# SELF-ADJOINT EXTENSION THEORY OF SINGULAR INTERACTIONS

by

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B.S., Mathematics, Boğaziçi University, 2014

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics

Boğaziçi University

2017

SELF-ADJOINT EXTENSION THEORY OF SINGULAR INTERACTIONS

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DATE OF APPROVAL: 04.08.2017

# ACKNOWLEDGEMENTS

First of all, words are not enough to express my acknowledgment to my advisor Olcay Coşkun. It is a great chance and genuinely honour for me to study with him. He encourages and affects me to apply for mathematics master program. His encouragements and comments were invaluable.

Equally, I owe my deepest gratitude to my co-advisor who is the master mind behind this thesis, Teoman Turgut. Without his guidance and persistent help, this thesis would not have been possible. Correspondingly, The door of Teoman Turgut office is always open for me. Whenever I run into a trouble spot, he gives me insightful suggestions.

Moreover, while I was typing the thesis, a fascinating event happened. Professor Loring W. Tu sent his original and inspirational book *An Introduction to Manifolds* [1] with his signature. I sincerely appreciate to him.

Withal, I am very grateful to Arzu Boysal not only for taking part my thesis committee but also being the advisor of me in undergraduate years. Special thanks to Ferit Öztürk who described the joy of doing mathematics and inspired me to study mathematics. I need to express my acknowledgment to Cem Yalçın Yıldırım. It was a privilege to take courses from him. I am thankful to Kamuran Saygılı and Levent Akant for participating on my thesis committee.

What is more, I received generous support from D. Can Doğan while I was writing this thesis. His support aided to me to stand up when I faced with some problems. I am much gratitude for Furkan Biten who gives me the unique way to resolve issues from a unique point of view. Besides, I should thank my flatmate Armağan Demir who has demonstrated the way of studying with patiently. He influenced me to devote myself to my academic ambitions.

Herewith, I ought to acknowledge Burak Kor for his genuine fellowship. He is always with me from the beginning of my academic life. His support matters a lot I want to thank Ulaş Yardımcı for exceptional companionship. We shared lots of unutterable moments, especially in our place. His friendship eases my life and improves my horizon of thinking. My sincere appreciation goes to Büşra Erarslan. Meeting with her was one of the milestones in my life. She triggered my passion for academics studies. Likewise, discussion with her always has been illuminating. Her suggestions always are intimate and unique.

Therewith, I am very grateful to Erdal Bilgiç. We spent a perfect time together in library. Furthermore, I would like to thank Deniz Yılmaz for enhancing my mathematical intellectuality. I need to thank Ömer Aktepe, Alperen Karan and Görkem Işın for their friendship during my education.

Furthermore, I should thank my manager Hikmet Karabacak and my colleagues Ümit Haydar Özgüler, Umut Onur, Duygu Çelik, Begüm Akın and the rest of current Verapi team members a long with Adem Yılmaz, Evren Toraman and the remainder of the former Verapi team members. Their understanding eases my study of this thesis.

Last but not the least, no matter these words are placed in the conclusion, family comes first. My expressions are inadequate to describe my feeling concerning my parents. I would like to show my greatest appreciation to my father Şencan Türköz and my mother Aysel Türköz. They have been exceptionally tolerant and supportive. Their continuous encouragement and constant support are priceless. Without their enthusiasm and will, I could not carry on the journey of my academic study.

# ABSTRACT

## SELF-ADJOINT EXTENSION THEORY OF SINGULAR INTERACTIONS

In this thesis, we study two-dimensional finite singular point interactions and three-dimensional finite singular interactions supported by curves on a compact Riemannian manifold. Our aim is to show that finite rank perturbations of self-adjoint operators on compact manifold agree with renormalization technique to describe singular Dirac delta potential on the compact Riemannian manifold. To achieve our goal, we rely on heat kernel techniques and the proper upper bounds. We review the heuristic construction of the resolvents via renormalization method. Later, we present rudiments of finite rank perturbation and self-adjoint extensions, suitable for our purposes. In the end, we prove our main results that singular interactions on compact manifold could be understood from the self-adjoint extension perspective.

# ÖZET

## TEKİL ETKŞİLEŞİMLERİN KENDİNE EŞ GENİŞLEME TEORİSİ

Bu tezde kompakt Riemannyan manifold üzerinde tanımlanmış iki boyutlu sonlu tekil noktasal etkileşim ve üç boyutlu eğrilerle desteklenmiş sonlu tekil etkileşimler çalışılmıştır. Tezin amacı tekil Dirac delta potansiyellerini tarif etmek etmek için kendine eş operatörlerin sonlu mertebe bozunumların kullanmasıyla renormalizasyon metodu ile aynı olduğunu göstermektir. Bu amaca ulaşmak için ısı çekirdeği tekniklerine güvenilmiştir. Ardından amaca uygun sonlu mertebeden bozunumların ve kendine eş genişlemerin temellerini sunulmaktadır. Son olarak, tezin ana sonucu olan kompakt Riemannyan manifold üzerinde tanımlanmış tekil etkileşimlerin kendine eş genişlemeler perspektifden anlaşılabileceği kanıtlanmıştır.

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# LIST OF SYMBOLS

| | |
|---|---|
| $\mathbb{C}$ | Complex Plane |
| $C^\infty(\mathcal{M})$ | The space of all smooth functions on manifold $\mathcal{M}$ |
| $d_g(x, y)$ | Geodesic distance between $x$ and $y$, $x, y \in \mathcal{M}$ |
| $d_g x$ | Riemannian volume element with the metric structure $g$ |
| $E$ | The bound state energy |
| $\mathscr{H}$ | Hilbert Space |
| $H$ | Hamiltonian |
| $\mathbf{H}^2$ | The Sobolev space |
| $K_t(x, y)$ | Heat Kernel on manifold $\mathcal{M}$ |
| $L^2(\mathcal{M})$ | The space of square integrable functions on manifold $\mathcal{M}$ |
| $Q(\cdot)$ | $Q$ - matrix function |
| $\mathbb{R}$ | Euclidian Space |
| $\mathcal{R}$ | Admissible matrix |
| $T_p(\mathcal{M})$ | The tangent space of manifold $\mathcal{M}$ at $x_p \in \mathcal{M}$ |
| $T_p^*(\mathcal{M})$ | The cotangent space of manifold $\mathcal{M}$ at $x_p \in \mathcal{M}$ |
| $V(\mathcal{M})$ | Volume of Riemannian manifold |
| $\delta_i^j$ | Kronecker delta |
| $\delta_g(x, a_i)$ | Dirac delta function in $\mathcal{M}$ at position $a_i$ with metric $g$ |
| $\epsilon$ | Cut off parameter in time |
| $\hbar$ | The Planck constant |
| $\Lambda$ | Cut off parameter in momentum |
| $\lambda$ | Coupling Constant |
| $\lambda(\epsilon)$ | Bare Constant |
| $\Phi_{ij}$ | Principle operator |
| $\nabla_g$ | Gradient on manifold $\mathcal{M}$ with metric structure $g$ |
| $\Delta_g$ | Laplacian on manifold $\mathcal{M}$ with metric structure $g$ |
| $\Re\, z$ | Real part of the complex number $z$ |
| $\Im\, z$ | Imaginary part of the complex number $z$ |

*To my family...*

# 1. INTRODUCTION

Quantum mechanics is dealing with the conduct of matter and light on atomics and subatomic level. In this scale, many equations of classical mechanics cease to be useful. The scientific investigation into wave idea of light commenced in $17th$ and $18th$ whereas Leonard Euler, Robert Hooke, and Christiaan Huygens proposed a wave theory of light based on experimental observation [2]. Thomas Young performed the double split experiment in 1803 [3]. This research played a vital role in the acceptance of the theory of light in scientific communities. Further, Micheal Faraday explored cathode rays in 1839 [4]. Likewise, Gustav Kirchhoff stated the black body radiation in 1859 [5]. Besides, Ludwig Boltzmann suggested that energy states of the physical system can be discrete in 1887 [6]. Afterward, the hypothesis of Max Planck claimed that energy is radiated and absorbed in discrete "quanta" (energy pocket), exactly coincide with observed patterns of black body radiation [7]. Congruently, Albert Einstein presented a quantum based theory to depict the photoelectric effect in 1905 [8]. Withal, the first scholars to deal with quantum phenomena in nature are Arthur Compton, C. V. Raman and Pieter Zeeman, each of whom has a quantum effect named after him [9,10]. Withal, Niels Bohr developed his theory of the atomic structure [11]. Besides, De Broglie discovered wave nature of electrons [12].

Mathematical foundations of quantum mechanics are developed by Paul Dirac [13], David Hilbert [14], John von Neumann [15], and Hermann Weyl [16]. The possible states are considered as a unit vector in complex Hilbert space. The formalism of quantum mechanics is constructed as the state of a system at a given time is described by a complex wave function addressed to as state vector in a complex vector space. Whereupon, matrix notation is called matrix mechanics is invented by Werner Heisenberg [17]. In the matrix formulation, the temporary condition of a quantum framework encodes the probabilities of its measurable properties or "observables" [18]. Energy, position, momentum, and angular momentum can be given as an example of observable. Observables can be discrete or continuous [19]. For instance, the position of a particle is continuous whereas the energy of an electron bound to a hydrogen atom

is discrete [20]. Schrödinger concocted wave wechanics, the answer for Schrödinger's equation [21]. The Schrödinger equation describes the time evolution of quantum state in which the operator corresponding to the total energy of the system (the Hamiltonian) generates the time evolution [22,23]. Deterministically, the time of evolution wave functions predicts the wave function will be at any time late [24]. Matrix Mechanics and Wave Mechanics, clearly contradictory, were appeared to be equivalent [21].

We provide a brief history of quantum mechanics and its mathematical foundations. Also, in quantum mechanics, two elegant methods regularization and renormalization are developed. There are some simple and informative examples of regularization and renormalization. One of them is two-dimensional Dirac delta potentials (zero range, contact or point interactions or Fermi pseudopotentials in the literature). The work of Kronig and Penney [25] commenced the study of one-dimensional point interactions. Dirac delta potentials in higher dimensions started with the review of Bethe and Peierls [26]. Thorn [27] stated that regularization and renormalization procedures provide the physical results. The detailed history of the development of point interaction Hamiltonians is given in the monograph [28]. The two-dimensional point interactions are fascinating since it is an instance of the dimensional transmutation and anomaly for a simple system which has symmetry. The two-dimensional Dirac delta potential in natural units ($\hbar = 2m = 1$) given by

$$-\Delta_g \psi(x) - \lambda \delta(x)\psi(x) = E\psi(x). \tag{1.1}$$

Then, the dimensional analysis for estimation of the energy does not apply because there is no natural scale for the problem, the coupling constant $\lambda$ is dimensionless. In physics, dimensionless coupling constant obstructs the determination of the bound state energy. This energy typically leads to an infinite result. This can be seen from a variational approach,

$$E[\psi] = \int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla\psi(x)|^2 - \lambda|\psi(0)|^2 dxdx \tag{1.2}$$

where $E[\psi]$ is the expected value of Hamiltonian with normalized wave function $\psi(x)$.

We introduce a wave function $\psi_\alpha(x) = \alpha\psi(\alpha x)$. As a result, one can show that $E[\psi_\alpha] = \alpha^2 E[\psi]$. Provided that we have a negative energy with wave function, $\psi$ which is square integrable, $\alpha^2$ is positive and arbitrary, thence we cannot keep the energy bounded from below $E \to -\infty$.

The way out is to use a regularization and renormalization procedure. Renormalization means that one introduces a scale into the problem by hand, consequently breaking the scale invariance of the Hamiltonian. These problems are related to the anomaly [29, 30] or symmetry breaking as well as dimensional transmutation [31–34]. The idea of renormalization is related to a function called a $\beta$ function whose roots determine the fixed points of the theory. Point interactions in quantum mechanics provide a nonperturbative solution to this problem. [35–37] and for this case the $\beta$ function has been computed accurately.

Another example of regularization and renormalization in quantum mechanics is singular interactions supported by curves. Many mathematicians and physicists pay attention to this subject. Behind the motivation of this attention is the desire to describe semiconductor wires as explored in [38, 39]. The curved-supported singular interactions describe electrons confined to a narrow tube-like region and this theory with self-adjoint extensions approach in $\mathbb{R}^3$ is constructed rigorously in [40–44]. Also, these works examined the ground state wave function and regularity of eigenfunctions. Mostly, this approach allows an infinite number of extensions. We choose a single extension parameter. Consequently, the value of binding establishes the theory.

It is natural to investigate the range of parameters that lead to bound states. The Schrödinger operator with singular interactions supported by curves embedded in

a Riemannian Manifold is studied which considers neither finite-length closed curves intersecting, or self-intersecting is investigated by B. Kaynak and T. Turgut [45].

The curve-supported interactions potential in natural units ($\hbar = 2m = 1$) can be given by

$$-\Delta_g \psi(x) - \frac{\lambda}{L} \int_\Gamma d_g s \delta_g(x, \gamma(s)) \int_\Gamma d_g s' \psi(\gamma(s')) = E\psi(x). \tag{1.3}$$

$\Gamma$ stands for the curves over which the integrals are taken, and $\lambda$ is the coupling constant. If the dimension of the manifold is greater than two dimension, regularization and renormalization method is applicable. Since singular interactions support by curves on a two dimension Riemannian manifold leads to finite bound-state energy, it is a simpler problem.

Berezin and Fadeev [46] developed an ingenious method for describing two and three-dimensional singular point interactions. In this way, the Hamiltonian is introduced rigorously as a self-adjoint operator derived from Krein's theory of symmetric operators [47]. The details can be found in the monograph [28]. There are two different ways to define self-adjoint operators. The first method is von Neumann's formula [15, 48, 49] and the second one is via using Krein's formula [50, 51].

In this thesis, we are focusing on a slightly different but related technique to deal with singular interactions on a compact manifold. This theory is called finite rank perturbation which is a purely mathematical subject. T. Kato stressed the importance of perturbation theory in his selected works [52–55]. Conjointly, T. Kato and S. T. Kuroda mentioned the definition of finite rank perturbation in their papers [56, 57]. Likewise, W. F. Donoghue showed the relation between the extension theory of symmetric operators and finite rank perturbations while he was studying the Schrödinger operator with zero range in three dimensions [58]. Also, Berezin and Fadeev [46] gave

an example of an unbounded rank one perturbations. To understand interactions on a compact manifold, we are using finite rank perturbations of self-adjoint operators. To comprehend finite rank perturbations, we will define rank one perturbations rigorously. This approach aids to get some intuitions and ideas to construct finite rank perturbations.

Rank one perturbations of self-adjoint positive operators $A$ are of the form

$$A_\alpha = A + \alpha(\varphi, \cdot)\varphi \tag{1.4}$$

where $\varphi$ is in general belongs to some Hilbert space. The generalization of rank-one perturbation is called finite rank perturbation.

Finite rank perturbations of self-adjoint positive operators $A$ are of the form

$$A_T = A + \sum_{i,j=1}^{N} t_{ji}\langle\varphi_i, \cdot\rangle\varphi_j \tag{1.5}$$

where the matrix $T$ is Hermitian as well as invertible and in general, some Hilbert spaces contain the vectors $\varphi_j$ are for $j = 1, 2, \cdots N$.

The work of Simon and Wolff [59] along with the one of the Gesztesy and Simon [60] examined $\mathscr{H}_{-1}$ perturbation of self-adjoint operators where $\mathscr{H}_{-1}$ is a dual Hilbert space to be introduced later. V. Koshmanenko and S. Albeverio gave some remarks on the Gesztesy and Simon version of rank one perturbations [61]. The paper of B. Simon [62] provided a complete explanation of unbounded $\mathscr{H}_{-1}$ perturbations which are defined via the standard perturbation technique [54, 63]. When it comes to $\mathscr{H}_{-2}$ perturbations, we should mention the paper by Javrian who examined the Weyl problem [64]. Kiselev and Simon commenced the abstract settings of $\mathscr{H}_{-2}$ perturbations based on the article of Berezin and Fadeev [46].

V. Koshmanenko developed the abstract quadratic form formula of $\mathscr{H}_{-2}$ perturbation for bounded below case [65–71]. The operator formulation appeared in the study of S. Albeverio, W. Karwowski, and V. Koshmanenko [72]. Additional studies of S. Albeverio and V. Koshmanenko are based on Krein's formula [66, 69, 70, 73–75]. The construction of $\mathscr{H}_{-2}$ perturbations has been used in the work of W. Karkowski, V. Koshmanenko and S. Ota for perturbation of the Laplacian in $\mathbb{R}^3$ via dynamics supported by the set of lower dimension [76].

S. Albeverio and P. Kurasov discussed $\mathscr{H}_{-2}$ rank one perturbation without renormalization of the coupling constant for semi-bounded self-adjoint operators [77], not necessarily semi-bounded operators [78]. The description of approximation of perturbed operator without renormalization of coupling constant is originated from an earlier study of P. Kurasov [79]. Congruently, an approximation of $\mathscr{H}_{-2}$ rank one perturbations is depicted by S. Albeverio and P. Kurasov [80]. Finite rank $\mathscr{H}_{-2}$ perturbations of non-semibounded operators are explored in [81].

S. Hassi and De Snoo [82, 83] showed that rank one perturbation could be examined without a semi-boundedness requirement, via analytic properties of rank one perturbation. Posilicano contributed the finite rank perturbation theory as well as its connection with Krein's formula [84–87]. S. T. Kurado suggested that varied $\mathscr{H}_{-2}$ perturbation method that does not require the criteria of bound from below perturbed operators [57, 88]. Furthermore, the suggestion of S. T. Kurado is developed by N. Nagatini and K. Watanabe [89–91]. Besides, S. Albeverio and Pankrashkin gave some remarks on Krein's formula and related to the boundary condition of finite rank perturbations [92].

Our primary goal in this thesis is to show that heuristically constructed singular interactions on a compact Riemannian manifold coincide with the theory of the finite rank perturbation of self-adjoint operators. Although there is no direct method to describe and to understand the behaviour of singular Dirac delta potentials on a compact Riemannian manifold, there is an elegant method which uses heat kernel to comprehend

these interactions on a compact manifold.

Heat Kernel plays very crucial role in differential geometry [1, 93], mathematical physics [94], geometric analysis [95–97] and quantum field theory [98]. We will now portray the content and the organization of this thesis:

Chapter 2 presents two vital upper bounds for the heat kernel on a compact Riemannian manifold. Upon that preliminary results are given to prove these bounds.

Chapter 3 consists of singular interactions on Riemannian manifolds. Section 3.1 indicates different quantization procedures [99]. In section 3.2, we present a heuristic renormalization and regularization of the model on the compact Riemannian manifold which contains a non-relativistic point particle living in two-dimensional Riemannian manifolds interacting with finitely many singular point interactions located on the manifold [100,101]. The rigorous construction of the model is performed using heat kernel in two dimensions is used to express the resolvent of the Hamiltonian of the system which is a finite and well-defined expression. After the implementation of renormalization, the resolvent has all the information concerning the spectrum. Thus, we provide an equation to determine the bound state energies. Moreover, Section 3.3 demonstrates a non-relativistic mass living in three-dimensional Riemannian manifold without boundary and interacting singularly by a potential supported on curves embedded in the manifold [45]. The model is constructed meticulously via a heat kernel technique because heat kernel in three dimensions is used to identify the resolvent of Hamiltonian of the system that is a finite and well-defined. The resolvent contains the knowledge about the spectrum after the renormalization is applied. In this manner, we give an equation to detect the bound state energies.

Chapter 4 examines the rank one perturbations. We produce some definitions in detail such as the scale of Hilbert spaces and important theorems of rank one perturbations concerning its self-adjoint extensions [47]. To fathom self-adjoint extensions, we introduce the resolvent of rank one perturbations.

Chapter 5 offers the finite rank perturbations. We present finite rank perturbations via Krein's resolvent formula. The method of S. Albeverio and P. Kurasov for rank-one perturbation of self-adjoint operators is explained and proved [47, 77].

Chapter 6 is devoted to our studies and original results. Initially, we define the singular Dirac delta potential model for $N$ points on the compact Riemannian manifold. We introduce a formal Hamiltonian as follows

$$\mathcal{H} = -\frac{\hbar^2}{2m}\Delta_g - \lambda\delta(x, a_i). \tag{1.6}$$

The details can be found in [102, 103]. Then, we show that $\Delta_g$ has a self-adjoint extension on domain $D = \{\psi \in \mathbf{H}^2 \mid \psi(a_i) = 0,\ i = 1, \ldots, N\}$ formally given by $\mathcal{H}$ via the technique of Albeverio and P. Kurasov for finite rank perturbation of self-adjoint operators [47, 77, 80, 81, 104]. Thereupon, the resolvent of the self-adjoint operator is computed. Thus, we mention that this resolvent coincides with the resolvent of renormalization technique.
Secondly, we introduce the interactions supported by curves on a three dimensional compact Riemannian manifold

$$\mathcal{H} = -\frac{\hbar^2}{2m}\Delta_g - \lambda\delta_{\Gamma_i} \otimes \delta_{\Gamma_i} \tag{1.7}$$

where $\Gamma$ stands for the arc-length parameterized closed curve of length $L$. Further details about this model can be obtained in [45]. We prove that $\Delta_g$ has a self-adjoint extension on domain $D = \{\psi \in \mathbf{H}^2 \mid \psi(\Gamma_i) = 0,\ i = 1, \cdots, N\}$ formally given by $\mathcal{H}$. After that we calculated the resolvent of the self-adjoint extension. Thence, we demonstrate that this resolvent corresponds with the resolvent of renormalization method.

Chapter 7 stands for the conclusion of this thesis. So as to, conclusion contains a summary of this work. Moreover, the future work of this thesis is mentioned.

In Appendix A, we summarize von Neumann Theory of self-adjoint extensions. In this regard, basic definitions of symmetric operators and self-adjoint operator are given. The properties of a required closed symmetric operator to have a self-adjoint extension is provided. Besides, we introduce and mention that proof of von Neumann's self-adjoint extension theorem. The details of this chapter can be found in the standard books [63, 105–109].

Appendix B contains the definition and some properties of Laplace operator on the Riemannian manifold in conjunction with the spectral attributes of the Laplacian operator. The further analysis of this chapter can be gained in these books [1,110–112].

In Appendix C, we present some fundamental definitions and related theorems about the heat kernel defined on Riemannian manifolds. The formal definition of heat kernel on Riemannian manifold together with its properties are covered. Addiional features concerning this chapter can be obtained in these studies [96, 113, 114].

# 2. UPPER BOUNDS OF THE HEAT KERNEL ON COMPACT RIEMANNIAN MANIFOLD

This chapter examines upper on-diagonal and off-diagonal bounds of the heat kernel on a compact Riemannian manifold. Furthermore, this part provides preliminary results.

Let $\mathcal{M}$ be a compact Riemannian manifold with metric $g$. For $x, y, z \in \mathcal{M}$ and $t > 0$,

$$K_t(x,y) = \int_{\mathcal{M}} K_{t/2}(x,z) K_{t/2}(z,y) d_g z \tag{2.1}$$

holds. This relation is obtained by semigroup property of the heat kernel $(C.11)$. Afterward, applying Cauchy-Schwarz inequality to (2.1) leads the following

$$\begin{aligned} K_t(x,y) &= \int_{\mathcal{M}} K_{t/2}(x,z) K_{t/2}(z,y) d_g z \\ &\leq \left[ \int_{\mathcal{M}} K^2_{t/2}(x,z) d_g z \right]^{1/2} \left[ \int_{\mathcal{M}} K^2_{t/2}(z,y) d_g z \right]^{1/2}. \end{aligned}$$

As a consequence,

$$K_t(x,y) \leq \sqrt{K_t(x,x) K_t(y,y)} \tag{2.2}$$

is obtained. The weighted integral of the heat kernel on compact manifold Riemannian manifold is provided by

$$E_D(x,t) = \int_{\mathcal{M}} K^2_t(x,z) \exp\left( \frac{d^2(x,z)}{Dt} \right) d_g z \tag{2.3}$$

where $D > 0$. If $\lim_{D\to\infty} E_D(x,t)$ is performed then (2.1) implies that

$$E_\infty(x,t) = K_{2t}(x,x). \tag{2.4}$$

Thus, (2.2) can be rewritten as follows

$$K_t(x,y) \le \sqrt{E_\infty(x,t/2)E_\infty(y,t/2)}. \tag{2.5}$$

**Lemma 2.1.** *Let $\mathcal{M}$ be a Riemannian manifold. Suppose that $x, y \in \mathcal{M}$. Then for any $D > 0$ and $t > 0$,*

$$K_t(x,y) \le \sqrt{E_D(x,t/2)E_D(y,t/2)}\exp\Big(\frac{-d^2(x,y)}{2Dt}\Big) \tag{2.6}$$

*holds.*

*Proof.* Suppose that $\alpha = d(y,z)$, $\beta = d(x,z)$ and $\gamma = d(x,y)$ for any $x, y, z \in \mathcal{M}$. The triangle inequality provides that $\alpha^2 + \beta^2 > \gamma^2/2$. Then,

$$\begin{aligned} K_t(x,y) &\le \int_{\mathcal{M}} K_{t/2}(x,z)e^{\beta^2/Dt}K_{t/2}(y,z)e^{\alpha^2/Dt}e^{-\gamma^2/2Dt} \\ &\le \Big(\int_{\mathcal{M}} K^2_{t/2}(x,z)e^{\beta^2/Dt}\Big)^{1/2}\Big(\int_{\mathcal{M}} K^2_{t/2}(y,z)e^{\alpha^2/Dt}\Big)^{1/2}e^{-\gamma^2/2Dt} \\ &\le \sqrt{E_D(x,t/2)E_D(y,t/2)}\exp\Big(-\frac{d^2(x,y)}{2Dt}\Big). \end{aligned}$$

Hence, proof is completed. □

**Definition 2.1.** *Regularity condition for a function $f$ is defined as follows. There are numbers $A \geq 1$ and $a > 1$ such that*

$$\frac{f(as)}{f(s)} \leq A\frac{f(at)}{f(t)} \quad \text{for all } 0 < s < t. \tag{2.7}$$

**Theorem 2.1.** *Let $\mathcal{M}$ be a compact Riemannian manifold. For some $x \in \mathcal{M}$, for all $t > 0$ and for each $C > 0$*

$$K_t(x, x) \leq \frac{C}{f(t)} \tag{2.8}$$

*along with*

$$E_D(x, t) \leq \frac{C'}{f(\delta t)} \tag{2.9}$$

*holds for $\delta > 0$ and for positive constant $C'$ where $f(t)$ is an increasing positive function on $(0, \infty)$ satisfying the regularity condition.*

*Proof.* Suppose that the integral identity $I(t)$ is given by

$$I(t) = \int_{\mathcal{M}} K_{t/2}(x, z)K_{t/2}(z, x)d_g z.$$

Semi-group property of heat kernel $(C.11)$ yields

$$I(t) = K_t(x, x).$$

Assume that $K_t(x,x)$ has a upper bound in such a manner,

$$K_t(x,x) \leq \frac{C}{f(t)} \tag{2.10}$$

where $f(t)$ is an an increasing positive function on $(0,\infty)$ satisfying regularity condition. Let $E_D(x,t)$ be weighted integral of the heat kernel so as,

$$E_D(x,t) = \int_{\mathcal{M}} K_t^2(x,z) \exp\Big(\frac{d(x,z)}{C't}\Big) d_g z.$$

Via Lemma 2.1,

$$E_D(x,t) \leq K_{2t}(x,x) \exp\Big(\frac{d(x,z)}{C't}\Big) \tag{2.11}$$

is derived. Combining of (2.10) and (2.11) concludes the following

$$E_D(x,t) \leq \frac{C'}{f(\delta t)}$$

for $\delta > 0$. □

**Corollary 2.1.** *Let $\mathcal{M}$ be a compact Riemannian manifold. Suppose that*

$$K_t(x,x) \leq \frac{C}{f(t)} \qquad \textit{and} \qquad K_t(y,y) \leq \frac{C}{g(t)} \tag{2.12}$$

*holds for some $x, y \in \mathcal{M}$ and for all $t > 0$ where $f(t)$ and $g(t)$ are increasing functions on $(0,\infty)$ satisfying the regularity condition.*

*Accordingly,*

$$K_t(x,y) \leq \frac{C'}{\sqrt{f(\delta t)g(\delta t)}} \exp\Big(\frac{-d^2(x,y)}{Dt}\Big). \tag{2.13}$$

*holds for all $t > 0$, $D > 1$ and for some $\delta > 0$.*

**Definition 2.2.** *Let $\mathcal{M}$ be a Riemannian manifold equipped with metric $g$. Suppose that $f : \Omega \to \mathcal{M}$ is a function on $\Omega \subset \mathcal{M}$ where $\Omega$ is open region in $\mathcal{M}$.*
*If the following holds*

$$d_g(f(x), f(y)) \leq C d_g(x,y) \tag{2.14}$$

*for some constant $C$ and for all $x, y \in \Omega$ where $\Omega$ is open region in $\mathcal{M}$. Then $f$ is called a Lipschitz function. Withal, the least constant $C$*

$$C = \sup\left\{ \frac{d_g(f(x), f(y))}{d_g(x,y)} \,\middle|\, x, y \in \Omega,\ x \neq y \right\}$$

*is called Lipschitz constant.*

**Definition 2.3.** *Let $\mathcal{M}$ be a Riemannian manifold equipped with metric $g$. A function $f : \Omega \to \mathcal{M}$ is called locally Lipschitz function provided that for all $\Omega_k$ is a compact subregion of $\Omega$ there exists $C_k$ in such a way,*

$$d_g(f(x), f(y)) \leq C_k d_g(x,y) \tag{2.15}$$

*holds for each $x, y \in \Omega_k$.*

The important property of locally Lipschitz function is differentiable at almost everywhere [115].

**Lemma 2.2.** *Let $\Omega \subset \mathcal{M}$ be a relatively compact region. Suppose that $u(x,t) \in C^2(\overline{\Omega})$ is a solution of heat equation in $\Omega_T$ and $u(x,t)|_{\partial\Omega} = 0$. Let $\xi(x,t)$ be a locally Lipschitz function on $(0,\infty) \times \mathcal{M}$ such that*

$$\frac{\partial}{\partial t}\xi + \frac{1}{2}|\nabla \xi|^2 \leq 0. \tag{2.16}$$

*Then, the following function*

$$J(t) = \int_\Omega u^2(x,t) e^{\xi(x,t)} d_g x \tag{2.17}$$

*is non-increasing.*

*Proof.* Observe that $J' \leq 0$, $u_t = \nabla_g u$ and $u_{tt} = \Delta_g u$. Before proceeding, notice that Stoke's theorem implies the following identity

$$\int_\Omega v \Delta_g u dt = -\int_\Omega (\nabla_g u, \nabla_n onnegativegv) dt \tag{2.18}$$

where $\Omega$ is relatively compact subset of compact Riemannian manifold $\mathcal{M}$. Via (2.16) with (2.18),

$$\begin{aligned} J'(t) &= \int_\Omega u^2 \xi_t e^\xi \, dt + 2\int_\Omega u u_t e^\xi dt \\ &\leq -\frac{1}{2}\int_\Omega u^2 |\nabla_g \xi|^2 e^\xi \, dt + 2\int_\Omega u \Delta_g u e^\xi \, dt \\ &= -\frac{1}{2}\int_\Omega u^2 |\nabla_g \xi|^2 e^\xi \, dt - 2\int_\Omega (\nabla_g u, \nabla_g \xi) e^\xi - 2\int_\Omega |\nabla_g u|^2 e^\xi dt \end{aligned}$$

$$J'(t) \leq -\frac{1}{2}\int_{\Omega}(u\nabla_g\xi + 2\nabla_g u)^2 e^{\xi}\, dt. \tag{2.19}$$

is obtained. (2.19) demonstrates that $J' \leq 0$. Thence, $J(t)$ is non-increasing. □

**Definition 2.4.** *Let $\mathcal{M}$ be a $N$-dimensional Riemannian manifold. For any nonnegative $f \in C_0^{\infty}(\mathcal{M})$ and $N > 2$,*

$$\left(\int_{\mathcal{M}} f^{\frac{2N}{N-2}} d_g x\right)^{\frac{N-2}{2}} \leq C\int_{\mathcal{M}} |\nabla f|^2 d_g x. \tag{2.20}$$

*This inequality* (2.20) *is called Sobolev inequality [96].*

The upper bound of heat kernel in $\mathcal{M}$

$$K_t(x,x) \leq \frac{C}{t^{N/2}} \tag{2.21}$$

for all $x \in \mathcal{M}$ and $t > 0$.

(2.21) is firstly showed by J. Nash from Sobolev inequality [116].

**Theorem 2.2.** *Let $\mathcal{M}$ be a $N$-dimensional manifold. Suppose that for any positive smooth function obeys Sobolev Inequality* (2.20). *Then,* (2.21) *holds for each $x \in \mathcal{M}$ and $t > 0$.*

*Proof.* It is enough to prove that (2.21) holds for each $x$ in relatively compact region $\Omega$ of $\mathcal{M}$ with smooth boundary. Therefore for fixed $y \in \Omega$, define $J(t)$ as follows

$$J(t) = \int_{\Omega} K_t^2(x,y) d_g x.$$

Henceforth, taking derivative of $J(t)$ so as to,

$$\begin{aligned} J'(t) &= 2\int_\Omega K_t(x,y)\frac{\partial}{\partial t}K_t(x,y)d_g x \\ &= \int_\Omega K_t(x,y)\Delta_g K_t(x,y)d_g x \\ &= -2\int_\Omega |\nabla_g K_t(x,y)|^2 d_g x \end{aligned} \tag{2.22}$$

shows that $J(t)$ is non-vashing. Notice that the Sobolev inequality extends to functions like $K_t(\cdot,\cdot)$ vanishing on $\partial\Omega$.
Then,

$$c\bigg(\int_\Omega K_t(x,y)^{\frac{N-2}{N}}d_g x\bigg)^{\frac{N-2}{N}} \leq \int_\Omega |\nabla K_t(x,y)|^2 d_g x \tag{2.23}$$

is obtained. By the Hölder inequality,

$$\int_\Omega K_t^2(x,y)d_g x \leq \bigg(\int_\Omega K_t^\alpha(x,y)\bigg)^{\frac{1}{\alpha-1}}\bigg(\int_\Omega K_t(x,y)d_g x\bigg)^{\frac{\alpha-2}{\alpha-1}} \tag{2.24}$$

is derived as long as $\alpha > 2$. Setting $\alpha = \frac{2n}{n-2}$ and from (2.23) provide

$$c\bigg(\int_\Omega K_t^2(x,y)d_g x\bigg)^{\frac{N+2}{N}}\bigg(\int_\Omega K_t(x,y)\bigg)^{\frac{-4}{n}} \leq \int_\Omega |\nabla_g K_t(x,y)|d_g x. \tag{2.25}$$

The last inequality is also known as Nash inequality. Observe that

$$\int_\Omega K_t(x,y)d_g x \leq 1.$$

(2.22) and (2.25) yield the following the differential inequality,

$$J' \leq -cJ^{\frac{N+2}{N}}.$$

Integrate both sides over time $t$ as follows

$$\int_0^t J'(t)dt \leq \int_0^t -cJ^{\frac{N+2}{N}}dt.$$

Thence,

$$J(t) \leq Ct^{\frac{-N}{2}}.$$

By semigroup property of heat kernel, $J(t) = K_{2t}(x,x)$. Whence, (2.21) is proved. □

**Theorem 2.3.** *Let $\mathcal{M}$ be $N$-dimensional compact Riemannian manifold with metric $g$. Henceforth, the following holds*

$$K_t(x,x) \leq \frac{1}{V(M)} + \frac{C}{t^{N/2}} \tag{2.26}$$

*for all $x \in \mathcal{M}$ and $t > 0$.*

*Proof.* Since $\mathcal{M}$ is compact then there exist the eigenfunction expansion via Spectral theorem $B.1$. In this manner,

$$\begin{aligned} K_t(x,x) &= \sum_{l=0}^{\infty} \exp(-\sigma_l t)\phi_l(x)\phi_l(x) \\ &= \exp(-\sigma_0 t)\phi_0(x)\phi_0(x) + \sum_{l=1}^{\infty} \exp(-\sigma_l t)\phi_l^2(x). \end{aligned}$$

The compactness of $\mathcal{M}$ brings out $\sigma_0 = 0$ and then $\phi_0(x) = \frac{1}{\sqrt{V(\mathcal{M})}}$. Hence,

$$K_t(x,x) = \frac{1}{V(\mathcal{M})} + \sum_{l=1}^{\infty} \exp(-\sigma_l t)\phi_l^2(x). \tag{2.27}$$

For any positive $f \in C_0^\infty(\mathcal{M})$ obeys Sobolev Inequality (2.20). Then, (2.21) holds for each $x \in \mathcal{M}$ and $t > 0$. Then, combining (2.27) and (2.21) provides the following,

$$K_t(x,x) \leq \frac{1}{V(\mathcal{M})} + \frac{C}{t^{N/2}}.$$

□

**Theorem 2.4.** *Let $\mathcal{M}$ be a $N$-dimensional compact Riemannian manifold equipped with metric $g$. Suppose that for any positive $f \in C_0^\infty(\mathcal{M})$ satisfies the regularity condition* (2.7) *in $\mathcal{M}$. Then, the following inequality holds*

$$K_t(x,y) \leq \Big(\frac{1}{V(\mathcal{M})} + \frac{C_1}{t^{N/2}}\Big) \exp\Big(\frac{-d_g^2(x,y)}{C_2 t}\Big) \tag{2.28}$$

*for all $x, y \in \mathcal{M}$ and $t > 0$.*

*Proof.* By corollary 2.1 and (2.21),

$$K_t(x,x,) \leq \frac{C}{f(t)} \quad \text{and} \quad K_t(y,y) \leq \frac{C}{g(t)}. \tag{2.29}$$

for $x, y \in \mathcal{M}$ and for $C > 0$ together with

$$K_t(x,y) \leq \frac{C}{\sqrt{f(t)g(t)}} \exp\big(\frac{-d_g^2(x,y)}{C't}\big) \tag{2.30}$$

are obtained for $C' > 0$ where $f(t)$ and $g(t)$ are increasing functions on $(0, \infty)$ satisfying the regularity condition. Whereby (2.20) and (2.29) arrange the following,

$$K_t(x,x) \leq \frac{1}{V(\mathcal{M})} + \frac{C}{t^{N/2}} \quad \text{and} \quad K_t(y,y) \leq \frac{1}{V(\mathcal{M})} + \frac{C}{t^{N/2}}. \tag{2.31}$$

The combination of (2.30) and (2.31) derives the following

$$K_t(x,y) \leq \frac{C}{\sqrt{t^{N/2}}\sqrt{t^{N/2}}} \exp\left(\frac{-d^2(x,y)}{C't}\right) \tag{2.32}$$

$$= \frac{C}{t^{N/2}} \exp\left(\frac{-d^2(x,y)}{C't}\right). \tag{2.33}$$

Hereupon, combining (2.31) and (2.33) complete the proof in such manner as to,

$$K_t(x,y) \leq \left(\frac{1}{V(\mathcal{M})} + \frac{C_1}{t^{N/2}}\right) \exp\left(\frac{-d^2(x,y)}{C_2 t}\right).$$

□

# 3. SINGULAR INTERACTIONS ON COMPACT RIEMANNIAN MANIFOLDS

## 3.1. Quantization on Manifold

The classical Hamiltonian of free moving particle on a curved space with given metric $g_{ij}$ is

$$H = \frac{1}{2} g^{ij}(x) p_i p_j.$$

The Hamiltonian for a particle in a curved space can be written in a local coordinate representation by

$$\langle x|H_0 = \left( -\frac{\hbar^2}{2m}\Delta_g + \xi\frac{\hbar^2}{m}R(x) \right) \langle x| \tag{3.1}$$

where $\xi$ can be taken arbitrary values and $R$ is the Ricci curvature of the manifold. A factor of $\frac{\hbar^2 R}{m}$ makes a difference from a physical point of view. Firstly, the issue of inquivalent Hamiltonian was mentioned by DeWitt [117–119] and there is no agreement concerning the quantization in curved space. For simplicity, $\xi$ can be chosen to be zero. This choice ensures that the spectrum of the free Hamiltonian is nonnegative. There is no trouble concerning the selection of $\xi = 0$ since this additional term is only to be determined by experiments [120, 121]. For the rest of this chapter, assume that the motion is restrained to the manifold a piori in order to prevent dealing with embedding space. The geometry intrinsic to the manifold provides all the properties of the system.

## 3.2. A Heuristic Renormalization and Regularization of Singular Point Interactions on A Compact Riemannian Manifold

A non-relativistic point particle of mass $m$ living in 2 dimensional Riemannian manifold $M$ without boundary interacts with a Dirac delta interaction located on the manifold. The bound state problem will be studied.

**Definition 3.1.** *The time independent Schrödinger equation on $M$ for bound state of a particle under the influence of $N$ attractive Dirac delta interaction reads*

$$\left[ -\frac{\hbar}{2m}\Delta_g - \sum_{i=1}^{N} \lambda_i \delta_g^2(x, a_i) \right] \psi(x) = -\nu^2 \psi(x) \tag{3.2}$$

*where $\lambda_i \in \mathbb{R}^+$ is strength of ith Dirac delta interaction at point $a_i \in M$ [24].*

The bound state energy $E$ has been parametrized as $-\nu^2$. Assume again, $a_i \neq a_j$ for $i \neq j$. By the Spectral Theorem (Theorem B.1) on a compact manifold, the problem can be solved formally

$$\sum_{l=0}^{\infty} \left[ \frac{\hbar^2}{2m} \sigma_l \, C_l - \sum_{i=1}^{N} A_i \, \lambda_i \, \phi_l(a_i) + \nu^2 C_l \right] \phi_l(x) = 0 \tag{3.3}$$

where $A_i \equiv \psi(a_i)$. See that $\phi_l$'s forming a complete orthonormal system allows to solve $C_l$.

$$C_l = \frac{1}{\frac{\hbar^2}{2m}\sigma_l + \nu^2} \sum_{i=1}^{N} A_i \, \lambda_i \, \phi_l(a_i). \tag{3.4}$$

Substituting (3.4) into the definition of $A_i$

$$A_i = \sum_{j=1}^{N} A_j \, \lambda_j \, \sum_{l=0}^{\infty} \frac{\phi_l(a_i)\phi_l(a_j)}{\frac{\hbar^2}{2m}\,\sigma_l + \nu^2} \tag{3.5}$$

and grouping the $A_i$ terms yield

$$\left[ \lambda_i^{-1} - \sum_{l=0}^{\infty} \frac{|\phi_l(a_j)|^2}{\frac{\hbar^2}{2m}\,\sigma_l + \nu^2} \right] A_i - \sum_{\substack{j=1 \\ j\neq i}}^{N} \left[ \frac{\lambda_j}{\lambda_i} \sum_{l=0}^{\infty} \frac{\phi_l(a_i)\phi_l(x)}{\frac{\hbar^2}{2m}\,\sigma_l + \nu^2} \right] A_j = 0 \; . \tag{3.6}$$

The linearity of $A_i$ is immediately recognized. Then, via matrix notation, (4.6) can be written

$$\sum_{j=1}^{N} \Phi_{ij}(E = -\nu^2) A_j = 0 \tag{3.7}$$

where

$$\Phi_{ij}(E) = \begin{cases} \lambda_i^{-1} - \sum_{l=0}^{\infty} \frac{|\phi_l(a_i)|^2}{\frac{\hbar^2}{2m}\,\sigma_l + \nu^2} & i \neq j, \\ -\frac{\lambda_j}{\lambda_i} \sum_{l=0}^{\infty} \frac{\phi_l(a_i)\phi_l(a_j)}{\frac{\hbar^2}{2m}\,\sigma_l + \nu^2} & \text{if } i \neq j \; . \end{cases} \tag{3.8}$$

Heuristically the expectation is that the diagonal term in (3.8) is divergent since its association in the flat space diverges. The large values of momentum correspond to short distances. The divergence structure of the problem is the same in the flat case since the interaction of the particle with the potential at a short distance causes this not convergence. The character of divergence on Riemannian manifold should be the same as in a flat space which is ensured by the definition of a manifold. Introduction of a cutoff to the upper bound of the infinite sum and choice of the bare coupling constant

is analogy to the well-known flat case leads to

$$\frac{1}{\lambda_i(\Lambda)} = \int_{\mathbb{R}^2} \frac{d^2p}{(2\pi\hbar)^2} \frac{\Theta_\Lambda(\mathbf{p})}{\frac{\mathbf{p}^2}{2m} + \mu_i^2} \longleftrightarrow \frac{1}{\lambda_i(\Lambda)} = \sum_{l=0}^{\Lambda} \frac{|\phi_l(a_i)|^2}{\frac{\hbar^2}{2m}\sigma_l + \mu_i^2} \tag{3.9}$$

where $-\mu_i^2$ in this case is the experimentally measured binding energy to a single Dirac delta interaction in $\mathcal{M}$. Then, take the limit $\Lambda \to \infty$. This should lead a finite result in two dimensions. Hence, the principal matrix at this limit becomes

$$\Phi_{ij}(E) = \begin{cases} \displaystyle\sum_{l=0}^{\infty} \frac{(\nu^2 - \mu_i^2)|\phi_l(a_i)|^2}{\left(\frac{\hbar^2}{2m}\sigma_l + \mu_i^2\right)\left(\frac{\hbar^2}{2m}\sigma_l + \nu^2\right)} & \text{if } i = j, \\ \\ \displaystyle -\sum_{l=0}^{\infty} \frac{\phi_l(a_i)\phi_l(a_j)}{\frac{\hbar^2}{2m}\sigma_l + \nu^2} & \text{if } i \neq j \end{cases} \tag{3.10}$$

thanks to the limit $\lim\limits_{\Lambda\to\infty} \dfrac{\lambda_j(\Lambda)}{\lambda_i(\Lambda)} = 1$. The convergence of these sums are not clear.

There is a very elegant method which is based on the Heat Kernel that is discussed in the previous chapter. Consider the separable Hamiltonians $H^\epsilon = H_0 - \sum_{i=1}^N |f_i^\epsilon\rangle\langle f_i^\epsilon|$, where $|f_i^\epsilon\rangle$ is a particular set of normalized vectors specified by its coordinate wave function $f_i^\epsilon(x)$. The resolvent formula of $H$ should be expressed in terms of the Hamiltonian $H_0$. Because of this, assume that two Dirac kets $|\psi\rangle$ and $|\rho\rangle$ are related by the means of $(H^\epsilon - E)|\psi\rangle = |\rho\rangle$ thus the following is satified

$$\left[H_0 - E - \sum_{j=1}^{N} |f_j^\epsilon\rangle\langle f_j^\epsilon|\right] |\psi\rangle = |\rho\rangle. \tag{3.11}$$

Assuming complex $E \notin Spec(H_0)$ and $\Re E < 0$. Acting with the operator $(H_0 - E)^{-1}$ on both sides and projecting it onto $\langle f_i^\epsilon|$,

$$\langle f_i^\epsilon|\psi\rangle = \langle f_i^\epsilon|(H_0 - E)^{-1}|\rho\rangle + \sum_{j=1}^{N}\langle f_i^\epsilon|(H_0 - E)^{-1}|f_j^\epsilon\rangle\langle f_j^\epsilon|\psi\rangle \tag{3.12}$$

is obtained or

$$\sum_{j=1}^{N} A_{ij}^\epsilon(E)\langle f_j^\epsilon|\psi\rangle = \langle f_i^\epsilon|\,(H_0 - E)^{-1}\,|\rho\rangle \tag{3.13}$$

where a matrix $A_{ij}^\epsilon(E)$ is defined as

$$A_{ij}^\epsilon(E) = \begin{cases} 1 - \langle f_i^\epsilon|\,(H_0 - E)^{-1}\,|f_i^\epsilon\rangle & \text{if } i = j, \\ -\langle f_i^\epsilon|\,(H_0 - E)^{-1}\,|f_j^\epsilon\rangle & \text{if } i \neq j. \end{cases} \tag{3.14}$$

After simple linear algebra calculation,

$$(H^\epsilon - E)^{-1} = (H_0 - E)^{-1} + (H_0 - E)^{-1}\left[\sum_{i,j=1}^{N}|f_i^\epsilon\rangle\left[A^\epsilon(E)\right]_{ij}^{-1}\langle f_j^\epsilon|\right](H_0 - E)^{-1} \tag{3.15}$$

while $[A^\epsilon(E)]_{ij}^{-1}$ exists. The heat kernels $\sqrt{\lambda_i(\epsilon)}K_{\epsilon/2}(x, a_i)$, centered at $x = a_i$ define the function $\langle x|f_i^\epsilon\rangle f_i^\epsilon(x)$'s here are chosen as a weighted natural Dirac delta sequence. Now, use $|f_i^\epsilon\rangle = \sqrt{\lambda_i(\epsilon)}|\tilde{f}_i^\epsilon\rangle$, where $\langle x|\tilde{f}_i^\epsilon\rangle = K_{\epsilon/2}(a_i, x)$, and write everything in terms of these pure Dirac delta sequences. The original Hamiltonian given in section 4.2, if the limit $\epsilon \to 0^+$

$$\langle x|f_i^\epsilon\rangle\langle f_i^\epsilon|\psi\rangle \longrightarrow \delta_g(x, a_i)\psi(x)\ . \tag{3.16}$$

via defining new operator $D_{ij}(\epsilon) = \sqrt{\lambda_i(\epsilon)}\delta_{ij}$, $F_{ij}(\epsilon) = |f_i^\epsilon\rangle\langle f_j^\epsilon|$ and similarly for $\tilde{F}_{ij}(\epsilon)$, write the operator in (3.15)

$$\begin{aligned}\sum_{i,j=1}^{N} |f_i^\epsilon\rangle \left[A^\epsilon(E)\right]^{-1}_{ij} \langle f_j^\epsilon| &= \mathrm{Tr}\left[F(\epsilon)\left[A^\epsilon(E)\right]^{-1}\right] = \mathrm{Tr}\left[D(\epsilon)\tilde{F}(\epsilon)D(\epsilon)\left[A^\epsilon(E)\right]^{-1}\right] \\ &= \mathrm{Tr}\left[\tilde{F}(\epsilon)(D^{-1}(\epsilon)A^\epsilon(E)D^{-1}(\epsilon))^{-1}\right] \\ &= \sum_{i,j=1}^{N} |\tilde{f}_i^\epsilon\rangle \left[D^{-1}(\epsilon)A^\epsilon(E)D^{-1}(\epsilon)\right]^{-1}_{ij} \langle \tilde{f}_j^\epsilon| \\ &= \mathrm{Tr}\left[\tilde{F}(\epsilon)\left[\Phi^\epsilon(E)\right]^{-1}\right] \end{aligned} \tag{3.17}$$

where a new operator $\Phi_{ij}(E)$ is defined.

$$\Phi^\epsilon_{ij}(E) = \sum_{k,l=1}^{N} D^{-1}_{ik}(\epsilon)A^\epsilon_{kl}(E)D^{-1}_{lj}(\epsilon) = \begin{cases} \lambda_i^{-1}(\epsilon) - \langle \tilde{f}_i^\epsilon|(H_0 - E)^{-1}|\tilde{f}_i^\epsilon\rangle & \text{if } i = j, \\ -\langle \tilde{f}_i^\epsilon|(H_0 - E)^{-1}|\tilde{f}_j^\epsilon\rangle\ , & \text{if } i \neq j. \end{cases} \tag{3.18}$$

putting the identity $1 = \int_M d_g x\ |x\rangle\langle x|$ between the operator $(H_0 - E)^{-1}$ and $\langle \tilde{f}_i^\epsilon|$ and the pair $|\tilde{f}_j^\epsilon\rangle$,

$$\Phi^\epsilon_{ij}(E) = \begin{cases} \lambda_i^{-1}(\epsilon) - \displaystyle\int_{\mathcal{M}}\int_{\mathcal{M}} d_g x\, d_g y\ K_{\epsilon/2}(a_i, x)R_0(x, y|E)K_{\epsilon/2}(y, a_i) & \text{if } i = j, \\ -\displaystyle\int_{\mathcal{M}}\int_{\mathcal{M}} d_g x\, d_g y\ K_{\epsilon/2}(a_i, x)R_0(x, y|E)K_{\epsilon/2}(y, a_j) & \text{if } i \neq j. \end{cases} \tag{3.19}$$

Using the formula for free resolvent expressed in terms of heat kernel then

$$\Phi_{ij}^{\epsilon}(E) = \begin{cases} \lambda_i^{-1}(\epsilon) - \displaystyle\int_0^\infty \frac{dt}{\hbar} \int_{\mathcal{M}} \int_{\mathcal{M}} d_g x \, d_g y \; K_{\epsilon/2}(a_i, x) K_t(x, y) e^{tE/\hbar} K_{\epsilon/2}(y, a_i) \\ \qquad\qquad \text{if } i = j, \\ -\displaystyle\int_0^\infty \frac{dt}{\hbar} \int_{\mathcal{M}} \int_{\mathcal{M}} d_g x \, d_g y \; K_{\epsilon/2}(a_i, x) K_t(x, y) e^{tE/\hbar} K_{\epsilon/2}(y, a_j) \\ \qquad\qquad \text{if } i \neq j \; . \end{cases} \tag{3.20}$$

Via semigroup property of the heat kernel regularized matrix whose elements are time integrals

$$\Phi_{ij}^{\epsilon}(E) = \begin{cases} \lambda_i^{-1}(\epsilon) - \displaystyle\int_0^\infty \frac{dt}{\hbar} K_{t+\epsilon}(a_i, a_i) \; e^{tE/\hbar} & \text{if } i = j, \\ -\displaystyle\int_0^\infty \frac{dt}{\hbar} K_{t+\epsilon}(a_i, a_j) \; e^{tE/\hbar} & \text{if } i \neq j. \end{cases} \tag{3.21}$$

is obtained. Note that diagonal entries of $\Phi_{ij}^{\epsilon}(E)$ is divergent when the limit $\epsilon \to 0^+$ is implemented since the asymptotic expansion of diagonal heat kernel in two dimensions. Upon that, divergence is managed by the first term of expression. The following may be chosen as the bare constant

$$\frac{1}{\lambda_i(\epsilon)} = \int_\epsilon^\infty \frac{dt}{\hbar} \; K_t(a_i, a_i) \; e^{-t\mu_i^2/\hbar}. \tag{3.22}$$

Thus, after short time limit $\epsilon \to 0^+$ is taken,

$$\Phi_{ij}(E) = \begin{cases} \displaystyle\int_0^\infty \frac{dt}{\hbar} \; K_t(a_i, a_i) \left[ e^{-t\mu_i^2/\hbar} - e^{tE/\hbar} \right] & \text{if } i = j, \\ -\displaystyle\int_0^\infty \frac{dt}{\hbar} \; K_t(a_i, a_j) \; e^{tE/\hbar} & \text{if } i \neq j. \end{cases} \tag{3.23}$$

After subtraction of (3.22) from diagonal of (3.23), convergence is granted. The renormalization can be performed via checking the momentum cut off $\Lambda$ for infinite sum that is introduced in previous section corresponding to the short time cut off $\epsilon$ for the lower bound of time integral in the heat kernel method. By the spectral theorem (theorem B.1) in (3.23) for $E = -\nu^2$,

$$\begin{aligned}\int_0^\infty \frac{dt}{\hbar}\, e^{\frac{tE}{\hbar}} K_t(a_i, a_j) \; &= \sum_{l=0}^{\infty} \phi_l(a_i)\phi_l(a_j) \int_0^\infty \frac{dt}{\hbar}\, e^{-\frac{t}{\hbar}\left(\frac{\hbar^2}{2m}\sigma_l - E\right)} \\ &= \sum_{l=0}^{\infty} \frac{\phi_l(a_i)\phi_l(a_j)}{\frac{\hbar^2}{2m}\sigma_l - E}.\end{aligned} \tag{3.24}$$

Notice that the sum and integral are interchangeable since the summation is uniformly convergent [93]. Taking the matrix elements in (3.15) via projecting onto position Dirac bracket $\langle x|$ and $|y\rangle$ and using equation (3.17), the regularized resolvent kernel in coordinate space $R^\epsilon(x, y|E) \equiv \langle x|(H^\epsilon - E)^{-1}|y\rangle$ is found.

$$\begin{aligned}R^\epsilon(x, y|E) = R_0(x, y|E) + \sum_{i,j=1}^{N} \int_{\mathcal{M}} \int_{\mathcal{M}} d_g x'\, d_g y'\, R_0(x, x'|E) K_{\epsilon/2}(x', a_i) \\ \times \left[\Phi^\epsilon(E)\right]^{-1}_{ij} K_{\epsilon/2}(a_j, y') R_0(y', y|E)\,. \end{aligned} \tag{3.25}$$

Take limit $\epsilon \to 0^+$ to get reguralized resolvent in terms of free resolvent,

$$\lim_{\epsilon\to 0^+} R^\epsilon(x, y|E) = R(x, y|E) = R_0(x, y|E) + \sum_{i,j=1}^{N} R_0(x, a_i|E)\, \Phi^{-1}_{ij}(E) R_0(a_j, y|E) \tag{3.26}$$

where $R_0(x, y|E)$ is the free resolvent kernel.

## 3.3. A Heuristic Renormalization and Regularization of Singular Interactions Supported by Curves on a Compact Riemannian Manifold

A non-relativistic mass of $m$ living in 3-dimensional Riemannian manifold $\mathcal{M}$ without boundary interacts singularly supported by curves is resided on the manifold. The bound state problem will be studied.

**Definition 3.2.** *Formally, the time independent Schrödinger equation in $\mathcal{M}$ for bound state of a singular interaction supported by curve reads*

$$-\frac{\hbar}{2m}\Delta_g\psi(x) - \frac{\lambda}{L}\int_\Gamma \delta_g(x,\gamma(s))d_g s \int_\Gamma \psi(\gamma(s))d_g s' = -\nu^2\psi(x) \tag{3.27}$$

*where $\gamma(s)$ is an arclength-parametrized closed curve $\Gamma$ of length $L$, over that integral is taken n and $\lambda \in \mathbb{R}^+$ is the coupling constant [24].*

The bound state energy $E$ has been parametrized as $-\nu^2$. In general, numerous non-intersecting and not self-intersecting curves are denoted by $\Gamma_i$ and $\lambda_i$ stands for corresponding coupling constant. Similar approach of previous section will be followed in order to get regularized principle operator as well as renormalized to obtain the regularized resolvent. A family of functions supported on curves can be defined as follows

$$\Gamma^\epsilon(x) = \int_{\Gamma_i} d_g s K_{\epsilon/2}(x,\gamma_i(s)). \tag{3.28}$$

If the limit $\epsilon \to 0^+$ is performed, a delta function supported on the curve is obtained whereby

$$\lim_{\epsilon\to 0}\Gamma^\epsilon(x) = \delta_\Gamma(x).$$

Moreover, $\langle\Gamma_i^\epsilon|\Gamma_i^\epsilon\rangle$ can given as follows

$$\langle\Gamma_i^\epsilon|\Gamma_i^\epsilon\rangle = \int_{\Gamma_i}\int_{\Gamma_i} d_g s d_g s' K_\epsilon(\gamma_i(s), \gamma(s')). \tag{3.29}$$

As long as $\epsilon > 0$, $K_\epsilon$ is finite. Hence, (3.29) is finite as well.

A regularized Schödinger equation can be rewritten for this family of functions as

$$(H_0 - E)|\psi\rangle = \sum_{i=1}^{N} \frac{\lambda_i}{L_i}|\Gamma_i^\epsilon\rangle\langle\Gamma_i^\epsilon|\psi\rangle, \tag{3.30}$$

where the integrals over curves concealed besides, the interaction can be expressed as projection since the case of non-intersection is considered. To get the resolvent, $\langle\tilde{\Gamma}_i^\epsilon|\psi\rangle$ is need to be solved in the following expression

$$|\psi\rangle = (H_0 - E)^{-1}|\tilde{\Gamma}_i^\epsilon\rangle\langle\tilde{\Gamma}_i^\epsilon|\psi\rangle + (H_0 - E)^{-1}|\varphi\rangle. \tag{3.31}$$

The $\langle\tilde{\Gamma}_i^\epsilon|\psi\rangle$ formula implies that $\sqrt{\frac{\lambda_i}{L_i}}$ is absorbed into the expansion,

$$\langle\tilde{\Gamma}_i^\epsilon|\psi\rangle = \left[\frac{1}{1 - \langle\tilde{\Gamma}_i^\epsilon|(H_o - E)^{-1}|\tilde{\Gamma}_i^\epsilon\rangle}\right]_{ij}\langle\tilde{\Gamma}_j^\epsilon|(H_0 - E)^{-1}|\varphi\rangle. \tag{3.32}$$

Plugging equation (3.32) into (3.31) and scaling by $\sqrt{\frac{\lambda_i}{L_i}}$ again therewith, the resolvent takes the following form

$$(H - E)^{-1} = (H_0 - E)^{-1} + \sum_{i,j=1}^{N} \frac{1}{\sqrt{L_i L_j}}(H_0 - E)^{-1}|\Gamma_i^\epsilon\rangle\Phi_{ij}^{-1}\langle\Gamma_j^\epsilon|(H_0 - E)^{-1} \tag{3.33}$$

where $\Phi_{ij}$ stands for the principal operator given by

$$\Phi_{ij}(E) = \begin{cases} \frac{1}{\lambda_i} - \frac{1}{L_i}\langle \Gamma_i^\epsilon | (H_0 - E)^{-1} | \Gamma_i^\epsilon \rangle, \text{ if } i = j, \\ -\frac{1}{\sqrt{L_i L_j}}\langle \Gamma_i^\epsilon | (H_0 - E)^{-1} | \Gamma_j^\epsilon \rangle \text{ if } i \neq j. \end{cases} \tag{3.34}$$

By $(C.23)$, the free resolvent can be expressed in terms of heat kernel

$$\langle x | (H^\epsilon - E)^{-1} | y \rangle = \int_\epsilon^\infty \frac{dt}{\hbar} e^{Et/\hbar} K_t(x, y). \tag{3.35}$$

Combining (3.34) and (3.35) gives the regularized principle operator in a coordinate bases

$$\begin{aligned} \Phi^\epsilon(E) = \frac{1}{\lambda(E)} - \frac{1}{L} \int_{\mathcal{M}} \int_{\mathcal{M}} d_g x d_g y \int_\Gamma \int_\Gamma d_g s d_g s' \\ \times \int_\epsilon^\infty \frac{dt}{\hbar} e^{Et/h} K_{\epsilon/2}(\gamma(s), x) K_t(x, y) K_{\epsilon/2}(y, \gamma(s')). \end{aligned} \tag{3.36}$$

Semi-group property of heat kernel $(C.11)$ and (3.35) yield the following

$$\Phi_{ij}^\epsilon(E) = \begin{cases} \frac{1}{\lambda_i(\epsilon)} - \frac{1}{L_i} \int_{\Gamma_i} \int_{\Gamma_i} d_g s d_g s' \int_\epsilon^\infty \frac{dt}{\hbar} e^{t(E-\epsilon)/\hbar} K_{t+\epsilon}(\gamma(s), \gamma(s')) \text{ if } i = j, \\ -\frac{1}{\sqrt{L_i L_j}} \int_{\Gamma_i} \int_{\Gamma_j} d_g s d_g s' \int_\epsilon^\infty \frac{dt}{\hbar} e^{t(E-\epsilon)/\hbar} K_{t+\epsilon}(\gamma(s), \gamma(s')) \quad \text{if } i \neq j. \end{cases} \tag{3.37}$$

Provided that the limit $\epsilon \to 0^+$ is implemented then

$$\Phi_{ij}(E) = \begin{cases} \dfrac{1}{\lambda_i(\epsilon)} - \dfrac{1}{L_i}\displaystyle\int_{\Gamma_i}\int_{\Gamma_i} d_g s d_g s' \int_0^\infty \frac{dt}{\hbar} e^{Et/\hbar} K_t(\gamma(s),\gamma(s')) \ \text{ if } \ i=j, \\ -\dfrac{1}{\sqrt{L_i L_j}}\displaystyle\int_{\Gamma_i}\int_{\Gamma_j} d_g s d_g s' \int_0^\infty \frac{dt}{\hbar} e^{Et/\hbar} K_t(\gamma(s),\gamma(s')) \quad \text{ if } \ i\neq j. \end{cases} \tag{3.38}$$

is derived. Notice that $\Phi_{ij}(E)$ is divergent if $i = j$. The asymptotic expansion of diagonal heat kernel allows to handle divergence via first term of expansion. Accordingly, bare constant can be chosen as follows

$$\frac{1}{\lambda_i(\epsilon)} = \frac{1}{L_i}\int_{\Gamma_i}\int_{\Gamma_i} d_g s d_g s' \int_\epsilon^\infty \frac{dt}{\hbar} e^{-t\mu_i^2/\hbar} K_t(\gamma(s),\gamma(s')). \tag{3.39}$$

Via (3.39) substuting into (3.37) together with taking short time limit $\epsilon \to 0^+$, the principle operator $\Phi_{ij}(E)$

$$\Phi_{ij}(E) = \begin{cases} -\dfrac{1}{L_i}\displaystyle\int_{\Gamma_i}\int_{\Gamma_i} d_g s d_g s' \int_0^\infty \frac{dt}{\hbar}\left[e^{Et/\hbar} - e^{-t\mu_i^2/\hbar}\right] K_t(\gamma(s),\gamma(s')) \ \text{ if } \ i=j, \\ -\dfrac{1}{\sqrt{L_i L_j}}\displaystyle\int_{\Gamma_i}\int_{\Gamma_j} d_g s d_g s' \int_0^\infty \frac{dt}{\hbar} e^{Et/\hbar} K_t(\gamma(s),\gamma(s')) \qquad \text{ if } \ i\neq j. \end{cases} \tag{3.40}$$

is obtained.

The regularized resolvent kernel $R^\epsilon(\gamma(s),\gamma(s')|E) \equiv \langle\gamma(s)|(H^\epsilon - E)^{-1}|\gamma(s)\rangle$ is found, as follows

$$\begin{aligned} R^\epsilon(x,y|E) = R_0(x,y|E) + \sum_{i,j=1}^N \int_{\mathcal{M}}\int_{\mathcal{M}} d_g x \ d_g y \ R_0(\gamma(s),x|E) K_{\epsilon/2}(x,\gamma(s)) \\ \times \left[\Phi^\epsilon(E)\right]_{ij}^{-1} K_{\epsilon/2}(\gamma(s'),y) R_0(y,\gamma(s')|E) \ . \end{aligned} \tag{3.41}$$

Since divergence issue is eliminated, the resolvent kernel is gained after implementation of the limit $\lim_{\epsilon \to 0^+} R^{\epsilon}(\gamma(s), \gamma(s')|E) = R(\gamma(s), \gamma(s')|E)$ such that

$$R(x, y|E) = R_0(x, y|E) + \sum_{i,j=1}^{N} R_0(x, \gamma(s)|E)\Phi_{ij}^{-1}(E)R_0(\gamma(s'), y|E). \tag{3.42}$$

**Remark 3.1.** *The free resolvent kernel and the principal matrix describe all geometric information about the resolvent kernel. Also, resolvent contains all information regarding spectrum.*

# 4. RANK ONE PERTURBATIONS

This chapter studies rank one perturbation, and the self-adjoint extension of rank one perturbations as well as its resolvent properties via Krein's formula.

**Definition 4.1.** *Let $A \in \mathscr{H}$ be self-adjoint operator with domain $D(A)$. Suppose that $\varphi$ is a vector in Hilbert space $\mathscr{H}$. A symmetric rank one bounded perturbation of $A$ is defined by*

$$A_\alpha = A + \alpha\langle\varphi, \cdot\,\rangle\varphi \tag{4.1}$$

*where $\langle\cdot,\cdot\rangle$ is the scalar product in Hilbert space $\mathscr{H}$.*

The rank one operator $\alpha\langle\varphi,\cdot\rangle\varphi$ is a bounded operator that is still contained in $\mathscr{H}$. Also, the operator $A_\alpha$ is well defined and self-adjoint in the domain of operator $A$. The spectral properties of the perturbed operator can be obtained by its resolvent. There is a method Krein's formula to obtain the resolvent. This formula links the resolvent of two self-adjoint extensions of one symmetric operator with finite deficiency indices.

Let $A$ be a self-adjoint operator and let $A_\alpha$ be rank one bounded perturbation of $A$. Actually, $A_\alpha$ and $A$ are two self-adjoint extensions of the symmetric operator $A^0$ being the restriction of the operator $A$ to the set of all functions orthogonal to the vector $\varphi$ :

$$D(A^0) = \{\psi \in D(A) : \langle\varphi, \psi\rangle = 0\}. \tag{4.2}$$

$A^0$ is a symmetric non-densely defined operator, since $\varphi \in \mathscr{H}$.

**Theorem 4.1.** *Suppose that $A$ is a self-adjoint acting in $\mathscr{H}$ and let $\varphi \in \mathscr{H}$ be an arbitrary vector. Then the resolvent of the original operator $A$ and its rank one perturbation $A_\alpha = A + \alpha\langle\varphi, \cdot\rangle$, $\alpha \in \mathbb{R}$, are related as follow for arbitrary $z$, $\Im z \neq 0$,*

$$\frac{1}{A_\alpha - z} - \frac{1}{A - z} = -\frac{\alpha}{1 + \alpha\left\langle \varphi, \frac{1}{A-z}\varphi \right\rangle} \left\langle \frac{1}{A - \overline{z}}\varphi, \cdot \right\rangle \frac{1}{A - z}\varphi. \tag{4.3}$$

*Proof.* In order to compute the resolvent of the self-adjoint operator $A_\alpha$,

$$h = (A_\alpha - z)f$$

is need to be solved for a given $h \in \mathscr{H}$ and $f \in D(A_\alpha) = D(A)$. Assume that $\Im z > 0$ and apply the operator $A_\alpha - z$ to the latter equality

$$\begin{aligned} h &= (A + \alpha\langle\varphi, \cdot\rangle\varphi - z)f \\ &= A - zf + \alpha\langle\varphi, f\rangle\varphi. \end{aligned}$$

By the resolvent of the original operator,

$$\frac{1}{A - z}h = f + \alpha\langle\varphi, f\rangle\frac{1}{A - z}\varphi \tag{4.4}$$

is obtained. The projection on vector $\varphi$ as follows

$$\left\langle \varphi, \frac{1}{A - z}h \right\rangle = \langle\varphi, f\rangle + \alpha\langle\varphi, f\rangle\left\langle \varphi, \frac{1}{A - z}\varphi \right\rangle.$$

The projection formula implies the following formula for $\langle\varphi, f\rangle$,

$$\langle\varphi, f\rangle = \frac{\langle\varphi, \frac{1}{A-z}h\rangle}{1+\alpha\langle\varphi, \frac{1}{A-z}\varphi\rangle}. \tag{4.5}$$

Thence, putting (4.5) into (4.4) leads the following,

$$h = (A_\alpha - z)\Big(\frac{1}{A-z}h - \frac{\alpha\langle\varphi, \frac{1}{A-z}h\rangle}{1+\alpha\langle\varphi, \frac{1}{A-z}\varphi\rangle}\frac{1}{A-z}\varphi\Big). \tag{4.6}$$

As a consequence, manipulation of (4.6) ends the proof so as,

$$\begin{aligned}
\frac{1}{A_\alpha - z}h &= \frac{1}{A-z}h - \frac{\alpha\langle\varphi, \frac{1}{A-z}h\rangle}{1+\alpha\langle\varphi, \frac{1}{A-z}\varphi\rangle}\frac{1}{A-z}\varphi \\
\frac{1}{A_\alpha - z} &= \frac{1}{A-z} - \frac{\alpha\langle\varphi, \frac{1}{A-z}\rangle}{1+\alpha\langle\varphi, \frac{1}{A-z}\varphi\rangle}\frac{1}{A-z}\varphi \\
\frac{1}{A_\alpha - z} &= \frac{1}{A-z} - \frac{\alpha}{1+\alpha\langle\varphi, \frac{1}{A-z}\varphi\rangle}\Big\langle\varphi, \frac{1}{A-z}\Big\rangle\frac{1}{A-z}\varphi \\
\frac{1}{A_\alpha - z} &= \frac{1}{A-z} - \frac{\alpha}{1+\alpha\langle\varphi, \frac{1}{A-z}\varphi\rangle}\Big\langle\varphi\frac{1}{A-\overline{z}}, \cdot\Big\rangle\frac{1}{A-z}\varphi.
\end{aligned}$$

□

The scale of Hilbert spaces belonging to self-adjoint operator $A$ acting in Hilbert space $\mathscr{H}$ will be defined via modulus $|A|$ of $A$, where

$$|A| \equiv (A * A)^{1/2}.$$

For $s \geq 0$, $\mathscr{H}_s(A)$ is $\mathrm{Dom}(|A|^{s/2})$ with norm equal to graph of operator

$$\|\psi\| = \|(|A|+1)^{s/2}\psi\|_{\mathscr{H}}. \tag{4.7}$$

The operator $|A|$ is self-adjoint, its domain coincides with the domain of $A$. These $\mathscr{H}_s$ with norm $\|\cdot\|_s$ is complete. The linear bounded functionals form the adjoint spaces $\mathscr{H}_{-s}(A) = \mathscr{H}_s(A)^*$. The norm in the space $\mathscr{H}_{-s}(A)$ is defined by the formula

$$\|\psi\|_{-s} = \Big\| \frac{1}{(|A|+1)^{s/2}}\psi \Big\|_{\mathscr{H}} \tag{4.8}$$

where the operator $\frac{1}{(|A|+1)^{s/2}}$ is well-defined. Assume that $\psi \in \mathscr{H}_{-s}(A)$ and $\eta \in \mathscr{H} = \mathscr{H}_0(A)$,

$$\Big\langle \frac{1}{(|A|+1)^{s/2}}\psi, \eta \Big\rangle = \Big\langle \psi, \frac{1}{(|A|+1)^{s/2}}\eta \Big\rangle.$$

Forthwith, it signifies that $\Big(\frac{1}{(|A|+1)^{s/2}}\Big)\psi$ is an element of Hilbert space $\mathscr{H}$ in conjunction with the norm of functional is given by the formula (4.8). $(|A|+1)^{t/2}$ defines an isometry from $\mathscr{H}_s(A)$ to $\mathscr{H}_{s-t}(A)$. All space $\mathscr{H}_{-s}(A)$ is equal to the completion of Hilbert space $\mathscr{H}$ in norm (4.8). Although traditionally, $\langle \cdot, \cdot \rangle$ denotes scalar products in Hilbert space. The action of the functionals is denoted by $\langle \cdot, \cdot \rangle$ as well. Then, define

$$\langle \psi, \eta \rangle = \Big\langle \frac{1}{(|A|+1)^{s/2}}\psi, (|A|+1)^{s/2}\eta \Big\rangle \tag{4.9}$$

where the bracket on right handside denotes the scalar product. The spaces $\mathscr{H}_s(A)$ form the following chain,

$$\cdots \subset \mathscr{H}_2(A) \subset \mathscr{H}_1(A) \subset \mathscr{H} = \mathscr{H}_0(A) \subset \mathscr{H}_{-1}(A) \subset \mathscr{H}_{-2}(A) \subset \ldots$$

The space $\mathscr{H}_s(A)$ coincides with domain of the operator $A$ and $\mathscr{H}_1(A)$ is the domain of $|A|^{1/2}$.

For every two $s, t$ such that $s \leq t$, the space $\mathscr{H}_t(A)$ is dense in $\mathscr{H}_s(A)$. The norm in Hilbert space $\mathscr{H}$ will be denoted by

$$\|\cdot\|_0 = \|\cdot\|_{\mathscr{H}}. \tag{4.10}$$

The norm in the space $\mathscr{H}_1(A)$ can be calculated as follows

$$\begin{aligned} \|\psi\|^2 &= \langle (|A|+1)^{1/2}\psi, (|A|+1)^{1/2}\psi\rangle \\ &= \langle \psi, (|A|+1)\psi\rangle \\ &= \langle (\sqrt{|A|}+i)\psi, (\sqrt{|A|}+i)\psi\rangle. \end{aligned} \tag{4.11}$$

Thus by analogy of (4.11),

$$\|\psi\|^2_{-1} = \Big\langle \frac{1}{(\sqrt{|A|}+i)}\psi, \frac{1}{(\sqrt{|A|}+i)}\psi \Big\rangle \tag{4.12}$$

is obtained. Moreover, the norms $\|\cdot\|^*_2$ and $\|\cdot\|^*_{-2}$ can be introduced in the spaces $\mathscr{H}_2(A)$ and $\mathscr{H}_{-2}(A)$. These norms are equivalent to standard norms in $\mathscr{H}_2(A)$ and $\mathscr{H}_{-2}(A)$ Hilbert spaces in such a manner,

$$\begin{aligned} \|\psi\|^{*2}_2 &= \|(A-i)\psi\|^2 = \Big\langle \psi, (A^2+1)\psi \Big\rangle \\ \|\psi\|^{*2}_{-2} &= \Big\| \frac{1}{A-i}\psi \Big\|^2 = \Big\langle \psi, \frac{1}{A^2+1}\psi \Big\rangle. \end{aligned} \tag{4.13}$$

In fact, the spaces $\mathscr{H}_s(A)$ are Hilbert spaces with scalar product associated with standard norm in the following way,

$$\langle \psi, \varphi \rangle_s = \langle \psi, (|A| + 1)^s \varphi \rangle.$$

**Lemma 4.1.** *Let $A$ be a self-adjoint operator acting in Hilbert space $\mathscr{H}$ and $A_\alpha$ be a rank one perturbation of $A$. Suppose $\varphi \in \mathscr{H}_{-2}(A)$. Then, the restriction of $A^0$ of the operator $A$ to the domain of functions $D(A^0) = \{\psi \in D(A) : \langle \varphi, \psi \rangle = 0\}$ defines symmetric operator with deficiency indices $(1, 1)$.*

*Proof.* Since $A$ is self-adjoint operator, $A$ is densely defined operator. Thenceforth, for every $f \in \mathscr{H}$ there exists $f_n \in D(A) = \mathscr{H}_2(A)$ converging to $f$ in $\mathscr{H}$-norm so as to,

$$\lim_{n \to \infty} \parallel f_n - g \parallel_{\mathscr{H}} = 0.$$

For the functional $\varphi \in \mathscr{H}$, there is a functional sequence $\varphi_n \in D(A) = \mathscr{H}_2(A)$ with unit norm $\|\varphi_n\|$ in a matter that $\langle \varphi, \varphi_n \rangle$ is divergent. In order to prevent unboundedness, pick the sequence $\tau$,

$$\tau_n = \frac{\langle \varphi, f_n \rangle}{\langle \varphi, \varphi_n \rangle} \qquad \text{such that} \quad \lim_{n \to \infty} \tau_n = 0.$$

Whence, set the sequence $\kappa_n = f_n - \tau_n$ is contained in the domain of restricted operator

$$\langle \varphi, \kappa_n \rangle = 0.$$

Withal, $\kappa_n$ converges to $f$ in Hilbert space norm,

$$\Big\| f_n - \frac{(\varphi, f_n)}{(\varphi, \varphi_n)}\varphi_n - f \Big\|_{\mathscr{H}} \leq \|f_n - f\|_{\mathscr{H}} + \Big| \frac{(\varphi, f_n)}{(\varphi, \varphi_n)} \Big|$$

$$\lim_{n\to\infty} \|f_n - f\|_{\mathscr{H}} + \Big| \frac{(\varphi, f_n)}{(\varphi, \varphi_n)} \Big| = 0.$$

Ergo, $A^0$ is densely defined. Deficiency element at the point $\lambda = i$ is $g_i = (A - i)^{-1}\varphi$. Wherefore for every $\psi \in \mathscr{H}$,

$$\begin{aligned}
\langle g_i, \psi \rangle &= \langle (A - i)^{-1}\varphi, \psi \rangle \\
&= \langle \varphi, (A + i)^{-1}\psi \rangle \\
&\leq \|\varphi\|_{-2} \Big( \Big\| \frac{|A|}{A + i}\psi \Big\|_{\mathscr{H}} + \Big\| \frac{1}{A + i}\psi \Big\|_{\mathscr{H}} \Big) \\
&\leq 2\|\varphi\|_{-2}\|\psi\|_0
\end{aligned}$$

is obtained. Suppose that $\psi \in D(A^0)$ whereby

$$\begin{aligned}
\langle \psi, A^* g_i \rangle = \langle A\psi, g_i \rangle &= \Big\langle \frac{A}{A + i}\psi, \varphi \Big\rangle \\
&= \Big\langle \frac{-i}{A + i}\psi, \varphi \Big\rangle = i\langle \psi, g_i \rangle
\end{aligned}$$

holds. Subsequently, $g_i$ is the deficiency element for the restricted operator that corresponds with the complex number $i$. Consequently, $A^0$ has deficiency indices $(1, 1)$. $\square$

It is a fact that every self-adjoint extension of $A^0$ can be constructed by Birman - Krein - Vishik Theory [50, 122–124].

Let $A^0$ be certain densely defined symmetric operator acting in Hilbert space $\mathscr{H}$. Without loss of generality, assume that the operator $A^0$ is closed. Also, suppose that the deficiency indices are equal to $(1, 1)$.

Let $g_i$ and $g_{-i}$ be two normalized deficiency elements at point $\sigma = \pm i$,

$$A^{0*} g_i = \pm i g_{\pm i}$$

such that

$$\|g_{\pm i}\|_{\mathscr{H}} = 1.$$

The domain of the adjoint operator is equal to

$$D(A^{0*}) = D(A^0_+) \oplus L\{g_i, g_{-i}\}$$

where $\oplus$ is direct sum and $L\{g_i, g_{-i}\}$ stands for the space spanned by $g_i$ and $g_{-i}$. Accordingly, for each $\psi$ in the domain of the adjoint operator posses the following representation

$$\psi = \hat{\psi} + a^+(\psi) g_i + a^-(\psi) g_{-i} \tag{4.14}$$

where $\hat{\psi} \in \mathrm{Dom}(A^0)$, $a^+(\psi)$ and $a^-(\psi)$ are complex numbers.

The adjoint operator $A^{0*}$ acts on every $\psi \in \mathrm{Dom}(A^{0*})$ as follows

$$A^{0*}(\hat{\psi} + a^{+}(\psi)g_i + a^{-}(\psi)g_{-i}) = A\hat{\psi} + ia^{+}(\psi)g_i - ia^{-}(\psi)g_{-i}. \tag{4.15}$$

All self-adjoint extensions of the operator $A^0$ can be parametrized by one univariate parameter $\nu$, $|\nu| = 1$ using von Neumann Theory of self-adjoint extensions.

To conclude, every self-adjoint extension $A_\nu$ coincides with the restriction of $A^{0*}$ to the domain $D(A_\nu) = \{\psi \in D(A^{0*}) : \ -\nu(\psi) = a^{+}(\psi)\}$.

Let $A$ denote the self-adjoint extension corresponding to $\nu = 1$. Therewith the deficiency element $g_i$ and $g_{-i}$ are related in following way,

$$g_{-i} = \frac{A - i}{A + i}. \tag{4.16}$$

The another representation for the functions $\psi \in \mathrm{Dom}(A^{0*})$,

$$\begin{aligned} \psi &= \tilde{\psi} + \frac{b(\psi)}{2}(g_i + g_{-i}) \\ &= \tilde{\psi} + b(\psi)\frac{A}{A + i}g_i \end{aligned} \tag{4.17}$$

where $\tilde{\psi} \in \mathrm{Dom}(A)$. This representation is linked to the (4.14) by the formulas

$$\begin{cases} \tilde{\psi} & = \hat{\psi} + \frac{a^{+}(\psi) - a^{-}(\psi)}{2}(g_i + g_{-i}) \\ & = \hat{\psi}(a^{+}(\psi) - a^{-}(\psi))\frac{i}{A+i}g_i \\ b(\psi) & = a^{+}(\psi) + a^{-}(\psi). \end{cases}$$

Via (4.17) the action of the adjoint $A^{0*}$ is given by

$$\begin{aligned} A^{0*} &= A^{0*}\Big(\tilde{\psi} + b(\psi)\frac{A}{A+i}g_i\Big) \\ &= A\tilde{\psi} - b(\psi)\frac{1}{A+i}g_i. \end{aligned} \tag{4.18}$$

Note that the latter formula is derived from (4.15). Calculate the boundary form $\langle A^{0*}\psi, \eta\rangle - \langle \psi, A^{0*}\eta\rangle$ of the adjoint operator $\psi, \eta \in D(A^{0*})$,

$$\begin{aligned} \langle A^{0*}\psi, \eta\rangle - \langle \psi, A^{0*}\eta\rangle &= \Big\langle A\tilde{\psi} - b(\psi)\frac{1}{A+i}g_i, \tilde{\eta} + b(\eta)\frac{A}{A+i}g_i\Big\rangle \\ &\quad - \Big\langle \tilde{\psi} + b(\psi)\frac{A}{A+i}g_i, A\tilde{\eta} - b(\eta)\frac{1}{A+i}g_i\Big\rangle \\ &= \langle A\tilde{\psi}, \tilde{\eta}\rangle - \tilde{b}(\psi)\Big\langle \frac{1}{A+i}g_i, \tilde{\eta}\Big\rangle \\ &\quad + b(\eta)\Big\langle A\tilde{\psi}, \frac{A}{A+i}g_i\Big\rangle - \tilde{b}(\psi)b(\eta)\Big\langle \frac{1}{A+i}g_i, \frac{A}{A+i}g_i\Big\rangle \\ &\quad - \langle \tilde{\psi}, A\tilde{\eta}\rangle - \tilde{b}(\psi)\Big\langle \frac{A}{A+i}g_i, A\tilde{\eta}\Big\rangle \\ &\quad + b(\eta)\Big\langle \tilde{\psi}, \frac{1}{A+i}g_i\Big\rangle + \tilde{b}(\psi)b(\eta)\Big\langle \frac{A}{A+i}g_i, \frac{1}{A+i}g_i\Big\rangle \\ &= \overline{\langle (A-i)g_i, \tilde{\psi}\rangle} b(\eta) - \overline{b(\psi)}\langle (A-i)g_i, \tilde{\eta}\rangle. \end{aligned}$$

The Hilbert space $\mathscr{H}$ contains the deficiency element $g_i$. Hence, the vector $(A-i)g_i$ is a bounded linear functional on the domain of operator $A$. The scalar products $\langle (A-i)g_i, \tilde{\psi}\rangle$ and $\langle (A-i)g_i, \tilde{\eta}\rangle$ on the latter formulas are well defined. There is an alternative way to parametrize the self-adjoint extension is to use real parameter $\gamma$ instead of unitary $\nu$ in the von Neumann theorem (theorem $A.3$). $A_\gamma$ denotes the restriction of operator $A^{0*}$ to the domain

$$D(A_\gamma) = \{\psi \in D(A^{0*}) : \langle (A-i)g_i, \tilde{\psi}\rangle = \gamma b(\psi)\}. \tag{4.19}$$

Furthermore, infinite values of parameters $\gamma : \gamma \in \mathbb{R} \cup \{\infty\}$ will be examined. For $\gamma = 0$,

$$D(A^{\infty}) = \{\psi \in D(A^{0*}) : b(\psi) = 0\} = D(A),$$

The operator $A^{\infty}$ dovetails with the operator $A$. The operator $A_{\nu}$ and $A_{\gamma}$ coincide if the parameters $\nu$ and $\gamma$ are related in the way,

$$\nu = \frac{\gamma + i}{\gamma - i} \text{ if and only if } \gamma = \frac{-i - i\nu}{1 - \nu}. \tag{4.20}$$

$\gamma$ runs over all real numbers including infinity. This means that the parameter $\nu$ runs over all unimodular complex numbers. Thence, every self-adjoint extension of $A^0$ is determined by a certain parameter $\gamma \in \mathbb{R} \cup \{\infty\}$.

**Theorem 4.2.** *Let $A$ and $B$ be two self-adjoint extensions of a certain symmetric densely defined operator $A^0$ with unit deficiency indices. Thusly, there exists a number $\gamma \in \mathbb{R} \cup \{\infty\}$ such that resolvents of $A$ and $B$ are related as follows*

$$\frac{1}{B - z} = \frac{1}{A - z} + \frac{1}{\alpha - \left\langle g_i, \frac{1 + Az}{A - z} g_i \right\rangle} \left\langle \frac{A - i}{A - \overline{z}} g_i, \cdot \right\rangle \frac{A - i}{A - z} g_i, \quad \Im z \neq 0 \tag{4.21}$$

*where $g_i$ is the normalized deficiency element for $A^0$ at the point $\sigma = i$. If $A = B$ then $\gamma = \infty$ and the resolvents of the self-adjoint operators coincide,*

$$\frac{1}{B - z} = \frac{1}{A - z}, \quad \Im\, z \neq 0. \tag{4.22}$$

*Proof.* $A$ is the self-adjoint extension of $A^0$. Accordingly, $((A-i)/(A+i))g_i$ is a deficiency element for $A^0$ at the point $\sigma = -i$. Additionally, pick

$$g_{-i} = \frac{A-i}{A+i} g_i.$$

(4.19) provides a description of the self-adjoint extensions of $A^0$. Wherefore, the operator $A$ coincides with the operator $A^\infty$.

The operator $B$ is a self-adjoint extension of $A^0$, too. Thus, there exists a certain real parameter $\gamma \in \mathbb{R} \cup \{\infty\}$ such that $B = A_\gamma$. Provided $\gamma = \infty$, the operators $A$ and $B$ coincide and (4.22) holds. Likewise,

$$h = (A_\gamma - z)f \tag{4.23}$$

holds for a given $h \in \mathscr{H}$.

Solving (4.23) is vital to end proof. This equation calculates the resolvent of operator $A_\gamma$. The function $f$ appearing in (4.23) is contained in domain of $A_\alpha$. Let $\gamma \neq 0$ and $\gamma \neq \infty$. Thereon, each $f \in D(A_\gamma)$ has the representation

$$f = \tilde{f} + \frac{1}{\gamma}\langle (A-i)g_i, \tilde{f}\rangle \frac{A}{A+i} g_i.$$

(4.18) together with (4.23) imply that

$$h = A\tilde{f} - z\tilde{f} - \frac{1}{\gamma}\langle (A-i)g_i, \tilde{f}\rangle \frac{1+Az}{A+i} g_i. \tag{4.24}$$

Applying the resolvent of the original operator $A$ to (4.24) and then projecting to the element $(A-i)g_i \in \mathscr{H}_{-2}(A)$,

$$\langle (A-i)g_i, \tilde{f}\rangle = \frac{\langle \frac{A-i}{A-\bar{z}}g_i, h\rangle}{1-\frac{1}{\gamma}\langle g_i, \frac{1+Az}{A-z}g_i\rangle} \tag{4.25}$$

is obtained. Thus,

$$\begin{aligned} f &= \frac{1}{A-z}h + \frac{1}{\gamma}\langle (A-i)g_i, \tilde{f}\rangle \frac{1+Az}{A-z}\frac{1}{A+i}g_i + \frac{1}{\gamma}\langle (A-i)g_i, \tilde{f}\rangle \frac{A}{A+i} \\ &= \frac{1}{A-z}h + \frac{1}{\gamma}\langle (A-i)g_i, \tilde{f}\rangle \frac{A-i}{A-z}g_i \\ &= \frac{1}{A-z}h + \frac{\langle \frac{A-i}{A-z}g_i, h\rangle}{\gamma - \langle g_i, \frac{1+Az}{A-z}g_i\rangle}\frac{A-i}{A-z}g_i \end{aligned}$$

is obtained. For the case $\gamma = 0$, the resolvent formula is satisfied. □

**Corollary 4.1.** *The resolvent of any self-adjoint extension $A_\gamma$ of $A^0$ is given by*

$$\frac{1}{A_\gamma - z} = \frac{1}{A-z} + \frac{1}{\gamma - \langle g_i, \frac{1+Az}{A-z}g_i\rangle}\left\langle \frac{A-i}{A-\bar{z}}g_i, \cdot \right\rangle \frac{A-i}{A-z}g_i, \ \Im\, z \neq 0. \tag{4.26}$$

# 5. FINITE RANK PERTURBATIONS

This chapter exhibits finite rank perturbations along with self- adjoint extensions of finite rank perturbations. Further, this part contains the relation between resolvents and self-adjoint extensions of finite rank perturbations thanks to Krein's formula.

**Definition 5.1.** *Let $A \in \mathscr{H}$ be a self-adjoint operator with domain $D(A)$. The symmetric finite rank perturbations of the operator $A$ is defined by*

$$A_T = A + \sum_{i,j=1}^{N} t_{ji} \langle \varphi_i, \cdot \rangle \varphi_j \tag{5.1}$$

*where $T$ is nonsigular Hermitian matrix, the vectors $\varphi_j \in \mathscr{H}_{-2}(A)$ for $j = 1, 2, \cdots N$ and $\langle \cdot, \cdot \rangle$ is the scalar product in $\mathscr{H}_{-2}$.*

Assume that the vectors $\varphi_j$ form an orthonormal system in the $\mathscr{H}_{-2}(A)$,

$$\left\langle \frac{1}{A-i}\varphi_i, \frac{1}{A-i}\varphi_j \right\rangle = \delta_i^j. \tag{5.2}$$

where $\delta_i^j$ stands for the Kronocker delta.

**Definition 5.2.** *The set of vectors $\varphi_j \in \mathscr{H}_{-2}(A)$ for $j = 1, 2, \cdots, N$ is $\mathscr{H}$ - independent if and only if*

$$\sum_{j=1}^{N} c_j \varphi_j \in \mathscr{H}, \quad c_j \in \mathbb{C}.$$

It means that

$$c_1 = c_2 = \cdots = c_n = 0.$$

Notice that even though $A_T$ in the domain $D(A)$ is not acting as operator in Hilbert space $\mathscr{H}$, the operator $A_T$ acts as an operator from $\mathscr{H}_2(A) = D(A)$ to $\mathscr{H}_{-2}(A)$. The self-adjoint extension operator corresponding to symmetric operator (5.1) coincides with one of the self-adjoint extension of the operator $A^0$ equal to the restriction of $A$ to the domain of $A^0$ so as,

$$D(A^0) = D(A) \cap \ Ker\ T.$$

**Lemma 5.1.** *Let $A$ be a self-adjoint operator acting in $\mathscr{H}$ and $A_T$ be a finite rank perturbation of $A$. Suppose that $\mathscr{H}$ - independent $\varphi_j \in \mathscr{H}_{-2}(A)$ for $j = 1, \cdots, N$ form an orthonormal system in $\mathscr{H}_{-2}(A)$. Thereupon, the restriction of $A$ to the domain of functions $D(A^0) = \{\psi \in D(A) :\ \langle \varphi_j, \psi \rangle,\ j = 1, 2, \cdots, N\}$ is densely defined operator with indices $(N, N)$.*

*Proof.* Let $A_l^0$ be a sequence of operators. Actually, each $A_l^0$ is the restriction of the operator $A$ for $l = 0, 1, 2, \cdots, N$ to the domains

$$D(A_l^0) = \{\psi \in D(A) :\ \langle \varphi_j, \psi \rangle = 0,\ j = 1, 2, \cdots, l\},$$

respectively. For all $l$, $A_{l+1}^0$ is the restriction of $A_l^0$. The first operator of the chain $A_0^0$ dovetails with the operator $A$. Also, the $N^{th}$ operator $A_N^0$ coincides with $A^0$. Hence,

$$D(A_{l+1}^0) = \{\psi \in D(A_l^0) :\ \langle \varphi_{l+1}, \psi \rangle = 0\}$$

holds. Via mathematical induction $A_l^0$ is densely defined. By lemma 4.1, $A_l^0$ has deficiency indices $(1,1)$. Assume that the operator $A_l^0$ is densely defined. On that account, there is a sequence $f_k \in D(A_n^0)$ converging to $f$ so as to,

$$\lim_{k\to\infty} \|f_k - f\| = 0.$$

The functional $\varphi_{l+1}$ is unbounded in the domain of $A_l^0$ $D(A_l^0)$. Besides, norm of $\varphi_{l+1}$ in $D(A_l^0)$ is equal to the standard norm in the Hilbert space. Let $\varphi_{l+1}$ be bounded on $D(A_l^0)$. Ergo, there exists a vector in Hilbert space $\mathscr{H}$ such that for any $\psi \in D(A_l^0)$,

$$\langle \varphi_{l+1}, \psi \rangle = \langle \varphi, \psi \rangle$$

holds. This leads to existence of $c_1, c_2, \cdots, c_l$ constants in a matter that

$$c_1\varphi_1 + c_2\varphi_2 + \cdots + c_l\varphi_l - \varphi_{l+1} = \varphi \in \mathscr{H}.$$

This cannot be occurred out of the fact that $\varphi_j$ vectors are $\mathscr{H}$ - independent. Thus, there exist $\psi_m \in D(A_l^0)$ with $\|\psi_m\| = 1$ and $\lim_{m\to\infty} \langle \varphi_{l+1}, \psi_m \rangle$ is divergent. By mimicking the proof of lemma 4.1, $A_{l+1}^0$ is densely defined as well. It follows that the operator $A^0 = A_n^0$ is densely defined and the deficiency indices of the symmetric operator $A^0$ are $(N, N)$. It suffices to prove that the $A^0$ has at least $N$ linearly independent deficiency elements corresponding the complex number $i$ in order to finish the proof. The vectors $\dfrac{1}{A-i}\varphi_j$ for $j = 1, 2, \cdots, N$ are deficiency elements for the operator $A^0$. $\dfrac{1}{A-i}\varphi_j$ vectors are linearly independent granted by (5.2). This completes the proof of lemma.

□

$\mathscr{H}$ - independent system $\{\varphi_j\}$ can be orthonormalized. This means that $\{\varphi_j\}_{j=1}^N$ system can be considered as orthonormal. Accordingly, each element $\psi \in D(A^{0*})$ can be expressed in the following form

$$\psi = \widehat{\psi} + \sum_{j=1}^{N} \Big( a_{+j}(\psi) \frac{1}{A-i} \varphi_j + a_{-j}(\psi) \frac{1}{A+i} \varphi_j \Big) \tag{5.3}$$

where $\widehat{\psi} \in D(A^0)$ and $a_{+j}(\psi), a_{-j}(\psi) \in \mathbb{C}$. The adjoint $A^{0*}$ acts on every $\psi \in D(A^{0*})$ as follows

$$\begin{aligned} A^{0*}\Big(\widehat{\psi} + \sum_{j=1}^{N} \Big( a_{+j}(\psi) + \frac{1}{A-i} \varphi_j + a_{-j}(\psi) \frac{1}{A+i} \varphi_j \Big) \\ = A + \widehat{\psi} + \sum_{j=1}^{N} \Big( a_{+j}(\psi) \frac{i}{A-i} \varphi_j + a_{+j}(\psi) \frac{-i}{A-i} \varphi_j \Big). \end{aligned}$$

Computing the boundary form $\langle A^{0*}\psi, \eta \rangle - \langle \psi, A^{0*}\eta \rangle$ of the adjoint operator for $\psi, \eta \in D(A^{0*})$ leads the following

$$\begin{aligned} \langle A^{0*}\psi, \eta \rangle - \langle \psi, A^{0*}\eta \rangle &= 2i \sum_{j,k=1}^{N} \Big( \overline{a_{-j}}(\psi) a_{-k}(\eta) - \overline{a_{+j}}(\psi) a_{+k}(\eta) \Big) \Big\langle \frac{1}{A-i} \varphi_j, \frac{1}{A-i} \varphi_k \Big\rangle \\ &= 2i \sum_{j=1}^{N} \Big( \overline{a_{-j}}(\psi) a_{-k}(\eta) - \overline{a_{+j}}(\psi) a_{+k}(\eta) \Big) \\ &= 2i \Big( \Big\langle \{a_{-j}\}_{j=1}^{N}(\psi), \{a_{-j}\}_{j=1}^{N}(\eta) \Big\rangle \\ &\quad - \Big\langle \{a_{+j}\}_{j=1}^{N}(\psi), \{a_{+j}\}_{j=1}^{N}(\eta) \Big\rangle \Big) \end{aligned}$$

where $\psi, \eta \in D(A^{0*})$ and the $\dfrac{1}{A-i}\varphi_j$ for $j = 1, 2, \cdots, N$ is the orthonormal system. $N$-dimensional unitary matrices via theory of the von Neumann theorem can parametrize the self-adjoint extensions of $A^0$.

Let $V = \{V_{jk}\}_{j,k=1}^{N}$ be an unitary matrix. The self-adjoint operator $A(V)$ corresponds to the restriction of the operator $A^{0*}$ to domain of operator $A(V)$ such that

$$D(A(V)) = \{\psi \in D(A^{0*}) : -V\{a_{-j}\}_{j=1}^{N}(\psi) = \{a_{+j}\}_{j=1}^{N}(\psi)\}.$$

The extension corresponding to $V = I$ matrix coincides with the original operator $A$. The norms in $\mathscr{H}_2(A)$ and $\mathscr{H}_{-2}(A)$ spaces are equal to the standard norms in these Hilbert spaces $\mathscr{H}$. Suppose that $\varphi_j$ vectors for $j = 1, 2, \cdots, N$ are $\mathscr{H}$ - independent. $\mathscr{H}_\varphi(A)$ stands for the domain of $A^{0*}$. Let $\psi$ be a vector of $\mathscr{H}_\varphi(A)$. Thus, the representation of $\psi$ in (5.3) can be written as

$$\psi = \widehat{\psi} + \sum_{j=1}^{N} a_{+j}(\psi)\Big(\frac{i}{A^2+1} + \frac{A}{A^2+1}\Big)\varphi_j + a_{-j}(\psi)\Big(\frac{-i}{A^2+1} + \frac{A}{A^2+1}\Big)\varphi_j \tag{5.4}$$

$$= \tilde{\psi} + \sum_{j=1}^{N} b_j(\psi)\frac{A}{A^2+1}\varphi_j \tag{5.5}$$

where

$$\tilde{\psi} = \widehat{\psi} + \sum_{j=1}^{N} a_{+j}(\psi)\frac{i}{A^2+1}\varphi_j - a_{-j}(\psi)\frac{i}{A^2+1}\varphi_j \in D(A), \tag{5.6}$$

$$b_j(\psi) = a_{+j}(\psi) + a_{-j}(\psi) \in \mathbb{C}. \tag{5.7}$$

The norm in $\mathscr{H}_\varphi(A)$ can be defined as follows

$$\|\psi\|_{\mathscr{H}_{\{\varphi\}}(A)} = \|\tilde{\psi}\|_2 + \sqrt{\sum_{j=1}^{N} |b_j(\psi)|^2}. \tag{5.8}$$

The space $\mathscr{H}_{\varphi}(A)$ can be seen as finite dimensional extension of $\mathscr{H}_2(A)$. In that way, $\mathscr{H}_{\varphi}(A)$ is isomorphic to $\mathscr{H}_2(A) \oplus \mathbb{C}^N$. The natural norm preserving embedding $\rho_N$ of $\mathscr{H}_2(A) \oplus \mathbb{C}^N$ into $\mathscr{H}_{\varphi}(A)$ is defined by (5.9). In addition, the dual space of $\mathscr{H}_{\varphi}(A)$ denoted by $\mathscr{H}_{-\varphi}(A)$ can be defined via analogy of (5.8).

Suppose that $T$ is a form bounded operator with respect to the operator $A$. Whence,

$$\langle \psi, T\psi \rangle \leq a\|\psi\|_1^2 + b\|\psi\|^2 \tag{5.9}$$

holds for every $\psi \in D(A)$ and $a, b \geq 0$. This supposition is valid if and only if each $\varphi_j$ for $j = 1, 2, \cdots, N$ is an element of $\mathscr{H}_{-1}(A)$. Therewith, assume that $\varphi_j \in \mathscr{H}_{-1}(A)$

$$\begin{aligned} |\langle \psi, T\psi \rangle| &= |\sum_{i,j=1}^{N} t_{ji} \langle \psi, \varphi_j \rangle \langle \varphi_i, \psi \rangle| \\ &\leq \Big( \sum_{i,j=1}^{N} |t_{ji}| \Big\| \frac{1}{\sqrt{|A|} + i} \varphi_j \Big\| \Big\| \frac{1}{\sqrt{|A|} + i} \varphi_j \Big\| \Big) \|(\sqrt{|A|} - i)\psi\|^2 \\ &= \Big( \sum_{i,j=1}^{N} |t_{ij}| \|\varphi_i\|_{-1} \|\varphi_j\|_{-1} \Big) \|\psi\|_1^2. \end{aligned}$$

Since the norms of the vectors $\dfrac{1}{|A| + i}\varphi_j$ can be calculated in Hilbert space $\mathscr{H}$ as follows

$$\Big\| \frac{1}{\sqrt{|A|} + i} \varphi_j \Big\|^2 = \Big\langle \varphi_j, \frac{1}{|A| + 1} \varphi_j \Big\rangle = \|\varphi_j\|_{-1}^2.$$

**Definition 5.3.** *Let $\mathcal{R}$ be the Hermitian matrix corresponding to the set of vectors $\varphi_j \in \mathscr{H}_{-2}(A)$ for $j = 1, 2, \cdots, N$. The self-adjoint operator $A$ is called admissible if*

$$\left\langle f, \frac{A}{A^2+1} g \right\rangle = \sum_{i,j=1}^{N} \overline{f_j} \mathcal{R}_{ji} g_i \tag{5.10}$$

This means that following conditions must be fulfilled. For every function $f$ and $g$ contained in the linear hull of the vectors such that

$$f = \sum_{j=1}^{N} f_j \varphi_j \quad \text{and} \quad g = \sum_{j=1}^{N} g_j \varphi_j.$$

Then the spectral measure $\mu_{f,g}$ corresponding to functions $f$ and $g$ determines the absolutely convergent integral

$$\int_{-\infty}^{\infty} \frac{|\lambda|}{\lambda^2+1} d|\mu_{f,g}| < \infty. \tag{5.11}$$

The selection of admissible matrix $\mathcal{R}$ determines the extension of all $\varphi_j \in \mathscr{H}_{-2}(A)$ to linear bounded functional on $\mathscr{H}_{\varphi}(A)$. In this regard,

$$\psi = \tilde{\psi} + \sum_{j=1}^{N} b_j(\psi) \frac{A}{A^2+1} \varphi_j \in \mathscr{H}_{\{\varphi\}}.$$

The scalar product of $\varphi_k$ and $\psi$ is as follows

$$\begin{aligned}\langle\varphi_k,\psi\rangle &= \langle\varphi_k,\tilde{\psi}\rangle + \sum_{j=1} b_j(\psi)\Big\langle\varphi_k, \frac{A}{A^2+1}\varphi_j\Big\rangle \\ &= \langle\varphi_k,\tilde{\psi}\rangle + \sum_{j=1}^{N} \mathcal{R}_{kj} b_j(\psi).\end{aligned}$$

The self-adjoint operator $A_T$ can be characterized via boundary conditions relating to the boundary $\langle\varphi_j,\tilde{\psi}\rangle$ and $b_j(\psi_j)$ for $j = 1, 2, \cdots, N$

$$\langle\varphi_j,\tilde{\psi}\rangle = \Gamma b_j(\psi) \tag{5.12}$$

where $\Gamma$ is the N-dimensional Hermitian matrix such that

$$\Gamma = -(T^{-1} + \mathcal{R}).$$

The Cayley transform links $\Gamma$ and $V$ matrices describing the same self-adjoint extensions,

$$\Gamma = -i\frac{I+V}{I-V} \quad \text{and} \quad V = \frac{\Gamma+I}{\Gamma-I}. \tag{5.13}$$

**Theorem 5.1.** *Let* $A_T = A + \sum_{i,j=1}^{N} t_{ji}\langle\varphi_i,\cdot\rangle\varphi_j$ *be the self-adjoint restriction of the operator* $A^{0*}$ *to the domain of functions satisfying the boundary conditions* (5.12) *in* $D(A^0)$. *Suppose that* $Q(\lambda)$ *denotes the* $Q$*-matrix function corresponding to the operator* $A$ *as well as vectors* $\varphi_j$

$$(Q(\lambda))_{jk} = \Big\langle\varphi_j, \frac{1+\lambda A}{A-\lambda}\frac{1}{A^2+1}\varphi_k\Big\rangle. \tag{5.14}$$

*As a result, the resolvent of the operator $A_T$ is given by*

$$\frac{1}{A_T-\lambda}=\frac{1}{A-\lambda}+\sum_{j,k=1}^{N}\frac{1}{A-\lambda}\varphi_j(\Gamma-Q(\lambda))^{-1}_{jk}\Big\langle\frac{1}{A-\overline{\lambda}}\varphi_k,\cdot\Big\rangle. \tag{5.15}$$

*Proof.* To compute the resolvent of the self-adjoint operator $A_T$,

$$f=\psi(A_T-\lambda)$$

equation needs to be solved. Thenceforth,

$$\psi=\frac{1}{A_T-\lambda}f=\tilde{\psi}+\sum_{j=1}^{N}b_j(\psi)\frac{A}{A^2+1}\varphi_j. \tag{5.16}$$

Likewise, rewrite (5.16) to express $f$ so as,

$$f=(A_T-\lambda)\tilde{\psi}+(A_T-\lambda)\sum_{j=1}^{N}b_j(\psi).\frac{A}{A^2+1}\varphi_j \tag{5.17}$$

Via plugging (5.1) into (5.17) such that

$$f=(A-\lambda)\tilde{\psi}-\sum_{j=1}^{N}b_j(\psi)\frac{1+\lambda A}{A^2+1}\varphi_j \tag{5.18}$$

is obtained. The resolvent of the original operator $A$ provides the following

$$\frac{1}{A-\lambda}f=\tilde{\psi}-\sum_{j=1}^{N}b_j(\psi)\frac{1+\lambda A}{A-\lambda}\frac{1}{A^2+1}\varphi_j \tag{5.19}$$

By projecting (5.19) onto vectors $\varphi_k$ for $k = 1, 2, \cdots, N$ leads the following

$$\Big\langle \varphi_k, \frac{1}{A-\lambda} f \Big\rangle = \langle \varphi_k, \tilde{\psi} \rangle - \sum_{j=1}^{N} b_j(\psi) \Big\langle \varphi_k, \frac{1+\lambda A}{A-\lambda} \frac{1}{A^2+1} \varphi_j \Big\rangle, \;\; k = 1, 2, \cdots, N. \quad (5.20)$$

The function $\psi$ agrees with the boundary condition leading to matrix notation as follows

$$\begin{aligned} \Big\{ \Big\langle \varphi_k, \frac{1}{A-\lambda} f \Big\rangle \Big\}_{k=1}^{N} &= \Big\{ \Big\langle \varphi_k, \tilde{\psi} \Big\rangle \Big\}_{k=1}^{N} - Q \Big\{ \Big\langle b_j(\psi) \Big\rangle \Big\}_{j=1}^{N} \\ &= \Big( \Gamma - Q \Big) \Big\{ \Big\langle b_j(\psi) \Big\rangle \Big\}_{j=1}^{N}. \end{aligned}$$

Then,

$$\Big\{ b_j(\psi) \Big\}_{j=1}^{N} = \Big\{ \Big\langle \varphi_k, \frac{1}{A-\lambda} f \Big\rangle \Big\}_{k=1}^{N} \Big( \Gamma - Q \Big)^{-1}. \quad (5.21)$$

The function $\psi$ is given by the following equation

$$\psi = \frac{1}{A_T - \lambda} f = \frac{1}{A-\lambda} f + \sum_{j=1}^{N} \frac{1}{A-\lambda} \varphi_j b_j(\psi). \quad (5.22)$$

Thence, (5.21) and (5.22) refer to (5.15). Consequently, proof is completed. □

# 6. MAIN RESULTS

This chapter contains the two main results of this thesis. The First one is proving that Laplacian formally given by Hamiltonian describing finite singular point interactions on a compact manifold has a self-adjoint extension. The second one is showing that Laplacian formally given by Hamiltonian demonstrating finite singular interactions supported by curves on a compact Riemannian manifold has a self-adjoint extension.

**Theorem 6.1.** *Let $\mathcal{M}$ be a two-dimensional compact Riemannian manifold equipped with metric $g$. Suppose that formal Hamiltonian $\mathcal{H}$ can be defined in $\mathcal{M}$ as follows*

$$\mathcal{H} = -\frac{\hbar^2}{2m}\Delta_g - \lambda\delta(x, a_i) \tag{6.1}$$

*where $\lambda \in \mathbb{R}^+$ is called coupling constant, $\delta$ is the Dirac delta distribution and $m$ is particle mass. Laplacian operator $\Delta_g$ can be defined as self-adjoint operator on $\boldsymbol{H}^2$. Thus, $\Delta_g$ formally given by Hamiltonian $\mathcal{H}$ is restricted to the domain $D = \{\psi \in \boldsymbol{H}^2 \mid \psi(a_i) = 0, \forall i = 1, \ldots, N\}$ has self-adjoint extension.*

*Proof.* For simplicity, assume that singular point interactions in natural unit ($\hbar = 2m = 1$). It is sufficient to show that $\delta(x, a_i) \in \mathscr{H}_{-2}(\Delta_g)$ by lemma 5.1. Then,

$$\|\delta(x, a_i)\|_{\mathscr{H}_{-2}(\Delta_g)} = \left\|\frac{1}{\Delta_g + \nu^2}\delta(\cdot, a_i)\right\|_{\mathscr{H}}. \tag{6.2}$$

$(C.23)$ together with the norm in Hilbert space yield the following

$$\left\|\frac{1}{\Delta_g + \nu^2}\delta(\cdot, a_i)\right\|_{\mathscr{H}} = \int_{\mathcal{M}} d_g x \int_0^\infty \int_0^\infty K_{t_1}(a_i, x) K_{t_2}(x, a_i) e^{-\nu^2(t_1+t_2)} dt_1 dt_2. \tag{6.3}$$

By semi-group property of heat kernel ($C$.11),

$$\left\| \frac{1}{\Delta_g + \nu^2} \delta(\cdot, a_i) \right\|_{\mathscr{H}} = \int_0^\infty \int_0^\infty K_{t_1+t_2}(a_i, a_i) e^{-\nu^2(t_1+t_2)} dt_1 dt_2 \tag{6.4}$$

is obtained.

Via change of variable $t = t_1 + t_2$ and $z = t_1 - t_2$ along with applying Fubini theorem [125, 126],

$$\left\| \frac{1}{\Delta_g + \nu^2} \delta(\cdot, a_i) \right\|_{\mathscr{H}} = \int_0^\infty \int_{-t}^{t} \frac{K_t(a_i, a_i)}{2} e^{-\nu^2 t} dz dt \tag{6.5}$$

$$= \int_0^\infty t K_t(a_i, a_i) e^{-\nu^2 t} dt \tag{6.6}$$

is obtained. Applying the on-diagonal upper bound for heat kernel (2.26) to (6.6) provides the following,

$$\int_0^\infty t K(a_i, a_i) e^{-\nu^2 t} dt \leq \frac{1}{V(\mathcal{M})} \int_0^\infty t e^{-\nu^2 t} dt + C \int_0^\infty e^{-\nu^2 t} dt. \tag{6.7}$$

Via Integration by parts, both $\int_0^\infty t e^{-\nu^2 t} dt$ and $\int_0^\infty e^{-\nu^2 t} dt$ are finite. Since $\mathcal{M}$ is finite and compact then $\frac{1}{V(\mathcal{M})}$ is finite. This concludes, the right hand-side of (6.7) is finite. Then $\int_0^\infty t K_t(a_i, a_i) e^{-\nu^2 t} dt$ is finite. Thus, $\|\delta(x, a_i)\|$ is finite. Consequently, $\delta(x, a_i) \in \mathscr{H}_{-2}(\Delta_g)$. □

Theorem 6.1 provides the existence of self-adjoint extension of $\Delta_g$ restricted to domain $D = \{\psi \in \mathbf{H}^2 \mid \psi(a_i) = 0, \forall i = 1, \ldots, N\}$ formally given by Hamiltonian $\mathcal{H}$ (6.1). The resolvent of the extension needs to be calculated to compare with finite singular interactions. The renormalized resolvent of heuristically constructed finite singular Dirac

delta potentials is computed previously (3.26).

Let $\Delta_T$ be a self-adjoint extension of $\Delta_g$ on $D = \{\psi \in \mathbf{H}^2 \mid \psi(a_i) = 0, \forall i = 1, \ldots, N\}$ formally given by Hamiltonian $\mathcal{H}$ provided in Theorem 6.1.

$$\frac{1}{\Delta_T + \nu^2} = \frac{1}{\Delta_g + \nu^2} \sum_{i,j=1}^{N} \frac{1}{\Delta_g + \nu^2} \delta(\cdot, a_i)(\Gamma - Q(\nu^2))^{-1}_{ij} \frac{1}{\Delta_g + \nu^2} \delta(\cdot, a_j). \tag{6.8}$$

is obtained where $\Gamma = -(I + \mathcal{R})$. $I$ stands for identity matrix and $\mathcal{R}$ is admissible matrix.
(6.8) can be expressed in terms of resolvent as follows

$$R(x, y|E) = R_0(x, y|E) + \sum_{i,j=1}^{N} R_0(x, a_i|E)(\Gamma - Q(E))^{-1}_{ij} R_0(a_j, y|E). \tag{6.9}$$

The resolvent formula (3.26) is related to (6.9). The difference stems from the choice of bare constant. In (5.9), bare constant is selected as follows

$$\frac{1}{\lambda_i(\epsilon)} = \int_{\epsilon}^{\infty} K_t(a_i, a_i) \Big[ \frac{e^{-it} + e^{it}}{2} \Big] dt - \gamma_i. \tag{6.10}$$

Furthermore, the bare constant (6.10) can be expressed in terms of resolvent with Dirac ket notation as follows

$$< a_i \Big| \Big( \frac{1}{2} \Big\{ \frac{1}{\Delta_g + i} + \frac{1}{\Delta_g - i} \Big\} - \frac{1}{\Delta_g - E} \Big) \Big| a_i) > . \tag{6.11}$$

The choice of bare constant (6.10) does not care physical meaning. In fact, every self-adjoint extension of $\Delta_g$ restricted to domain $D = \{\psi \in \mathbf{H}^2 \mid \psi(a_i) = 0, \forall i = 1, \ldots, N\}$ can be constructed by Birman - Krein - Vishik Theory [50, 122–124].

As a consequence, finite rank perturbations of self-adjoint operators on compact manifold agrees with renormalization technique to describe finite singular point interactions.

**Theorem 6.2.** *Let $M$ be a compact manifold equipped with metric $g$. Let $\mathcal{H}$ be Hamiltonian*

$$\mathcal{H} = -\frac{\hbar^2}{2m}\Delta_g - \lambda\delta_{\Gamma_i} \otimes \delta_{\Gamma_i} \tag{6.12}$$

*in $\mathcal{M}$ where $\Gamma_i$ stands for the arc-length parametrized close curve of length $L$. $\lambda \in \mathbb{R}^+$ is called the coupling constant, $\delta$ is the Dirac delta distribution defined supported over curve $\Gamma_i$ and $m$ is particle mass. Laplacian operator $\Delta_g$ can be defined as self-adjoint operator on $\boldsymbol{H}^2$. Hence, the restriction of $\Delta_g$ formally given by Hamiltonian $\mathcal{H}$ to the domain $D = \{\psi \in C_0^\infty(M) \mid \psi(\Gamma_i) = 0,\ \forall i = 1, \cdots, N\}$ has self-adjoint extension.*

*Proof.* For simplicity, assume that formally given Hamiltonian $\mathcal{H}$ has natural units ($\hbar = 2m = 1$). It is suffice to show that $\delta_{\Gamma_i} \in \mathscr{H}_{-2}(\Delta_g)$ by lemma 5.1.

$$\|\delta_{\Gamma_i}\|_{\mathscr{H}_{-2}(\Delta_g)} = \left\| \frac{1}{\Delta_g + \nu^2}\delta_{\Gamma_i} \right\|_{\mathscr{H}}.$$

($C$.23) together with the norm in Hilbert space provides the following

$$\begin{aligned}
\left\| \frac{1}{\Delta_g + \nu^2}\delta_{\Gamma_i} \right\|_{\mathscr{H}} &= \int_{\mathcal{M}} dz \int_0^\infty K(x,y)e^{-\nu^2 t}dt \int_{\Gamma_i} \delta_g(y - \gamma(s))d_g s \\
&= \int_{\mathcal{M}} dz \int_{\Gamma_i} \frac{d_g s}{\sqrt{L}} \int_{\Gamma_i} \frac{d_g s'}{\sqrt{L}} \int_0^\infty K_{t_1}(x, \gamma(s))e^{-\nu^2 t_1}dt_1 \\
&\qquad \times \int_0^\infty K_{t_2}(x, \gamma(s'))e^{-\nu^2 t_2}dt_2 \\
&= \frac{1}{L}\int_{\Gamma_i}\int_{\Gamma_i} d_g s d_g s' \int_{\mathcal{M}} dz \\
&\times \int_0^\infty \int_0^\infty K_{t_1}(x, \gamma(s))K_{t_2}(x, \gamma(s'))e^{-\nu^2(t_1+t_2)}dt_1 dt_2
\end{aligned}$$

Symmetry property ($C$.12) and semi-group property of heat kernel ($C$.11) lead

$$\left\| \frac{1}{\Delta_g + \nu^2} \delta_{\Gamma_i} \right\|_{\mathscr{H}} = \frac{1}{L} \int_{\Gamma_i} \int_{\Gamma_i} d_g s d_g s' \int_0^\infty \int_0^\infty K_{t_1+t_2}(\gamma(s), \gamma(s')) e^{-\nu^2(t_1+t_2)} dt_1 dt_2. \tag{6.13}$$

The change of variable properties of heat kernel as $t = t_1 + t_2$, $x = t_1 - t_2$ and applying Fubini theorem [125, 126],

$$\begin{aligned} \left\| \frac{1}{\Delta_g + \nu^2} \delta_{\Gamma_i} \right\|_{\mathscr{H}} &= \frac{1}{L} \int_{\Gamma_i} \int_{\Gamma_i} d_g s d_g s' \int_0^\infty \int_{-t}^t \frac{K_t(\gamma(s), \gamma(s'))}{2} e^{-\nu^2 t} dx dt \\ &= \frac{1}{L} \int_{\Gamma_i} \int_{\Gamma_i} d_g s d_g s' \int_0^\infty t K_t(\gamma(s), \gamma(s')) e^{-\nu^2 t} dt. \end{aligned}$$

By the off-diagonal upper bound of heat kernel on compact manifold provides (2.28), the following,

$$\int_0^\infty t K_t(\gamma(s), \gamma(s')) e^{-\nu^2 t} dt \leq \Big( \frac{1}{V(\mathcal{M})} \int_0^\infty t e^{-\nu^2 t} dt + C \int_0^\infty \frac{1}{\sqrt{t}} e^{-\nu^2 t} dt \Big) e^{\frac{-d^2(\gamma(s), \gamma(s'))}{C' t}} \tag{6.14}$$

is obtained. $e^{-d^2(\gamma(s),\gamma(s'))/C't}$ can be arranged as $e^{-d^2(\gamma(s),\gamma(s'))/C't} \approx 1$ via setting $\gamma(s)$ and $\gamma(s')$ sufficiently close. Thenceforth, (6.14) turns out the following,

$$\int_0^\infty t K_t(\gamma(s), \gamma(s')) e^{-\nu^2 t} dt \leq \frac{1}{V(\mathcal{M})} \int_0^\infty t e^{-\nu^2 t} dt + C \int_0^\infty \frac{1}{\sqrt{t}} e^{-\nu^2 t} dt \tag{6.15}$$

By multiplying $\frac{1}{L} \int_{\Gamma_i} \int_{\Gamma_i} d_g s d_g s'$ to both sides of (6.15),

$$\begin{aligned} \frac{1}{L} \int_{\Gamma_i} \int_{\Gamma_i} d_g s d_g s' \int_0^\infty t K_t(\gamma(s),\gamma(s')) e^{-\nu^2 t} dt \leq \frac{1}{L} \int_{\Gamma_i} \int_{\Gamma_i} d_g s d_g s' \frac{1}{V(\mathcal{M})} \int_0^\infty t e^{-\nu^2 t} dt \\ + \frac{1}{L} \int_{\Gamma_i} \int_{\Gamma_i} d_g s d_g s' C \int_0^\infty \frac{1}{\sqrt{t}} e^{-\nu^2 t} dt \end{aligned} \tag{6.16}$$

is derived. Via Integration by parts, both $\int_0^\infty te^{-\nu^2 t}dt$ and $\int_0^\infty \frac{1}{\sqrt{t}}e^{-\nu^2 t}dt$ are finite. Since $\mathcal{M}$ is finite dimensional compact Riemannian manifold then $V^{-1}(\mathcal{M})$ is finite. Likewise, $\frac{1}{L}\int_{\Gamma_i}\int_{\Gamma_i} d_g s d_g s'$ is finite, too. This implies that the right handside of (6.16) is finite. This means that $\frac{1}{L}\int_{\Gamma_i}\int_{\Gamma_i} d_g s d_g s' \int_0^\infty tK_t(\gamma(s),\gamma(s'))e^{-\nu^2 t}dt$ is finite. Whence, $\delta_{\Gamma_i} \in \mathscr{H}_{-2}(\Delta_g)$. □

The existence of self-adjoint extension of the restriction of formally given by Hamiltonian $\mathcal{H}$ (6.12), $\Delta_g$ to domain $D = \{\psi \in C_0^\infty(M) \mid \psi(\Gamma_i) = 0,\ \forall i = 1, \cdots, N\}$ is provided by Theorem 6.2. The resolvent of the extension of $\Delta_g$ formally given by Hamiltonian $\mathcal{H}$ (6.12) to the domain $D = \{\psi \in C_0^\infty(M) \mid \psi(\Gamma_i) = 0,\ \forall i = 1, \cdots, N\}$ provided by Theorem 6.2 should be computed. Heuristically constructed renormalized resolvent of finite singular interactions supported by curves is already calculated in (3.42). Thence, these resolvents can be compared.

Let $\Delta_T$ be a self-adjoint extension of $\Delta_g$ on $D = \{\psi \in C_0^\infty(M) | \psi(\Gamma_i) = 0, i = 1, \cdots, N\}$ formally given by Hamiltonian $\mathcal{H}$ provided in Theorem 6.2. Thus,

$$\frac{1}{\Delta_T + \nu^2} = \frac{1}{\Delta_g + \nu^2} + \sum_{i,j=1}^{N} \frac{1}{\Delta_g + \nu^2}\delta_{\Gamma_i}(\Gamma - Q(\nu^2))^{-1}_{ij}\frac{1}{\Delta_g + \nu^2}\delta_{\Gamma_j} \tag{6.17}$$

is derived where $\Gamma = -(I + \mathcal{R})$. $I$ stands for identity matrix and $\mathcal{R}$ denotes admissible matrix.

(6.17) becomes the following in terms of resolvent as follows

$$R(x, y|E) = R_0(x, y|E) + \sum_{i,j=1}^{N} R_0(x, \gamma(s)|E)(\Gamma - Q(E))^{-1}_{ij} R_0(\gamma(s'), y|E). \tag{6.18}$$

The resolvent formula (6.18) relates to (3.42). The difference comes from the bare constant choice. In (6.18) bare constant is picked as follows

$$\frac{1}{\lambda_i(\epsilon)} = \frac{1}{L_i}\int_{\Gamma_i}\int_{\Gamma_i} d_g s d_g s' \int_{\epsilon}^{\infty} dt K_t(\gamma(s),\gamma(s'))\Big[\frac{e^{-it}+e^{it}}{2}\Big] - \gamma_i. \tag{6.19}$$

Alternatively, the bare constant can be represented in terms of resolvent with Dirac ket notation as follows

$$< \Gamma_i(s)\Big|\Big(\frac{1}{2}\Big\{\frac{1}{\Delta_g + i} + \frac{1}{\Delta_g - i}\Big\} - \frac{1}{\Delta_g - E}\Big)\Big|\Gamma_i(s') > . \tag{6.20}$$

The choice of bare constant (6.19) does not care physical meaning. In fact, every self-adjoint extension of $\Delta_g$ restricted to domain $D = \{\psi \in C_0^{\infty}(M) \mid \psi(\Gamma_i) = 0,\ \forall i = 1, \cdots, N\}$ can be constructed by Birman - Krein - Vishik Theory [50, 122–124].

Consequently, finite rank perturbations of self-adjoint operators on compact manifold coincides with renormalization technique to describe finite singular interactions supported by curves. $D = \{\psi \in C_0^{\infty}(M) \mid \psi(\Gamma_i) = 0,\ \forall i = 1, \cdots, N\}$

# 7. CONCLUSION

This thesis was intended as an attempt to describe quantum mechanical models of finite singular point interactions and finite singular interactions supported by curves on the compact Riemannian manifold by finite rank perturbation point of view.

Firstly, upper bounds of heat kernel are discussed since heat kernel plays a crucial role to fathom singular interactions. Then, heuristic renormalization and regularization of singular point interactions on a compact Riemannian manifold and finite singular interactions supported by curves on a compact Riemannian manifold are presented. Later, rank one perturbations are discussed to get an intuition of finite rank perturbations. In this sense, finite rank perturbations can be regarded as a generalization of rank one perturbations. Afterward, finite rank perturbation and its self-adjoint extensions are offered.

To conclude, finite rank perturbations of self-adjoint operators on compact manifold agrees with renormalization technique to describe singular Dirac delta potential on the compact Riemannian manifold are provided as main results.

In future work, explanation singular Dirac delta potential on the compact Riemannian manifold will be examined via purely von Neumann Theory of self-adjoint extension approach.

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# APPENDIX A: VON NEUMANN THEORY OF EXTENSIONS

**Definition A.1.** *The graph of linear transformation $T$ is the set of pairs such that*

$$\{\langle \varphi, T\varphi \rangle | \varphi \in D(T)\}. \tag{A.1}$$

*The graph of $T$ is a subset of $\mathscr{H} \times \mathscr{H}$ which is denoted by $\Gamma(T)$. This is inner product in Hilbert space $\mathscr{H}$*

$$\big(\langle \varphi_1, \psi_1 \rangle, \langle \varphi_2, \psi_2 \rangle\big) = (\varphi_1, \varphi_2) + (\psi_1, \psi_2). \tag{A.2}$$

*$T$ is called a closed operator if $\Gamma(T)$ is a closed subset of $\mathscr{H} \times \mathscr{H}$.*

**Definition A.2.** *Let $T'$ and $T$ be operator in Hilbert space $\mathscr{H}$. $T'$ is called be an extension of $T$ and write $T' \supset T$ if $\Gamma(T') \supset \Gamma(T)$. Equivalently, $D(T') \supset D(T)$ and $T'\varphi = T\varphi$ if and only if $T' \supset T$.*

**Definition A.3.** *Let $T$ be an operator acting in Hilbert space $\mathscr{H}$. If an operator $T$ has a closed extension then it is called a closable. Every closable operator has smallest closed extension which is called its closure and it is denoted by $\overline{T}$.*

**Definition A.4.** *A linear operator $T$ is defined on $\mathscr{X} \subset \mathscr{H}$ is s called densely defined if $D(T)$ is dense in $\mathscr{X}$ such that*

$$\overline{D(T)} = \mathscr{X}. \tag{A.3}$$

**Definition A.5.** *Let $T$ be a densely defined operator in Hilbert space $\mathscr{H}$. Let $D(T^*)$ be set of $\varphi \in \mathscr{H}$ for which there is an $\eta \in \mathscr{H}$ with*

$$(T\psi, \varphi) = (\psi, \eta) \; for \; all \; \psi \in D(T). \tag{A.4}$$

*Provided that for each $\varphi \in D(T^*)$ $T^*\varphi = \eta$ then $T^*$ is called adjoint of $T$. By the Reisz lemma [127], $\varphi \in D(T^*)$ if and only if $|T\psi, \varphi| \leq C\|\psi\|$ for all $\psi \in D(T)$.*

**Definition A.6.** *Let $T$ be a densely defined operator in Hilbert space. $T$ is called symmetric (or Hermitian) if and only if*

$$(T\varphi, \psi) = (\varphi, T\psi) \tag{A.5}$$

*for all $\varphi, \psi \in D(T)$. Moreover, if $T = T^*$ and $D(T) = D(T^*)$ then $T$ is called self-adjoint.*

Since $D(T^*) \supset D(T)$ is dense in $\mathscr{H}$, a densely defined symmetric operator $T^*$ is closed extension of $T$ if $T$ is symmetric. Thus, smallest closed extension of $T$ should be in $T^*$, denoted by $T^{**}$. Thus, for a symmetric operator $T$,

$$T \subset T^{**} \subset T^*$$

holds. Likewise, for a closed symmetric operators,

$$T = T^{**} \subset T^*$$

holds. Additionally, for a self-adjoint operators,

$$T = T^{**} \subset T^{*}$$

is obtained. Thenceforth, a closed symmetric operator $T$ is self-adjoint if and only if $T^{*}$ is symmetric.

**Theorem A.1.** *Let $T$ be a symmetric operator acting in Hilbert space $\mathscr{H}$ then followings are equivalent,*

(*i*) *$T$ is self-adjoint.*
(*ii*) *$T$ is closed and $Ker(T^{*} \pm i) = \{0\}$.*
(*iii*) *$Ran(T \pm i) = \mathscr{H}$.*

*Proof.* Assume that $T$ is self-adjoint. Then, $T$ is closed since $T$ is densely defined and symmetric. Besides, $T = T^{*}$ and $\pm i \neq \sigma(T)$. Thence, $Ker(T^{*} \pm i) = \{0\}$.

Suppose that $T$ is closed and $Ker(T^{*} \pm i) = \{0\}$. $Ran(T - i)$ is dense in $\mathscr{H}$. Whence, $T$ is symmetric then following property holds

$$\|(T - i)\phi\|^{2} = \|T\phi\|^{2} + \|\phi\|^{2}. \tag{A.6}$$

If $\phi_n$ is a sequence in $D(T)$ such that $(T + i)\phi_n$ converges to some $\psi_\infty$. By $(A.6)$, $\phi_n$ is a Cauchy sequence, then there exist $\phi_n$ such that

$$\phi_n \to \phi_\infty \in \mathscr{H}.$$

Yet, $T\phi_n = (T+i)\phi_n - i\phi_n$ is convergent and the fact that graph is closed. By this fact, $\phi_\infty \in D(T)$ is showed and $T\phi_\infty = \psi_\infty - i\phi_\infty$. Then,

$$Ran(T^* \pm i) = \{0\}.$$

Assume that $Ran(T \pm i) = \mathscr{H}$. Let $\phi \in D(T^*)$ and $\eta \in D(T)$. Accordingly,

$$(T-i)\eta = (T^*-i)\phi$$

because $T$ is symmetric and $(T^*-i)(\eta-\phi) = 0$. But if $(T+i)$ is surjective, then $(T^*-i)$ is injective and $\phi = \eta$. This concludes $\phi \in D(T)$. That is to say, $T$ is self-adjoint. □

**Definition A.7.** *Let $T$ be a closed operator in Hilbert space $\mathscr{H}$. A complex number $\sigma$ is in the resolvent set $\rho(T)$ if $\sigma I - T$ is a bijection of $D(T)$ onto $\mathscr{H}$ with a bounded inverse. If $\sigma \in \rho(T)$, $R_\sigma(T) = (\sigma I - T)^{-1}$ is called the resolvent of $T$ at $\sigma$. If $\sigma \neq \rho(T)$ then $\sigma$ is said to be in spectrum $\sigma(T)$ of $T$.*

Assume that $A$ is a symmetric operator. Let

$$\mathscr{K}_+ = Ker(i - A^*) = Ran(i + A)^\perp, \tag{A.7}$$

$$\mathscr{K}_- = Ker(i + A^*) = Ran(-i + A)^\perp. \tag{A.8}$$

$\mathscr{K}_+$ and $\mathscr{K}_-$ are called deficiency subspaces of A. The pair of numbers $n_+$, $n_-$ given by $n_+(A) = dim[\mathscr{K}_+]$, $n_-(A) = dim[\mathscr{K}_-]$ are called the deficiency indices of $A$. Suppose that $B$ is a closed extension of $A$. For $\varphi \in D(B^*)$, $(\psi, B^*\varphi) = (B\psi, \varphi) = (A\psi, \varphi)$ for

all $\psi \in D(A)$. Thence, $\varphi \in D(A^*)$ and $B^*\varphi = A^*\varphi$ such that

$$A \subseteq B \subseteq B^* \subseteq A^*. \tag{A.9}$$

Now, sesquilinear forms on $D(A^*)$ as follows

$$(\varphi, \psi)_A = (\varphi, \psi) + (A^*\varphi, A^*\psi) \tag{A.10}$$

$$[\varphi, \psi]_A = (A^*\varphi, \psi) - (\varphi, A^*\psi). \tag{A.11}$$

A subspace of $D(A^*)$ such that $[\varphi, \psi]_A = 0$ for all $\varphi$ and $\psi$ in the subspace is called as A-symmetric. Subspace of $D(A^*)$ as A-closed or A-orthogonal refers that $(\varphi, \psi)_A = 0$ for all $\varphi$ and $\psi$ in $D(A^*)$.

**Lemma A.1.** *Suppose that $A$ is closed symmetric operator,*

(*i*) *The closed extensions of $A$ are the restrictions of $A^*$ to $A$-closed, $A$- symmetric subspace.*

(*ii*) *$D(A)$, $\mathscr{K}_+$ and $\mathscr{K}_-$ are $A$-closed, mutually $A$-orthogonal subspaces of $D(A^*)$ and*

$$D(A^*) = D(A) \oplus_A \mathscr{K}_+ \oplus_A \mathscr{K}_-. \tag{A.12}$$

(*iii*) *There is an one to one correspondence between $S$ which is $A$-closed and $A$-symmetric subspace of $D(A^*)$ that contains $D(A)$ and $S'$ that is $A$-closed and $A$-symmetric subspace of $\mathscr{K}_+ \oplus_A \mathscr{K}_-$ given by $S = D(A) \oplus_A S'$.*

*Proof.* From (A.9), every symmetric extension of $A$ is in domain of $A^*$. Besides, its domain $A$-closed and the extension is symmetric if and only if the extension is closed if and only if its domain is A-symmetric.

Notice that $A$ is closed, $\mathscr{K}_+$ and $\mathscr{K}_-$ are A-closed. Thereupon, $D(A)$ is $A$-closed in the weaker topology provided by the usual inner product. On that account, three subspaces are orthogonal.
Let $\psi \in D(A^*)$ such that $\psi \perp_A D(A) \oplus_A \mathscr{K}_+ \oplus \mathscr{K}_-$. For $\varphi \in D(A)$,

$$\begin{aligned}(\varphi, \psi) + (A^*\varphi, A^*\psi) &= (\varphi, \psi)_A \\ &= 0.\end{aligned}$$

Thus,

$$(\varphi, \psi) = -(A\varphi, A^*\psi). \tag{A.13}$$

$(A.13)$ implies that $A^*\psi \in D(A^*)$ and $A^*A^*\psi = -\psi$ since $(A^* + i)(A^* - i)\psi = (A^*A^* + I)\psi = 0$. $A^* - i\psi \in \mathscr{K}_-$. But if $\varphi \in \mathscr{K}_-$,

$$\begin{aligned}i(\varphi, (A^* - i)\psi) &= (\varphi, \psi) + (A^*\varphi, A^*\psi) \\ &= (\varphi, \psi)_A \\ &= 0.\end{aligned}$$

$\psi \perp_A \mathscr{K}_-$ leads to $(A^* - i)\psi = 0$. This means that $\psi \in \mathscr{K}_+$. Thus, $\psi \perp_A \mathscr{K}_+$ provides that $\varphi = 0$.

Let $S'$ be $A$-closed and $A$-symmetric subspace of $\mathscr{K}_+ \oplus \mathscr{K}_-$. Assume that $\varphi = \varphi_0 + \varphi_1$ and $\psi = \psi_0 + \psi_1$ together with $\varphi_0, \psi_0 \in D(A)$ as well as $\varphi_1, \psi_1 \in S'$. $A$ is symmetric

with $[\psi, \varphi] = 0$ along with that $S'$ is $A$-symmetric therefore $[\psi_0, \varphi_0] = 0$. Furthermore,

$$\begin{aligned} [\varphi_0, \psi_1]_A &= (A^*\varphi_0, \psi_1) - (\varphi_0, A^*\psi_1) && \text{(A.14)} \\ &= (A\varphi_0, \psi_1) - (\varphi_0, A^*\psi_1) && \text{(A.15)} \\ &= 0, \end{aligned}$$

for $\varphi_0 \in D(A)$ as well as $\psi_1 \in D(A^*)$.

$$\begin{aligned} [\varphi_1, \psi_0]_A &= (A^*\varphi_1, \psi_0) - (\varphi_1, A^*\psi_0) && \text{(A.16)} \\ &= (A\varphi_1, \psi_0) - (\varphi_1, A^*\psi_0) && \text{(A.17)} \\ &= 0, \end{aligned}$$

as long as $\varphi_1 \in D(A)$ and $\psi_0 \in D(A^*)$. It follows that

$$[\varphi, \psi]_A = [\varphi_0, \psi_0]_A + [\varphi_1, \psi_0]_A + [\varphi_0, \psi_1]_A + [\varphi_1, \psi_1]_A = 0.$$

Therefor $S = D(A) \oplus_A S'$ is a symmetric subspace. $D(A)$ and $S'$ are $A$-closed as well as $A$-orthogonal. This derives that $S$ is A-closed since. Conversely, suppose that $S'$ is an $A$-closed, $A$-symmetric subspace of $D(A^*)$ containing $D(A)$.

Let $S' = \mathscr{K}_+ \oplus_A \mathscr{K}_-$. Then $S'$ is A-closed and A-symmetric. Suppose $\varphi \in S$, accordingly $\varphi$ can be expressed as $\varphi = \varphi_0 + \varphi_1$ uniquely where $\varphi_0 \in D(A)$ and $\varphi_1 \in \mathscr{K}_+ \oplus_A \mathscr{K}_-$. It is the fact that $D(A) \subset S$ so as $\varphi_0 \in S$ implying $\varphi_1 \in S$ in conjunction with $S = D(A) \oplus_A S'$. □

**Theorem A.2.** *Let $A$ be a closed symmetric operator. The closed symmetric extensions of $A$ are in one to one correspondence with the set of partial isometries (in the usual inner product) of $\mathscr{K}_+$ into $\mathscr{K}_-$. If $U$ is such an isometry with initial space*

*$I(U) \subseteq \mathscr{K}_+$, then the corresponding closed symmetric extension $A_\nu$ has domain*

$$D(A_\nu) = \{\varphi + \varphi_+ + U\varphi_+ \mid \varphi \in D(A),\ \varphi_+ \in I(U)\} \tag{A.18}$$

*and*

$$A_u(\varphi + \varphi_+ + U\varphi_+) = A\varphi + i\varphi_+ - i\varphi_+. \tag{A.19}$$

*If dim $I(U)$ is finite, the deficiency indices of $A_\nu$ are $n_\pm(A_\nu) = n_\pm(A) - dim\ [I(U)]$ [48].*

*Proof.* Let $A'$ be a closed extension of $A$. Via the lemma A.1, $D(A') = D(A) \oplus_A S'$ where $S'$ is an $A$-closed, $A$-symmetric subspace of $\mathscr{K}_+ \oplus \mathscr{K}_-$ If $\varphi \in S'$, it can be expressed uniquely $\varphi = \varphi_+\varphi_-$. $S'$ is $A$-symmetric then

$$\begin{aligned} 0 &= (A^*\varphi, \varphi) - (\varphi, A^*\varphi) \\ &= 2i(\varphi_-, \varphi_-) - 2i(\varphi_+, \varphi_+). \end{aligned}$$

This implies that

$$\|\varphi_+\|^2 = \|\varphi_-\|^2. \tag{A.20}$$

Then, $\varphi_+ \mapsto \varphi_-$ is a well-defined isometry from a subspace of $\mathscr{K}_+$ into $\mathscr{K}_-$ since $S'$ is a subspace of $\mathscr{K}_+ \oplus_A +\mathscr{K}_-$ and (A.20) says the corresponding partial isometry $U$. Thence,

$$D(A') = \{\varphi + \varphi_+ + U\varphi_+ \mid \varphi_+ \in I(U)\} \tag{A.21}$$

and

$$A'(\varphi + \varphi_+ + U\varphi_+) = A^*(\varphi + \varphi_+ + U\varphi_+) = A\varphi + i\varphi_+ - iU\varphi_+. \tag{A.22}$$

Conversely, assume that $U$ is an isometry from a subspace of $\mathscr{K}_+$ into $\mathscr{K}_-$. Define $D(A')$ and $A'$ via (A.21) and (A.22). Then $D(A')$ is $A$-closed, $A$-symmetric subspace of $D(A^*)$ thus via lemma A.1, $A'$ is a closed symmetric extension of $A$.

$$n_\pm(A_\nu) = n_\pm(A) - dim\ [I(U)] \tag{A.23}$$

is provided by checking ranges of $i + A'$ and $i - A'$ on $D(A')$ if $dim\ I(U) <$ is finite. □

**Corollary A.1.** *Let $A$ be a closed symmetric operator with deficiency indices $n_+$ and $n_-$. Then,*

*a) $A$ is self-adjoint if and only if $n_+ = 0 = n_-$.*

*b) $A$ has self-adjoint if and only if $n_+ = n_-$. There is one to one correspondence between self-adjoint extension of $A$ and unitary maps $\mathscr{K}_+$ and $\mathscr{K}_-$.*

*c) If either $n_+ = 0 \neq n_-$ or $n_- = 0 \neq n_+$, then $A$ has non-trivial symmetric extensions. (such operators called maximal symmetric).*

**Definition A.8.** *An antilinear map such that $C(\alpha\varphi + \beta\psi) = \bar{\alpha}C\varphi + \bar{\beta}C\psi$ is called a conjugation map if it is norm-preserving and $C^2 = I$.*

**Theorem A.3** (Von Neumann's Theorem)**.** *Let $A$ be a symmetric operator and suppose there exists a conjugation map $C : D(A) \to D(A)$ and $AC = CA$. Then, $A$ has the equal deficiency and therefore, has self-adjoint extensions [49].*

*Proof.* Via $C^2 = I$ and $CD(A) \subseteq D(A)$, $CD(A) = D(A)$ is obtained.

Assume that $\varphi_+ \in \mathscr{K}_+$ and $\psi D(A)$. Then,

$$\begin{aligned} 0 = (\overline{\varphi_+, (A+i)\psi}) &= (C\varphi_+, C(A+i)\psi) \\ &= (C\varphi_+, (A-i)C\psi). \end{aligned}$$

$C$ maps from $D(A)$ onto $D(A)$, $C\varphi_+ \in \mathscr{K}_-$. Hence, $C : \mathscr{K}_+ \to \mathscr{K}_-$. Suppose that $\varphi_-$ is element of $\mathscr{K}_-$ and $D(A)$ contains $\psi$. Therewith,

$$\begin{aligned} 0 = (\overline{\varphi_-, (A-i)\psi}) &= (C\varphi_-, C(A-i)\psi) \\ &= (C\varphi_-, (A+i)C\psi). \end{aligned}$$

Similarly, $C$ takes $D(A)$ onto $D(A)$. $C\varphi_- \in \mathscr{K}_+$, so $C : \mathscr{K}_- \to \mathscr{K}_+$.
Since $C$ preserves norm, $dim\ [\mathscr{K}_+] = dim\ [\mathscr{K}_-]$. □

# APPENDIX B: LAPLACIAN ON RIEMANNIAN MANIFOLD

**Definition B.1.** *Let $\mathcal{M}$ be a smooth $N$-dimensional manifold. A map $\eta : C^{\infty}(\mathcal{M}) \to \mathbb{R}$ at a point $x_p \in \mathcal{M}$ is called $\mathbb{R}$-differentiation if*
*(i) $\eta$ is linear.*
*(ii) $\eta$ satisfies the Leibnez rule such that*

$$\eta(fg) = \eta(f)g(x_p) + \eta(g)f(x_p) \text{ for all } f, g \in C^{\infty}(\mathcal{M}) \tag{B.1}$$

**Definition B.2.** *Let $\mathcal{M}$ be a smooth manifold with $N$-dimensional. The tangent space of $\mathcal{M}$ at $x_p \in \mathcal{M}$ which is denoted by $T_p(M)$ is the set of all $R$-differentiations at $x_p$. Also, every element of $T_p(\mathcal{M})$ is called tangent vector at $x_p$.*

Suppose that $\mathcal{M}$ is a smooth manifold with $N$-dimension. Thus, $T_p(\mathcal{M})$ is a linear $N$-dimensional vector space [97]. Then, each vector can be expressed by local coordinates $x^1, x^2, \cdots, x^N$ in chart $U$ located in neighbourhood at $x_p$. Any vector $\eta \in T_p(\mathcal{M})$ of the form

$$\eta = \eta^i \frac{\partial}{\partial x^i}. \tag{B.2}$$

**Definition B.3.** *The directional derivative of the smooth function $f$ along the direction $\eta$ is given by*

$$\eta(f) = \eta^i \frac{\partial f}{\partial x^i}. \tag{B.3}$$

**Definition B.4.** *For any $x \in \mathcal{M}$, a vector field on smooth manifold is family of tangent vectors $\{X(x)\}_{x\in\mathcal{M}}$ such that $\{X\} \in T_x(\mathcal{M})$.*

$$X(x) = X^i(x)\frac{\partial}{\partial x^i}$$

*where $x^1, \cdots, x^N$ are local coordinates. $X(x)$ is called smooth if every function $X^i(x)$ is smooth on any chart.*

**Definition B.5.** *Assume fixed $x_p \in M$ and $f \in C^\infty(M)$ in neighbourhood of $x_p$. The linear functional, $df$ at $x_p$ in $T_{x_p}(\mathcal{M})$ is provided by*

$$\langle df, \eta\rangle = \eta(f) \tag{B.4}$$

*for any $\eta \in T_{x_p}(\mathcal{M})$ where the pairs of a linear functional on $T_{x_p}(\mathcal{M})$ and a vector from $T_p(\mathcal{M})$ is denoted by $\langle\cdot,\cdot\rangle$.*

**Definition B.6.** *The dual space of $T_{x_p}(\mathcal{M})$, so called the cotangent space is denoted by $T^*_{x_p}(\mathcal{M})$. Each element of $T^*_{x_p}(\mathcal{M})$ is called a covector. Additionaly, $df$ $(B.4)$ is a element of the cotangent space $T^*_{x_p}(\mathcal{M})$.*

Every elements of $T^*_{x_p}(\mathcal{M})$ can be expressed as $\{dx^i\}_{i=0}^N$ corresponding to $\left\{\dfrac{\partial}{\partial x^i}\right\}_{i=0}^N$ basis of $T_{x_p}(\mathcal{M})$ Then, $df$ can be represented as

$$df = \frac{\partial f}{\partial x^i}dx^i \tag{B.5}$$

provided that $\langle e^i, e_j\rangle = \delta^i_j$ where $\{e^1, \cdots, e^N\}$ is any basis of $T_x(\mathcal{M})$ and $e_1, \cdots, e_N$ is basis of $T^*_x(\mathcal{M})$.

Besides, $\delta^i_j$ is a Kronecker delta which means if $i = j$ then $\delta^i_j = 1$. Otherwise, $\delta^i_j = 0$.

**Definition B.7.** *Let $\mathcal{M}$ be a smooth manifold with $N$ dimension. $\{g(x)\}_{x\in\mathcal{M}}$ is a family such that each family members $g(x)$ is a symmetric, positive definite and bilinear form on the tangent space $T_x(\mathcal{M})$.*

As a consequence, a new pairing $\langle\cdot,\cdot\rangle_g$, can be defined of the form

$$\langle \varsigma, \eta\rangle_g = g_{ij}\varsigma^i\eta^j$$

in local coordinates where $\eta, \varsigma \in T_x(\mathcal{M})$. $\{g(x)\}_{x\in\mathcal{M}}$ is called a Riemannian metric. Likewise, a smooth manifold $\mathcal{M}$ equipped with a Riemannian metric $g$ is called a Riemannian manifold.

**Definition B.8.** *Let $\mathcal{M}$ be a Riemannian manifold equipped with metric $g$ and $f$ be a smooth function in $\mathcal{M}$, then*

$$\langle \nabla_g f, \eta\rangle_g = \eta(f) \quad \textit{for all} \quad \eta \in T_p(\mathcal{M}) \tag{B.6}$$

*defines the gradient of the function $f$.*

In local coordinates,

$$(\nabla_g f)^i = g^{ij}\frac{\partial f}{\partial x^i}. \tag{B.7}$$

**Definition B.9.** *Let $\mathcal{M}$ be $N$-dimensional Riemannian manifold with metric $g$. The volume form of an oriented Riemannian manifold*

$$d(vol) = d_g x\sqrt{det\ g} = dx^1 \wedge \cdots \wedge dx^N \sqrt{det\ g} \tag{B.8}$$

*where $g = (g_{ij})$ and $\wedge$ denotes the wedge product [1].*

**Definition B.10.** *The volume of finite dimensional compact Riemannian manifold $\mathcal{M}$ is defined by*

$$V(\mathcal{M}) = \int_{\mathcal{M}} d_g x \tag{B.9}$$

**Definition B.11.** *The divergence of smooth vector field $X(x)$ denoted by* $\mathrm{div}X$ *on a Riemannian manifold $M$ is a unique smooth function provided that*

$$\int_{\mathcal{M}} f(\mathrm{div}X) d_g x = -\int_{\mathcal{M}} \langle X, \mathrm{div} f\rangle d_g x \tag{B.10}$$

*holds for any $f \in C_0^\infty(\mathcal{M})$. $f \in C_0^\infty(\mathcal{M})$ is a subspace of $C^\infty(\mathcal{M})$ containing functions that have compact support.*

The divergence in local coordinates on a Riemannian manifold can be obtained as follows

$$\mathrm{div}X = \frac{1}{det\ g}\partial_i\big(g^{ij}\sqrt{det\ g}\ X^i\big) \tag{B.11}$$

where $\partial_i = \frac{\partial}{\partial x^i}$ for simplicity.

**Definition B.12.** *The divergence of the gradient of a smooth function on any Riemannian manifold $M$ is called the Laplacian operator $\Delta_g$ such that*

$$\Delta_g\ f = -\mathrm{div}(\nabla_g\ f). \tag{B.12}$$

*Also, $\Delta_g f$ is a smooth function in $\mathcal{M}$.*

In local coordinate $\Delta_g\ f$ in $\mathcal{M}$ can be seen as

$$\Delta_g\ f = -\frac{1}{\sqrt{det\ g}} \sum_{i,j=1}^{N} \partial_i\big(g^{ij}\sqrt{det\ g}\ \partial_j\big). \tag{B.13}$$

**Definition B.13.** *Let $\mathcal{M}$ be a Riemannian manifold equipped with metric $g$. The space of square integrable functions in $\mathcal{M}$, $L^2(\mathcal{M})$ is defined as*

$$L^2(\mathcal{M}) = \Big\{f : \int_{\mathcal{M}} |f|^2 d_g x < \infty\Big\}. \tag{B.14}$$

Inner product of $L^2(\mathcal{M})$ is defined as follows

$$\langle\psi_1(x), \psi_2(x)\rangle_{L^2(\mathcal{M})} = \int_{\mathcal{M}} \psi_1(x)\overline{\psi_2(x)} d_g x. \tag{B.15}$$

Let $\mathcal{M}$ be a compact Riemannian manifold without boundary equipped with a metric $g$. The closed eigenvalue problem $\mathcal{M}$ is defined by

$$\Delta_g u(x) = \sigma u(x) \tag{B.16}$$

where $u(x) \in C^2(\mathcal{M})$ and $\sigma \in \mathbb{R}$. The eigenvalue problem investigates the range of numbers $\sigma$ for which a nontrivial solution $u \in C^2(\mathcal{M})$ exists.

**Theorem B.1** (Spectral Theorem [128])**.** *Let $\mathcal{M}$ be a compact Riemannian manifold equipped with metric $g$. Then, there is a complete orthonormal basis of smooth eigenfunctions $\{\phi_l\}_{l=0}^{\infty}$ of $\Delta_g$ in $L^2(\mathcal{M})$. Hence,*

$$\sigma_0 < \sigma_1 < \cdots < \sigma_l \to \infty \quad as \quad l \to \infty. \tag{B.17}$$

*First eigenvalue is $\sigma_0 = 0$. This leads to constant eigenfunction*

$$\phi_0 = \frac{1}{\sqrt{V(\mathcal{M})}}. \tag{B.18}$$

**Corollary B.1.** *Let $\mathcal{M}$ be a compact Riemannian manifold equipped with metric $g$. $f(\Delta_g)$ can be defined where $f$ is continuous a function on spec$(\Delta_g)$. It turns out the important operator $\exp(-t\Delta_g)$ where $t$ is real parameter. In the basis $\{\phi_k\}$, it has the matrix representation*

$$\exp(-t\Delta_g) = diag(e^{-t\sigma_1}, e^{-t\sigma_2}, \cdots, e^{-t\sigma_k}, \cdots). \tag{B.19}$$

*Thus, if $t \geq 0$ then $\exp(-t\Delta_g)$ is a bounded self-adjoint operator in $L^2(\mathcal{M})$.*

**Proposition B.1.** *Let $M$ be Riemannian Manifold with equipped metric $g$. $\Delta_g$ is nonnegative and symmetric in $L^2(\mathcal{M})$.*

*Proof.* Let $\varphi \in L^2(\mathcal{M})$. Consider the inner product of $\varphi$ and $\Delta_g\varphi$.

$$\begin{aligned}\langle \varphi(x), \Delta_g\varphi(x)\rangle_{L^2(\mathcal{M})} &= \int_{\mathcal{M}} \varphi(x)\overline{\Delta_g\varphi(x)}d_gx \\ &= \int_{\mathcal{M}} \varphi(x)\mathrm{div}\overline{\nabla_g\,\varphi(x)}d_gx \\ &= \int_{\mathcal{M}} \nabla_g|\varphi(x)|^2 d_gx > 0, \end{aligned} \tag{B.20}$$

Thus $\Delta_g$ is nonnegative. Suppose that $\eta \in L^2(\mathcal{M})$ and examine the inner product of $\eta$ and $\Delta_g\varphi$,

$$\begin{aligned}\langle \Delta_g\eta(x), \varphi(x)\rangle &= \int_{\mathcal{M}} \Delta_g\eta(x)\overline{\varphi(x)}d_gx \\ &= \sum_{j=1}^{N}\int_{\mathcal{M}} \partial_{x_j}^2\eta(x)\overline{\varphi(x)}d_gx \\ &= \sum_{j=1}^{N}\int_{\mathcal{M}} \eta(x)\overline{\partial_{x_j}\varphi(x)}d_gx \\ &= \langle \eta, \Delta_g\varphi(x)\rangle_{L^2(\mathcal{M})}. \end{aligned} \tag{B.21}$$

This concludes that $\Delta_g$ is symmetric. □

Actually more is true, $\Delta_g$ is an unbounded self-adjoint operator. The spectral theorem provides an expansion of any function $\varphi(x) \in L^2(\mathcal{M})$ with eigenfunction basis.

$$\psi(x) = \sum_{l=0}^{\infty} C_l \phi_l(x) \tag{B.22}$$

where $C_l$ is Fourier expansion coefficient.
On compact Riemannian manifold, the orthogonality means that

$$\int_{\mathcal{M}} \phi_k(x)\phi_l(x) d_g x = \delta_k^l. \tag{B.23}$$

The completeness property on compact manifold is the following

$$\sum_{l=0}^{\infty} \phi_l(x)\phi_l(a_i) = \delta_g(x, a_i). \tag{B.24}$$

This means $\phi_l$ functions are smooth on Riemannian manifold $\mathcal{M}$.

Let $\mathcal{M}$ be a $N$ - dimensional Riemannian manifold equipped with metric $g$. $N$-dimensional Dirac-delta function at point $a_i \in \mathcal{M}$, $\delta(x, a_i)$ can be defined in terms of a well behaved test function $f(x)$ as follows

$$\int_{\mathcal{M}} \delta_g(x, a_i) f(x) d_g x = f(a_i). \tag{B.25}$$

# APPENDIX C: THE HEAT KERNEL ON RIEMANNIAN MANIFOLD

**Definition C.1.** *Let $\mathcal{M}$ be a $N$-dimensional Riemannian manifold. The parabolic differential heat operator in $\mathcal{M}$ can be defined as follows*

$$L = \Delta_g - \frac{\partial}{\partial t}. \tag{C.1}$$

The importance of the heat operator $L$ is generating the heat equation when $Lu = 0$ where $u$ is defined in $\mathcal{M}$ as a function of $x$ and $t$.

**Definition C.2.** *Let $\mathcal{M}$ be a Riemannian manifold. The heat equation can be defined in $\mathcal{M}$ as follows*

$$\Delta_g u - \frac{\partial u}{\partial t} = 0. \tag{C.2}$$

The heat equation describes the conduction of heat through a given region over time. Thus, on a Riemannian manifold $\mathcal{M}$, $u$ can be considered as the temperature at the point $x \in \mathcal{M}$ at time $t$.

The heat equation on Riemannian manifold supplemented with initial condition

$$\begin{aligned} \Delta_g \, u &= \frac{\partial u}{\partial t} \\ u(x,0) &= f(x) \end{aligned} \tag{C.3}$$

is called the Cauchy problem. The equation with initial condition above explains the conduction of heat through the medium without source or sink of heat, if the initial temperature everywhere is specified.

**Definition C.3.** *Let $\mathcal{M}$ be a Riemannian manifold. A fundemental solution of heat equation is called heat kernel and it is denoted by $K_t(x, y)$.*

$K_t(x, y)$ is a continuously defined on $(0, \infty) \times \mathcal{M} \times \mathcal{M}$ and $f \in L^2(\mathcal{M})$ that satisfies following

$$\begin{aligned} \Delta_g K_t(x, y) - \frac{\partial}{\partial t} K_t(x, y) &= 0 \\ \lim_{t \to 0^+} K_t(\cdot, y) &= \delta_g(\cdot, y). \end{aligned} \tag{C.4}$$

Besides, for all bounded functions $f \in L^2(\mathcal{M})$,

$$\lim_{t \to 0^+} \int_{\mathcal{M}} K_t(x, y) f(x) d_g x = f(y)$$

for all $y \in \mathcal{M}$.

Thus, if $u(x, 0) = f(x)$ then the solution of Cauchy problem is provided by

$$u(x, y) = \int_{\mathcal{M}} K_t(x, y) f(y) d_g y. \tag{C.5}$$

Another expression of Cauchy problem is solution given by

$$u(x,t) = \exp(-\Delta_g t) f(x). \tag{C.6}$$

The function $f \in L^2(\mathcal{M})$ has an expansion in basis $\{\phi_k\}$ gruanteed by the Spectral Theorem $B.1$,

$$f(x) = \sum_{k=0}^{\infty} a_k \phi_k(x)$$

where $a_k = \int_{\mathcal{M}} f(y)\phi_k(y) d_g y$. Then via (B.19),

$$\begin{aligned}
\exp(-\Delta_g t) f(x) &= \sum_{k=0}^{\infty} a_k \phi_k(x) \\
&= \sum_{k=0}^{\infty} \int_{\mathcal{M}} \exp(-\Delta_g t) f(y) \phi_k(x) \phi_k(y) d_g y \\
&= \int_{\mathcal{M}} \sum_{k=0}^{\infty} \exp(-\Delta_g t) \phi_k(x) \phi_k(y) f(y) d_g y.
\end{aligned}$$

is derived. Henceforth,

$$K_t(x,y) = \sum_{k=0}^{\infty} \exp(-\Delta_g t) \phi_k(x) \phi_k(y) \tag{C.7}$$

$$= \sum_{k=0}^{\infty} \exp(-\sigma_k t) \phi_k(x) \phi_k(y). \tag{C.8}$$

Assume that $\mathcal{M}$ is a compact Riemannian manifold. Consider heat kernel with eigenfunction expansion

$$\begin{aligned} K_t(x,y) &= \sum_{l=0}^{\infty} \exp(-\sigma_l t)\phi_l(x)\phi_l(y) \\ &= \exp(-\sigma_0 t)\phi_0(x)\phi_0(y) + \sum_{l=1}^{\infty} \exp(-\sigma_l t)\phi_l(x)\phi_l(y). \end{aligned}$$

By the Theorem $B.1$, $\sigma_0 = 0$ and its eigenfunction is constant (B.18) $\phi_0 = \frac{1}{\sqrt{V(\mathcal{M})}}$.

$$\begin{aligned} K_t(x,y) &= \frac{1}{\sqrt{V(\mathcal{M})}}\frac{1}{\sqrt{V(\mathcal{M})}} + \sum_{l=1}^{\infty} \exp(-\sigma_l t)\phi_l(x)\phi_l(y) \\ &= \frac{1}{V(\mathcal{M})} + \sum_{l=1}^{\infty} \exp(-\sigma_l t)\phi_l(x)\phi_l(y) \end{aligned} \tag{C.9}$$

so that

$$K_t(x,y) = \frac{1}{V(\mathcal{M})} + \mathcal{O}(e^{-\sigma_l t}).$$

Equivalently,

$$K_t(x,y) = \frac{1}{V(\mathcal{M})} \quad \text{as} \quad t \to \infty. \tag{C.10}$$

**Proposition C.1** (Semi-group Property)**.** *Let $\mathcal{M}$ be a compact Riemannian manifold with Riemannian metric $g$. Then, the heat kernel $K_t(x,y)$ in $\mathcal{M}$ has semi-group property as follows*

$$\int_{\mathcal{M}} K_{t_1}(x,z)K_{t_2}(z,y) = K_{t_1+t_2}(x,y) \tag{C.11}$$

*for* $x, y, z \in \mathcal{M}$ *and* $t_1, t_2 > 0$.

*Proof.* Compactness of a Riemannian manifolds brings the following expansions

$$\begin{aligned}
\int_{\mathcal{M}} K_{t_1}(x,z)K_{t_2}(z,y) &= \sum_{l=0}^{\infty} \exp(-\sigma_l t_l)\phi_l(x)\phi_l(z) \sum_{k=0}^{\infty} \exp(-\sigma_k t_k)\phi_k(z)\phi_k(y) d_g z \\
&= \sum_{l=0}^{\infty} \exp(-\sigma_l(t_l+t_2))\phi_l(x))\phi_l(y) \int_{\mathcal{M}} \phi_l(z)\phi_k(z) d_g z \\
&= K_{t_1+t_2}(x,y) \int_{\mathcal{M}} \phi_l(z)\phi_k(z) d_g z \\
&= K_{t_1+t_2}(x,y).
\end{aligned}$$

□

**Proposition C.2** (Symmetry Property)**.** *Let* $\mathcal{M}$ *be a compact Riemannian manifold. Hence, the heat kernel* $K_t(x,y)$ *defined in* $\mathcal{M}$ *is symmetric*

$$K_t(x,y) = K_t(y,x) \tag{C.12}$$

*for all* $x, y \in \mathcal{M}$ *and* $t \geq 0$.

*Proof.* Eigenfunction expansion of $K_t(x, y)$ as follows

$$\begin{aligned} K_t(x,y) &= \sum_{n=0}^{\infty} \exp(-\sigma_n t)\phi_n(x)\phi_n(y) \\ &= \sum_{n=0}^{\infty} \exp(-\sigma_n t)\phi_n(y)\phi_n(x) \\ &= K_t(y,x). \end{aligned}$$

□

**Definition C.4.** *Let $\mathcal{M}$ be a compact Riemannian manifold equipped with metric $g$ and $\Omega$ be relatively compact region of $\mathcal{M}$. Parabolic boundary $\partial_p\Omega_T$ is defined by*

$$\partial_p\Omega_T = \partial\Omega_T \backslash \{(x,t) : \ t = T\} \tag{C.13}$$

*where $\Omega_T = (0,T) \times \Omega$ is subregion of $[0,\infty) \times \mathcal{M}$.*

**Proposition C.3** (Maximum Principle)**.** *Let $u(x,t) \in C^2(\Omega_T) \cap C(\overline{\Omega})$ be a solution of the heat equation in $\Omega_T$. Then*

$$\sup_{\Omega_T} u = \sup_{\partial_p\Omega_T} u. \tag{C.14}$$

**Proposition C.4** (Minimum Principle)**.** *Let $u(x,t) \in C^2(\Omega_T) \cap C(\overline{\Omega})$ be a solution of the heat equation in $\Omega_T$. Then*

$$\inf_{\Omega_T} u = \inf_{\partial_p\Omega_T} u. \tag{C.15}$$

**Proposition C.5** (Strong Maximum Principle)**.** *Let $u(x,t) \in C^2(\Omega_T) \cap C(\overline{\Omega})$ be a solution of the heat equation in $\Omega_T$ amd $\Omega$ be a connected region if there exist a point $(x_0, t_0) \in \Omega_T$ such that*

$$\max_{\Omega_T} u = \max_{\partial_p \Omega_T} u. \tag{C.16}$$

*Then $u$ is constant in $\overline{\Omega}_{t_0}$.*

**Proposition C.6** (Strong Minimum Principle)**.** *Let $u(x,t) \in C^2(\Omega_T) \cap C(\overline{\Omega})$ be a solution of the heat equation in $\Omega_T$ amd $\Omega$ be a connected. Then there exist a point $(x_0, t_0) \in \Omega_T$ such that*

$$\min_{\Omega_T} u = \min_{\partial_p \Omega_T} u. \tag{C.17}$$

*Thus, $u$ is constant in $\overline{\Omega}_{t_0}$.*

Note that proof of Maximum/Minimum Principle and their strong versions can be found in [97, 129].

**Proposition C.7** (Positivity Property)**.** *Let $\mathcal{M}$ be a compact Riemannian manifold with metric $g$. Then, the heat kernel $K_t(x,t)$ is positive in $\mathcal{M}$,*

$$K_t(x,y) > 0 \tag{C.18}$$

*for all $x, y \in \mathcal{M}$ and $t > 0$.*

*Proof.* Assume that $f$ is nonnegative satisfying $(C.3)$. Then the minimum principle $(C.15)$ implies that fundamental solution $u(x,t)$ is nonnegative. Smoothness of $\partial\Omega$ implies that $u = e^{t\Delta_\Omega} f$ is a solution of Cauchy problem. Thence,

$$e^{t\Delta_\Omega} f \geq 0, \quad \forall f \in C_0^\infty(\Omega), \text{ for } f > 0.$$

Consequently, applying the strong minimum principle $(C.16)$. $e^{t\Delta_g f}$ is nonnegative in $[0,\infty) \times \mathcal{M}$. On that account, $K_t(x,y) > 0$ for all $x, y \in \mathcal{M}$ and $t > 0$. □

**Lemma C.1.** *Let $\mathcal{M}$ be a compact Riemannian manifolds. Therein, the heat kernel $K_t(x,y)$ defined in $\mathcal{M}$ has the following integral identity,*

$$\int_{\mathcal{M}} K_t(x,y) d_g x \leq 1 \tag{C.19}$$

*for all $x, y \in \mathcal{M}$.*

*Proof.* Assume that $f = 1$ then $u = e^{t\Delta_\Omega}$ solves the Cauchy problem then $\sup_{\Omega_T} u \leq 1$. Via maximum principle $\sup_{\Omega_T} u \leq 1$. This means $u \leq 1$ in $\mathcal{M}$. Thus,

$$e^{t\Delta_\Omega} \leq 1. \tag{C.20}$$

$(C.20)$ and $(C.5)$ together with strong maximum principle implies that

$$\int_{\mathcal{M}} K_t(x,y) d_g x \leq 1.$$

□

Another beneficial and significant matter hereof heat kernel is its relation with the resolvent. Firstly realized by Fock [130]. This relationship is useful to resolvent kernel as an integral over time of the heat kernel. To show the connection between heat kernel and resolvent kernel, the following operator identity is an essential step.

**Proposition C.8.** *Let $\mathcal{M}$ be a compact Riemannian manifold equipped with metric $g$ and $\Delta_g$ a Laplacian defined in $\mathcal{M}$. Then, the following holds for $\Re(\sigma) < 0$.*

$$\frac{1}{\Delta_g - \sigma} = \int_0^\infty e^{-t(\Delta_g - \sigma)} dt \tag{C.21}$$

This identity $(C.21)$ is proved rigorously via semigroup theory technique. This argument can be found in Kato's book [55]. This result can be extended analytically to the largest set in the entire complex plane $E$. Then $(C.21)$ becomes

$$(\Delta_g - E)^{-1} = \int_0^\infty e^{-t(\Delta_g - E)} dt. \tag{C.22}$$

To sum up, the integral representation of the free resolvent kernel is in terms of the heat kernel as follows

$$R_0(x, y|E) = \langle x|(\Delta_g - E)^{-1}|y\rangle = \int_0^\infty e^{Et} K_t(x, y) dt. \tag{C.23}$$