BOUNDARY EFFECTS ON BOSE-EINSTEIN CONDENSATION IN ULTRA-STATIC SPACE-TIMES

by

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We have no knowledge except

what You have taught us...

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ABSTRACT

BOUNDARY EFFECTS ON BOSE-EINSTEIN CONDENSATION IN ULTRA-STATIC SPACE-TIMES

In this theses relativistic boson anti-boson system, inside d dimensional box B with volume V and surface area A, which is a sub-manifold of d dimensional manifold M is studied. Mellin-Barnes Transformations and short time asymptotic of heat kernel are used as mathematical tools. Both Neumann and Dirichlet boundary conditions are considered. Analysis are done for the free energy and charge. Charge is expressed as a function of temperature. Effects of boundaries on temperature is calculated perturbatively. Same methods are employed for studying the neutral gas; free energy and number of particles are studied and again the effect of boundary on temperature is calculated. Apart from these, upper and lower bounds for depletion coefficient of ideal relativistic bose gas in thermodynamic limit is given with the help of Li-Yau and Colbois-Maerten bounds.

ÖZET

ULTRA DURAĞAN UZAY-ZAMANLARDA SINIRLARIN BOSE-EINSTEIN YOĞUŞMASINA ETKİSİ

Bu tezde d boyutlu M çokkatlısının yine aynı boyutta alt çokkatlısı olan, V hacmine, A yüzeyalanına sahip B kutusundaki rölativistik boson anti-boson sistemi incelenmiştir. Bu incelemede Mellin-Barnes Dönüşümü ve ısı çekirdeiğinin kısa zaman için olan asimtot yaklaşımı matematiksel yöntemler olarak kullanılmıştır. Hem Neumann hem de Dirichlet sınır koşulları göz önünde bulundurulmuştur. Analizler serbest enerji ve yük için yapılmıştır.Yük sıcaklığın bir fonksiyonu olarak ifade edilmiş, pertürbatif yöntemle sınırların sıcaklığa olan etkisi hesaplanmıştır. Aynı metodlarla nötür sistemin serbest enerjisi ve parçacık sayısı için de çalışılmış yine sınırların sıcaklık üzerine etkisi hesaplanmıştır. Bunların dışında, Li-Yau ve Colbois-Maerten'in bir Riemann çok katlısındaki Laplace operatörünün özdeğerleri için verdiği alt ve üst sınırlardan istifade edilererek termodinamik limitte yoğuşuğun tükenimi için alt ve üst sınırlar verilmiştir.

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LIST OF SYMBOLS

c	Velocity of light in free space
$g_{\mu\nu}$	Metric tensor
$\hbar$	Reduced Planck's constant
k_B	Boltzmann constant
Tr	Trace
z	fugacity
$\mathcal{Z}$	Grand Partition function
β	Temperature Parameter
γ_{ij}	space components of metric tensor
δ_{ij}	Identity matrix
ϵ_σ	Energy eigenstate
λ_{dB}	De Broglie wavelength
μ	chemical potential

LIST OF ACRONYMS/ABBREVIATIONS

BE	Bose-Einstein
BEC	Bose-Einstein Condensation
FP	Finite Part
JILA	Joint Institute for Laboratory Astrophysics
MIT	Massachusetts Institute of Technology
KG	Klein Gordon Equation
MOT	Magneto Optic Trapping
i.e.	id est (it is; in other words)
etc.	et cetera (and the other things)

1. INTRODUCTION

In 1924, Indian physicist Satyendra Nath Bose rederived Planck's black body radiation by considering light as a gas of identical particles. His idea was rejected for publication in some physics journals. Bose sent his idea to Einstein. His motivation on sending the idea to Einstein was about using Plank's quantization to explain photo electric effect. Einstein was impressed by the Bose's idea. He translated paper into German, submited it on Zeitschrift für Physik and added the note "Bose's derivation of Planck's formula signifies an important advance. The method used also yields the quantum theory of ideal gases which I will work out in detail elsewhere." [1]. After receiving inspiration from Bose, Einstein generalized the idea to massive particles and developed one of the two important quantum statistics of indistinguishable particles, called Bose-Einstein(BE) statistics, which is valid for bose particles i.e. particles with integer spin numbers whose wave function is symmetric under particle interchange. From this statistics Einstein foresaw that below a certain temperature, called the critical temperature, non-interacting bose particles tend to occupy the lowest energy state. This phenomenon is called Bose-Einstein Condensation(BEC).

After Einstein's theoretical prediction, it took a long time to observe BEC in laboratory. Bose particles are preferred to be in gas phase to neglect interactions between them, however almost all particles become liquid or solid when the system is cooled down to temperatures required for BEC. First signs of BEC are noticed in 1938 by Fritz London [2] with superfluid ^{4}He, a liquid discovered by Pyotr Kapitsa, John Allen and Don Misener [3] [4]. The reason under partial observation was the strong inter-particle interaction in liquid phase.

Trapping spin polarized hydrogen in a liquid helium box was a second important attempt to observe BEC. The reason behind using spin polarized hydrogen is that it remains in gaseous phase even at very low temperatures. The pioneers of the attempt are Silvera and Walraven [5]. The method for cooling down the hydrogen inside is based on cooling down the Helium walls. But three body recombination problem caused by

high density(required for BEC) disallows full BEC. Cooling the system to very low temperatures is not enough to observe BEC, trapping also plays a very important role to break the relation of the sample with the walls.

Ambition to observe quantum phenomenon in macro world propelled scientists to develop more cooling and trapping techniques. Thanks to Magneto Optical Trapping(MOT) followed by Evaporative Cooling, BEC is observed. MOT is composed of magnetic trapping and laser cooling systems. The set up for magnetic trapping is basically, two Helmholtz coils, one of them carriying current in clockwise and other in counter-clockwise. Naturally net magnetic force at the middle of set up remains zero. Main principle behind the Laser Cooling is based on Doppler Shift. Six circularly polarized laser beams are directed to trapped particles. The technique reduces the average velocity of the particles by transferring the momentum of photons to the bosons moving in the opposite direction. When the evaporative cooling is applied after MOT, energetic particles are removed from trap, and the system is cold enough to observe BEC. Eric Cornell, Carl Wieman, and co-workers at JILA on 5 June 1995 achieved BEC on^{87}Rb by using MOT and the evaporative cooling techniques [6]. Soon after, Wolfgang Ketterle at MIT condensed ^{23}Na [7].

The purpose of this theses is to study the boundary effects of non-interacting relativistic bose gas in curved space, inside a d dimensional box B having an arbitrary shape, with volume V, surface area A and boundary ∂B. This box is a submanifold of d dimensional manifold M which stands to represent space part of an ultra static space time $\mathbf{R} \times \mathbf{M}$. Induced metric is in the form of

$$ds^2 = -dt^2 + \gamma_{ij} dx^i dx^j. \tag{1.1}$$

If we set $\gamma_{ij} = \delta_{ij}$ it is obvious that Minkowski is a special case of this study. It is assumed that A/V is a finite and small quantity. Advances in trapping techniques provided to trap very large amount of particles thus for some systems it becomes important to consider boundary effects.

Since the mathematics behind statistical mechanics requires to count number of states and particles, the main expressions describing physical quantities are in the form of sums.Expressing density of states in flat space enables us to convert sums into integrals in thermodynamic limit. Studying in curved space makes density of states inexpressible. Thus there seems no way to convert sums into integrals. Studying statistical mechanics in curved space requires alternative mathematical methods to cope with these sums.

In order to study the bose gas, we start with expressing the free energy and depletion coefficient as harmonic sums involving the trace of heat kernel. Since the particles are very energetic, BEC can occur at high temperature in relativistic case. Thus high temperature expansion of relativistic bose gas is studied, $mc^2 \ll k_B T$ is assumed. βm is the expansion parameter. In general, zeta function regularization is used for high temperature expansion [8–15]. In this theses, Mellin transform technique, which is a more rigorous way, is employed to obtain high temperature expansion [16]. The Mellin transform technique is a powerful tool that gives direct correspondence between the poles and residues of transformed function and the asymptotics of the original function [17–19]. The calculations are done for three dimensions and generalized to d dimensions. Depletion coefficients are calculated via perturbative techniques assigning A/V as the small quantity. For charged boson, the leading finite size corrections involve logarithmic βm term, in neutral case the corrections are in the form of powers of βm. The results for free energy are coherent with the ones derived in [12, 14, 15]. Boundedness and large time decay properties of heat kernel together with its short time asymptotic play very important roles in our calculations. Former ones provide convergent Mellin integrals and the latter contains information about the geometry of the box. Besides that, eigenvalue estimates of heat kernel in thermodynamic limit is given. With this estimates upper and lower bounds of depletion coefficient for the charged relativistic bose gas is given and these estimates are in turn used to show the occurrence of the condensation.

The analysis in ultra-static space time can be considered as a motivation to study static space times. [16]

Quantum field theory approach to relativistic boson-antiboson system at high temperature is also very significant that, it is related with spontaneous symmetry breaking. When the temperature of the system is above critical temperature i.e.

$$\mu^2 < \epsilon_0 + m^2 \tag{1.2}$$

(here ϵ_0 is the smallest eigenstate of the system.) it is always possible to have a zero vacuum expectation value this corresponds to unbroken symmetry. As the temperature falls, chemical potential increases and reaches to a critical value i.e. $\mu_c^2 = \epsilon_0 + m^2$. This condition determines the critical temperature. At critical temperature it is possible to have non-zero vacuum expectation value which corresponds to a broken symmetry. Thus, phase transition in BEC can be interpreted as spontaneous symmetry breaking. [14, 15, 20]

It is aimed to cover following topics in this theses. In chapter 2 the derivation of Bose Einstein Condensation is given via microcanonical ensemble. Chapter 3 gives the derivation of heat kernel from heat equation and a sense on the short time asymptotic, upper and lower bounds of heat kernel. Chapter 4 deals with the expression for free energy of charged boson in curved space. Subordination identity and Mellin-Barnes transformation for harmonic sums are used in deriving the high temperature expansion. In chapter 5, charge is calculated and the relation between charge and temperature is given. In chapter 6 and 7 same calculations are done for the neutral boson system. In chapter 8 upper and lower bounds of depletion coefficient for relativistic ideal gas in thermodynamic limit is given. In chapter 9 some commentaries are provided for this work. As a last step, in the appendixes A, B and C; technical details are given for the mathematical tools used in the text.

This theses relies heavily on the article "Boundary effects on Bose-Einstein condensation in ultra-static space-times" [16] and partly on the article "Bose-Einstein condensation on a manifold with non-negative Ricci curvature" [21].

2. BOSE EINSTEIN CONDENSATION

Quantum indistinguishablity of bosons with their tendency to occupy same quantum state is at heart of Bose Einstein Condensation(BEC). To analyse statistics of N_s bosons with M_s number of quantum states, we use M_s-1 walls to describe M_s quantum states and N_s number of balls to describe indistinguishable bose particles. Arranging $(N_s + M_s - 1)$ balls and walls in a row in any order can be done in

$$W_s = \frac{(N_s + M_s - 1)!}{N_s!(M_s - 1)!} \tag{2.1}$$

different ways.
Consider a particle in free space, with periodic boundary conditions

$$\psi(x, y, z) = \psi(x + L_x, y, z) = \psi(x, y + L_y, z) = \psi(x, y, z + L_z) \tag{2.2}$$

The solution of the Schrödinger equation reads

$$\psi(r) = \frac{1}{V^{1/2}} e^{ik.r} \quad , \quad \epsilon_k = \frac{\hbar^2 k^2}{2m} \quad \text{with} \quad k = \left(\frac{2\pi\, n_x}{L_x}, \frac{2\pi\, n_y}{L_y}, \frac{2\pi\, n_z}{L_z} \right) \tag{2.3}$$

The distance between any consecutive $k_{x,y,z}$ vectors is $\frac{2\pi}{L_{x,y,z}}$. Therefore the infinitesimal volume $d^3k = dk_x dk_y dk_z$ in k-space has $\frac{V}{(2\pi)^3} d^3k$ quantum states. Thus inside a shell of radius k_s and thickness δk_s we have

$$M_s = 4\pi k_s^2 \delta k_s \frac{V}{(2\pi)^3} \tag{2.4}$$

quantum states. In terms of single particle energy ϵ_s ,

$$M_s = \frac{V m^{3/2} \epsilon^{1/2}}{\sqrt{2}\pi^2 \hbar^3} \delta\epsilon_s = V g(\epsilon_s) \delta\epsilon_s \tag{2.5}$$

from which one can easily pass to the density of states,

$$g(\epsilon) = \frac{m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\epsilon^{1/2} \tag{2.6}$$

At that point we use the definition definition of entropy $S = k_B \ln W$ where k_B is the Boltzmann's constant and W is the number of available configurations of quantum states for a given energy E. This quantity is calculated as the multiplication of possible quantum states for each shell mentioned above;

$$W = \prod_s W_s = \prod_s \frac{(N_s + M_s - 1)!}{N_s!(M_s - 1)!} \tag{2.7}$$

Entropy is;

$$S = k_B \ln W = k_B \sum_s \left[(N_s + M_s)\ln(N_s + M_s) - N_s \ln N_s - M_s \ln M_s\right] \tag{2.8}$$

It is appropriate to use Lagrange multipliers method to maximize the entropy, with constraints $N = \sum_s N_s$ and $U = \sum_s \epsilon_s N_s$ and Lagrange multipliers $-k_B\beta\mu$ and $k_B\beta$.

$$\frac{\partial S}{\partial N_s} - k_B\beta\frac{\partial U}{\partial N_s} + k_B\beta\mu\frac{\partial N}{\partial N_s} = 0 \tag{2.9}$$

Taking the differentials and calculating for N_s gives;

$$N_s = \frac{M_s}{e^{\beta(\epsilon_s - \mu)} - 1} \tag{2.10}$$

Please note that in this theses we study the relativistic boson anti-boson system in which the constraints are;

$$U = \sum_s \epsilon_s(N_{s_1} + N_{s_2}) \quad and \quad Q_s = \sum_s N_{s_1} - N_{s_2} \tag{2.11}$$

second one is called the charge of the system. Here N_{s_1} is the number of bosons and N_{s_2} is the number of anti-bosons in energy state s. For this systems introduce the Lagrange multipliers as $-k_B\beta\mu_1$, $-k_B\beta\mu_2$, $k_B\beta$. Entropy can be calculated analogously,

$$\begin{aligned} S &= k_B \sum_s (N_{s_1} + N_{s_2} + M_s) \ln(N_{s_1} + N_{s_2} + M_s) \\ &- N_{s_1} \ln N_{s_1} - N_{s_2} \ln N_{s_2} - M_s \ln M_s \end{aligned} \tag{2.12}$$

Maximizing the entropy via Lagrange multipliers with respect to the the variables N_{s_1}, N_{s_2} gives;

$$\frac{\partial S}{\partial N_{s_1}} - k_B\beta \frac{\partial U}{\partial N_{s_1}} + k_B\beta\mu_1 \frac{\partial Q_s}{\partial N_{s_1}} = 0 \tag{2.13}$$

and

$$\frac{\partial S}{\partial N_{s_2}} - k_B\beta \frac{\partial U}{\partial N_{s_2}} + k_B\beta\mu_2 \frac{\partial Q_s}{\partial N_{s_2}} = 0 \tag{2.14}$$

From last two relations we have the result;

$$\mu_1 = -\mu_2 \tag{2.15}$$

Returning back to the neutral system, Bose-Einstein distribution can be introduced as;

$$f_{BE}(\epsilon) = \frac{1}{e^{\beta(\epsilon-\mu)} - 1} \tag{2.16}$$

To understand the physical meanings of Lagrange multipliers we consult to the first law of thermodynamics,

$$dU = TdS - PdV + \mu dN \tag{2.17}$$

which can also be written as

$$dS = \frac{1}{T}(dU + PdV - \mu dN) \tag{2.18}$$

On the other hand writing entropy as a differential gives;

$$dS = \sum_s \frac{\partial S}{\partial N_s} dN_s = k_B\beta \sum_s \left(\frac{\partial U}{\partial N_s} - \mu \frac{\partial N}{\partial N_s} \right) = k_B\beta(dU - \mu dN) \tag{2.19}$$

Comparing with the second form of first law of thermodynamics gives $\beta = \frac{1}{k_B T}$ and constant μ is the chemical potential.

The above distribution i.e. Eq. 2.16 can also be derived via grand canonical ensemble. Since grand canonical ensemble is used throughout next sections, we give a brief discussion of it here. Consider a small system inside a heat bath composed of N variable particles with variable energy E inside a fixed volume V. Energy and particle exchange between the heat bath and the small system is allowed. Large system which is the combination of heat bath and the small system, is composed of N_0 number of fixed particles with fixed energy E_0 inside a fixed volume V_0. If the small system has N particles then it can be in one of the below energy states;

$$E_{N_1} \leq E_{N_2} \ldots \leq E_{N_j} \leq \ldots \tag{2.20}$$

Let the small system be in the E_{N_j} energy state, then the macrostate of heat bath can be described with, $E_0 - E_{N_j}$ energy, $N_0 - N$ particles and $V_0 - V$ volume. There are many possible microstates to describe this macrostate. Averaging postulate of statistical mechanics states that all microstates are equally probable. Based on this postulate, we can conclude that number of microstates corresponding to a certain macrostate is proportional to probability of observing this macrostate. Define the statistical weight of the macrostate as;

$$\Omega = \Omega(E_0 - E_{N_j}, V_0 - V, N_0 - N) \tag{2.21}$$

Then the probability of observing this macrostate is;

$$P_{N_j} = G \times \Omega \tag{2.22}$$

where G is the normalization constant. Entropy of the system is;

$$S = k_B \ln \Omega \tag{2.23}$$

Probability can be expressed as;

$$P_{N_j} = G\, e^{S/k_B} \tag{2.24}$$

Since the small system is very small compared to the heat bath, i.e.

$$V \ll V_0, \qquad N \ll N_0, \qquad E_{N_j} \ll E_0 \tag{2.25}$$

entropy of heat bath can be Taylor expanded as;

$$\begin{aligned} S(E_0 - E_{N_j}, V_0 - V, N_0 - N) &= S(E_0, V_0, N_0) - E_{N_j} \left(\frac{\partial S}{\partial E_0}\right)_{V_0,N_0} \\ &- V \left(\frac{\partial S}{\partial V_0}\right)_{E_0,N_0} - N \left(\frac{\partial S}{\partial N_0}\right)_{E_0,V_0} \end{aligned} \tag{2.26}$$

Using the the thermodynamic definitions;

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E_0}\right)_{V_0,N_0} \qquad \mu = -T \left(\frac{\partial S}{\partial N_0}\right)_{E_0,V_0} \tag{2.27}$$

and remembering that there is no volume change, we can write S as;

$$S = S(E_0, V_0, N_0) - \frac{E_{N_j}}{T} + \frac{\mu N}{T} \tag{2.28}$$

Now with the help of Eq. 2.24 we can write the probability of observing the small system with N particles at energy E_{N_j} is;

$$P_{N_j} = \frac{e^{\beta(\mu N - E_{N_j})}}{\mathcal{Z}} \tag{2.29}$$

where $\mathcal{Z}$ is sum over all possible number of particles and all energy states i.e.

$$\mathcal{Z} = \sum_{N=0}^{N_0} \left(\sum_{j=1}^{\infty} e^{\beta(\mu N - E_{N_j})} \right) \tag{2.30}$$

This is called grand partition function. Now let us carry these definitions to a more microstate by expressing everything in terms of single particle states and define the probability distribution function with relations;

$$E_{N_j} = \sum_i \epsilon_i n_i, \qquad N = \sum_i n_i, \qquad P_{N_j} = \prod_i p_i(n_i) \tag{2.31}$$

the grand partition function becomes;

$$\prod_i \mathcal{Z}_i \quad where \quad \mathcal{Z}_i = \sum_{n_i} e^{\beta(\mu - \epsilon_i) n_i} \tag{2.32}$$

Since there is no restriction on particle number occupying a single state for bose particles, i.e. $n_i = 0, 1, 2, 3...$ one can express the grand partition function as;

$$\mathcal{Z} = \prod_i \frac{1}{1 - e^{\beta(\mu - \epsilon_i)}} \tag{2.33}$$

At this point we define grand potential which is a useful expression to study the thermodynamics of the system;

$$-\frac{1}{\beta} \ln \mathcal{Z} = -\frac{1}{\beta} \sum_i \ln(1 - e^{\beta(\mu - \epsilon_i)}) \tag{2.34}$$

one can also write;

$$p_i(n_i) = \frac{e^{\beta(\mu-\epsilon_i)n_i}}{\mathcal{Z}_i} \tag{2.35}$$

which is the probability of finding n_i particles in energy state ϵ_i. In order to get mean occupation number we use the relations;

$$< n_i >= \sum_{n_i=0}^{\infty} n_i \, p_i(n_i) = \frac{1}{\beta}\left(\frac{\partial \ln \mathcal{Z}_i}{\partial \mu}\right) \tag{2.36}$$

After a careful algebra one can find the Bose-Einstein distribution as;

$$< n_i >= \frac{1}{e^{\beta(\epsilon_i-\mu)} - 1} \tag{2.37}$$

Now we are ready to derive BEC, which is a very interesting phenomena observed when the Bose particles are at sufficiently low temperatures. BEC occurs in the momentum space even in the absence of any interaction so it is very different from classical condensation of vapour.

When a particle becomes lighter and slower wave nature of particle becomes more and more dominant than the particle nature that de Broglie wave length λ_{dB} of the particle extends. In a cold gas of atoms if the inter-particle distance is comparable with λ_{dB} then wavelengths of neighbouring atoms start to overlap and form a giant single matter wave and BEC occurs.

From Bose-Einstein distribution total number of particles in a box is,

$$N = \sum_k \frac{1}{e^{\beta(\epsilon_k-\mu)} - 1} \tag{2.38}$$

In the thermodynamic limit ($V \to \infty$) we can replace summation with an integration

$$\sum_k \to \int \frac{V}{(2\pi)^3} d^3k \tag{2.39}$$

Then Eq.2.38 can be written as;

$$N = \frac{V}{(2\pi)^3} \int \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1} \, d^3k \tag{2.40}$$

expressing the particle density N/V in terms of density of states gives,

$$n = \int_0^\infty \frac{1}{e^{\beta(\epsilon - \mu)} - 1} \, g(\epsilon) \, d\epsilon \tag{2.41}$$

Most of the time it is easier to determine particle density and the temperature of the system rather than the chemical potential μ. That is why it is convenient to consider Eq.2.41 as $\mu(n, T)$ although it is in the form of $n(T, \mu)$. We introduce two new dimensionless variables fugacity, $z = e^{\beta\mu}$ and $x = \beta\epsilon$ and express Eq.2.41 with these variables as;

$$n = \frac{(mk_BT)^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty \frac{ze^{-x}}{1 - ze^{-x}} \, x^{1/2} dx \tag{2.42}$$

Expanding the first term of integrand as;

$$\frac{ze^{-x}}{1 - ze^{-x}} = ze^{-x}(1 + ze^{-x} + z^2e^{-2x} + ...) = \sum_{p=1}^{\infty} z^p e^{-px} \tag{2.43}$$

and inserting into Eq.2.42 gives

$$\sum_{p=1}^{\infty} z^p \int_0^\infty e^{-px} x^{(1/2)} dx \tag{2.44}$$

After change of variables integral part becomes equal to the $\Gamma(3/2)$ and Eq.2.42 becomes;

$$n = \left(\frac{mk_BT}{2\pi\hbar^2}\right)^{3/2} g_{3/2}(z) \tag{2.45}$$

where

$$g_{3/2}(z) = \sum_{p=1}^{\infty} \frac{z^p}{p^{3/2}} \tag{2.46}$$

This series is convergent if $|z| < 1$ and divergent if $|z| > 1$ At $z = 1$,

$$g_{3/2}(1) = \sum_{p=1}^{\infty} \frac{1}{p^{3/2}} = \zeta\left(\frac{3}{2}\right) = 2.612 \tag{2.47}$$

where

$$\zeta(s) = \sum_{p=1}^{\infty} \frac{1}{p^s} \tag{2.48}$$

is the Riemann zeta function. In order to determine chemical potential we write the Eq.2.45 as;

$$g_{3/2}(e^{\beta}\mu) = \left(\frac{2\pi\hbar^2}{mk_BT}\right)^{3/2} n \tag{2.49}$$

High temperature and low density corresponds to small values of $g_{3/2}(z)$ function and for small z expansion we have $g_{3/2}(z) \approx z + ...$ For chemical potential we have;

$$\mu \approx -\frac{3}{2}k_BT \ln\left(\frac{mk_BT}{2\pi\hbar^2 n^{2/3}}\right) \tag{2.50}$$

Cooling the gas to lower temperatures the value of z increases and becomes 1, at that point $\mu = 0$. The temperature where this happens(for fixed density) defines the critical

temperature, T_c,

$$T_c = \frac{2\pi\hbar^2}{k_B m}\left(\frac{n}{2.612}\right)^{2/3} \tag{2.51}$$

At that point BEC starts and if we go on to cool down the gas μ does not change and stay 0. Out of N particles macroscopic number N_0 start to occupy the $\epsilon_k = 0$ quantum state. By using Bose Einstein distribution number of particles sitting in the state $\epsilon_k = 0$ can be calculated as;

$$N_0 = \frac{1}{e^{-\beta\mu} - 1} \tag{2.52}$$

Writing the same equation for μ gives;

$$\mu = -k_B T \ln\left(1 + \frac{1}{N_0}\right) \sim -k_B T \frac{1}{N_0} \tag{2.53}$$

To concern the remaining particles out of condensate we may write;

$$N = N_0 + \sum_{k\neq 0} \frac{1}{e^{\beta\epsilon_k} - 1} \tag{2.54}$$

Writing in terms of densities and replacing the k summation by an integral gives;

$$n = n_0 + \frac{(mk_B T)^{3/2}}{\sqrt{2}\pi^2\hbar^3} \int_0^\infty \frac{e^{-x}}{1 - e^{-x}} x^{1/2} dx \tag{2.55}$$

Integral part is equal to $\Gamma(3/2)\,\zeta(3/2)$. For $T < T_c$,

$$n = n_0 + 2.612\left(\frac{mk_B T}{2\pi\hbar^2}\right)^{3/2} \tag{2.56}$$

and,

$$\frac{n_0}{n} = 1 - \left(\frac{T}{T_c}\right)^{3/2} \tag{2.57}$$

When $T = 0$ all of the particles condensate $n_0 = n$, when $T = T_c$ $n_0 = 0$, in between various amounts of condensation occurs. See figure 2.1.

Please also note that, when the exponential in Eq.2.52 is Taylor expanded and divided by V we have;

$$n_0 = -\frac{1}{V\beta\mu} \tag{2.58}$$

Just at the critical temperature when n_0 is 0 if the volume is finite we need a divergent μ. Below critical temperature ($\mu = 0$) macroscopic occupation of ground sate is observed.From the above relation one can see that we need $V \to \infty$ limit to get a finite ground state number density. Thus at finite volume BEC does not occur [22, 23].

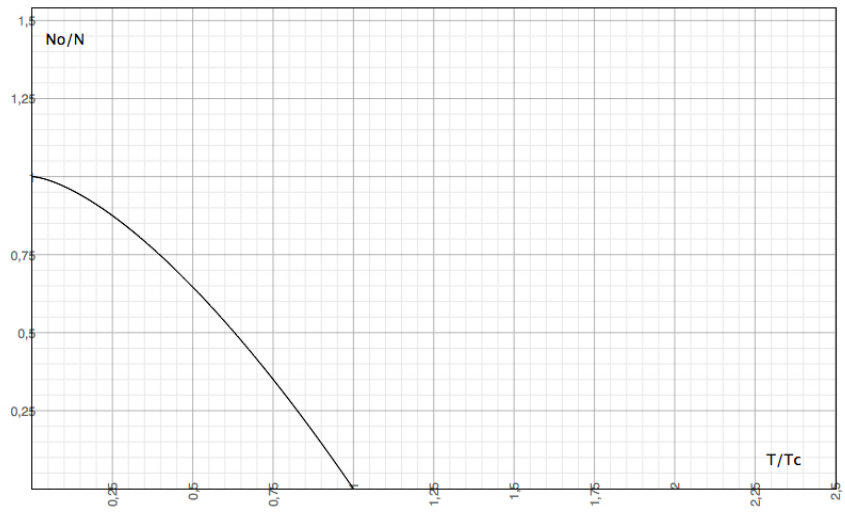

Figure 2.1. Condensate fraction versus temperature .

3. HEAT EQUATION AND THE HEAT KERNEL

In this theses short time asymptotics of heat kernel on a manifold is used. To reach this asymptotics we first study heat equation and the heat kernel in flat space.

3.1. Heat Equation

Heat equation sometimes known as the diffusion equation is;

$$u_t - \Delta u = 0 \tag{3.1}$$

Here u is the heat density and the equation describes the evolution of this density in time. Here the Laplacian Δ takes the derivatives with respect to space coordinates only. $t > 0$ and $x \in O$ where $O \subset \mathbb{R}^n$. The idea behind the heat equation is simple and can be considered as the motion of a dye drop inside a stationary liquid. Fick's first law of diffusion states that the dye moves from higher concentration regions to lower concentration regions with an amount proportional to the gradient of concentration. Let $V \subset O$ be the region of interest. Fick's second law states that, rate of change with respect to time of the total quantity within V equals to the negative of net flux through ∂V. i.e.

$$\frac{d}{dt}\int_V u\,dx = -\int_{\partial V} k(\mathbf{n}.\nabla u)\,dS \tag{3.2}$$

Where k is the proportionality constant. By Stoke's theorem and unit proportionality constant we have the heat equation;

$$u_t = \Delta u \tag{3.3}$$

Now the aim is to find the fundamental solution i.e. the heat kernel. The equation tells us the first order time derivative of solution is equivalent to second order space

derivative. With this information one can suggest that if $u(x,t)$ is a solution then by chain rule $u(\lambda x, \lambda^2 t)$ is also a solution for $\lambda \in \mathbb{R}$. This scaling implies a solution $v(y)$ where $y = \frac{|x|^2}{t}$. By direct substitution of $v(y)$ one can find that $\frac{1}{t^{n/2}} e^{\frac{-|x|^2}{4t}}$ solves the heat equation (3.3).

The translation of solution $u(x,t)$ i.e. $u(x-y,t)$ is also a solution for a fixed y. When the initial value is indicated for the problem;

$$u_t - \Delta u = 0 \qquad \text{in } \mathbb{R}^n \times (0,\infty) \tag{3.4}$$

$$u = g \qquad \text{on } \mathbb{R}^n \times (t=0) \tag{3.5}$$

by convolution;

$$u(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{(x-y)^2}{4t}} g(y)\, dy \quad (x \in \mathbb{R}^n,\ t>0) \tag{3.6}$$

is the solution for the initial value problem [24, 25].

$$K(x,y,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{\frac{-|x-y|^2}{4t}} & (x,y \in \mathbb{R}^n, t>0) \\ \delta(x,y) & (x,y \in \mathbb{R}^n, t \to 0) \end{cases} \tag{3.7}$$

is called the heat kernel on $\mathbb{R}^n$, with the Euclidean metric and is also known as transition probability density of Brownian motion. To get 1 from integral of the heat kernel over $\mathbb{R}^n$ the prefactor $\frac{1}{(4\pi)^{n/2}}$ is set.

The above analysis for heat equation is a special case where the space is flat without a boundary, when a boundary is introduced the solution depends on the shape of the system. But when the short time asymptotics of heat kernel is considered, this dependence has a universal form through the heat kernel coefficients, which are given in terms of curvature scalars of the bulk and the boundary. When the heat kernel in curved space with a boundary is considered above analysis become very complicated. In the next section some information, used in the text, about the short time asymptotics

of heat kernel is summarized without proofs.

3.2. Short-Time Asymptotics of the Heat Kernel

$$Tr\, e^{t\Delta} = \int_M dV\, K(x,x,t) \tag{3.8}$$

and

$$Tr\, e^{t\Delta} = \sum_{\sigma=0}^{\sigma=\infty} e^{-t\epsilon_\sigma} \tag{3.9}$$

are two important relations used in the text. From these relations one can see the relation between the energy eigenstates and the heat kernel. Here Δ is the Laplacian with eigenvalues $-\epsilon_\sigma$ in the corresponding manifold.

As an application we can use the above properties to study total number of bose particles in a box. After a little algebra we can see in three dimensional flat space;

$$\sum_{\sigma=0}^{\infty} e^{-t\epsilon_\sigma} = \frac{V}{(4\pi t)^{3/2}} \tag{3.10}$$

For total number of particles derived from Bose-Einstein distribution note that;

$$N = \sum_{\sigma=0}^{\infty} \frac{1}{e^{\beta(\epsilon_\sigma - \mu)} - 1} = \sum_{\sigma=0}^{\infty} \sum_{k=1}^{\infty} e^{k\beta\mu}\, e^{-k\beta\epsilon_\sigma} \tag{3.11}$$

combining the last two equations we can get a result consistent with the one in previous chapter. i.e.

$$n = \frac{1}{(4\pi\beta)^{3/2}}\, g_{3/2}(e^{\beta\mu}) \tag{3.12}$$

where n is the density of particles and

$$g_{3/2}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^{3/2}} \tag{3.13}$$

is the Bose-Einstein function.

In this theses high temperature regime is considered. As we will see in the next sections, high temperature corresponds to small time in the heat kernel. Heat kernel with a boundary has the expansion of the form;

$$\int_M dV\, K(x,x,t) \sim t^{(-d/2)} \left(a_0 + a_{1/2} t^{1/2} + a_1 t + \ldots\right) \tag{3.14}$$

as $t \to 0$, where d is the dimension of the manifold and a's are known as heat kernel coefficients or Hadamard - Minakshisundaram - DeWitt - Seeley (HMDS) coefficients. Here are only the first few of these coefficients for the operator

$$-\Delta + U \tag{3.15}$$

in the presence of Dirichlet and Neumann boundary conditions;

$$\begin{aligned} a_0 &= \frac{1}{(4\pi)^{d/2}} \int_B dV \\ a_{1/2} &= r \frac{1}{4(4\pi)^{\frac{d-1}{2}}} \int_{\partial B} dS \\ a_1 &= \frac{1}{6(4\pi)^{d/2}} \left[\int_B dV(-6U + R) + 2 \int_{\partial B} dS\, K_{aa} \right] \end{aligned} \tag{3.16}$$

Here r is 1 for Neumann boundary condition and -1 for Dirichlet boundary condition, R is the Ricci scalar, K_{ab} is the extrinsic curvature of ∂B and summation convention is assumed. Please note that these coefficients are related to the geometry of the system.

For the theorems and details behind the above information one can consult [26–

29].

3.3. Eigenvalue Bounds for the Heat Kernel

In this section we give a brief discussion about the eigenvalue bounds of the heat kernel, which is used to discuss to consider relativistic ideal bose gas in thermodynamic limit.

In order to study eigenvalue bounds of heat kernel on a Riemannian manifold one should introduce some more restrictions on the geometry.

Let (S, g) be a d dimensional Riemannian manifold with metric g and nonnegative Ricci curvature. M is an open, connected submanifold of S. Eigenvalues and square-integrable eigenfunctions of Laplacian in the presence of Neumann boundary conditions is described as;

$$-\Delta f_\sigma = \epsilon_\sigma f_\sigma, \quad \hat{n} \cdot \nabla f_\sigma|_{\partial M} = 0 \tag{3.17}$$

Eigenvalues are ordered as;

$$\epsilon_0 < \epsilon_1 \leq \ldots \to \infty \tag{3.18}$$

The ground state is

$$f_0 = \frac{1}{\sqrt{V}} \tag{3.19}$$

with eigenvalue $\nu_0 = 0$. For the geometry described above Lie and Yau bounds exist:

$$\frac{1}{(4\pi t)^{d/2}} V \leq \operatorname{Tr} e^{\Delta t} \leq \tilde{C}(d) g(t) \tag{3.20}$$

where $\tilde{C}(d)$ is a positive constant which depends only on the dimension d, and $g(t)$ is a function described as;

$$g(t) = \begin{cases} \left(\frac{D_M}{\sqrt{t}}\right)^d & \text{if } \sqrt{t} \leq D_M \\ 1 & \text{if } \sqrt{t} \geq D_M \end{cases}$$

Here D_M is the diameter of manifold M.
Considering the upper bound of Eq. 3.20 one can write;

$$\sum_{\sigma=0}^{\infty} e^{-\epsilon_\sigma t} \leq \tilde{C}(d)g(t) \tag{3.21}$$

equivalently;

$$\left(1 + e^{-\epsilon_1 t} + e^{-\epsilon_2 t} + \ldots + e^{-\epsilon_\sigma t} + \ldots\right) \leq \tilde{C}(d)g(t) \tag{3.22}$$

$$(k+1)e^{-\epsilon_k t} \leq \left(1 + e^{-\epsilon_1 t} + e^{-\epsilon_2 t} + \ldots + e^{-\epsilon_\sigma t}\right) \leq \tilde{C}(d)g(t) \tag{3.23}$$

also holds. As a result we have;

$$(\sigma + 1) \leq e^{\epsilon_\sigma t}\tilde{C}(d)g(t) \tag{3.24}$$

This inequality holds for all t we get the best upper bound by minimizing the right hand side;

$$\epsilon_\sigma g(t_0) + g'(t_0) = 0 \tag{3.25}$$

Considering the conditions given on function g, we see that $\sqrt{t_0} < D_M$ and result for t_0 is;

$$t_0 = \frac{d}{2\epsilon_\sigma} \tag{3.26}$$

Combining this result with (3.24) we get the bound

$$\epsilon_\sigma \geq \frac{C(d)}{D_M^2}\,(\sigma+1)^{2/d} \geq \frac{C(d)}{D_M^2}\,\sigma^{2/d} \tag{3.27}$$

[30]. We also have the following upper bound of Colbois and Maerten [31] for the eigenvalues

$$\epsilon_\sigma \leq B(d)\left(\frac{\sigma}{V}\right)^{2/d} \tag{3.28}$$

Again $B(d)$ is a positive constant that depends only on the dimension. After exponentiating the bounds (3.27, 3.28) and summing up all the terms, one can write the expression for the heat kernel $\mathrm{Tr}' e^{\Delta t}$ (Please note that the ground state is omitted);

$$\frac{1}{V}\sum_{\sigma=1}^{\infty} e^{-\frac{tB}{V^{2/d}}\,\sigma^{2/d}} \leq \frac{1}{V}\sum_{\sigma=1}^{\infty} e^{-t\epsilon_\sigma} \leq \frac{1}{V}\sum_{\sigma=1}^{\infty} e^{-\frac{tC}{D_M^2}\,\sigma^{2/d}} \tag{3.29}$$

When the above Riemann sums compared with integrals we get stronger integral bounds;

$$\frac{1}{V}\int_1^{\infty} dx\, e^{-tBx^{2/d}} \leq \frac{1}{V}\,\mathrm{Tr}' e^{\Delta t} \leq \frac{D_M^d}{V}\int_0^{\infty} dx\, e^{-tCx^{2/d}} \tag{3.30}$$

when the integrals are evaluated explicitly we have the result;

$$\frac{1}{B^{d/2}}\,\frac{d}{2}\,\Gamma\left(\frac{d}{2}, \frac{t\,B}{V^{2/d}}\right)\left(\frac{1}{t}\right)^{d/2} \leq \frac{1}{V}\,\mathrm{Tr}' e^{\Delta t} \leq \frac{D_M^d}{V\,C^{d/2}}\,\Gamma\left(\frac{d}{2}+1\right)\left(\frac{1}{t}\right)^{d/2} \tag{3.31}$$

Here $\Gamma(x, y)$ is the incomplete gamma function.

4. FREE ENERGY OF CHARGED BOSONS

Consider the Klein-Gordon equation (KG) with an external potential U on $\mathbf{R}\times\mathbf{M}$ manifold with an induced ultra-static metric $ds^2 = -dt^2+\gamma_{ij}dx^i dx^j$, in which $\partial_t\gamma_{ij} = 0$.

$$(-\Box + m^2 + U)\psi = 0 \tag{4.1}$$

where $\Box$ is the D'alembertian for the scalar field defined as;

$$\Box = \frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu \tag{4.2}$$

If we separate the time derivative, we can write the KG as;

$$-\frac{\partial^2\psi}{\partial t^2} = \left[-\Delta + U + m^2\right]\psi \tag{4.3}$$

where

$$\Delta = \frac{1}{\sqrt{\gamma}}\partial_i\sqrt{\gamma}\gamma^{ij}\partial_j \tag{4.4}$$

is the Laplacian on $\mathbf{M}$. Separating time and space part of wave function and substituting into KG as,

$$\psi = e^{it(\lambda_\sigma+m^2)^{1/2}}\chi(x) \tag{4.5}$$

gives the relation;

$$(\lambda_\sigma + m^2)\psi = (-\Delta + U + m^2)\psi \tag{4.6}$$

Obviously, λ'_σs are eigenvalues of the operator $-\Delta + U$ which is assumed to be non negative and we define the single particle Hamiltonian as

$$H = -\Delta + U + m^2 \tag{4.7}$$

which is naturally positive by the above assumption.
In grand canonical ensemble Free Energy of the charged bosons is expressed as;

$$A = \frac{1}{\beta} \sum_\sigma \left[\log\left(1 - e^{-(\beta^2\lambda_\sigma + \beta^2 m^2)^{1/2} + \beta\mu}\right) + \log\left(1 - e^{-(\beta^2\lambda_\sigma + \beta^2 m^2)^{1/2} - \beta\mu}\right)\right] \tag{4.8}$$

After applying the identity below, which is easily derived after integrating the geometric series expansion of $1/(1-x)$, to the Eq.4.8

$$-\log(1-x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \tag{4.9}$$

we get;

$$A = \sum_\sigma \sum_{k=1}^{\infty} 2\cosh(k\beta\mu)\, \frac{e^{-k(\beta^2\lambda_\sigma + \beta^2 m^2)}}{k\beta} \tag{4.10}$$

at that point we use the subordination identity (for proof consult Appendix A) i.e.

$$e^{-b\sqrt{x}} \;=\; \frac{b}{2\sqrt{\pi}} \int_0^\infty \frac{dt}{t^{3/2}}\, e^{-\frac{b^2}{4t}}\, e^{-tx} \tag{4.11}$$

for the exponential part and adapt the right hand side of the following notation;

$$\sum_\sigma e^{\frac{-t}{m^2}\lambda_\sigma} = Tr\, e^{\frac{t}{m^2}(\Delta - U)} \tag{4.12}$$

and change the integration variable to $t = s\beta^2 m^2$ in (4.8) we express the free energy of the system as;

$$A = -\sum_{k=1}^{\infty} \frac{m}{\sqrt{\pi}} \cosh(k\beta\mu)\, F(k\beta m) \tag{4.13}$$

where

$$F(x) = \int_0^{\infty} \frac{dt}{t^{3/2}}\, e^{\frac{-x^2}{4t}} e^{-t}\, (Tr\, e^{\frac{t}{m^2}(\Delta - U)}) \tag{4.14}$$

The right hand side of (4.13) is an harmonic sum. General form of an harmonic sum is:

$$G(x) = \sum_k \lambda_k g(\mu_k x) \tag{4.15}$$

where λ_k are the amplitudes, μ_k are the frequencies, and $g(x)$ is the base function(more discussion is given in Appendix C).
We can express harmonic sum as;

$$A(x) = -\frac{m}{\sqrt{\pi}} \sum_{k=1}^{\infty} g(kx) \tag{4.16}$$

with $\mu_k = k$, $\lambda_k = 1$ and base function;

$$g(x) = \cosh(\xi x)\, F(x), \quad x = \beta m, \quad \xi = \frac{\mu}{m} \tag{4.17}$$

$x \to 0$ asymptotic of an harmonic sum can be determined from Mellin transform of it. The Mellin transform of a function $f(x)$ is defined as

$$\widetilde{f}(s) = (\mathcal{M}f)(s) = \int_0^{\infty} dx\, x^{s-1} f(x) \tag{4.18}$$

After attempting to integrate term by term together with a simple change of variable one can show Mellin transform of 4.16 is;

$$(\mathcal{M}A)(s) = -\frac{m}{\sqrt{\pi}}\zeta(s)(\mathcal{M}g)(s) \tag{4.19}$$

Here $\zeta(s)$ is the Riemann zeta function which is a meromorphic function on the whole complex plane with a single simple pole at $s = 1$. In order to use Mellin transform we must be sure about the existence of inverse Mellin transform defined as;

$$(\mathcal{M}^{-1}\widetilde{f})(x) = \int_{c-i\infty}^{c+i\infty} ds\, \widetilde{f}(s)x^s \tag{4.20}$$

more about Mellin transform and its validity for our case is discussed in a detailed way in the Appendix C. The correspondence between the poles of $(\mathcal{M}f)(s)$ and the asymptotics of $f(x)$ is given by a dictionary:

$$(\mathcal{M}f)(s) \asymp \sum_{w,k} \frac{A(w,k)}{(s-w)^{k+1}} \tag{4.21}$$

where $\asymp$ refers to the singular part of a function, then

$$f(t) \sim \sum_{w,k} A(w,k)\frac{(-1)^k}{k!}t^{-w}(\log t)^k \tag{4.22}$$

This dictionary tells, residues that belong to singular expansion of the Mellin transform of the function $f(x)$ are related with the $x \to 0$ asymptotics of $f(x)$. Note that logarithmic contributions are coming from multiple poles. Now our aim is to get the singular expansion of the Mellin transform of Eq 4.16. In order to perform this, we need the $x \to 0$ singular asymptotics of $F(x)$, which is needed to determine the poles of transformation. Singular part of $F(x)$ is calculated in the Appendix B and the result is;

$$F(x) \asymp \sum_{n=1}^{4} b_{-n}x^{-n} + b' \log(x/2) \tag{4.23}$$

We also need the $x \to \infty$ behaviour of $F(x)$ which is also studied in the Appendix B;

$$F(x) = O\left(x^{-1}e^{-x}\right), \quad x \to \infty \tag{4.24}$$

In order to make Mellin integrals similar to the Gamma function, with which analytic continuation is familiar, we introduce a unity in the form of $e^{-\alpha kx}\, e^{\alpha kx}$ in front of base function $g(kx)$ and write A as;

$$A = -\frac{m}{\sqrt{\pi}} \sum_{k=1}^{\infty} e^{-\alpha kx} g_\alpha(kx) \tag{4.25}$$

where;

$$g_\alpha(x) = e^{\alpha x} \cosh(\xi x) F(x) \tag{4.26}$$

Here α is an arbitrary parameter with $0 < \alpha < 1-\xi$, which will be used as a regularizing parameter in Mellin integrals. One can observe α independent results at the end.

Using (4.23) we see that as $x \to 0$

$$g_\alpha(x) \sim d_{-4}x^{-4} + c_{-3}x^{-3} + d_{-2}x^{-2} + d_{-1}x^{-1} + d_0 + \dots \tag{4.27}$$

these d coefficients are;

$$\begin{aligned}
d_{-4} &= b_{-4} \\
d_{-3} &= b_{-3} + \alpha b_{-4} \\
d_{-2} &= b_{-2} + \alpha b_{-3} + \frac{(\alpha^2+\xi^2)}{2} b_{-4} \\
d_{-1} &= b_{-1} + \alpha b_{-2} + \frac{(\alpha^2+\xi^2)}{2} b_{-3} + \frac{(\alpha^3+3\alpha\xi^2)}{6} b_{-4} \\
d_0 &= \alpha b_{-1} + \frac{(\alpha^2+\xi^2)}{2} b_{-2} + \frac{(\alpha^3+3\alpha\xi^2)}{6} b_{-3} + \frac{(\alpha^4+6\alpha^2\xi^2+\xi^4)}{24} b_{-4}
\end{aligned} \tag{4.28}$$

and $x \to \infty$ behaviour of $g_\alpha(x)$ is;

$$g_\alpha(x) = O\left(x^{-1}\, e^{-(1-\xi-\alpha)x}\right) \tag{4.29}$$

Mellin transform of A can be written as;

$$(\mathcal{M}A)(s) = \frac{-m}{\sqrt{\pi}}\zeta(s)(\mathcal{LM}g_\alpha)(s,\alpha) \tag{4.30}$$

where $\mathcal{LM}$ is the Laplace-Mellin transform defined as;

$$(\mathcal{LM}g_\alpha)(s,\alpha) = \int_0^\infty dx\, x^{s-1}\, e^{-\alpha x} g_\alpha(x) \tag{4.31}$$

At that point the reason behind introducing α parameter is more obvious. In this integral upper limit is not problematic due to the large x behaviour of $g_\alpha(x)$. Considering the lower limit, if α is non-negative the integral is convergent for $\Re s > 4$. Now we will subtract divergent terms of $g_\alpha(x)$ term by term from the integral and add them again to observe the poles.

$$\begin{aligned}(\mathcal{LM}g_\alpha)(s,\alpha) &= \int_0^\infty dx\, x^{s-1}\, e^{-\alpha x}\left[g_\alpha(x) - d_{-4}x^{-4}\right] - \int_0^\infty dx\, x^{s-1}\, e^{-\alpha x} d_{-4}x^{-4} \\ &= \int_0^\infty dx\, x^{s-1}\, e^{-\alpha x}\left[g_\alpha(x) - d_{-4}x^{-4}\right] + d_{-4}\alpha^{4-s}\Gamma(s-4) \end{aligned} \tag{4.32}$$

In the above expression the first integral is now convergent for $\Re s > 3$, from the second expression we get a pole at $s = 4$. Subtracting and adding the remaining divergent terms of $g_\alpha(x)$ we get the meromorphic continuation of $(\mathcal{LM}g_\alpha)(s,\alpha)$. Meromorphic extension of $(\mathcal{LM}g)(s,\alpha)$ into the half-plane to the left of $\Re s = 4$ is;

$$\begin{aligned}(\mathcal{LM}g_\alpha)(s,\alpha) \sim\, & d_{-4}\alpha^{4-s}\Gamma(s-4) + d_{-3}\alpha^{3-s}\Gamma(s-3) + d_{-2}\alpha^{2-s}\Gamma(s-2) \\ & + d_{-1}\alpha^{1-s}\Gamma(s-1) + d_0\alpha^{-s}\Gamma(s)\ldots \end{aligned} \tag{4.33}$$

In this expression all the poles are simple. One can read the poles from;

$$\Gamma(s-n) \asymp \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \frac{1}{s-n+m} \tag{4.34}$$

In the first term in 4.33 we have a pole coming from $s = 4$, for the second term we have two poles coming from $s = 4, 3$ etc. The main expression that we are concerning is $(\mathcal{M}A)(s) = -m\pi^{-1/2}\zeta(s)(\mathcal{L}\mathcal{M}g)(s, \alpha)$. When Equation 4.33 is multiplied with $\zeta(s)$ we encounter with a double pole at $s = 1$. Pole structure of of $\zeta(s)$ behaves like;

$$\zeta(s) \sim \frac{1}{s-1} + \gamma, \quad s \to 1 \tag{4.35}$$

Here γ is the Euler-Mascheroni constant.
Now after a little algebra we are able to analyse singular expansion of $(\mathcal{M}A)(s)$ and its poles, thus get the asymptotic expansion of A. Introduce the notation:

$$A \sim A^{(4)} + A^{(3)} + \dots \tag{4.36}$$

where $A^{(m)}$ represents the contribution from the pole at $s = m$.

Singular expansion for $s = 4$ is;

$$(\mathcal{M}A)(s) \asymp \frac{-m}{\sqrt{\pi}} \zeta(4) b_{-4} \frac{1}{s-4} \tag{4.37}$$

with the help of dictionary introduced, the corresponding asymptotic is;

$$A^{(4)} = \frac{-m}{\sqrt{\pi}} \zeta(4) b_{-4} (\beta m)^{-4} \tag{4.38}$$

For $s = 3$

$$(\mathcal{M}A)(s) \asymp \frac{-m}{\sqrt{\pi}} \frac{\zeta(3)}{s-4} b_{-3} \tag{4.39}$$

The asymptotic is

$$A^{(3)} = \frac{-m}{\sqrt{\pi}}\zeta(3)b_{-3}(\beta m)^{-3} \tag{4.40}$$

For $s = 2$

$$(\mathcal{M}A)(s) \asymp \frac{-m}{\sqrt{\pi}}\frac{\zeta(2)}{s-2}\quad\left(\frac{\xi^2}{2}b_{-4} + b_{-2}\right) \tag{4.41}$$

The asymptotic is

$$A^{(2)} = \frac{-m}{\sqrt{\pi}}\zeta(2)\left(\frac{\xi^2}{2}b_{-4} + b_{-2}\right)(\beta m)^{-2} \tag{4.42}$$

For $s = 1$ which corresponds to a double pole

$$\begin{aligned}(\mathcal{M}A)(s) &\sim \frac{-m}{\sqrt{\pi}}\left(\frac{1}{s-1}+\gamma\right)\left(FP(\mathcal{L}\mathcal{M}g)(s)|_{s\to 1} + \frac{\left[\frac{\xi^2}{2}b_{-3}+b_{-1}\right]}{s-1}\right)\\ &\asymp \frac{-m}{\sqrt{\pi}}\left[\frac{\left[\frac{\xi^2}{2}b_{-3}+b_{-1}\right]}{(s-1)^2}\right.\\ &+ \left.\frac{FP(\mathcal{L}\mathcal{M}g)(s=1,\alpha)+\gamma\left[\frac{\xi^2}{2}b_{-3}+b_{-1}\right]}{s-1}\right]\end{aligned} \tag{4.43}$$

FP stands for shortening "finite part" and is equal to;

$$\begin{aligned}FP(\mathcal{L}\mathcal{M}g)(s=1,\alpha) &= (\mathcal{L}\mathcal{M}g)(s=1,\alpha) - \frac{\left[\frac{\xi^2}{2}b_{-3}+b_{-1}\right]}{s-1}\\ &= \int_0^\infty dx\, x^{s-1}\cosh(\xi x)F(x) - \frac{\left[\frac{\xi^2}{2}b_{-3}+b_{-1}\right]}{s-1}\end{aligned} \tag{4.44}$$

and

$$\begin{aligned} A^{(1)} &= \frac{m}{\sqrt{\pi}}\left[\frac{\xi^2}{2}b_{-3}+b_{-1}\right](\beta m)^{-1}\log(\beta m)^{-1} \\ &\quad +\frac{-m}{\sqrt{\pi}}\left(\gamma\left[\frac{\xi^2}{2}b_{-3}+b_{-1}\right]+FP(\mathcal{LM}g)(1,\alpha)\right)(\beta m)^{-1} \end{aligned} \tag{4.45}$$

Rewriting all the b coefficients in (4.28) in terms of heat kernel coefficients gives:

$$\begin{aligned} A^{(4)} &= \frac{-16}{\sqrt{\pi}}\zeta(4)m^4a_0(\beta m)^{-4} \\ A^{(3)} &= -4\zeta(3)m^3a_{1/2}(\beta m)^{-3} \\ A^{(2)} &= -\frac{4m^2}{\sqrt{\pi}}\zeta(2)\left[(2\mu^2-m^2)a_0+a_1\right](\beta m)^{-2} \\ A^{(1)} &= \left[2(\mu^2m-m^3)a_{1/2}+\frac{2}{\pi}ma_{3/2})\right](\beta m)^{-1}\log(\beta m)^{-1} \\ &- \left(\gamma\left[2(\mu^2m-m^3)a_{1/2}+\frac{2}{\pi}ma_{3/2}\right]+\frac{m}{\sqrt{\pi}}FP(\mathcal{LM}g)(1,\alpha)\right)(\beta m)^{-1} \end{aligned} \tag{4.46}$$

There is no α dependence in the result. One can observe the same result in [12,14,15].

Calculations can be generalized to $d>3$ dimensions by changing the power of t in front of short time asymptotic of heat kernel from $3/2$ to $d/2$. Result is then;

$$\begin{aligned} A &\sim -\zeta(d+1)\frac{(2m)^{d+1}}{\sqrt{\pi}}\Gamma\left(\frac{d+1}{2}\right)a_0\,(\beta m)^{-(d+1)} \\ &- \zeta(d)\frac{(2m)^d}{\sqrt{\pi}}\Gamma\left(\frac{d}{2}\right)a_{1/2}\,(\beta m)^{-d} \\ &- \zeta(d-1)\frac{(2m)^{d-1}}{\sqrt{\pi}}\Gamma\left(\frac{d-1}{2}\right)\left(\left[-m^2+(d-1)\,\mu^2\right]a_0+a_1\right)(\beta m)^{-(d-1)} \end{aligned} \tag{4.47}$$

5. CHARGE

In this chapter we will do the same calculations in the previous chapter for the net charge of the excited bosons which is expressed as

$$Q_e = \sum_{\sigma\neq 0}\left[\frac{1}{e^{\beta(\lambda_\sigma+m^2)^{1/2}-\beta\mu}-1}-\frac{1}{e^{\beta(\lambda_\sigma+m^2)^{1/2}+\beta\mu}-1}\right] \tag{5.1}$$

$$\tag{5.2}$$

in the canonical ensemble. After expressing the terms in square brackets as geometric series we get;

$$Q_e = \sum_{\sigma\neq 0}\sum_{k=0}^{\infty} e^{k\beta(\lambda_\sigma+m^2)^{1/2}}\, 2\sinh(k\beta\mu) \tag{5.3}$$

and using the subordination identity Eq.4.11 gives;

$$Q_e = \sum_{k=1}^{\infty}\frac{(k\beta m)\sinh(k\beta\mu)}{\sqrt{\pi}}F_1(k\beta m) \tag{5.4}$$

where

$$F_1(x) = \int_0^\infty \frac{dt}{t^{3/2}}\, e^{\frac{-x^2}{4t}}\, e^{-t}\, (Tr'\, e^{\frac{t}{m^2}\Delta}) \tag{5.5}$$

Tr' represents the trace without the ground state.

$$Tr'\, e^{t\Delta} = Tr\, e^{t\Delta} - e^{-\lambda_0 t} \tag{5.6}$$

λ_0 is the lowest eigenvalue of $-\Delta$. Heat kernel expansion is in the same form

$$Tr'\, e^{t\Delta} \sim \sum_{k=0}^{\infty} a'_{k/2}\, t^{\frac{k-3}{2}} \tag{5.7}$$

with different coefficients. The relation between previous and this coefficients is

$$a'_{2l+3} = a_{2l+3} - (-1)^l \lambda_0^l / l! \tag{5.8}$$

and

$$a'_k = a_k \tag{5.9}$$

otherwise. Thus all the way in these calculations heat kernel coefficient are equal to each other As in the previous case, small x asymptotic expansion of F_1 is given by (B.9). Please also note that, due to the condition $F_1(x) < F(x)$, the estimates (B.14) and (B.15) for $F(x)$ are also valid for $F_1(x)$.

We will derive the high temperature expansion starting with the expression

$$Q_e = \frac{1}{\sqrt{\pi}} \sum_{k=1}^{\infty} e^{-\alpha k x} f_\alpha(kx) \tag{5.10}$$

where

$$f_\alpha(x) = e^{\alpha x} x \sinh(\xi x) F(x) \tag{5.11}$$

Remaining analysis is similar to the free energy case . The small x asymptotic of f_α is obtained by combining the asymptotic of $F(x)$ with the Taylor expansions of $\sinh(\xi x)$ and $e^{\alpha x}$:

$$f_\alpha(x) \sim c_{-2} x^{-2} + c_{-1} x^{-1} + c_0 + \dots \tag{5.12}$$

c coefficients in terms of b coefficients are;

$$\begin{aligned} c_{-2} &= \xi b_{-4} \\ c_{-1} &= \xi b_{-3} + \alpha \xi b_{-4} \\ c_0 &= \xi b_{-2} + \alpha \xi b_{-3} + \left(\frac{\xi \alpha^2}{2} + \frac{\xi^3}{6} \right) b_{-4} \end{aligned} \tag{5.13}$$

On the other hand using (B.15), for $x \to \infty$ we get

$$f_\alpha(x) = O\left(e^{-(1-\xi-\alpha)x}\right) \tag{5.14}$$

Mellin transform of Q_e is given by

$$(\mathcal{M}Q_e)(s) = \frac{1}{\sqrt{\pi}}\zeta(s)(\mathcal{L}\mathcal{M}f_\alpha)(s,\alpha) \tag{5.15}$$

$$\begin{aligned}(\mathcal{L}\mathcal{M}f_\alpha)(s,\alpha) &= \int_0^\infty dx\, x^{s-1}\, e^{-\alpha x}\left[f(x) - c_{-2}x^{-2}\right] + \int_0^\infty dx\, x^{s-1}\, e^{-\alpha x} c_{-2}x^{-2} \\ &= \int_0^\infty dx\, x^{s-1}\, e^{-\alpha x}\left[f(x) - c_{-2}x^{-2}\right] + c_{-2}\alpha^{2-s}\Gamma(s-2) \\ &= \ldots. \end{aligned} \tag{5.16}$$

Performing the meromorphic continuation as in the free energy case we get the expansion;

$$(\mathcal{L}\mathcal{M}f_\alpha)(s,\alpha) \sim c_{-2}\alpha^{2-s}\Gamma(s-2) + c_{-1}\alpha^{1-s}\Gamma(s-1) + c_0\alpha^{-s}\Gamma(s) + \ldots \tag{5.17}$$

Again all the poles are simple. The pole at $s = 2$ comes from the c_{-2} term, while the one at $s = 1$ comes from both c_{-2} and $c = -1$ terms, etc.
For $s = 2$:

$$(\mathcal{M}Q_e)(s) \asymp \frac{1}{\sqrt{\pi}}\zeta(2)(\xi b_{-4})\,\frac{1}{s-2} \tag{5.18}$$

The asymptotic is

$$Q_e^{(2)} = \pi^{-1/2}\zeta(2)(\xi b_{-4})(\beta m)^{-2} \tag{5.19}$$

For $s = 1$:

$$\begin{aligned}(\mathcal{M}Q_e)(s) &\sim \pi^{-1/2}\left(\frac{1}{s-1}+\gamma\right)\left(FP(\mathcal{L}\mathcal{M}f)(s)\mid_{s\to 1}+\frac{\xi b_{-3}}{s-1}\right)\\ &\asymp \pi^{-1/2}\left[\frac{\xi b_{-3}}{(s-1)^2}+[\gamma\xi b_{-3}+FP(\mathcal{L}\mathcal{M}f)(s=1,\alpha)]\frac{1}{s-1}\right]\end{aligned} \tag{5.20}$$

Finally for $s = 0$:

$$(\mathcal{M}Q_e)(s) \asymp \frac{1}{\sqrt{\pi}}\zeta(0)\left[\frac{\xi^3}{6}b_{-4}+\xi b_{-2}\right]\frac{1}{s} \tag{5.21}$$

Thus we arrive at the high temperature expansion

$$Q_e \sim Q_e^{(2)} + Q_e^{(1)} + \dots \tag{5.22}$$

with

$$Q_e^{(2)} = \pi^{-1/2}\zeta(2)\xi b_{-4}(\beta m)^{-2} \tag{5.23}$$

$$\begin{aligned}Q_e^{(1)} &= -\pi^{-1/2}(\xi b_{-3})(\beta m)^{-1}\log(\beta m)^{-1}+\\ &\quad +\pi^{-1/2}\left[\gamma(\xi b_{-3})+FP(\mathcal{L}\mathcal{M}f_\alpha)(1,\alpha)\right](\beta m)^{-1}\end{aligned} \tag{5.24}$$

Now expressing everything in terms of heat kernel coefficients by using Eq.(B.9) we get;

$$\begin{aligned}Q_e^{(2)} &= \frac{16}{\sqrt{\pi}}\zeta(2)\mu m^2 a_0(\beta m)^{-2}\\ Q_e^{(1)} &= -\left(4m\mu a_{1/2}\right)(\beta m)^{-1}\log(\beta m)^{-1}+\\ &\quad +\left[\gamma 4m\mu a_{1/2}+\frac{FP(\mathcal{L}\mathcal{M}f_\alpha)(1,\alpha)}{\sqrt{\pi}}\right](\beta m)^{-1}\end{aligned} \tag{5.25}$$

And for the higher dimensional case, i. e. for $d > 3$, we have;

$$\begin{aligned} Q_e &\sim \zeta(d-1)\,\frac{(2m)^{d-1}}{\sqrt{\pi}}\,4\mu\,\Gamma\left(\frac{d+1}{2}\right)\,a_0\,(\beta m)^{-(d-1)} \\ &+ \zeta(d-2)\,\frac{(2m)^{d-2}}{\sqrt{\pi}}\,4\mu\,\Gamma\left(\frac{d}{2}\right)\,a_{1/2}\,(\beta m)^{-(d-2)} \\ &+ \zeta(d-3)\,\frac{(2m)^{d-3}}{\sqrt{\pi}}\,4\mu\,\Gamma\left(\frac{d-1}{2}\right)\left[\left(-m^2+\frac{(d-1)}{3}\mu^2\right)a_0+a_1\right]\,(\beta m)^{-(d-3)} \end{aligned} \tag{5.26}$$

Now we will consider the critical temperature. First we will calculate T as a function of Q_0, charge of the ground state, for a fixed Q. We will study for $d = 3$ case. Inserting the calculated heat kernel coefficients into leading boundary contributions in (5.25) gives;

$$\frac{2V}{\pi^2}\zeta(2)\mu m^2(\beta m)^{-2} - \frac{A}{4\pi}\mu m(\beta m)^{-1}\log(\beta m)^{-1} = Q - Q_0 \tag{5.27}$$

During the condensation for Q_0 we have;

$$Q_0 = \frac{1}{e^{\beta(m-\mu)}-1} - \frac{1}{e^{\beta(m+\mu)}-1} \tag{5.28}$$

Taylor expanding the exponentials and calculating for μ gives;

$$\mu = \pm m - \frac{1}{\beta q_0 V} + O(V^{-2}) \tag{5.29}$$

Here and below $q_0 = Q_0/V$ and $q = Q/V$. If we set $y = k_B T/m$ and divide both sides of (5.27) by V we get;

$$\frac{2}{\pi^2}\zeta(2)\mu m^2 y^2 - \frac{\varepsilon}{4\pi}\mu m y\log y = q - q_0 \tag{5.30}$$

Here $\varepsilon = A/V$ is the small parameter in terms of which we can expand the solution perturbatively

$$\frac{k_B T}{m} = y = y_0 + \varepsilon y_1 + \dots . \tag{5.31}$$

Sign of $q - q_0$ depends on the sign of μ; if $q > q_0$ then $\mu > 0$ and if $q < q_0$ then $\mu < 0$. We can study two cases simultaneously. Replacing μ by $\pm m$ is acceptable because from Eq.5.29 we see that the error is $O(V^{-1})$ (here we are assuming that the geometry of our box B is regular in the sense that $1/V << A/V$). Then we can write the Eq.5.30 as;

$$\frac{2}{\pi^2}\zeta(2)m^3y^2 - \frac{\varepsilon}{4\pi}m^2 y \log y = |q - q_0| \tag{5.32}$$

Perturbatively solving gives the result;

$$y_0 = \left(\frac{\pi^2(|q - q_0|)}{2\zeta(2)m^3}\right)^{1/2} \tag{5.33}$$

$$y_1 = \frac{\pi}{32m\zeta(2)} \log \frac{\pi^2(|q - q_0|)}{2\zeta(2)m^3} \tag{5.34}$$

Conventional procedure for a general BEC system is to set $q_0 = 0$ to determine the critical temperature, but in our case, since we are studying in a box with a finite volume V it is not possible to take the thermodynamic limit. This implies a divergent μ. A similar argument is discussed in [32] for non-relativistic case. For finite size systems it is convenient to calculate temperature as a function of the condensate density q_0.

In equation (5.31) small parameter $\varepsilon = A/V$ is in the same order with $1/L$, where L stands for typical length of the system. This small parameter stands in front of the first perturbative term y_1 and suggests that if typical length is in the same order with the boson mass m, y_1 term becomes important. In other words, when the typical length is comparable with de Broglie wavelength we encounter with size effects in depletion coefficient.

Higher dimensional can be calculated again by substituting

$$y = (\beta m)^{-1}, \qquad \epsilon = A/V, \qquad \mu = m \tag{5.35}$$

and using perturbative technique used above (5.31). Results for d dimensional charged boson is ;

$$y_0 = \left[\frac{(|q - q_0|)\,(\sqrt{\pi})^{d+1}}{m\,2^d\,\zeta(d-1)\,\Gamma\left(\frac{d+1}{2}\right)} \right]^{\frac{1}{d-1}} \tag{5.36}$$

$$y_1 = \pm \frac{\sqrt{\pi}}{4m(d-1)} \frac{\zeta(d-2)}{\zeta(d-1)} \frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)} \tag{5.37}$$

.

Please note that the above calculations are done for Neumann boundary conditions, in which the ground state energy $\lambda_0 = 0$. Dirichlet problem can also be studied, by considering the 'r' sign in front of the heat kernel coefficient. Other than this fact, it is not possible to set $\mu = m$ in (5.30) because of the non zero ground state energy. In Dirichlet problem, depletion coefficient depends also on the ground state energy.

6. FREE ENERGY OF NEUTRAL BOSONS

Using the expansion of logarithm and the subordination identity (4.11) as used in chapter 4 we can write the free energy for a gas of neutral bosons

$$A = \frac{1}{\beta}\sum_{\sigma}\left[\log\left(1 - e^{-(\beta^2\lambda_\sigma+\beta^2m^2)^{1/2}+\beta\mu}\right)\right] \tag{6.1}$$

as

$$\begin{aligned} A &= -\sum_{k=1}^{\infty}\frac{m}{2\sqrt{\pi}}e^{k\beta\xi m}F(k\beta m) \\ &= \frac{-m}{2\sqrt{\pi}}\sum_{k=1}^{\infty}e^{-\alpha kx}g(kx) \end{aligned} \tag{6.2}$$

where

$$g_\alpha(x) = e^{(\alpha+\xi)x}F(x) \sim d_{-4}x^{-4} + d_{-3}x^{-3} + \ldots \quad x \to 0 \tag{6.3}$$

In this case d coefficients in terms of b coefficients are;

$$\begin{aligned} d_{-4} &= b_{-4} \\ d_{-3} &= b_{-3} + b_{-4}(\alpha+\xi) \\ d_{-2} &= b_{-2} + b_{-3}(\alpha+\xi) + b_{-4}\frac{(\alpha+\xi)^2}{2} \\ d_{-1} &= b_{-1} + b_{-2}(\alpha+\xi) + b_{-3}\frac{(\alpha+\xi)^2}{2} + b_{-4}\frac{(\alpha+\xi)^3}{6} \\ d_0 &= b_{-1}(\alpha+\xi) + b_{-2}\frac{(\alpha+\xi)^2}{2} + b_{-3}\frac{(\alpha+\xi)^3}{6} + b_{-4}\frac{(\alpha+\xi)^4}{24} \end{aligned} \tag{6.4}$$

Again the Mellin transform of the Harmonic sum can be written as;

$$(\mathcal{M}A)(s) = \frac{-m}{2\sqrt{\pi}}\zeta(s)(\mathcal{L}\mathcal{M}g_\alpha)(s,\alpha) \tag{6.5}$$

where

$$(\mathcal{LM}g)(s,\alpha) = \int_0^\infty dx\, x^{s-1} e^{-\alpha x} g_\alpha(x) \tag{6.6}$$

Singular asymptotic calculation is very similar to the one in the charged case. We directly write the result as;

$$(\mathcal{LM}g)(s,\alpha) \sim d_{-4}(\alpha)^{4-s}\Gamma(s-4) + d_{-3}(\alpha)^{3-s}\Gamma(s-3) + \dots. \tag{6.7}$$

Again the poles are at $s = 4, 3, 2, 1, \dots$ with the same manner in the charged case. We will use the same notation to express the free energy;

$$A \sim A^{(4)} + A^{(3)} + A^{(2)} + A^{(1)} + \dots \tag{6.8}$$

where

$$\begin{aligned}
A^{(4)} &= \frac{-m}{2\sqrt{\pi}}\zeta(4)(16m^3 a_0)(\beta m)^{-4} \\
A^{(3)} &= \frac{-m}{2\sqrt{\pi}}\zeta(3)(16\mu m^2 a_0 + 4\sqrt{\pi} m^2 a_{1/2})(\beta m)^{-3} \\
A^{(2)} &= \frac{-m}{2\sqrt{\pi}}\zeta(2)\left[4(2m\mu^2 - m^3)a_0 + 4\sqrt{\pi}\mu m a_{1/2} + 4m a_1\right](\beta m)^{-2} \\
A^{(1)} &= \frac{m}{2\sqrt{\pi}}\left[\left(\frac{8}{3}\mu^3 - 4m^2\mu\right)a_0 + 2\sqrt{\pi}(\mu^2 - m^2)a_{1/2} + 4\mu a_1 + \frac{2}{\sqrt{\pi}}a_{3/2}\right] \\
&\times \ (\beta m)^{-1}\log(\beta m)^{-1} \\
&- \ \frac{m}{2\sqrt{\pi}}\left(\gamma\left[\left(\frac{8}{3}\mu^3 - 4m^2\mu\right)a_0 + 2\sqrt{\pi}(\mu^2 - m^2)a_{1/2} + 4\mu a_1 + \frac{2}{\sqrt{\pi}}a_{3/2}\right]\right. \\
&+ \ \left. FP(\mathcal{LM}g_\alpha)(1,\alpha)\right)(\beta m)^{-1}
\end{aligned} \tag{6.9}$$

higher dimensional case, i.e. for $d > 3$, gives the result;

$$\begin{aligned} A &\sim -\zeta(d+1)\,\frac{(2m)^{d+1}}{2\sqrt{\pi}}\,\Gamma\left(\frac{d+1}{2}\right)\,a_0\,(\beta m)^{-(d+1)} \\ &- \zeta(d)\,\frac{(2m)^{d}}{2\sqrt{\pi}}\left[2\mu\,\Gamma\left(\frac{d+1}{2}\right)\,a_0 + \Gamma\left(\frac{d}{2}\right)\,a_{1/2}\right]\,(\beta m)^{-d} \\ &- \zeta(d-1)\,\frac{(2m)^{d-1}}{2\sqrt{\pi}}\,\Gamma\left(\frac{d-1}{2}\right)\left[\left(-m^2+(d-1)\mu^2\right)\,a_0\right. \\ &- \left. 2\mu\,\Gamma\left(\frac{d}{2}\right)\,a_{1/2} + a_1\right]\,(\beta m)^{-(d-1)} \end{aligned} \tag{6.10}$$

7. NUMBER OF NEUTRAL BOSONS

Expressing the summand as a geometric sum and using the subordination identity Eq(4.11). for a gas of neutral Bosons we get the result;

$$\begin{aligned} N_e &= \sum_{\sigma\neq 0} \frac{1}{e^{\beta(\lambda_\sigma+m^2)^{1/2}-\beta\mu}-1} && (7.1)\\ &= \sum_{k=1}^{\infty} \frac{(k\beta m)e^{k\beta\xi m}}{2\sqrt{\pi}} F(k\beta m) \\ &= \frac{1}{2\sqrt{\pi}} \sum_{k=1}^{\infty} e^{k\xi x} f(kx) && (7.2) \end{aligned}$$

where

$$f_\alpha(x) = e^{(\alpha+\xi)x} xF(x) \sim d_{-4}x^{-3} + d_{-3}x^{-2} + \ldots \quad x \to 0 \tag{7.3}$$

Again the Mellin transform of the Harmonic sum can be written as;

$$(\mathcal{M}N_e)(s) = \frac{1}{2\sqrt{\pi}}\zeta(s)(\mathcal{LM}f)(s,\alpha) \tag{7.4}$$

where

$$(\mathcal{LM}f_\alpha)(s,\alpha) = \int_0^\infty dx\, x^{s-1}e^{\alpha x} f(x) \tag{7.5}$$

Singular asymptotic calculation is very similar to the previous cases. We directly write the result as;

$$(\mathcal{LM}f_\alpha)(s,-\xi) \asymp d_{-4}(\alpha)^{3-s}\Gamma(s-3) + d_{-3}(\alpha)^{2-s}\Gamma(s-2) + \ldots. \tag{7.6}$$

Poles are at $s = 3, 2, 1, \ldots$. Again the pole at $s = 3$ comes from d_{-4} term, the pole at $s = 2$ comes from both d_{-4} and d_{-3} terms, etc. We use the same notation to express

high temperature expansion;

$$N_e \sim N_e^{(3)} + N_e^{(2)} + N_e^{(1)} + \dots \tag{7.7}$$

where

$$\begin{aligned}
N_e^{(3)} &= \frac{1}{2\sqrt{\pi}}\zeta(3)(16m^3a_0)(\beta m)^{-3} \\
N_e^{(2)} &= \frac{1}{2\sqrt{\pi}}\zeta(2)(16\mu m^2a_0 + 4\sqrt{\pi}m^2a_{1/2})(\beta m)^{-2} \\
N_e^{(1)} &= \frac{-1}{2\sqrt{\pi}}\left(4(2m\mu^2 - m^3)a_0 + 4\sqrt{\pi}\mu m a_{1/2} + 4ma_1\right)(\beta m)^{-1}\log(\beta m)^{-1} \\
&+ \frac{1}{2\sqrt{\pi}}(\gamma\left(4(2m\mu^2 - m^3)a_0 + 4\sqrt{\pi}\mu m a_{1/2} + 4ma_1\right) \\
&+ FP(\mathcal{LM}f_\alpha)(1,\alpha))(\beta m)^{-1}
\end{aligned} \tag{7.8}$$

For higher dimensional case, i.e. for $d > 3$, the result is;

$$\begin{aligned}
N_e &\sim \zeta(d)\,\frac{(2m)^d}{\sqrt{\pi}}\,\Gamma\left(\frac{d+1}{2}\right)\,a_0\,(\beta m)^{-d} \\
&+ \zeta(d-1)\,\frac{(2m)^{d-1}}{\sqrt{\pi}}\,\left[2\mu\,\Gamma\left(\frac{d+1}{2}\right)\,a_0 + \Gamma\left(\frac{d}{2}\right)\,a_{1/2}\right]\,(\beta m)^{-d+1} \\
&+ \zeta(d-2)\,\frac{(2m)^{d-2}}{\sqrt{\pi}}\Gamma\left(\frac{d-1}{2}\right)\left[\left(-m^2 + (d-1)\mu^2\right)\,a_0 +\right. \\
&+ \left. 2\mu\,\Gamma\left(\frac{d}{2}\right)\,a_{1/2} + a_1\right](\beta m)^{-d+2}
\end{aligned} \tag{7.9}$$

Now we will calculate the relation between the temperature and n_0 ($n = N/V$ and $n_e = N_e/V$) using this high temperature expansion. First consider the case $d = 3$. We again set $y = k_BT/m$, $\varepsilon = A/V$ and during condensation we have the relation;

$$N_0 = \frac{1}{e^{\beta(m-\mu)} - 1} \tag{7.10}$$

which gives the result;

$$\mu \sim m - \frac{1}{\beta n_0 V} \tag{7.11}$$

Again it is not problematic to replace μ by m due to the difference between them is of $O(1/V)$ which is smaller than $\varepsilon = A/V$. So we can write the relation as;

$$\frac{1}{\pi^2}\zeta(3)m^3y^3 + \zeta(2)\left[\frac{1}{\pi^2}m^3 \pm \frac{\varepsilon}{8\pi}m^2\right]y^2 = n - n_0 \tag{7.12}$$

Perturbative expansion by writting $y = y_0 + \varepsilon y_1 + \ldots$ gives;

$$\zeta(3)y_0^3 + \zeta(2)y_0^2 = \frac{\pi^2(n-n_0)}{m^3} \tag{7.13}$$

and

$$\frac{3\zeta(3)m^3}{\pi^2}y_0^2y_1 + \frac{2\zeta(2)m^3}{\pi^2}y_0y_1 \pm \frac{\zeta(2)m^2}{8\pi}y_0^2 = 0 \tag{7.14}$$

From the second equation we can express y_1 in terms of y_0

$$y_1 = \mp\frac{\zeta(2)\pi}{8m}\frac{y_0}{3\zeta(3)y_0 + 2\zeta(2)} \tag{7.15}$$

To solve the first equation we will use the detailed balance method [33]. We will assume m is small. Let us write it as

$$\zeta(3)y_0^3 + \eta\zeta(2)y_0^2 = \frac{\pi^2(n-n_0)}{m^3} \tag{7.16}$$

Here η is an expansion parameter which will be set equal to 1 at the end of the calculation. Now plugging in the perturbation expansion $y_0 = z_0 + \eta z_1 + \ldots$ we get

$$y_0 = \frac{((\zeta(3))^{-1}\pi^2(n-n_0))^{1/3}}{m} - \frac{\zeta(2)}{3\zeta(3)} + \ldots. \tag{7.17}$$

The leading term above is in accord with the result obtained in [34].
Again as in the charged case, when the typical length is comparable with the de Broglie wavelength of bosons, we encounter with size effects in depletion coefficient.

Performing the same calculations for the higher dimensional case, i.e. for $d > 3$ gives the result

$$y_0 = \left[\frac{(n - n_0)\,(\sqrt{\pi})^{d+1}}{m^d\,\zeta(d)\,\Gamma\left(\frac{d+1}{2}\right)} \right]^{\frac{1}{d}} - \frac{\zeta(d-1)}{d\,\zeta(d)}, \tag{7.18}$$

$$y_1 = \frac{\sqrt{\pi}}{4m}\,\frac{\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(\frac{d+1}{2}\right)}\,\frac{\zeta(d-1)\,y_0}{[d\,\zeta(d)\,y_0 + (d-1)\,\zeta(d-1)]} \tag{7.19}$$

In this chapter calculations are done for Neumann problem as in the charged case. For Dirichlet problem discussion at the end of chapter 5 is valid for neutral gas too.

8. RELATIVISTIC IDEAL GAS IN THERMODYNAMIC LIMIT

Up to now, all calculations are done for finite volume. In this section thermodynamic limit is considered. Interval for heat kernel eigenvalues, calculated in thermodynamic limit (3.31) is used. $T \to 0$ limit of upper bound decreases, which implies BEC. Instead of charge we consider the charge density;

$$q_e(T) = \frac{1}{V} \sum_{\sigma \neq 0} \left[\frac{1}{e^{(\beta^2 \lambda_\sigma + \beta^2 m^2)^{1/2} - \beta\mu} - 1} - \frac{1}{e^{(\beta^2 \lambda_\sigma + \beta^2 m^2)^{1/2} + \beta\mu} - 1} \right] \tag{8.1}$$

After converting the above expressions into geometric series and using the subordination identity (4.11) as in the previous cases, charge density (8.1) can be expressed as;

$$q_e(T) = \sum_{k=1}^{\infty} \int_0^{\infty} \frac{ds}{s^{3/2}} \frac{k}{\sqrt{\pi}} e^{-\frac{k^2}{4s}} \sinh(k\beta\mu) \frac{1}{V} (\mathrm{Tr}' e^{s\beta^2 \Delta}) e^{-s\beta^2 m^2} \tag{8.2}$$

Plugging the upper and lower limits in (3.31) and taking the integrals with the help of information

$$\int_0^{\infty} \frac{ds}{s^3} e^{-\frac{k^2}{4s} - s\beta^2 m^2} = \frac{8\beta^2 m^2}{k^2} K_2(k\beta m) \tag{8.3}$$

one gets the result;

$$\frac{3m^2}{B^{3/2}\beta} \sum_{k \neq 0} \frac{e^{k\beta\mu}}{k} K_2(k\beta m) \leq q_e(T) \leq \frac{3Am^2}{C^{3/2}\beta} \sum_{k \neq 0} \frac{e^{k\beta\mu}}{k} K_2(k\beta m) \tag{8.4}$$

Here K_2 is the modified Bessel function of the second kind. General integral representation of K_ν for $\nu > 0$ and $x > 0$ is,

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{1/2} \frac{e^{-x}}{\Gamma(\nu + 1/2)} \int_0^{\infty} ds e^{-s} s^{\nu - 1/2} \left(1 + \frac{s}{2x}\right)^{\nu - 1/2} \tag{8.5}$$

If we set $\nu = 2$ and calculate the last term in parenthesis when the power is equal to 2, we get an upper bound for $K_2(x)$,

$$K_2(x) < \frac{4}{3\sqrt{2x}} e^{-x} \left[\Gamma(5/2) + \frac{\Gamma(7/2)}{x} + \frac{\Gamma(9/2)}{4x^2} \right] \tag{8.6}$$

If negative k terms are neglected in (8.4) we get a stronger upper bound. Due to the decreasing property of Bessel function, we can rewrite the upper bound as;

$$\begin{aligned} q_e(T) \;\; < \;\; & \frac{3Am^2}{C^{3/2}\beta} e^{\beta\mu} K_2(\beta m) \\ & + \frac{4Am^2}{C^{3/2}\beta} \int_1^\infty dx \frac{e^{-x\beta(m-\mu)}}{(\beta m)^{1/2} x^{3/2}} \left[\Gamma(5/2) + \frac{\Gamma(7/2)}{x\beta m} + \frac{\Gamma(9/2)}{4(\beta m x)^2} \right] \end{aligned} \tag{8.7}$$

In thermodynamic limit i.e. as $\mu \to m$ all terms in integral (8.7) give finite results. After that, when $\beta \to \infty$ limit is considered, we see that the upper bound of depletion coefficient tends to go to 0. This limit suggests that BEC occurs in three dimension.

On the other hand, one can consider the bounds for $d = 2$ case;

$$\frac{m^{3/2}}{B\beta^{1/2}} \sum_{k=1} \frac{\sinh(k\beta\mu)}{k^{1/2}} K_{3/2}(k\beta m) \leq q_e(T) \leq \frac{Am^{3/2}}{C\beta^{1/2}} \sum_{k=1} \frac{\sinh(k\beta\mu)}{k^{1/2}} K_{3/2}(k\beta m) \tag{8.8}$$

where;

$$K_{3/2}(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} [1 + \frac{1}{x}] \tag{8.9}$$

To get a stronger lower bound for q_e observe the below condition for Bessel function $K_{3/2}$,

$$\left(\frac{\pi}{2x}\right)^{1/2} e^{-x} [1 + \frac{1}{x}] > \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} \tag{8.10}$$

Again due to the decreasing property of Bessel function, lower bound of depletion coefficient can be replaced with the following integral from 1 to ∞;

$$\frac{m^{3/2}}{B\beta}\int_1^\infty dx\sinh(x\beta\mu)\frac{e^{-x\beta m}}{x} \leq q_e(T) \tag{8.11}$$

The integral (8.11) in thermodynamic limit i.e. as $\mu \to m$ for a constant β is divergent, that suggests in two dimension BEC does not occur as stated in Hohenberg-Mermin-Wagner theorem [35, 36].

9. CONCLUSION

Due to the hardness of expressing density of states in curved space, it becomes impossible to convert sums for thermodynamic expressions (including trace of heat kernel) into integrals. In order to cope with these sums various mathematical methods are developed and used.

To sum up, in this theses, charged and neutral Bose gas in curved space in a box are studied. Once the sum expressions for thermodynamic variables are expressed as the harmonic ones the dictionary for Mellin-Barnes transform is used. This dictionary states the direct correspondence between the poles of transformed sum and the asymptotics of the original function.

The asymptotics of the heat equation with Dirichlet and Neumann boundary conditions on a manifold with boundary are already known. The coefficients of this expansion known as heat kernel coefficients include expressions about the geometry of the manifold. Geometry dependence of these coefficients helps us to see size effects on various quantities including heat kernel. First the expression for free energy, second the charge of the system are analysed. Charge is expressed in terms of temperature but when calculating the critical temperature we encounter with a divergent chemical potential. The reason for divergence is the finite volume of the system. In finite volume when we cool down the system particles both occupy the ground state and the excited states. Calculating the charge(number) in terms of temperature perturbatively we encounter with the size effect in the second term. In order this term to be noticeable we have the condition $A/V \backsim m$. In this expression left hand side is related with the typical length of the system and right hand side is related with the de Broglie wavelength of bosons and the relation states that when the typical wavelength is comparable with the de Broglie wavelength of bosons, size effects become important.

Apart from these calculations, depletion coefficient in thermodynamic limit is also considered. For this case eigenvalue bounds of heat kernel for Neumann problem,

suggested by Li-Yau and Colbois-Maerten is used. By considering the upper bound in three dimension, BEC is shown. For two dimension lower bound is studied and it is shown that BEC does not occur in two dimension.

As a next work, in perturbative calculations, instead of leading two contributions one can consider leading three contributions of asymptotic expansion of N in thermodynamic limit. In such a case second term in perturbative solution obviously vanishes. In the third term we encounter with heat kernel coefficient a_1 in which scalar curvature and extrinsic curvature of manifold is hidden.

Moreover, one can specify the metric and calculate the geometric properties of the box. This is studied in [16]. In mentioned work static space-times: Schwarzschild, Reissner-Nordström (RN), extreme RN and dilatonic black holes are studied and a universal expression is found for their entropy at Hawking temperature T_H. Fermionic systems can also be studied as a next work.

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APPENDIX A: SUBORDINATION IDENTITY

One can observe the equivalence of the integrals

$$\int_{-\infty}^{\infty} f(y)\,dy = \int_{-\infty}^{\infty} f(y - \frac{1}{y})\,dy \tag{A.1}$$

by dividing the right hand side into two integrals as; first from $-\infty$ to 0 and second from 0 to ∞ and substituting $y - \frac{1}{y}$ with u.
Consider the well known integral

$$\int_{-\infty}^{\infty} e^{-ty^2}\,dy = \sqrt{\frac{\pi}{t}} \tag{A.2}$$

with the above equivalence where $f(y) = e^{-ty^2}$. After inserting $t = \frac{b\sqrt{x}}{2}$ and changing the variable as $y^2 = \frac{b}{2\sqrt{x}s}$ we obtain the result:

$$e^{-b\sqrt{x}} = \frac{b}{2\sqrt{\pi}} \int_0^{\infty} \frac{ds}{s^{3/2}}\, e^{b^2/4s}\, e^{-sx} \tag{A.3}$$

known as the subordination identity.

APPENDIX B: ASYMPTOTIC EXPANSION OF $F(x)$

We are looking for the divergent part of the asymptotic expansion, of the function $F(x)$ as $x \to 0$. Where;

$$F(x) = \int_0^\infty \frac{dt}{t^{3/2}}\, e^{\frac{-x^2}{4t}} e^{-t}\, (Tr\, e^{\frac{t}{m^2}\Delta}) \tag{B.1}$$

Due to the e^{-t}, terms and boundedness of $Tre^{t\Delta}$ as $t \to \infty$, we don't have any problem with the upper bound of the integral. To analyse the lower bound of integral we use the short time asymptotics of $Tre^{t\Delta}$ which is;

$$Tre^{t\Delta} \sim \frac{1}{t^{3/2}} \sum_{k=0}^{\infty} a_{k/2} t^{k/2} \tag{B.2}$$

If we write the series term by term and take the integral, we see that first five terms of expansion gives divergent results. Then if we rewrite the function $F(x)$ as convergent+divergent part we get,

$$\begin{aligned} F(x) \;=\; & \int_0^\infty dt\, t^{-3/2}\, e^{-\frac{x^2}{4t}-t} \left[Tr'\, e^{\frac{\Delta}{m^2}t} - (tm^{-2})^{-3/2} \sum_{n=0}^{4} a_{n/2} (tm^{-2})^{n/2} \right] \\ & + \int_0^\infty dt\, t^{-3/2}\, e^{-\frac{x^2}{4t}-t} (tm^{-2})^{-3/2} \sum_{n=0}^{4} a_{n/2} (tm^{-2})^{n/2} \end{aligned} \tag{B.3}$$

Divergent part of asymptotic of $F(x)$ as $x \to 0$ can be expressed as

$$\sum_{n=0}^{4} m^{3-n}\, a_{n/2} F_n(x) \tag{B.4}$$

where

$$F_n(x) = \int_0^\infty dt\, t^{(n/2)-3}\, e^{-\frac{x^2}{4t}-t} = 2 \left(\frac{x}{2}\right)^{-(2-\frac{n}{2})} K_{2-\frac{n}{2}}(x) \tag{B.5}$$

Here K's are the modified Bessel functions. Using the below series expansion for non-integer ν [37]

$$K_\nu(x) = \frac{\pi \csc(\nu\pi)}{2}\left[\sum_{k=0}^{\infty}\frac{1}{\Gamma(k-\nu+1)k!}\left(\frac{x}{2}\right)^{2k-\nu} - (\nu \to -\nu)\right] \tag{B.6}$$

and for integer n

$$\begin{aligned} K_n(x) &= \frac{1}{2}\sum_{k=0}^{n-1}(-1)^k\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k-n} + \\ &\quad +(-1)^{n+1}\sum_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k+n}}{k!(n+k)!}\left[\log\frac{x}{2} - \frac{1}{2}\psi(k+1) - \frac{1}{2}\psi(n+k+1)\right] \end{aligned} \tag{B.7}$$

collecting the relevant terms we get

$$\begin{aligned} F(x) &\asymp m^3 a_0\left(\frac{x}{2}\right)^{-4} + m^2\frac{\sqrt{\pi}}{2}a_{1/2}\left(\frac{x}{2}\right)^{-3} + (-m^3a_0 + ma_1)\left(\frac{x}{2}\right)^{-2} \\ &= (-m^2\sqrt{\pi}a_{1/2} + \frac{1}{\sqrt{\pi}}a_{3/2})\left(\frac{x}{2}\right)^{-1} + (-m^3a_0 + 2ma_1 - m^{-1}a_2)\log\frac{x}{2} \end{aligned} \tag{B.8}$$

From this we get the b coefficients as;

$$\begin{aligned} b_{-4} &= 16m^3a_0 \\ b_{-3} &= 4\sqrt{\pi}m^2a_{1/2} \\ b_{-2} &= 4(-m^3a_0 + ma_1) \\ b_{-1} &= 2(-\sqrt{\pi}m^2a_{1/2} + \frac{1}{\sqrt{\pi}}a_{3/2}) \\ b' &= -m^3a_0 + 2ma_1 - m^{-1}a_2 \end{aligned} \tag{B.9}$$

These a coefficients are heat kernel coefficients. First few are;

$$\begin{aligned} a_0 &= \frac{V}{(4\pi)^{3/2}} \\ a_{1/2} &= r\frac{A}{16\pi} \\ a_1 &= \frac{1}{6(4\pi)^{3/2}}\left[\int_B d\mu_g(-R) + \int_{\partial B} 2K\right] \end{aligned} \tag{B.10}$$

Which are expressed in terms of R, K, $d\mu_g$ that are the Riemann curvature of $\mathbf{M}$, the extrinsic curvature of the surface ∂B of the box, and the Riemann volume element on $\mathbf{M}$ respectively. In $a_{1/2}$, r can take the values $1, -1$ which correspond to either Neumann or Dirichlet boundary conditions, respectively.

On the other hand in our analysis we also need $x \to \infty$ behaviour of $F(x)$. We will first start with the boundedness of $Tr\, e^{-Ht}$. Assigning λ as the first non zero eigenvalue of H, one can see the relation;

$$Tre^{-H\,t} \leq Tr(e^{-H\,t_0})\, e^{-\lambda(t-t_0)} \leq Tre^{-H\,t_0} \quad \text{for} \quad t \geq t_0 \tag{B.11}$$

Considering the most dominant term in heat kernel asymptotics we can write;

$$Tre^{-H\,t_0} \sim \frac{V}{(4\pi t_0)^{3/2}} \quad t_0 \to 0 \tag{B.12}$$

After performing asymptotic analysis, choosing t_0 small enough and using the above inequality we get;

$$Tre^{-H\,t} \leq \frac{3}{2}\frac{V}{(4\pi t_0)^{3/2}} \tag{B.13}$$

Inserting the largest value for $Tre^{-H\,t}$ in Eq.B.1 we get the bound;

$$F(x) \leq \frac{V}{(4\pi t_0)^{3/2}}\frac{3\sqrt{2}}{\sqrt{x}}K_{1/2}(x) = \frac{V}{(4\pi t_0)^{3/2}}\frac{3\sqrt{\pi}}{x}e^{-x} \tag{B.14}$$

And the result is;

$$F(x) = O\left(x^{-1}e^{-x}\right) \quad as \quad x \to \infty \tag{B.15}$$

APPENDIX C: MELLIN TRANSFORM

Mellin transform of a function $f(x)$ is defined as;

$$\mathcal{M}[f(x);s] := f^*(s) = \int_0^\infty f(x)x^{s-1}dx \tag{C.1}$$

where $f^*(s)$ is the transformed complex function. If the integral in Eq.C.1 is convergent for a strip call $\langle\alpha,\beta\rangle$ and if this is the largest strip then this is called the fundamental strip. Changing the variable as $x = e^{-u}$ gives Laplace transform, and $x = e^{i\omega}$ gives Fourier transform. Mellin transform is used in a very wide range of areas like digital structures, probabilistic algorithms, and communication theory. It is also useful to study asymptotics of Gamma related functions, with Mellin transform. In this theses we use Mellin transform to study asymptotic analysis of the sums in the form of;

$$G(x) = \sum_k \lambda_k\, g(\mu_k x) \tag{C.2}$$

i. e. harmonic sums as $x \to 0$. Here λ_k, μ_k and $g(x)$ are called amplitudes, frequencies and the base function respectively. Mellin transform has many functional properties. One of them is;

$$\mathcal{M}[f(\mu x);s] = \mu^{-s} f^*(s) \quad s \in \langle\alpha,\beta\rangle \tag{C.3}$$

known as rescaling rule. Thanks to this rule we have the relation;

$$\mathcal{M}\left[\sum_k \lambda_k f(\mu_k x);s\right] = \left(\sum_k \frac{\lambda_k}{\mu_k^s}\right).f^*(s) \tag{C.4}$$

for harmonic sums. For $\lambda_k = 1$ one gets the $\zeta(s)$ for the sum above [18].

Now we will give the validity of the dictionary introduced in chapter 4, using the following discussion and the theorem introduced;

Inverse Mellin transform is defined as;

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f^*(s)x^{-s}ds = f(x) \tag{C.5}$$

Fundamental strip for $f^*(s)$ is $a < \Re s < b$ and $a < c < b$. Assume that analytic continuation of $f^*(s)$ into a region $\Re s \leq a$ with poles at $s = w_1 \geq w_2 \geq \ldots$ is possible. In order to calculate inverse Mellin C.5 integral we close the counter with a vertical line passing through two consecutive poles, w_r and w_{r+1} with the condition $\Re s = \sigma_0 < c$.

Theorem C.1. *Suppose that $f^*(s)$ is analytic in a left hand plane, $\sigma \leq c$, apart from poles at $s = \omega_r$, $r = 0, 1, 2, ...$; let the principal part of the Laurent expansion of $f^*(s)$ about $s = \omega_r$ be given by*

$$\sum_{n=0}^{k} A(\omega_r, k_r)\frac{(-1)^n n!}{(s+\omega_r)^{n+1}} \tag{C.6}$$

Assume that $|f^*(\sigma + i\tau)| \to 0$ as $|\tau| \to \infty$ for $\sigma_0 \leq \sigma \leq c$, and that $|f^*(\sigma_0 + i\tau)|$ is integrable for $|\tau| < \infty$. Then, if σ_0 can be chosen so that $-\Re(\omega_{r+1}) < a' < -\Re(\omega_r)$, $f(x)$ has the asymptotic expansion [38].

$$f(x) \sim \sum_{i=1}^{r}\sum_{k}^{k_r} A(w_r, k_r)x^{-w_r}(\log x)^{k_r} \quad as \quad x \to 0 \tag{C.7}$$

The above background information suggests that if

$$\lim_{|\tau|\to\infty} |f^*(\sigma + i\tau)| = 0 \quad for \quad \sigma_0 \leq \sigma \leq c \tag{C.8}$$

and

$$\int_{-\infty}^{\infty} d\tau\, |f^*(\sigma_0 + i\tau)| < \infty \tag{C.9}$$

is satisfied then the dictionary described by Equations 4.21,4.22 is valid.
For our case $f = A$ and

$$A^*(s) = \frac{1}{\sqrt{\pi}}\zeta(s)(\mathcal{LM}g_\alpha)(s,\alpha) \tag{C.10}$$

where $s = \sigma + i\tau = |s|e^{i\phi}$. We should analyse the behaviour of

$$\zeta(s)\left[\int_0^\infty dx\, x^{s-1}e^{-\alpha x}\left[g_\alpha(x) - \sum_{j=-4}^{-1} c_j x^j\right] + \sum_{j=-4}^{-1} c_j \alpha^{-j-s}\Gamma(s+j)\right] \tag{C.11}$$

as $\tau \to \infty$.
$|\zeta(\sigma + i\tau)|$ is an algebraically increasing function as $|\tau| \to \infty$. With the information

$$|\Gamma(\sigma + i\tau)| = O(|\tau|^{\sigma-\frac{1}{2}}e^{-\frac{1}{2}\pi|\tau|}) \quad as \quad \tau \to \infty \tag{C.12}$$

[37] one can observe the convergence for the second term in square parentheses. For the first part, if we consider the magnitude of the integrand we encounter with a phase, $|x^{i\tau}|$, with which we loose information in $\tau \to \infty$ limit. In order to cope with first term define $\hat{s} = s/|s| = p + iq$ and rotate the contour $(0, \infty)$ by $\phi < \pi/2$. Along this new contour C, the first term in integral (C.11) becomes;

$$\int_C dz \ldots = \hat{s}^s \int_0^\infty dx\, x^{s-1}e^{-\alpha\hat{s}x}[g(\hat{s}x) - \sum_{j=-4}^{-1} c_j x^j] \tag{C.13}$$

Assuming that the condition $g_\alpha(x) - \sum_{j=-4}^{-1} c_j x^j = O(1)$ is also valid when x is replaced by $\hat{s}x$ and hence,

$$g(\hat{s}x) - \sum_{j=-4}^{-1} c_j x^j = O(1) \tag{C.14}$$

Plugging the result into (C.13) together with the below comparison gives the result;

$$\hat{s}^s \int_0^\infty dx\, x^{s-1} e^{-\alpha \hat{s} x}\, C \leq C|\hat{s}^s| \int_0^\infty dx\, x^{\sigma-1} e^{-\alpha x p} = C|\hat{s}^s| \frac{\Gamma(\sigma)}{(\alpha p)^\sigma} \tag{C.15}$$

Also note the equivalence for $|\hat{s}^s|$ that;

$$|\hat{s}^s| = \frac{|s^s|}{|s|^\sigma} = e^{-\tau\phi} = e^{-|\tau||\phi|} \tag{C.16}$$

With the above information, one can see the decay of the first term as $\tau \to \infty$ as well.

Now we should consider the contribution coming from the big circular arc C_R of radius R, that we encounter after rotating the original contour by an angle ϕ. On this arc $z = Re^{i\theta}$.

After changing the variable t to xu in (4.14) one gets;

$$F(x) = \frac{1}{x^{1/2}} \int_0^\infty \frac{du}{u^{3/2}}\, e^{-\frac{x}{4u}} e^{-xu}\, Tr e^{\frac{x\Delta}{m^2} u} \tag{C.17}$$

Analytically continuing F to $\Re z > 0$ gives;

$$g_\alpha(z) = e^{\alpha z} \cosh(\xi z) F(z) \tag{C.18}$$

Using triangle inequality for cosh term gives;

$$\mid g_\alpha(z) \mid = e^{\alpha R \cos\theta} \cosh(\xi R \cos\theta) \mid F(z) \mid \tag{C.19}$$

Note that another inequality, again depends on triangle inequality due to the trace inside the integral is,

$$|F(z)| \leq R^{-1/2} F(R\cos\theta) \tag{C.20}$$

Combining the two inequality gives the bound

$$|g_\alpha(z)| \leq R^{-1/2}\, e^{\alpha R\cos\theta} \cosh(\xi R\cos\theta) F(R\cos\theta) \tag{C.21}$$

Considering large R behaviour of F via the result (B.15) gives;

$$|g_\alpha(z)| \leq cR^{-1/2}\, e^{\alpha R\cos\theta} \cosh(\xi R\cos\theta)(R\cos\theta)^{-1/2} e^{-R\cos\theta} \tag{C.22}$$

Since $-1+\alpha+\xi<0$, the product of exponential terms is less than 1. Thus we arrive at the result;

$$|g_\alpha(z)| \leq c(R)^{-1}(\cos\theta)^{-1/2} \tag{C.23}$$

Now along C_R note the comparison;

$$\begin{aligned}\left|\int_{C_R} dz\ldots\right| &\leq R^\sigma \int_0^\phi d\theta e^{-\alpha R\cos\theta}\left[|g_\alpha(z)| + \sum_{j=-4}^{-1}|c_j|R^j\right] \\ &\leq R^\sigma e^{-R\cos\phi}\int_0^\phi d\theta\left[c(R)^{-1}(\cos\theta)^{-1/2} + \sum_{j=-4}^{-1}|c_j|R^j\right] \end{aligned} \tag{C.24}$$

which shows that the integral along C_R vanishes as $R\to\infty$.