) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{A.17}$$

These matrices show how each basis element acts on the Lie algebra through the adjoint. The Killing form is a symmetric bilinear form on a Lie algebra g. For two elements $X, Y \in \mathfrak{g}$, it is defined as

$$B(X, Y) = \mathrm{Tr}(\mathrm{ad}(X) \cdot \mathrm{ad}(Y)) \tag{A.18}$$

Tr denotes the trace of the matrix product. The Killing form provides important information about the structure of the Lie algebra, such as detecting whether it is semisimple (the Killing form is non-degenerate) or identifying Cartan subalgebras.

A Lie algebra $\mathfrak{g}$ is simple if it is non-abelian and has no non-trivial ideals other than $\{0\}$ and $\mathfrak{g}$ itself. A semisimple Lie algebra can be written as a sum of simple Lie algebras.

APPENDIX B: HYPERGEOMETRIC FUNCTIONS AND THEIR IDENTITIES

B.1. Elliptic Gamma

The general framework of elliptic hypergeometric integrals. For any complex number x and a base p in the complex plane such that $|p| < 1$, the infinite product is defined as

$$\Gamma(x;p,q) \prod_{n_1,n_2\geq 0} \frac{1-x^{-1}p^{n_1+1}q^{n_2+1}}{1-xp^{n_1}q^{n_2}}, \tag{B.1}$$

where we define $(x;p)_\infty = \prod_{j=0}^{\infty}(1-xp^j)$ and the function holds followings: The symmetry

$$\Gamma(z;p,q) = \Gamma(z;q,p), \tag{B.2}$$

the reflection equation

$$\Gamma(z;p,q)\Gamma(pq/z;p,q) = 1, \tag{B.3}$$

the limiting relations

$$\lim_{p\to 0}\Gamma(z;p,q) = \frac{1}{(z;q)_\infty}, \quad \lim_{z\to 1}(1-z)\Gamma(z;p,q) = \frac{1}{(p;p)_\infty(q;q)_\infty}. \tag{B.4}$$

the asymptotic limit

$$\Gamma\left(\mathrm{e}^{-2\pi v u};\mathrm{e}^{-2\pi v\omega_1},\mathrm{e}^{-2\pi v\omega_2}\right) \underset{v\to 0^+}{=} \mathrm{e}^{-\pi\frac{2u-\omega_1-\omega_2}{12v\omega_1\omega_2}}\gamma^{(2)}\left(u;\omega_1,\omega_2\right). \tag{B.5}$$

The shift from the elliptic gamma function to the hyperbolic version occurs uniformly across compact sets.

B.2. Barnes Double Gamma

The function $\Gamma_b(x)$, closely related to the double gamma function explored in [48], can be defined using the integral representation provided in [49]. The fundamental component for all entities that arise in the framework of the quantum algebra $\mathcal{U}_q(sl(2))$ is Barnes' double Gamma function

$$\log\Gamma_b(x) = \int_0^\infty \frac{dt}{t}\left(\frac{e^{-xt}-e^{-Qt/2}}{(1-e^{-bt})(1-e^{-t/b})} - \frac{(Q-2x)^2}{8e^t} - \frac{Q-2x}{t}\right). \tag{B.6}$$

where $Q = \mathrm{b} + \frac{1}{\mathrm{b}}$ and significant relations of $\Gamma_b(x)$ are functional relation

$$\Gamma_b(x+b) = \sqrt{2\pi} b^{bx-\frac{1}{2}} \Gamma^{-1}(bx)\Gamma(x), \tag{B.7}$$

$\Gamma_b(x)$ is defined as meromorphic function where

$$\text{poles: } x = -nb - mb^{-1}, n, m \in \mathbb{Z}^{\geq 0}. \tag{B.8}$$

B.3. Hyperbolic Hypergoemetric Gamma

For a deeper analysis, we introduce the parametrization $z = e^{-2\pi\mu x}$, $q = e^{-2\pi\mu\omega_1}$, and $p = e^{-2\pi\mu\omega_2}$, and explore the asymptotic behavior of the elliptic gamma function in the limit as $\mu \to 0$. This limit is significant for understanding the decoupling mechanisms and dimensional reduction in theories such as 4D $\mathcal{N} = 1$ supersymmetric models. Specifically, the elliptic gamma function $\Gamma(e^{-2\pi\mu x}; e^{-2\pi\mu\omega_1}, e^{-2\pi\mu\omega_2})$ exhibits a significant asymptotic relation, simplifying to [50]

$$e^{-\pi(2x-\omega_1-\omega_2)/12\mu\omega_1\omega_2}\gamma^{(2)}(x;\omega_1,\omega_2), \tag{B.9}$$

as $\mu \to 0$. This relation underscores the function's critical role in the study of supersymmetric gauge theories and their lower-dimensional limits, as discussed in the work of [39]. It holds the following properties. Asymptotic behavior results in Bernouille polynomials

$$\begin{aligned}\gamma^{(2)}(x;\omega_1,\omega_2) : \lim_{x\to\infty} e^{\frac{i\pi}{2}B_{2,2}(x;\omega_1,\omega_2)}\gamma^{(2)}(x;\omega_1,\omega_2^{\circ}) = 1, \arg\omega_1 < \arg x < \arg\omega_2 + \pi,\\ \lim_{x\to\infty} e^{-\frac{i\pi}{2}B_{2,2}(x;\omega_1,\omega_2)}\gamma^{(2)}(x;\omega_1,\omega_2) = 1, \arg\omega_1 - \pi < \arg x < \arg\omega_2.\end{aligned} \tag{B.10}$$

The second order Bernoulli polynomial $B_{2,2}(y;\omega_1,\omega_2)$ defined by

$$B_{2,2}(x;\omega_1,\omega_2) = \frac{x^2}{\omega_1\omega_2} - \frac{x}{\omega_1} - \frac{x}{\omega_2} + \frac{1}{6}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right) + \frac{1}{2}. \tag{B.11}$$

It encapsulates the leading terms of the asymptotic expansion. This is crucial for understanding how functions behave in

$$\gamma^{(2)}(y;\omega_1,\omega_2) = \exp\left(-\int_0^\infty \left(\frac{\sinh(2y-\omega_1-\omega_2)x}{2\sinh(\omega_1 x)\sinh(\omega_2 x)} - \frac{2y-\omega_1-\omega_2}{2\omega_1\omega_2 x}\right)\frac{dx}{x}\right). \tag{B.12}$$

It satisfies the following functional relation

$$\begin{aligned}\frac{\gamma^{(2)}(x+\omega_1;\omega_1,\omega_2)}{\gamma^{(2)}(x;\omega_1,\omega_2)} = 2\sin\frac{\pi x}{\omega_2}, \quad \frac{\gamma^{(2)}(x+\omega_2;\omega_1,\omega_2)}{\gamma^{(2)}(x;\omega_1,\omega_2)} = 2\sin\frac{\pi x}{\omega_1},\\ \gamma^{(2)}(-x;\omega_1,\omega_2) = \gamma^{(2)}(x+\omega_1;\omega_1,\omega_2)^{-1}.\end{aligned} \tag{B.13}$$

reflection property

$$\gamma^{(2)}\left(-x;\omega_1,\omega_2\right)\gamma^{(2)}\left(x+\omega_1;\omega_1,\omega_2\right)=1. \tag{B.14}$$

B.4. Double Sine Function

The double sine function is denoted as $s_b(x)$; it is known for its ability to capture the symmetry of the gauge group and the interactions within the chiral and vector multiplet contributions to the partition function. It is closely related to Faddeev's non-compact quantum dilogarithm[10] $e_b(x)$ [8]

$$e_b(x)=\exp\left\{-\int_{\mathbb{R}+\mathrm{i}0}\frac{dt}{4t}\frac{e^{-2itx}}{\sinh bt\sinh\frac{t}{b}}\right\}, \tag{B.15}$$

where the integration contour goes around the pole $t=0$ in the upper half-plane and b is squash parameter $b=Q+Q^{-1}$. $s_b(x)$ is then related to $e_b(x)$ as follows

$$s_b(x)=e^{\frac{\mathrm{i}\pi}{2}x^2+\frac{\mathrm{i}\pi}{24}\left(b^2+b^{-2}\right)}e_b(x). \tag{B.16}$$

And another representation can be given as

$$s_b(x)=e^{-\frac{i\pi}{2}x^2}\prod_{j=1}^{\infty}\frac{1+e^{2\pi bx+2\pi ib^2\left(j-\frac{1}{2}\right)}}{1+e^{2\pi b^{-1}x+2\pi ib^{-2}\left(\frac{1}{2}-j\right)}}, \tag{B.17}$$

The double sine function $s_b(x)$ exhibits the following properties: Functional properties

$$\frac{s_b\left(x+\frac{\mathrm{i}}{2}b^{\pm1}\right)}{s_b\left(x-\frac{\mathrm{i}}{2}b^{\pm1}\right)}=2\cosh\left(\pi b^{\pm1}x\right), \tag{B.18}$$

reflection property

$$s_b(x)s_b(-x)=1, \tag{B.19}$$

complex conjugation

$$\overline{s_b(x)}=s_b(-\bar{x}), \tag{B.20}$$

zeros/poles

$$\left(s_b(x)\right)^{\pm1}=0\Leftrightarrow\pm x\in\left\{\mathrm{i}\frac{Q}{2}+nb+mb^{-1};n,m\in\mathbb{Z}^{\geq0}\right\}, \tag{B.21}$$

residue

$$\mathrm{Res}_{x=-i\frac{Q}{2}}\,s_b(x)=\frac{\mathrm{i}}{2\pi}, \tag{B.22}$$

asymptotics

[10] The double sine function can be viewed as a non-compact or quantum deformation of the classical dilogarithm function.

$$s_b(x) \sim \begin{cases} e^{-\frac{i\pi}{2}\left(x^2+\frac{1}{12}\left(b^2+b^{-2}\right)\right)} & \text{for } |x| \to \infty, |\arg(x)| < \frac{\pi}{2}, \\ e^{+\frac{i\pi}{2}\left(x^2+\frac{1}{12}\left(b^2+b^{-2}\right)\right)} & \text{for } |x| \to \infty, |\arg(x)| > \frac{\pi}{2} \end{cases}. \tag{B.23}$$

and we can define a function S_b as

$$S_b(x) = s_b\left(\mathrm{i}x - \frac{\mathrm{i}}{2}Q\right) \tag{B.24}$$

normalizes the function, often leading to more symmetric and convenient expressions. We have following relations

$$S_b(x) = \frac{\Gamma_b(x)}{\Gamma_b(Q+-x)}. \tag{B.25}$$

where we define Double Sine in terms of the Barnes double gamma $\Gamma_b(x)$ self-duality

$$S_b(x) = S_{b^{-1}}(x), \tag{B.26}$$

functional relation

$$S_b\left(x+b^{\pm 1}\right) = 2\sin\left(\pi b^{\pm 1}x\right) S_b(x), \tag{B.27}$$

reflection identity

$$S_b(x)S_b(Q-x) = 1. \tag{B.28}$$

The behavior of $S_b(x)$ when $b \to 0$ is described by

$$S_b\left(\frac{\nu}{2\pi b}\right) = e^{-\frac{\mathrm{i}}{2\pi b^2}\left(\frac{1}{4}\nu^2-\frac{\pi}{2}\nu+\frac{1}{6}\pi^2\right)} \exp\left(-\frac{1}{2\pi \mathrm{i} b^2}\mathrm{Li}_2\left(e^{\mathrm{i}\nu}\right)\right)\left(1+\mathcal{O}\left(b^2\right)\right) i \tag{B.29}$$

APPENDIX C: USEFUL RELATIONS

To maintain consistent definitions and avoid confusion, defining all relations with a single function is challenging. Therefore, the functions $G_b, g_b, \phi_b, S_b, s_b, e_b,$ and $\gamma^{(2)}$ are interrelated as described below

$$g_{\mathrm{b}}(x) = \frac{\lambda_{\mathrm{b}}}{G_{\mathrm{b}}\left(\frac{Q}{2} + \frac{1}{2\pi i b}\log x\right)}, \quad \text{with } \lambda_{\mathrm{b}} = \exp\left[\frac{i\pi}{4} + \frac{i\pi}{12}\left(b^2 + b^{-2}\right)\right], \tag{C.1}$$

where

$$G_{\mathrm{b}}(x) = e^{-\frac{i\pi}{2}x(Q-x)}S_{\mathrm{b}}(x), \qquad S_{\mathrm{b}}(x) = \frac{\Gamma_{\mathrm{b}}(x)}{\Gamma_{\mathrm{b}}(Q-x)}, \tag{C.2}$$

It is related to quantum dilogarithm Φ_{b}

$$\begin{aligned} \Phi_{\mathrm{b}}(x) &= AG_{\mathrm{b}}^{-1}\left(-ix + \frac{Q}{2}\right), \\ A &= e^{-i\pi\left(1-4c_{\mathrm{b}}^2\right)/12}, \quad c_{\mathrm{b}} = iQ/2. \end{aligned} \tag{C.3}$$

It is related to $\gamma^{(2)}$ via

$$\gamma^{(2)}\left(-\mathrm{i}\sqrt{\omega_1\omega_2}\,(x + c_b)\,;\omega_1,\omega_2\right) = \frac{e^{\mathrm{i}\pi x^2/2}}{\sqrt{\zeta_{inv}}\Phi_b(x)}. \tag{C.4}$$

where we set $b = \sqrt{\frac{\omega_1}{\omega_2}}$ and $\zeta_{inv} \equiv e^{\pi\mathrm{i}\left(1+2c_b^2\right)/6} = e^{\pi\mathrm{i}c_b^2}\zeta_o^2, \quad \zeta_o \equiv e^{\pi\mathrm{i}\left(1-4c_b^2\right)/12}$. Define S_b is related to double sine up to a shift

$$S_b(x) = s_b\left(\mathrm{i}x - \frac{\mathrm{i}}{2}Q\right) \tag{C.5}$$

APPENDIX D: PROOF OF INTEGRAL PENTAGON IDENTITY

There are various definitions of Ramanujan integral identity, and it can be written in terms of the Quantum dilogarithm function, and details can be found in [40]

$$\Psi_{u,v}(\omega) = \int_{\mathbb{R}} \frac{\Phi_b(z+u)}{\Phi_b(z+v)} e^{2\pi i\omega z} dz. \tag{D.1}$$

The commutation relation of the self-adjoint operators X and P

$$[\mathrm{P}, \mathrm{X}] = \frac{1}{2\pi i}. \tag{D.2}$$

The operator pentagon identity holds

$$\Phi_b(\mathrm{P})\Phi_b(\mathrm{X}) = \Phi_b(\mathrm{X})\Phi_b(\mathrm{P}+\mathrm{X})\Phi_b(\mathrm{P}). \tag{D.3}$$

where the parameter b is element of $\mathbb{C}$. The Fourier transformation of the function

$$F_+(\omega) = \int_{\mathbb{R}} \Phi_b(z) e^{2\pi i\omega z} dz = \Psi_{0,v}(\omega)\big|_{v\to-\infty} = e^{-i\pi\omega^2 + i\pi\left(1-4k^2\right)/12} e_b\left(\omega + k\right). \tag{D.4}$$

Inverse Fourier transformation of QDL

$$(\Phi_b(x)) = \int_{\mathbb{R}} F(z) e^{-2\pi i x z} dz. \tag{D.5}$$

By using the inverse Fourier transform and commutator of X and P on Pentagon relation D.3.

$$e^{-2\pi i x \mathrm{X}} e^{-2\pi i y \mathrm{P}}. \tag{D.6}$$

We can equate the above equation's coefficients, and thus, we get the following integral identity

$$F(x)F(y)e^{2\pi i x y} = \int_{\mathbb{R}} F(z)F(x-z)F(y-z)e^{i\pi z^2} dz. \tag{D.7}$$

Multiply both sides with $e^{-2\pi I\eta y}$ and take the integral over y. We get the following form

$$F(x)\Phi_b(y-x) = \Phi_b(y) \int_{\mathbb{R}} F_+(z)F_+(x-z)e^{i\pi z^2 - 2\pi i y z} dz. \tag{D.8}$$

Use the Fourier transform D.4 analytic proof can be found in [40]

$$\begin{aligned} &\frac{\Phi_b(y-x)}{\Phi_b\left(-x-k\right)\Phi_b(y)} e^{-i\pi\left(1-4k^2\right)/12} \\ &\quad = \int_{\mathbb{R}} \frac{\Phi_b\left(z+k\right)}{\Phi_b\left(z-x-k\right)} e^{-2\pi i z(y+k)} dz = \Psi\left(k, -x-k, -y-k\right). \end{aligned} \tag{D.9}$$

The LHS and RHS of the equation are consistent.