INVESTIGATION OF PRESERVICE MATHEMATICS TEACHERS' ALGEBRAIC THINKING THROUGH TRANSLATIONS AMONG MULTIPLE REPRESENTATIONS

by

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ABSTRACT

INVESTIGATION OF PRESERVICE MATHEMATICS TEACHERS' ALGEBRAIC THINKING THROUGH TRANSLATIONS AMONG MULTIPLE REPRESENTATIONS

The objective of this explanatory study was to gain insight into thinking process of preservice secondary school mathematics teachers as they approached problems that required for translations from graphical representations to model representation. Previous research has indicated preservice teachers' weakness on transition of multiple representations as well as their tendency to use symbolic demonstrations and procedures. Algebraic thinking is a cohesive thread of exploring mathematical relationships, formulating generalizations and reasoning and connecting of multiple representations. In this aspect, this study strongly emphasized preservice teachers' patterns of behavior that elicits algebraic thinking while translation process are required between graphical representation and model representation. For this purpose, explanatory research design (Creswell, 2008) was implemented. Semi-structured and video-recorded interviews with four preservice teachers who recruited according to their relatively different positions in the study constituted the main data sources of this study. The findings demonstrated preservice teachers' relatively weak conceptual understanding on the concept of functions and difficulties with coordinating and interpreting dependent and independent variables. Besides, some participating teachers elicited their complexity between perception of the shape of the graph and attributes of the graph, which pointed their limited knowledge on mathematical model representations. These results may indicate importance of addressing translations among multiple representations in functions during preservice teacher education.

ÖZET

ADAY MATEMATİK ÖĞRETMENLERİNİN CEBİRSEL DÜŞÜNME YETİLERİNİN FONKSİYONLARDA ÇOKLU TEMSİLLERİ KULLANABİLME BECERİLERİ ÜZERİNDEN İNCELENMESİ

Bu açıklayıcı araştırmanın amacı, son sınıf matematik öğretmen adaylarının grafik gösteriminden model gösterimine geçerken geliştirdikleri düşünce süreçlerini incelemektir. Yapılan çalışmalar, öğretmen adaylarının çoklu temsil geçişlerindeki yaşadıkları güçlükleri gösterirken cebirsel gösterimi kullanmaya yönelik yatkınlıklarını da ortaya çıkarmaktadır. Cebirsel düşünme; matematiksel ilişkileri keşfetme, genellemeleri formüle etme, akıl yürütme ve çoklu temsiller yardımıyla düşüncelerini ifade etmeyi birbirine bağlayan bir matematiksel düşünce sistemini ifade eder. Bu bağlamda bu araştırma grafik gösterimi ile model gösterimi arasında geçiş yapan öğretmen adaylarının düşünce biçimlerine odaklanmaktadır. Bu amaçla açıklayıcı (Creswell, 2008) araştırma deseni uygulanmıştır. Çalışmanın amacına yönelik olarak seçilen dört öğretmen adayı ile yapılan yarı yapılandırılmış ve video kaydına alınmış görüşmeler araştırmanın temel veri kaynağını oluşturmaktadır. Araştırma sonucu öğretmen adaylarının fonksiyon kavramı ile ilgili zayıf kavramsal bilgilerini ortaya çıkarırken, bağımlı ve bağımsız değişkende oluşan değişimleri koordine etmekte ve yorumlamakta zorlandıklarını göster-miştir. Ayrıca, bazı öğretmen adaylarının model gösterimine geçerken grafiğin şekli ile grafiğin davranışı arasında yaşadığı karmaşa, matematiksel model gösterimi ile ilgili kısıtlı bilgilerine işaret etmektedir. Bu araştırmanın sonuçları; öğretmen eğitiminde çoklu temsiller arası geçiş kurabilme becerisi kazandırmanın önemini göstermektedir.

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LIST OF ACRONYMS/ABBREVIATIONS

CCSSI	Common Core State Standards Initiative
MONE	Ministry of National Education in Turkey
NCTM	National Council of Teachers of Mathematics
MRFQ	Multiple Representation of Functions Questionnaire
NRC	National Research Council
PCK	Pedagogical Content Knowledge
SSPC	Student Selection Placement Center in Turkey
TIMMS	Trends in International Mathematics and Science Study
TTKB	Board of Education in Turkey

1. INTRODUCTION

"If there is a heaven on for school subjects, algebra will never go there. It is the one subject in the curriculum that has kept children from finishing high school, from developing their special interests and from enjoying much of their home study work. It has caused more family rows, more tears, more heartaches, and more sleepless night than any other subjects." (Kilpatrick and Izsak, 2008; p.3)

Algebra has always been a fundamental subject matter in mathematics curricula. Considering the recognition of algebra as gatekeeper for further mathematical study as well as academic passport (Schoenfeld, 1995) for future educational opportunities and promising career paths (Kaput, 1998; Moses and Cobb, 2001; National Research Council, 1998), algebra has become one of the major components of mathematics education all around the world (Leung *et al.*, 2014). Algebra standards proposed by National Council of Teachers of Mathematics (NCTM, 2000) outlines understanding patterns, relations, and functions as well as representing and analysing mathematical situations and structures using algebraic symbols. Though seen similar to algebra, algebraic thinking is a more comprehensive concept (Gürbüz and Şahin, 2015), that encompasses many different types of thought compromising of exploration of mathematical patterns, generalization of mathematical ideas and argumentation about those generalized idea. Algebraic thinking compromises of competency of using symbolic manipulations, processing algebraic relationships, designing multiple embodiments of a single concept and, formulating generalizations (Kaput and Blanton, 1999).

The algebra standards proposed by National Council of Teachers of Mathematics (NCTM, 2000) has emphasized ways of representing mathematical relationships in multiple ways and stated detailed expectations across grades. Elementary and middle school students throughout grade 8 should be able to "use various representations such as graphs, tables and equations to draw conclusions" (p.158). In grades 9-12, students are required to "identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships, use symbolic

expressions, including iterative recursive forms, to represent relationships arising from various contexts" (p. 296). In this respect, many researchers have concluded that representations of function concept in multiple ways, including graphical, tabular, symbolic and model to support students' understanding of algebra. Given the emphasis on use of various representations rather than predominant use of symbolic representations as a vehicle for algebraic thinking (Coulombe and Berenson, 2001), it is concluded that one way to help students promote their learning about algebra and algebraic thinking is teaching through multiple representations (Friedlander and Tabach, 2001).

In addition to applying multiple representations, the process of creating mathematical model representation is another fundamental components of mathematical education. Jacobson (2014) described the ability to create a model representation as requiring students to "use their understanding of arithmetic operations to make mathematical sense of problem situations and to relate this sense making to functions represented by equations, tables, and graphs" (p. 155). NCTM Standards (2010) for school mathematics emphasized that mathematical model representations encourage students learn how mathematics works in the real world and sense about what they have done during the mathematical process. Furthermore, investigating preservice teachers' mathematical model representations helps researchers to a better understanding of how one simplifies, constructs and works mathematically on a problem, which are key elements of algebraic reasoning (Deal, 2015).

The main purpose of this study is to investigate preservice teachers' repertoire of knowledge and skills on using multiple representations, particularly, transition from graphical representation to model representation in the domain of functions. Throughout the process of formation of modeling, preservice teachers' use of multiple representations to build mathematical ideas were used as the lens to investigate their algebraic thinking.

2. LITERATURE REVIEW

2.1. Algebra and Algebraic Thinking

The word "algebra" comes from the Latin translation of the book title "Hisab al-jabr w'al-muqabala", or "Calculation by Restoration and Reduction", written by Al-Khwarizmi, a Persian mathematician who lived in Baghdad around 780 to 850 DC (Puig and Rojano, 2004). This book was designed as a combination of mathematical ideas emerged from Babylonian, Indian and Greek as well as source of problems of inheritance and property division (Suggate, Davis and Goulding, 2006). In his book, Al- Khwarizmi defined "al-jabr" as eliminating with subtractions and proposed this method as a way for eliminating linear or quadratic equations into simpler form. This systematic approach to solving an equation has become the name of one of the field of mathematics (Drijvers, Goddijn and Kindt, 2010) and algebra has been considered as "a highly compact and efficient set of tools for solving problems (practical and theoretical)" (Suggate, Davis and Goulding, 2006, p. 127).

Throughout mathematics history, algebra was denoted as a system holding up procedures and notations as well as an instrument for manipulating symbols and solving problems. Considering modern algebra, since the mid-1980s, approach to algebra had diversified into several directions over time. It was seen as a collection of rules which; (1) transforms and solves equations, (2) puts word problems into equations and solving specific problems, (3) generalizes mathematical models, (4) introduces and uses the concepts of variable and function and (5) the study of algebraic structures (Bednarz, Kieran and Lee, 1996). Similarly, Bell (1996) interpreted algebra as a vehicle for expressing generalizations, relations, formulas as well as translating real world situations into mathematical problems by using equations with unknowns and handle them. Furthermore, Kaput (2008), pointed out that generalization of mathematical concepts (numbers, quantities, relations and functions) and transforming them into tangible demonstrations by using different kinds of mathematical reasoning lie down at the most fundamental level of algebra. Additionally, Usiskin (1995) described algebra as a

language including five major aspects; unknowns, formulas, generalized patterns, place holders and relationships. Further, Howe (2005) addressed the relationship between arithmetic patterns and algebraic generalities and asserted that algebra is a work concerning of variables and doing arithmetic operations with these variables in order to create new mathematical structures such as polynomials and rational equations to make models or represent abstract situations. In the light of discussion above encompassing different components of algebra, algebra can be defined as the language of mathematics (Usiskin, 1995) that deals with numbers, relations and processes (Carreher and Schlienann, 2007) as well as analyses of relationship between quantities (Kieran, 1996; Usiskin, 1999), derivation of mathematical models to represent abstract concept and generalizability of mathematical structures (Kaput, 2008; Mason, 1996; Usiskin, 1999).

Algebra as a way to explore, analyze and represent mathematical concept and ideas (NCTM 2000) shows that definition of algebra cannot be thought without considering doing algebra. Classification of what constitutes "doing algebra" has been discussed and different approaches to algebra are heightened in many publications (Kaput, 2008; Kieran, 2007; NCTM, 1997).It is stated that doing algebra becomes a process rather than a subject of mathematics. This process can be considered as making mathematical generalizations, manipulating mathematical structure and symbolizing mathematical relationships, and deriving meaning from different representations. For example, MacLane and Birkhoff (1999) stated that algebra as the art of manipulation on operations and numbers (as cited in Usiskin, 1999). The rules for these manipulations hold for all number, so the manipulations may be carried out with letters standing for the numbers. It then appears that the same rules even apply to things, which are not numbers at all, and this leads to generalized arguments for algebra.

According to NCTM (2000), "Algebra is a way of thinking and a set of concepts and skills that enable students to generalize, model, and analyse mathematical situations. Algebra provides a systematic way to investigate relationships; help to describe, organize and understand the world (p. 1)". When algebra is a collection of content to be made sense of mathematics, algebraic thinking, on the other hand, necessitates understanding of patterns, relations and functions, representing and analysing mathe-

matical situations by using algebraic symbols, using mathematical models to represent and understand quantitative relationships and analysing change in various contexts (NCTM, 2000). While it is thought that being able to use mathematical reasoning must lie at the heart of mathematical learning (Russell, 1999), algebraic thinking is one of the most essential form of mathematical reasoning with respect to constructing mathematical knowledge for understanding.

2.1.1. Algebraic Thinking

Algebra and algebraic thinking remain at the cornerstone of the mathematics curriculum and deep into the bedrock of the mathematics (Burke *et al.*, 2001). Since fitting the idea of algebraic thinking into a single conception oversimplifies its real value in mathematics, researchers have been focusing on different aspects of this phenomenon, notably, various definition and discussions are made in the spirit of respect for each different features of algebra.

Lins' (1992) framework on algebraic thinking focused on three different modes; thinking arithmetically, thinking internally and thinking analytically. First, the arithmeticity in algebraic thinking is originated from numbers and arithmetical operations. This mode of thinking promotes modeling in numbers, comprehensibly, proper use of arithmetical operations and numbers in order to represent problems via model representations. Second, the notion of internality feature of algebraic thinking is the "characteristics of algebraic thinking that arithmetical operations become objects and being used as tools" (p. 14). Finally, the analyticity of algebraic thinking enables learners to recognize a wide range of subjective analyses of objective data, and to evaluate how well each analysis might meet learners needs (Lins, 1992).

Accordingly, Sierpinska (1993) focused on the independency of algebra from particular numbers and arithmetical expressions. Similarly, Usiskin (1999) expressed algebra as language of generalizations since repeating a progress over and over again equip someone a simple language for explaining their work without using particular numbers, which is language of algebra. For example, it is noted that students who thinks

arithmetically, beholds the idea that 3 + 5 = 8 and this is the end of the story. For algebraically thinking students, 3 + 5 is a special case of x + y. When one started to concentrate on the structure of an equation/function/operations, rather than specific numbers in particular cases, it points developed algebraic thinking in those students. This perspective towards to algebra, in continuous back and forth of movement from specific numbers to generalized expression representing whole situations, fosters pupils' algebraic thinking (Cusi, Malara and Navarra, 2011). Similarly, Vygotsky (1986) asserted that students who has mastered algebraic concepts attains "a new higher plane of thought" (p. 202) a level of abstraction and generalization that transforms the meaning of the lower (arithmetic) level. The critical role of abstractions and generalizations in algebraic thinking is also emphasized by Sierpinska (1993) as her quote of "algebraic thinking develops upon the arithmetic thinking and transcends it through generalization" (p. 105-106).

Researchers suggest identification of eloquent teaching process to strengthen robust algebraic habits of mind through traditional arithmetic topics (Cai and Knuth, 2011; Jacobs *et al.*, 2007). This is the habit of analytic and logical ways to looking for a pattern, discovery of mathematics in this pattern, and generalization of whole process (algebraic expression or equation) that helps students think beyond the rules. Driscoll (1999) proposed that algebraic thinking is promoted by certain habits of mind. According to Driscoll (1999), promotion of algebraic thinking includes being able to think about functions and relations as well as their working principles and to think about the impact that a system's structure has on calculations. In this respect, he introduced three habits that facilitates algebraic thinking; capability of working backwards (doing-undoing), formulating well defined functional rules and abstracting from computations (see in Figure 2.1.)

First of all, as indicated by Driscoll (p.1, 1999), "Effective algebraic thinking sometimes involves reversibility (i.e., being able to undo mathematical processes as well as do them)." In other words, students and teachers are expected to work backward to starting point of the problem as well as complete necessary algebraic procedures to obtain answers. For example, ideally, students are asked to find the roots of the

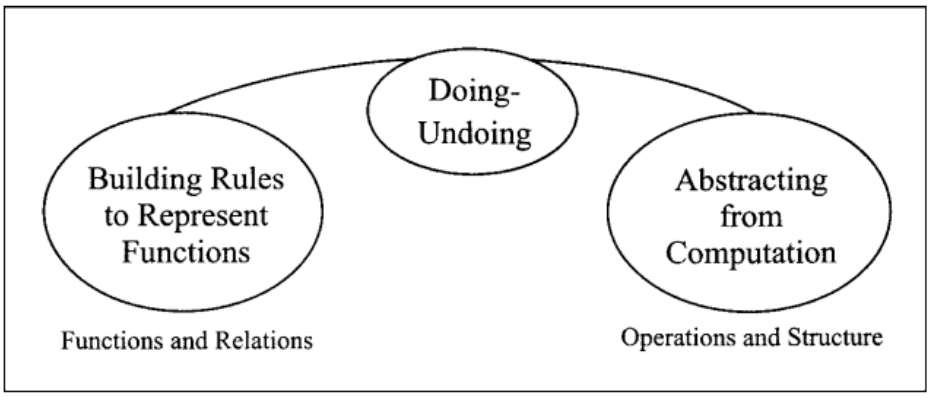


Figure 2.1. Three algebraic habits of mind

equation $x^2 - 16 = 0$ should also provide an answer for the question, which equation has roots of 4 and -5? (Driscoll, 1999).

Secondly, the algebraic habit of mind of building rules to represent functions is critical in algebraic thinking since it is directly related to recognize patterns and organize the given data in order to form well-defined functions representing the current situations. This ability also helps students to characterize how a functional rule works and observe reversibility of the process, which further helps manipulate data to work abstractly (Areyan, 2015; Driscoll, 1999).

Third, Driscoll (1999) define algebraic thinking as one's ability to embody quantitative expressions in order to make variables observable. In a similar manner, Kieran (1996) proposed that algebraic thinking is about constructing relations for quantitative situations through using representations. In the light of discussion above, algebraic thinking essentially involves one's ability to represent quantitative relationships and being able to think abstractly about the quantitative situations. At this point, multiple embodiment of functions comes to play its pivotal role in development of robust algebraic thinking. Many teachers and researchers agree on that presentation of algebra as the study of expressions and equations can pose serious obstacles in the process of effective and meaningful learning (Kieran, 1992). In this respect, NCTM proposed that algebraic ideas should be presented in a way that patterns are verbally described and represented by different representations such as tables, symbols and graphs as well as model representations, and variables are used to express a pattern, generalization, or

situation for developing a algebraic thinking (NCTM, 2000).

2.1.1.1. Functions as a Concept of Algebraic Thinking. The concept of functions is one of the most critical and unifying theme throughout the full range mathematics courses. Eisenberg and Dreyfus (1994) and NCTM (2000) stressed that having a good understanding of multiple representations of functions is an important facet of mathematics that enables someone to understand and connect different mathematical ideas. Within this context, Leinhardt, Zaslavsky, and Stein (1990) emphasized the use of graphical representations of functions. These researchers claimed that graphical representation exhibited simultaneous correlated changes between numbers and relationships between two pairs. To illustrate, in symbolic representation, such as $y = 5x + 20$, the dependent variable (y) is given in terms of some algebraic notations involving the independent variable (x) and one can observe how values of x and y changed together. As Kaput (cited in Pitts, 2003) stated, graphical representation is "very efficient and explicit representation of both the quantitative relationship and the variation [between the x and y values]" (p.170). In other words, in case of graphical representations, not only does one recognize the whole idea behind the representation that cannot be easily noticed by just looking numerical data (Bell and Janvier, 1981), also change of varying values of two quantities can be observed.

With this conception of graph, it is important to discuss covariational thinking at this point. Simply stated, covariational reasoning entails thinking about co-occurrence of two variables or quantities. In particular, covariational thinking refers "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson *et al.*, 2002, p.354). As Thompson stated (2011), covariation refers to pattern of correspondence between varying quantities rather than numerical patterns of two sets (Confrey and Smith, 1995). Several researchers (Confrey and Smith, 1995; Zeytun, Çetinkaya and Erbaş, 2010) emphasized the role of idea of covariation in understanding of functions. Confrey (1988) focuses on coordination of value of two variables as they change and in this context (Confrey and Smith, 1995) distinguished covariation conception of functions

from correspondence between variables as follows;

A covariation approach, entails being able to move operationally from ym to $ym + 1$ coordinating with movement from xm to $xm + 1$. For tables, it involves the coordination of the variation in two or more columns as one moves down (or up) the table. (Confrey and Smith, 1994, p. 137)

To illustrate, consider the relationship between x and y values represented in Figure 2.2.

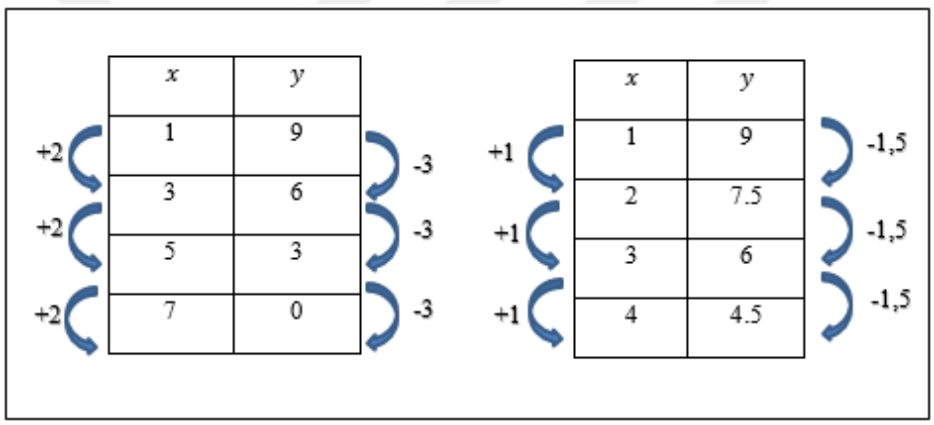


Figure 2.2. Two example of Confrey's covariational thinking

One's way of thinking to construct the pattern in the tabular representation in each subsequent row, the value of x increases from 1 to 3 as the value of y diminished by 3. The linear change between x and y can be easily generalized in a way that y decreases 1,5, $\frac{1}{2}$ of 3 units for every increase of 1 unit of x values. In other words, the change in one variable (e.g. increasing the value of x by 2) are coordinating with changes in another variable (e.g. decreasing the value of y by 1.5). From the covariational perspective, this is not a static correspondence between dependent and independent variables. Instead, this table represents a dynamic process between two quantities whose values vary simultaneously. It is important to note that a person who has limited covariational thinking skills might not focus on simultaneously accumulating quantities. Rather, s/he constructed a new ordered pair appropriate to tabular representations by generalizing the relationships between the change in the value of x and change in the value of y (Frank, 2017). Covariational thinking, on the other hand, involves

coordinating changes in one variable with the other variable beyond the idea of recursive patterning.

Many researchers have highlighted the importance of covariational thinking in comprehension of graphical representation of several mathematical discipline such as rate of change (Johnson, 2015) and graphical representations of two quantities (Carlson *et al.*, 2002; Moore and Thompson, 2015). Notwithstanding its importance of learning essential ideas in several mathematics concepts, there are many research eliciting students and teachers' limited covariational thinking abilities. To illustrate, Carlson, Larsen and Lesh (2003) encouraged 22 preservice teachers to construct graphical representation of height of a bottle with respect to increasing amount of water, and notified teachers' limited understanding of covariational thinking and interpreting the model representations. Furthermore, in their study with high performing 2nd semester calculus students, Carlson *et al.* (2002) elicited students' difficulty coordinating the instantaneous rate of change with fixed and continuous changes in dependent variables.

Carlson (1998) conducted a research with undergraduate students required to interpret dynamic graphical functions and observed their noticeable behaviors when analyzing the questions. Carlson *et al.* (2002) developed a covariational framework in which those behaviors are classified into five separated mental actions and five corresponding levels of covariational thinking (see Table 2.1.). This framework enables researcher to categorize mental actions demonstrated by participants as well as classify their level of covariational thinking as they engage in a task including covariation in graph representation.

Table 2.1. Mental actions of Carlson's covariation framework

Action	Description of mental action	Behaviors
MA1	Coordinating the value of one variable with changes in the other	Labeling the axes with verbal indications of coordinating the two variable (e.g., y changes with changes in x)
MA2	Coordinating the direction of change of one variable with changes in the other variable	Constructing an increasing straight line Verbalizing an awareness of direction of change of the output while considering) changes in the output)
MA3	Coordinating the amount of change of one variable with changes in the other variable	Plotting points/constructing secant lines Verbalizing an awareness of amount of change of the output while considering) changes in the output)
MA4	Coordinating the average rate of change of the function with uniform increments of change in the input variable	Constructing contiguous secant straight line Verbalizing an awareness of the rate of change of output while considering uniform increments of the input)
MA5	Coordinating the instantaneous rate of change of the function with continuous variable in the independent domain of the -function	Constructing a smooth curve with clear indications of concavity changes Verbalizing an awareness of instantaneous changes in the rate of change for the entire) domain of the functions)

Calrson *et al.* (2002) highlighted that one's thinking must consists of mental actions related to assigned level as well as other actions associated with all lower levels in order to deserve that level of covariational thinking. For example, to be able to assign a person to level 3, researcher should have supportive arguments for existence of behaviors related to mental actions 1, 2 and 3.

In this study, covariational framework proposed by Carlson *et al.* (2002) was used as an analytical tool for describing and classifying mental actions related to covariational thinking of participants in the context of a specific real world problem.

2.1.2. Algebraic Thinking in Teaching Mathematics

Algebra has been used as the entry requirement for universities starting from 1820 in USA (Kilpatrick, Mesa and Sloane, 2007). The role of algebra as prerequisite for entrance to the college lead growing trend in college preparatory courses in regard of shaping their curriculum policies in order to provide rigorous learning and crucial foundation for advanced algebra study in mathematics. Additionally, Kaput (1998; 2002) accentuated the essentiality of algebra not only because of its central position in college admissions success and also its gatekeeper role in advancing mathematics and science.

Given its gatekeeper role as well as students' unsatisfactory results from both national and international examinations such as TIMMS (Cai, Ng and Moyer, 2011), the need for integration of algebraic ideas into mathematics curriculum in the earlier grades become central points for curriculum developers around the world (Cai and Knuth, 2011). In several countries (e.g. USA, Singapore, Russia and South Korea), national mathematics curricula introduce some algebraic concepts to younger children and are designed to help students develop algebraic thinking initiated from elementary grades. For example, curriculum developers in China and Singapore created appropriate algebraic practices for younger learners in curriculum in order to help them experience formal study of algebra and develop algebraic thinking in early grades (Cai *et al.*, 2011). In this sense, many researchers nominated early algebra acquisition that

begins from elementary grades as a keystone in secondary mathematics education (Carraher, Martinez and Schlieman, 2008. Taylor-Cox, 2003; Usiskin, 1999; Van Amerom, 2002).

Principles and Standards for School Mathematics published by NCTM in 2000 outlined the importance of comprehensive and meaningful themes in school curriculum for algebra learning and classified four core algebra learning expectations that students satisfy until the end of their education period. What students must come to understand and operate are enumerated by NCTM (2000) standards as follows; (1) Understand patterns, relations and functions; (2) Represent and analyze mathematical situations and structures using algebraic symbols; (3) Use mathematical models to represent and understand quantitative relationships; and (4) Analyze change in various contexts (p.37). A different framework was proposed by Cai *et al.*, (2011) which focused on three components of algebraic knowledge; (1) Representing situations, (2) Equation solving and (3) Making generalizations. They asserted that mastering these three components requires the acquisition of the fundamental habits of mind involved thinking algebraically. In order for algebra to be accessible for everyone, NCTM proposed that algebraic ideas should emerge and be investigated as students; "(a) identify or build numerical and geometric patterns; (b) describe patterns verbally and represent them with tables or symbols; (c) look for and apply relationships between varying quantities to make predictions; (d) make and explain generalizations that seem to always work in particular situations; (e) use graphs to describe patterns and make predictions; (f) explore number properties; and (g) use invented notation, standard symbols, and variables to express a pattern, generalization, or situation" (NCTM, 2000, p. 159).

In Turkey, mathematics curriculum has been bombarded by calls for development and implementation of effective and comprehensive mathematics program since 2003 (Ubuz *et al.*, 2010). Ministry of National Education (MoNE) indicated that the vision of the mathematics program based on the principle of "Every child can learn mathematics" (MoNE 2009, p. 7). This idea resembles NCTM's "equity principles" (2000, p.12) emphasizing students' equal access to substantive mathematics education regardless of their background, interests and needs. Then, along with the curriculum

reform presented by the Ministry of National Education in 2015, the concept of algebraic thinking in elementary school curriculum has a place for the first time in the elementary school mathematics curriculum that attempts to integrate algebraic thinking across all grades. In elementary school curriculum, algebraic thinking is proposed as a strand blended with numbers, operations that aims of recognition and generalization of patterns and number sequences, under the title of "introduction to algebra" (MoNE, 2015). As students moves into the middle grades, they encounter algebra as one of the five learning domain through grades 8th to grade 6th grade. Students are expected to interpret and calculate algebraic expressions, grasp the concept of equivalence, formulate and solve equations with one and two unknowns (MoNE, 2013). In the high school mathematics curriculum from 9th to 12th grade that was promulgated in 2013, numbers and algebra, geometry and data, counting and probability generate three learning domains of the curriculum. The role of conceptual understanding rather than procedural knowledge is dignified in high school mathematics curriculum.

Principles and standards proposed by NCTM (2000) stated that "Instructional programs from pre-kindergarten through grade 12 should enable all students to understand patterns, relations, and functions; represent and analyze mathematical situations and structures using algebraic symbols; use mathematical models to represent and understand qualitative relationships; and analyze change in various contexts" (p.8). In this regard, "algebra for everyone" was announced as a worthy slogan which states the fundamental goal in mathematics education. This expectation lean on the premise holding the idea that algebra is a thinking mechanism, a tool for problem solving, a key to understand and communicate in mathematics language which everyone should have. Instead of seeing algebraic thinking a separate strand of mathematics and disconnected set of skills and ideas to be gained, it is important that instructional programs allow algebraic thinking to evolve and continue throughout mathematics education by actively engaging with subject matters.

2.2. Multiple Representations of Functions

Representations are mental and physical constructs (Tripathi, 2008); in a sense as a language to express abstract mathematical idea and relationship (NCTM, 2000; Scwartz, 1984) in order to build up appropriate concept images (Tall, 1983). Graphs, diagrams, verbal expression, tables are denoted as "useful tools for building understanding and for communicating both information and understanding" (NCTM, 2000, p.64) when incorporated with the curriculum materials and classroom instructions (Monk, 2003).

The term "representation" is denoted as both internal and external demonstrations of mathematics (Pape and Tchoshanov, 2001). Pape and Tchoskanov (2001) highlighted two classes of representations in which people represent the world: (a) internal representations; a cognitive schemata and mental structure in people's head (Zhang, 2000) and (b) external representations; which are the focus of this thesis, knowledge and structure that is physically displayed through pictures, symbols, graphs, etc. The dynamic linkage among internal and external representations is demonstrated by Pape and Tchoskanov (2001) and their framework clearly display the distinction between internal and external representations and their interaction on the basis of child's understanding of the concept of the numeracy (see in Figure 2.3).

The outer space of the small circle demonstrate five distinct external representations of "five". The verbal and written expression of five, a set of five equivalent cubes (manipulative), five children (drawing) and the sign of "5" illustrates children's externalization of the number of "five". Having looked through the inner circle, children's internal manifestations or mental image of "five" and the zone of interaction of internal and external representations can be observed. Pape and Tchoskanov (2001) emphasized the mutual relationships between the two forms of representation and asserted that "the development of students 'representational thinking is a two-sided process, an interaction of internalization of external representations and externalization of mental images" (p.119).

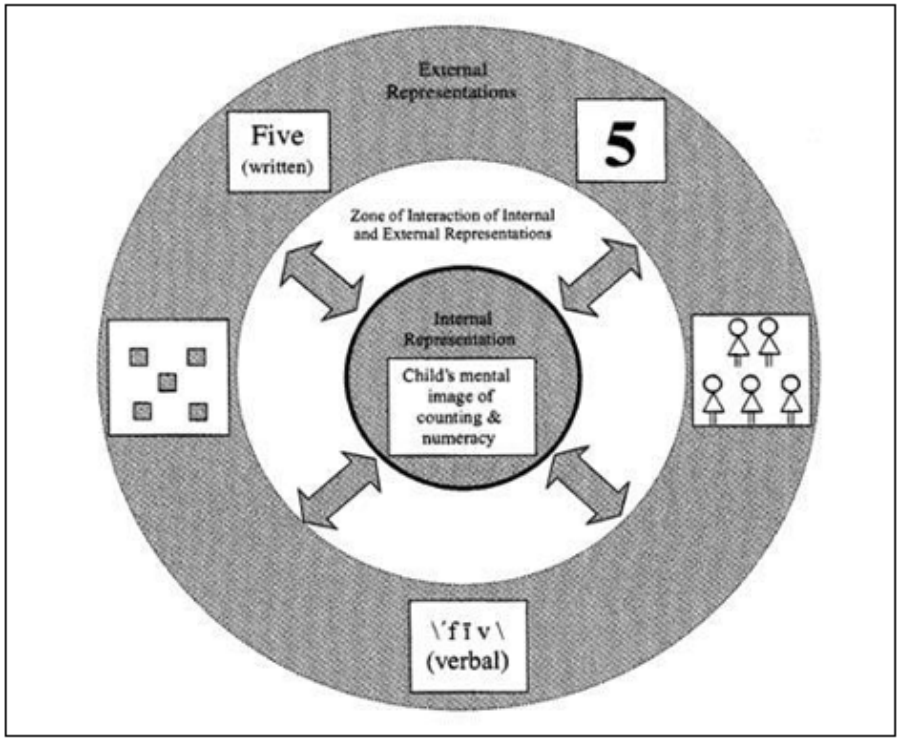


Figure 2.3. The relationship between internal and external representations in developing child's understanding of the concept of the numeracy

Pimm (1995) argued that representations "re-present something of the world using something else" (p. 119). Particularly, representations are concrete manifestation of a person's internal construction of the external world (Von Glaserfeld, 1987).Therefore, the way learners sense, think and rationalize mathematical problems and abstract concepts may vary widely. Representations are fundamental tools that foster one's capacity to think mathematically by facilitating linkages between the real world and mathematics (NCTM, 2000). The abstract nature of mathematics requires communication of mathematical ideas via multiple representations (Kilpatrick, Swafford and Findell, 2001) and thus, multiple representations as a tool for communicating help to facilitate the flow of the thought, recognize and interpret the ideas as well as communicate among different individuals (Zazkis and Liljedahl, 2004).

Ainsworth (1999, 2006) emphasized power of appropriate use of multiple representations in understanding complex concepts in mathematics. In addition, he addressed

three different purposes of learning with multiple representations; complement, constrain and construct. Firstly, multiple representations play a complementary role by taking advantages of the differences between representations (Ainsworth, 2014). For example, equations display quantitative relationship between two concepts and invite computational processes. Tables allow a quick overview of the aggregate data by explicitly writing information a more organized format, emphasize important points and find a way to a pattern in a variety of contexts without requiring calculation (Ainsworth, 2006). Each representation carries different aspect of the external world more clearly (Friedlander and Tabach, 2001). The combined use of these representations helps learners to realize unique features and different inferences of the same situation according to different perspectives that complements each other.

Secondly, multiple representations may serve as a prompt for further interpretation of complex forms of embodiments. Representations that are more familiar and easy to interpret can facilitate the analysis of other more complicated representation since various representations that theoretically involve the same information, constrain interpretation of a second representation (Ainsworth, 2006; Schwonke, Berthold and Renkl, 2009). To illustrate, the common confusion regarding horizontal lines in the velocity time graph that misinterpreted the objects being at rest, can be eliminated via displaying a iconic moving objects with a constant speed accompanied by velocity-time graph. Hence, learners may understand that the horizontal line in velocity-time graph refers to uniform motion rather than no motion (Ainsworth, 2008).

Third, multiple representations allow learners to construct and assist deeper understanding and appreciation of concepts when information of invariant features of a domain and properties of individual representations is integrated (Ainsworth, 2006, 2008). Kaput (1989) has stated that "the cognitive linking of representations creates a whole that is more than the sum of its parts" (p. 179). In this respect, it is emphasized the holistic approach towards multiple representation that grasp a structural whole so that integration and coordination of various representation of information promote understanding.

Table 2.2. Janvier's approach to multiple representations

From \ To	Verbal	Tables	Graphs	Formulae
Verbal		Measuring	Skecthing	Modeling
Tables	Reading		Plotting	Fitting
Graphs	Interpretation	Reading off		Curve fitting
Formulae	Parameter recogn.	Computing	Sketching	

According to Janvier (1987a), four prominent ways in which functions are represented are graphs, formula, table, verbal descriptions. He further served star like iceberg as a useful analogy to understand working principle of representations see in Figure 2.4.

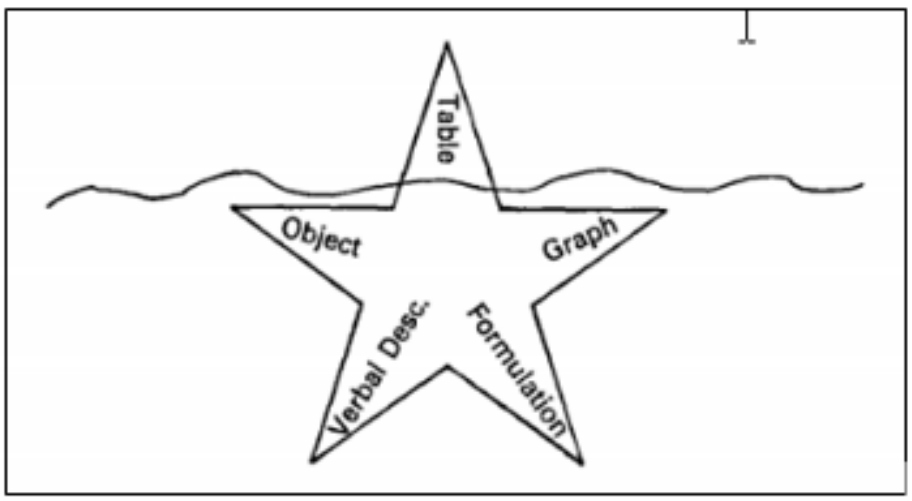


Figure 2.4. Janvier's analogy for representations

Janvier (1987b) presented representation as "the psychological processes involved in going from one mode of representation to another; for example, from an equation to a graph" (p. 27). The translation process consists of two modes of representations and Janvier defines this process communication between one sources of representation to another mode of representations. In this respect, he illustrates this translation process between two pairs of representations in the following Table 2.2.

Janvier's works of 4x4 table in order to illustrate transition process between different types of representations and relevant to implementing a multiple representation approach to concept of functions. Throughout this research, verbal description refers model representation as well as formulae represents symbolic or algebraic representation in order to help reader make sense of the mathematical ideas discussed in this study. According to Janvier framework, for example, in translating from formula or symbolic representation to graphical representation, one must fit a curve in a graph. When moving from tabular representation to model representation, one must read x and corresponding y value in the graph. A simple example for illustrating the different demonstrations of a same problem is presented in Figure 2.5.

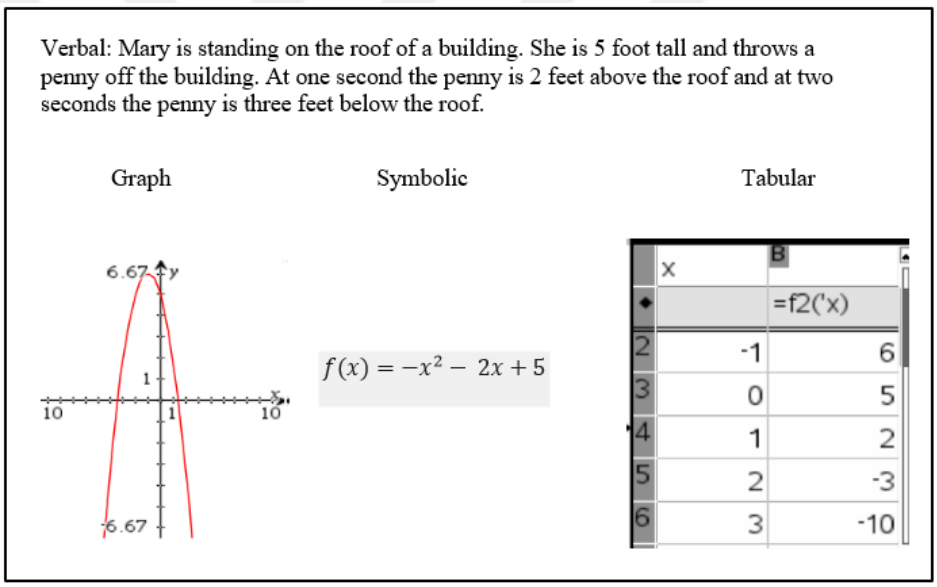


Figure 2.5. Different representation of the same function problem

Employing multiple embodiments of the concept is essential in mathematics not only its role in exemplifying abstractions but also creating a solid conceptual understanding of a single concept by illustrating interconnectedness between the representations. While the problems are analyzed, using different mathematical demonstrations such as creating table and graph or using algebraic expressions efficiently play a central role in the process of conceptual understanding as well as application of inquiry. In this respect, Friedlander and Tabach (2001) call attention to consider both advantages and disadvantages of each representation since in the case mathematics learning, the use of symbolic, tabular, graphical and modeling representations aids learners to create

their mental constructions and lead the way to effective and meaningful learning.

The symbolic representation refers to use of algebraic expressions such as letters and numbers to represent relationship between variables and what those variables are. The relationship between Celsius and Fahrenheit scale are symbolically represented as $T(inF) = 9/5T(inC) + 32$. The use of symbolic representation leads organization of information and effective demonstration of patterns and different models in mathematics however, excessive use of symbolic representation may delimitate building meaning for the interpretation of the results and restrict conceptual understanding (Friedlander and Tabach, 2001).

The tabular representation is a form of documenting the inputs and corresponding outputs of a function and allows large amount of data to be sorted in a neat format. For example, Fahrenheit to Celsius (°F to °C) conversion is illustrated with the following table (Table 2.3.).

Table 2.3. Table representation of Celsius to Fahrenheit

°F	-11	32	68	131
°C	-24	0	20	55

Tables with their columns and rows of information help learners to look up all elements and list different variables in order to catch general trend of the functions. But, it is less effective in showing the relationship between variables.

The graphical representations are a multidimensional drawing interrelatedness among two or more variables and it is the best way to display a large quantity of data (Figure 2.6). To illustrate; the following graph help learners draw general characteristics of data and observe relationship between variables.

Graphical representations deliver a compact way of presenting information and give clear demonstrations of some important features of data such as slope, maximum and minimum points of the graph etc. However, the graphical representations may

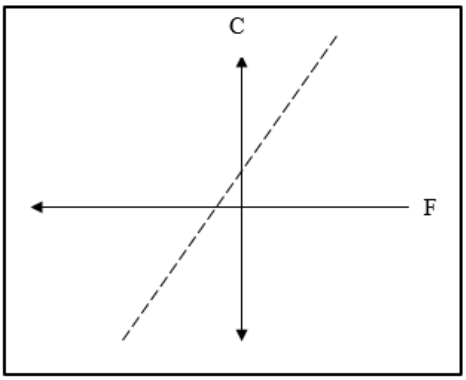


Figure 2.6. Graph of relationship between Celcius and Fahrenheit

be influenced by some external factors such as scaling, thus it causes inaccuracy of interpretation of the data (Friedlander and Tabach, 2001).

The modeling representation is method of simulating real life situation in order to describe the behavior of a mathematical system (Figure 2.7). For example, the thermometers including both Celsius and Fahrenheit scale are input/output in functions and a model representation.

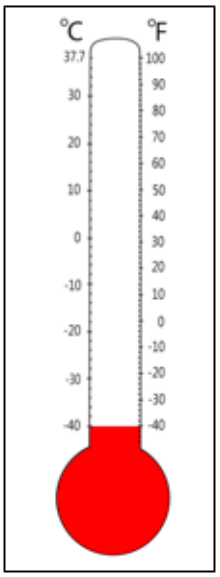


Figure 2.7. Model representation of relationship between Fahrenheit and Celcius

Use of mathematical model representation allows students to remember more easily a phenomenon rather than formulated by an equation and makes it easier to build connections. However, creation of appropriate mathematical model representations for

describing a phenomenon is not so easy and this process take more time than the traditional approach.

2.2.1. Multiple Representations in Mathematics Curricula

Considering mathematics education, curriculum standards created and presented by NCTM (2000) define the knowledge and skills that students gain from prekindergarten through high school in regard of mathematical representations as follows; "(1) Create and use representations to organize, record, and communicate mathematical ideas, (2) Select, apply and translate among mathematical representations to solve problems, and (3) Use representations to model and interpret physical, social and mathematical phenomena." (NCTM, 2000, p. 67). The standards called for students to use mathematical representations as an effective tool for creating and refining personal representation system that helped them to organize their thinking, to learn relative strength and weakness of each representation system in order to perform the most appropriate form of representation in different context and to create and interpret models of physical, social and mathematical phenomenon (NCTM, 2000).

Accordingly, the recent curriculum changes in elementary grades 1-4 of Turkey sets out major elements for effective use of multiple representation throughout grades. The Turkish elementary school curriculum suggests introducing mathematical concepts and relationship with their different embodiments in earlier grades in order to allow students develop their conceptual worlds and understanding. This emphasis on using multiple representations is accompanied by the objectives that student will be able to represent mathematical concepts in different ways (MoNE, 2015, p.5) as well as students will be able to interpret and make inferences on information represented in table and graph (MoNE, 2015, p.36). Similarly, Turkish mathematics curriculum for grade 5-8 puts emphasis on proper attainment of multiple representations such as model, shape, picture, graph, table to express mathematical ideas as well as interpretations and translations of representations (MoNE, 2013). When students move into high school, the demands of curriculum increase in terms of using multiple representations. The word of "representation" is addressed 26 times in high school curriculum (MoNE,

2013) and acquisition of interpreting and translating of multiple representations is denoted as one of the most fundamental goal of the curriculum. It is emphasized that students' process of construction of information must be accompanied by multiple representations and lectures. Throughout high school, lectures and classroom activities must incorporate symbolic, verbal, tabular and graphical representation of concepts (MoNE, 2013).

Even though it's increasing emphasis on literature of mathematics education, many scholars draw attention that students fall below their expected level while making transition between model, tabular, graphical and algebraic representations and suffered from being incompetent in this subject. Gagatsis and Shiakalli (2004) emphasized cognitive obstacles that university students confronted when they pass from one representation to another and translate different representations of the concept of function, especially among graphical and model translation. Besides, MacGregor and Stacey (1993) and Pawley *et al.*, (2005) revealed preservice teachers' weaknesses in translating verbal representation into an equation. Similarly, Galbraith and Haines (2000) highlighted students' fragmented knowledge on multiple representation in which connections are poorly made among various representations and draw attention to students' vulnerable understanding in translation between algebraic and graphical representation. In this line, it is revealed that a limited number of students have reached well-constructed understanding in multiple representation and most of students failed to grasp fundamental concepts presented in various representations.

2.2.2. Model Representations of Functions

The idea of integration of multiple representation in comprehension and manipulation of mathematical concepts has emphasized in mathematics learning (Çıkla, 2004; Leinenbach and Raymond, 1996). In addition to applying multiple representations, the process of mathematical modeling representations is other fundamental components of mathematical education. Jacobson (2014) described the ability to create model representations as requiring students to "use their understanding of arithmetic operations to make mathematical sense of problem situations and to relate this sense making to

functions represented by equations, tables, and graphs" (p. 155). NCTM Standards (2010) for school mathematics emphasized that mathematical model representations encourage students learn how mathematics works in the real world and sense about what they have done during the mathematical process. According to NCTM;

> "to think algebraically; one must be able to understand patterns, relations, and functions; represent and analyze mathematical situations and structures using algebraic symbols; use mathematical models to represent and understand quantitative relationships; and analyze change in various contexts. Each of these components evolves as students grow and mature." (NCTM, 2000, p. 64)

Concurring with this claim, Lins (2000) asserted the idea that to become competent in thinking algebraically is associated with adopting "a specific intention" (p. 12) which search for different solutions and models for the same problem as well as discuss the characteristics of each model. In a similar manner, NCTM (2000) proposed that flexible use of multiple embodiments of representations in order to understand and communicate about real life situations is an important ability to be gained in meaningful mathematics learning. In other words, algebraic thinking compromises of organization of the world via making models, representations and manipulations of these models.

3. SIGNIFICANCE OF STUDY

The concepts of function recognized as the central and fundamental objects of algebra education that provide students a consistent way to connect phenomenon and prepare them for advanced mathematical ideas (Yerushalmy and Shternberg 2001) that lead to robust algebraic thinking. In this respect, Lloyd and Wilson (1998) emphasized that functions are 'multi-faceted idea' (p.250) that cannot be fully understood working in a single representation environment. In a similar vein, the role of multiple representations is emphasized for profound and deep-rooted understanding of functions in mathematics education. In addition to importance of perceiving functions in multiple ways, competency with functions is also related to flexibly making translation between different modes of representations (Cunnigham, 2015).

With the emphasis on construction of algebraic ideas from early grades in national and international curriculum (NCTM, 2000; MoNE, 2013, 2015), it is important to determine preservice teachers' algebraic thinking process in regard of their ability to translate between different modes of representation in functions. The comprehensive evaluation of the preservice teachers' translation ability from one type of representation to another will help researcher understand underlying algebraic ideas such as variable, covariation etc. behind the translation procedure. When looking from the literature perspective, there has been no study focusing directly on translation between graphical representations to model representation. Most of the literature involving translation among multiple representations emphasized symbolic representations and its translation pairs. Particularly, this study not only filled this gap in the literature with regard to conceptions and aspects related to algebraic thinking when preservice teachers engage in a problem requiring translation from graphical representation to model representation. It will also be able to provide insight how preservice teachers' thinking would promote or inhibit their translation skills when solving problems.

4. STATEMENT OF PROBLEM

This study aimed to reveal preservice mathematics teachers' knowledge and capabilities with respect to use of multiple representations when working with mathematical functions. In particular, this study aimed to explain preservice mathematics teachers' algebraic thinking behaviors with respect to their ability to translate from graphical representation to model representation when participants face a word problem that leads to a quadratic graph. The research question investigated in this study were;

(i) What are the thought processes of preservice mathematics teachers with regard to translating a given function from graphical representation to model representation?

(ii) What are the characteristics of model representations constructed by participating preservice teachers' when translating from a graphical representation?

5. METHODOLOGY

The purpose of this chapter is to establish methodological background of this study to address research questions. This chapter consists of research design, sample of the study, instrument of the study, descriptions of the pilot study and data collection and data analysis.

5.1. Research Design

The major mission of this study was to investigate algebraic thinking of preservice teachers' while they engaged in multiple representations related problems embedded in mathematical function. For this purpose, the present study was designed as explanatory research design that qualitative approach was used. In this research, researcher first collected data from questionnaire and then, conducted task-based interviews with participants who were selected based on their answers on questionnaire. A qualitative approach to researching participants' thought processes allowed the privilege of an advocacy, participatory and self-reflective approach (Creswell, 2009).

The researcher first conducted a questionnaire with senior year preservice mathematics teachers in order to reveal teachers' conceptions in regard of translating from graphical representation to model representation, which would help the researcher to identify the best candidate for being the participant in our interviews. Afterwards, task-based interviews with four senior year teachers were conducted to examine participants' algebraic thinking in depth. Task-based interviews enabled the researcher to elaborate preservice teachers' algebraic thinking in depth in terms of multiple representations and make inferences of preservice teachers' taught process. The purpose of individual task-based interviews was to pave the way for more robust analysis and gain in breadth and depth of understanding preservice teachers' thinking (Goldin, 1998, 2000).

The data from both students' individual responses to the translation activity in interviews, videotaped and transcribed narratives were the main sources of this study and triangulated to enhance the quality of results.

5.2. Sample of the Study

The participants of this study were four preservice teachers studying in one public university in Istanbul. These four participants were not randomly selected, rather chosen purposefully based upon preliminary results of multiple representation of functions questionnaire (see Figure 5.1). Maxwell (2005) supported, "Selecting those times, settings, and individuals that can provide you with the information that you need in order to answer your research questions is the most important consideration in qualitative selection decisions" (p. 88). Since the intent of this study was to obtain depth of understanding for preservice teachers' use of mathematical representations and translations, researchers' first priority was to reach "information-rich" (Patton, 2002) cases in order to be investigated. In this respect, the target sample for this investigation was drawn from all senior secondary school preservice mathematics teachers (n=24) studying in a public university in Istanbul. Secondary school mathematics education programme is a five year program by the time the data was collected. In order to maximize return rates and ensure data trustworthiness and credibility, the researcher was in charge of recruiting and collecting data from questionnaire. In this sense, data collection process took place within one hour research method course while the preservice teachers were attending the course early in the academic year under researcher's supervision. All participants were given multiple representations of functions questionnaire (MRFQ) in this period of time.

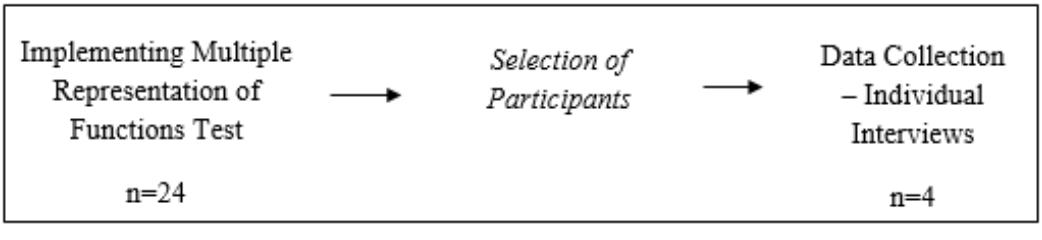


Figure 5.1. Participant selection model of the study

After collection of 24 preservice teachers' results in the questionnaire, researchers decided to evaluate questionnaire data quantitatively, in which participants were scored by criterion referenced standards in three themes; (1) total achievement score (2) total translation score and (3) total model translation score. Total achievement score indicates preservice teachers' general academic proficiency in the questionnaire which ranges from a minimum score of 0 to maximum score 10. Total translation score reflects the sum of the translation score of participants and obtained by preservice teachers' score in translating among tabular, graphical, algebraic and model representations between a minimum score of 0 (indicating no translation activity), a score of 1 (indicating a partial translation activity) and maximum of 2 (indicating ability to translate among multiple representations). Total model translation score is based on how accurate and suitable their mathematical models were in the questions requiring translation from and to model representations. Participants were given a minimum of 0 (indicating no modeling activity), a score of 1 (indicating a partial modeling activity) and a maximum of 2 (indicating proper use of mathematical models). After that, participant preservice teachers' scores were grouped in the following ways;

Table 5.1. Rubric for the Questionannaire

	Total Modeling Score	Total Translation Score	Total Achievement Score
High	8-12	16-24	80-120
Middle	4-7	8-15	40-79
Low	0-3	0-7	0-39

According to Yin (2008), case study research design has five primary rationale for selection of sample; (1) critical case, (2) extreme or unusual case, (3) revelatory case, (4) common case (5) longitudinal case. The selection of the individuals for the task-based interviews was based on a desire to find out information-rich cases (Patton, 2002) which enabled me understand and gain insight into their process. In our case, I chose critical case sampling technique that is particularly beneficial when researchers possess a small number of cases (Strewig and Stead, 2001; Patton, 1990). These critical cases were the ones who were more likely to have the greatest impact on our development of

knowledge about research questions and provide wealth of information (Patton, 2002).

In order to explain sample selection method, a complete list of questionnaire results was presented in Figure 5.2. My aim was to select cases that would maximize diversity relevant to the process of translation of model representation.

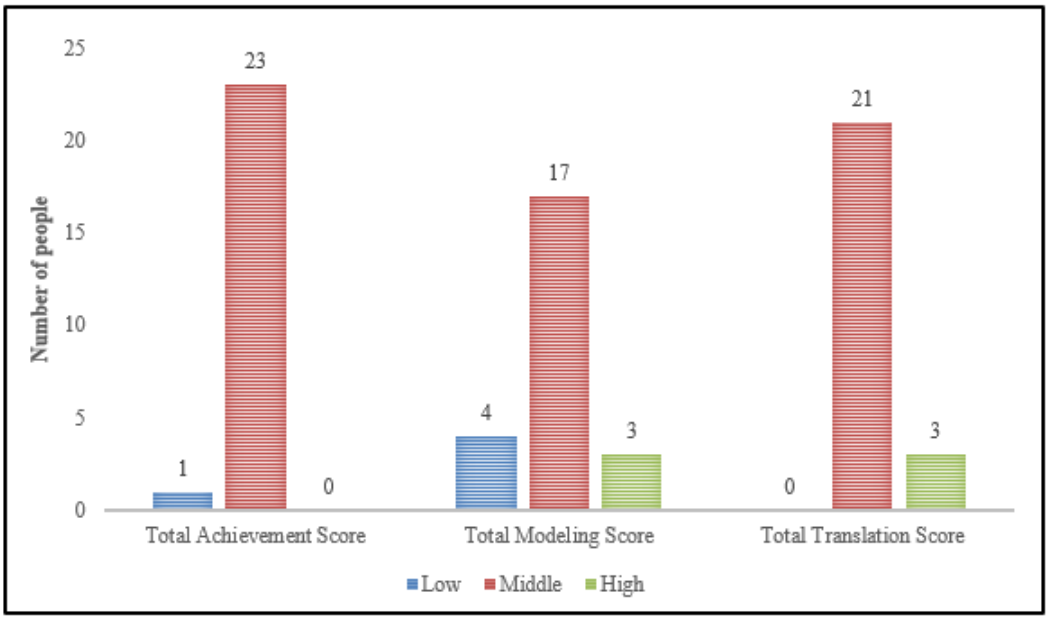


Figure 5.2. Preservice mathematics teachers' total achievement, total modeling and total translation score

According to questionnaire results, no participant preservice teacher could get enough score to become high achiever with respect to all three criteria; total achievement, total translation score and total model translation score. It is important to highlight that nearly all participating preservice teachers (23 out of 24 people) received middle achievement score, which inhibited our desire to understand thought process of a preservice teacher who got high achievement score in the questionnaire. As selection of a participant, achieving a high level of expertise in three criteria was not possible, we wanted to emphasize other possible critical cases.

The first case drawn to my attention was Tarık who had the highest results of total achievement, total translation and total model translation score in the questionnaire. Interview with Tarık helped me fulfill the need to understand a high-achieving

preservice teachers' algebraic thinking process when dealing with translation between representations.

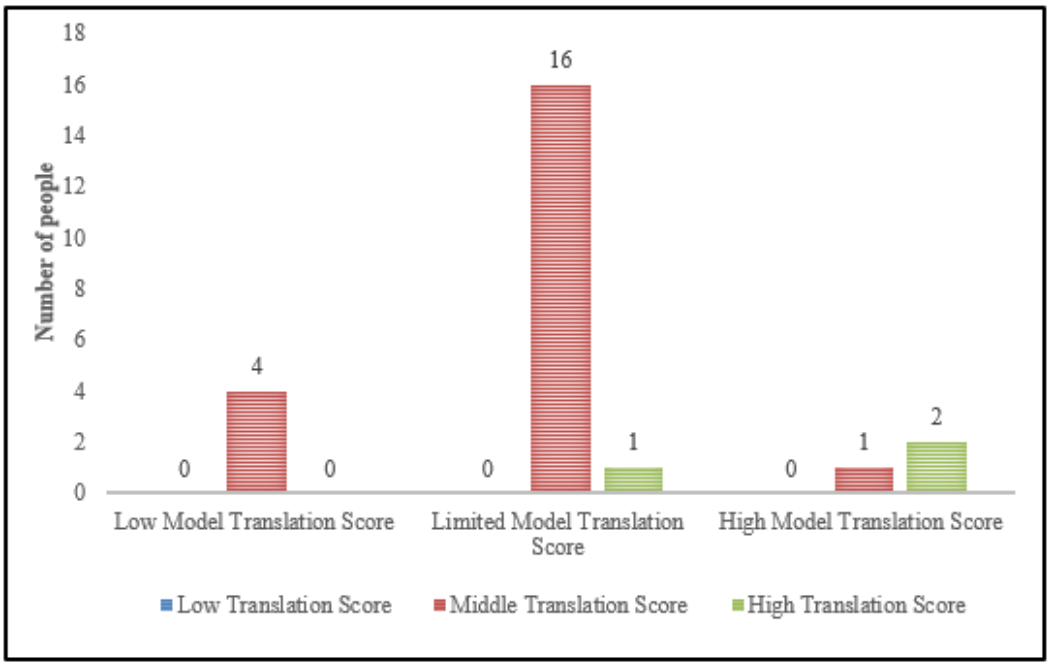


Figure 5.3. Comparison of preservice mathematics teachers' total achievement, total modeling and total translation score according to their total translation score

The other case drawn attention was Çınar, managing to be high achiever according to her model translation score. She separated from other two preservice teachers, earning high model score (Table 5.3) with respect to her middle total translation score. Whereas she satisfied lower bound of high model translation score, her total translation score indicated that she was partially proficient in that skill and got middle translation score. Interview with Çınar would enabled the researcher to understand thinking patterns of another high achieving participant who had relatively low levels of translation skills between different modes of representations comparing to other peers.

There was one participant receiving low achievement score in the questionnaire. When I checked other scores of this participant and found out that this preservice teacher also has the lowest model representation and translation score among 24 people as shown in Table 5.4.This participant was given a name "Bilge" and selected for the task-based interviews as one of the low achiever participants of the study.

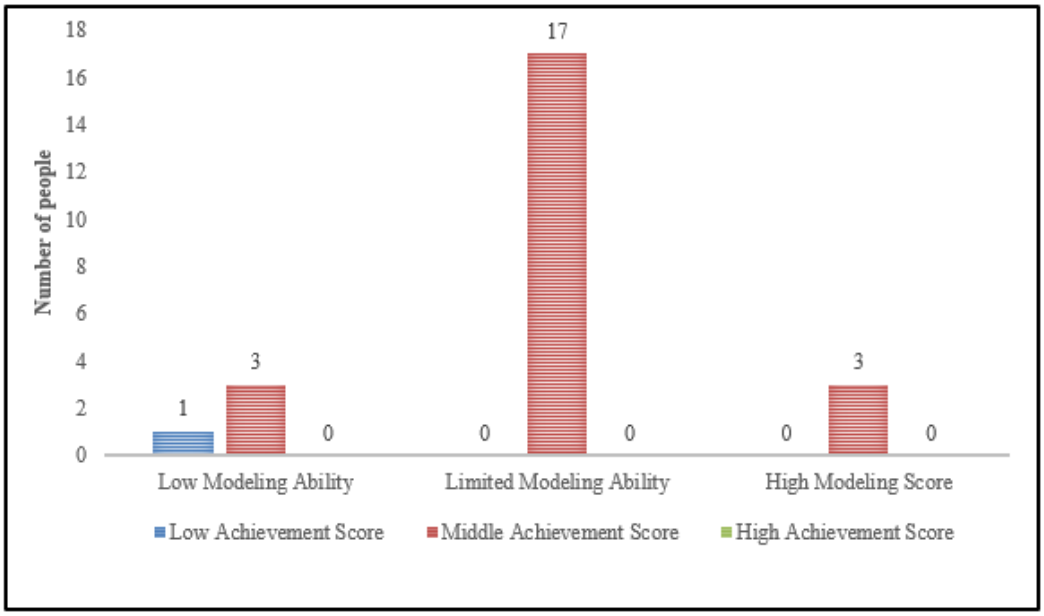


Figure 5.4. Comparison of preservice mathematics teachers' total modeling score according to their total achievement score

Our last interviewee was Deniz who reflected to average preservice teachers participating in this study since she was one of the preservice teachers who got middle score in three criteria among 16 participating preservice teachers (shown in Figure 5.3). According to Patton (2002), "any common patterns that emerge from great variation are of particular interest and value in capturing the core experiences and central, shared dimensions of a setting or phenomenon" (p.234). She was also a low achieving preservice teacher in this study since her total translation and total model translation score caught lower bound score for being middle achiever in those areas.

Table 5.2. Participants chosen for the task based interviews

	Total modeling score	Total translation score	Total achievement score
Tarık	High	High	Middle
Çınar	High	Middle	Middle
Bilge	Low	Middle	Low
Deniz	Middle	Middle	Middle

As shown in Table 5.2., from pool of 24 participants, one preservice teacher with high levels of total model translation score and total translation score as well as three preservice teachers with middle levels of translation model score, who separated according to their total model translation score, were selected for qualitative interviews.

5.3. Instruments

The data were collected by (i) the means of Multiple Representation of Functions Questionnaire, (ii) transcripts of the video records of task-based interviews.

5.3.1. Multiple Representations of Functions Questionnaire

A questionnaire was used as an instrument to collect data from preservice mathematics teachers in order to reveal their algebraic thinking through the use of multiple representations. Multiple representations of functions questionnaire that was used in this study is given in Appendix C. Most of the items of the questionnaire was derived from several research manuscripts (Bush and Leinwand, 2002; Carlson *et al.* 2002; Leinhardt, Zaslavsky and Stein, 1990; Nilklad L., 2004; Shell Centre for Mathematical Education (University of Nottingham), 1985; You, 2006, see Appendix-A) and adopted in Turkish by the researcher. In addition to questions adopted from these sources, four questions were also written by the researcher.

The questionnaire consists of two main parts. The first part of the multiple representations of functions questionnaire includes demographic characteristics of participants such as gender, age, academic year, year of teaching experience, mathematics courses that are taken and current course credit load. The second part of the questionnaire will be used to collect information about preservice mathematics teachers' abilities to recognize different aspects of multiple representations, interpret and translate among various representations of functions.

In order to understand preservice teachers' algebraic thinking when working with function problems requiring translation of one kind of representation to another as well

as assess their ability to move back and forth across different kinds of mathematical representations, 12 questions were formulated according to Janvier translation model discussed in page 19.

There were two items per translation between two forms of representations and therefore questionnaire contained 12 problems (see Appendix-B) with a time of 1 hour for working on the tasks. To illustrate, one of the items adopted from Carlson *et al.*, (2002) which requires preservice teachers to prepare a tabular representation that shows height of water with respect to time, is as follows in Figure 5.5.

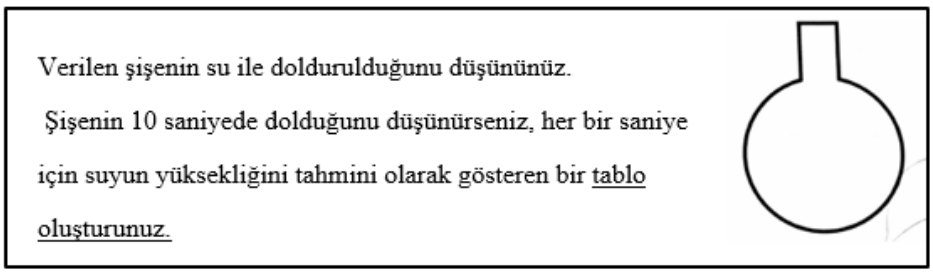


Figure 5.5. A task in the Multiple Representation of Functions Questionnaire

To test to content validity of the MRFQ, five mathematics education scholars who had extensive teaching experiences at school and universities checked the questions. Five experts examined items one by one, then gave very detailed comments, and suggested improvements for content appropriateness. Moreover, a Turkish language teacher examined the items of questionnaire in terms of grammar, punctuation and word selection. Furthermore, three experienced mathematics teachers marked four participants' responses in questionnaire, which were randomly selected, in order to ensure interrater reliability.

The pilot study was the initial step for practical application of multiple representations of functions questionnaire. The trail rune of the full-scale study was conducted with senior undergraduate preservice mathematics teachers (n=21) in another faculty of education located in İstanbul. The pilot study served three purposes: (1) to assess the feasibility of the study (2) to collect preliminary data (3) to develop and revise research questions and research plan. According to results of the pilot study, some

questions were refined in order to increase the effectiveness of the items.

5.3.2. Task Based Interviews

In the current study, our purpose was to gain an understanding of preservice teachers' thinking process. For this purpose, we adopted a question The usefulness of face-to-face semi structured interviews with sample in our study is obvious since researcher and interviewee engage in a conversation in which researcher could guide interview and both sides focused on the details related to research study (DeMarrais, 2004).

Preservice teachers' questionnaire results demonstrated their inability to proceed graph to model representations and utilize a valid approach when making translations. In this sense, researchers decided to explore preservice teachers' argumentation during solving a mathematical task requiring translation from graphical representation to model representation as well as evaluate the characteristics of model representations created by participants. Case study promoted by an open-ended question was developed in order to reveal preservice teachers' process of translating between representations. The question stated that the given graph intersects with two points on x-axis when zooming out (see Appendix D). Participants were asked to complete the graph and provide a mathematical model as it would fit their graphical representation. It is important to note that even though it is possible to generate model representations between some intervals, there is no real world model representation that can fully fit into the graph values. This task was chosen on purpose as it was intended to investigate how preservice teacher think and evaluate how much preservice teachers can generate ideas in unorthodox situation. This task was supposed to be out of ordinary in order to prevent memorized responses. Hence, researcher could assess how preservice teachers think algebraically while translating between two representations.

The theoretical framework to investigate characteristics of participants' model was emerged from various theoretical framework and perspectives (e.g., Blum and Leiss, 2005, 2007; Common Core Standards Initiative (CCSSI), 2010; NCTM, 2000;

Niss, Blum and Galbraith, 2007; Pollak, 2003). The task was cognitively demanding while it allowed the researcher to assess the characteristics of the model representations. The researchers identified three criteria in order to examine characteristics of model.

The framework assessing the characteristics of models in this study comprised of following three criteria;

(i) The mathematical model derives from a realistic situation.
(ii) The mathematical model aligns with the mathematical properties of the graph representation.
(iii) The dependent and independent variables of the function are reasonable and clearly identified.

One semi-structured one to one task-based interviews with each of the four preservice mathematics teachers, in which their algebraic thinking process when translating graphical representation to model representation was audio taped and video recorded, comprise the primary data collection of this study. The task chosen for the interview (see Appendix-D) is open-ended in that participants' responses could differ depending on how the question was structured and what kind of variables were used throughout the study. Besides, the interviews were semi-structured so that researcher managed the flow of the discussion in a way that participants were encouraged to elaborate their thinking process. Throughout the interviews, subjects were required to talk aloud about their thinking process and provide written explanations to justify their arguments.

The pilot study was done for semi-structured interviews with a preservice teacher who purposefully selected, four days ago before actual interviews. Pilot interview did not only allow me practice in interviewing, but also helped me recognize which questions were confusing and needed modifications, which questions produced redundant or meaningful data.

5.4. Data Collection

Data were collected in two major ways to gain an understanding of the conceptions of multiple representations: the administration of Multiple Representations of Functions Questionnaire (MRFQ) and task-based interviews. Data collection process for questionnaire and task-based interviews took place in the department of secondary school mathematics education of the university at the beginning week of the fall semester of 2016-2017 academic years. More specifically, this time period was purposefully chosen, as participant preservice teachers were not taught anything specific to model representation and hence intervention effects on thinking process were minimized. To reach all senior mathematics teachers, the course of "teaching methods in mathematics" was chosen. The answers of four preservice teachers who completed MRFQ were scored by three mathematics teachers. Pairwise comparisons among raters were examined and the level of agreement was calculated. Three raters were blind to the students' identities, as well as the other rater's scores. Three raters came to a high agreement for total achievement, total translation and total model translation score.

The interview process began with an open-ended mathematical task. Since interviews were planned as semi-structured, the flexibility of asking follow up questions for potentially sensitive areas was well suited to answering why questions (Miles and Gilbert, 2005) and developed a keen understanding on the topic of research interest. All interviews were conducted at the preservice teachers' current faculty, in the meeting room of Department of Mathematics Education. The interviews were audio-taped and then transcribed. The interviews lasted about 40 minutes with high achieving participants and about 20 minutes with low achiever participants and no second interviews were planned.

5.5. Data Analysis

Creswell (2009) states that in qualitative research, data collection and data analysis should proceed simultaneously. In this respect, I began to make sense of data and do some rudimentary analysis as soon as the first pieces of the data were gathered

in this research. The overall process of data analysis was used a road map to reach potential answers for my research question. These answers can be named as categories, themes, or findings (Merriam, 2009).

In order to answer the research question, I first categorized all actions demonstrated by the preservice mathematics teachers in task-based interviews one by one and listed what steps occurred in the translation cycle. Then I refined and established several coding derived from transcriptions of every participant's interviews, which formed basic units of my analysis. After specifying each code, I sorted each of the coded excerpts by themes in order to identify major elements of their thinking. These codes, themes and their explanation will be found in findings part. I also provided excerpts and analysis of student interviews to illustrate participants' thinking.

5.6. Role of Researcher

Qualitative researchers "deploy a wide range of interconnected interpretive practices, hoping always to get a better understanding of the subject matter at hand" (Denzin and Lincoln, 2000, p. 4). To fulfill this role, my main responsibilities in this research was to collect and analyze the data from the questionnaire, select the appropriate participants according to research aim and conduct the task-based interviews with participating preservice teachers.

During the task-based interviews, I was the mediator that I informed participants about the purpose of the research and allowed interviewee to clarify any doubts about the interviews. The task-based interviews in this research with preservice teachers were semi-structured, therefore I could ask probing questions, then listen and think and ask more probing questions in order to get deeper into preservice teachers' understandings. Throughout the interviews, I avoided using leading or short answer questions and being judgmental. Instead my follow-up questions usually centered around why and how questions in order to receive essential and clear information about their understandings.

5.7. Trustworthiness of the Study

Regardless of the type of research, validity and reliability are two fundamental issues for the usefulness of study within research context and appropriateness of findings of the research. Since this study was designed as a qualitative research, the researcher must provide readers enough details to show that procedures are faithfully followed and related findings makes sense (Firestone, 1987). In qualitative research, internal validity, external validity, reliability and objectivity of the study is framed in terms of trustworthiness (Merriam, 2009). Many scholars asserted the idea that validity and reliability must be approached in a way that congruent with the philosophical assumptions underlying the qualitative research paradigm. In this respect, Guba and Lincoln (1981) and Lincoln and Guba (1985) proposed credibility, transferability, dependability and confirmability as more suitable terms for internal validity, external validity, reliability and objectivity.

Internal validity or credibility is one of most fundamental criteria dealing with the congruency of findings with reality. Credibility was established in this study through peer scrutiny of the research (Shenton, 2004), analyst and theoretical triangulation of the data sources (Silverman, 2000; Merriam, 2009). In order to examine and review data obtained from transcribed video records, I have met my advisor weekly throughout the data analysis. Over the duration of this period, observable actions were reclassified by my advisor to be able to compare their compatibleness with my existing codes. The experiences and perceptions of my advisor allowed me widen my perspective in terms of discussing alternative approaches and develop interpretations of derived codes. Besides, Silverman (2000) highlighted the importance of relevancy of research findings to existing body of knowledge in related issues in qualitative research. In this respect, I tried to provide theoretical foundation for every assumptions I have made.

External validity or transferability deals with in which extend generalizability of findings of a study is possible. In qualitative study, it is important to note that the value of research lays in "the particular description and themes developed in context of" the specific conditions in which the study is conducted (Creswell, 2009, p. 193).

In this line, I believe that I presented sufficient description about research methods, participant selection, data collection and analysis procedures. Hence, other researcher can determine the extent to which their circumstances match the research context and therefore, it is possible to transfer findings of the research in other context.

Dependability or reliability refers to exact replication of research and findings. For this study, researchers traced the following procedures recommended by Gibbs (2007) in order to strength the reliability of the research;

(i) The interview transcripts were checked multiple times to be sure that the written record correctly matched the video-records
(ii) Detailed theoretical perspectives behind data analysis was presented in detail.
(iii) As codes extracted from the original sources, two researcher constantly compared derived codes with collected information in order to guarantee that there is no deviation in the meaning of the codes throughout the analysis.
(iv) Through peer debriefing (Creswell, 2009), researcher and her thesis advisor continuously shared their ideas. By doing this, advisor would question, challenge and probe researcher's assumptions whereas researcher could discuss and defend her hypothesis and analysis.

Confirmability or objectivity refers that research findings are the results of the experiences and information of participants, rather than researchers' preferences. On order to strength confirmability, process notes was used as audi trail (Lincoln and Guba, 1985) that I could present an account of theoretical and methodological underpinnings for data constructions and research decisions.

Furthermore, to exercise confidentiality, participants were asked to write last letter of their surname accompanied by last four digit of their school number when researcher conducted multiple representation of functions questionnaire. After four participants were selected for interviews, they were given pseudonyms and these names are used consistently throughout the text.

6. FINDINGS

The purpose of this case study was to explore what kind of algebraic thinking skills preservice mathematics teachers use when working with multiple representations in functions. In particular, I aimed to understand how preservice teachers integrate algebraic thinking into tasks required them to make translation between graph to model and model to other representations. In this respect, Tarık and Çınar were considered as high achieving preservice teachers whereas Deniz and Bilge were identified low achieving according to results in the questionnaire. In this part, I present the results of the analysis of preservice teachers' data in two main sections. The first sections presents results describing participants' observable actions of modeling when translating from graphical representation. The second part includes analysis of preservice teachers' model representations according to the framework derived from several studies (e.g., Blum and Leiss, 2005, 2007; Common Core Standards Initiative (CCSSI), 2010; NCTM, 2000; Niss, Blum and Galbraith, 2007; Pollak, 2003). In order to answer the research questions, I first categorized all actions demonstrated by the preservice mathematics teacher one by one and listed what steps occurred in the translation cycle. Then I refined and established several coding derived from transcriptions of every participant's interviews, which formed basic units of my analysis. After specifying each code, I sorted each of the coded excerpts by themes in order to identify major elements of their thinking. I will also provide excerpts and analysis of student interviews to illustrate participants' thinking. In the following extracts, the abbreviations R used to represents researcher who was also interviewer in this study. In order to ensure confidentiality of the participants, they were given pseudo names and these names are used consistently throughout the text.

In this section, I will report participants' translations process between representations by using themes that emerged from interviews. Each participating preservice teachers' emergent codes and themes as well as explanation of each themes and example from the interviews were given at the beginning of their translation process.

6.1. Case of Tarık

Tarık was one of the high achiever participant of this study. He earned highest total translation score and highest total model translation score on the questionnaire. Besides, He got total 77 points out of 100 in the MRFQ, which is the highest total achievement score among participating preservice teachers. When looking from the questions related to translation of model representation, it can be seen that Tarık got 2 points for the questions that required translations from symbolic to graph, tabular to symbolic, symbolic to model, symbolic to tabular, graphical to model, graphical to symbolic and tabular to model. The only question that he could not manage to get translation point was about translation from tabular representation to graphical representation.

Tarık's emergent codes, themes, explanation of each themes and example from the interviews were summarized in the following tables.

Table 6.1. Tarık's emergent codes and corresponding themes

RELEVANT CODES	THEME
Graph representation Graph completion	Interpretation of graph
Thinking about quadratic functions Thinking about quadratic functions in real life Thinking about quadratic functions in physics	Interpretation of function
Finding a satisfactory answer to what question requires	Make sense of model repr.

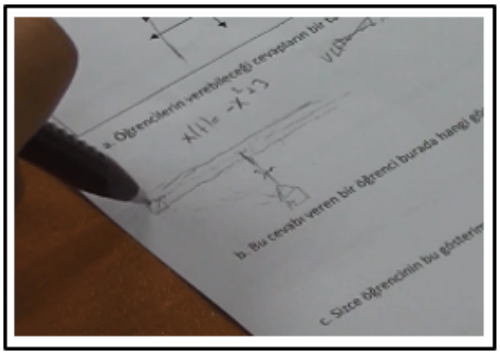
Table 6.1. Tarık's emergent codes and corresponding themes (cont.)

RELEVANT CODES	THEME
IV → time, DV → location Find an x value that can fit negative values on x-axis Thinking about money interest situation IV → time, DV → location IV → time, DV → location for projectile motion	Searching for variable
Writing $-x^2 + 3$ as a representative equation of graph Manipulating $-x^2 + 3$ as it would be a perfect square like $(x - 2)^2$ Searching for an familiar equation Thinking about $\frac{1}{2}at^2$ in projectile motion	Writing equation
Drawing a model to explain location-time example	Drawing a model
Looking interaction between his drawn model and given graph Finding conflict when matching his graph and his model	Alignment of model-graph

Table 6.2. Tarık's theme categories and their definitions

THEME	EXPLANATION	EXAMPLE
Interpretation of graph	Determining the characterization of the graph based on information given in the explanation part	T: I will get a parabolic shape when zooming out the graph since it intersects two points on x-axis.
Interpretation of function	Emphasizing quadratics functions and where it can be observed in daily life	T: Where can I find a quadratic function in real life? Physics come to my mind. I remember that velocity and time have a quadratic relationship. For example, I remember $\frac{1}{2}at^2$.
Make sense of model representation	Searching for meaning of modeling with mathematics	T: What am I supposed to do here if I am required to write a model?
Searching for variable	Looking for different possible variable options for given graph	T: What about location-time graph? R: What do you mean? T: Well.. I can state one of my variable as time and look for change of location in graph.
Writing equation	Writing equations that can represent the given graph	T: If this is a graph looking downward, then its equation may be $-x^2 + 3$

Table 6.2. Tarık's theme categories and their definitions (cont.)

THEME	EXPLANATION	EXAMPLE
Drawing a model	Demonstrating his model by a simple drawing	
Alignment of model-graph	Checking whether written model matches with the given graphical representation	T:I summarized what I have done here. I modelled this graph as motion of ball thrown upward from a point 3 m above the ground with an initial velocity. In addition, this graph shows the change depending on the time. It will go upward and come down. There are two moments that are same as the ground level and those are two points that intersect x-axis on the given graph. The y values would be zero there.

6.1.1. Translation Process of Tarık

Interpretation of the Graph: Tarık started interpreting the graph and completed it as would intersect with two point on x-axis. He used the given information requiring him to intersect x-axis in two different points and identified the graph as quadratic functions. He explained his thinking as *"this [refers the given line] must be one of the legs of the parabola, so the other leg comes from here. We close this parabola so much that we saw here as it is a straight line"* and draw a parabola shown in Figure 6.1.

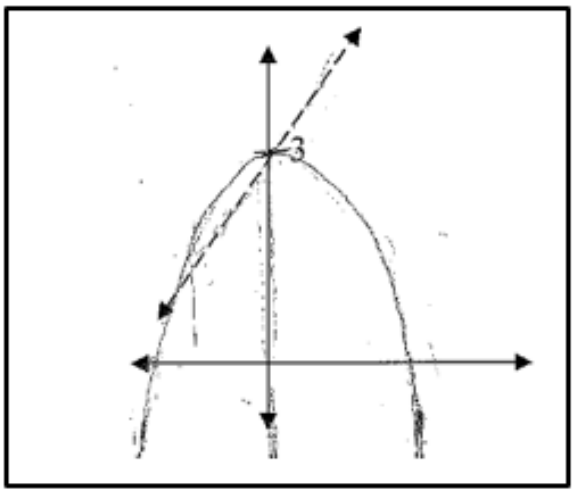


Figure 6.1. Tarık's drawing on quadratic function

Finding Meaning for Model: After that, Tarık asked me what question requires him to do by writing a model representation. I reminded him that the question wants him to write a real world situation.

Interpretation of Function: He started to construct a model representation by first focusing on and remembering properties of quadratic functions since he could infer that the corresponding parabola has two roots and hence it is a quadratic function. He then focused on finding a problem situation that could fit the parabola. It was observed that he tried to recall his previous experience with quadratics and find familiar quadratic problem.

Searching for Variable: He stated that *"I can adapt a simple plan here... location-time graph..."* and then emphasized on creating quadratic equations of parabola by thinking about its' y value, two roots and downward legs.

Writing Equation: He expressed *"If I accept this as a graph going downwards and point here [showing the point 3 on y-axis] as vertex of graph, it could be* $y = -x^2 + 3$*"* and created a symbolic equation in the form of $y = -x^2 + 3$ which was assumed to satisfy the graph.

Searching for Variable: After writing symbolic representation, Tarık tried to assign variables into the equation $y = -x^2 + 3$ from real life so that this model could represent a situation. What he wanted to do is basically match two variables from a real life scenario so that they would be suitable when interpreting via predetermined symbolic equation. When he was asked about his thinking process on deciding relationship between the variables and mathematical terms in his model representation, he answered: *"Since the beginning, I am always thinking in this way; I replaced these $x's$ [showing the equation $y = -x^2 + 3$] with different numbers and it will give me corresponding y values that fit this graph... In other words, I want a variable that will change y values so; I can write a model according to these changing values."*

After this explanation, the first thing he thought was about profit because of the negative values on the y-axis. Tarık proposed that values below the x-axis refer to loss of the money, yet he could not relate negative values on x-axis in his profit-time model representation. Therefore, he could not build up a model representation about profit. In this sense, he again focused on the graphical representation of the data and tried to form an appropriate variable that could be assigned to negative values. Then, he further tried to endeavor to have a coherent and appropriate mathematical model representation, which also could correspond negative values on x-axis. In his next attempt, he focused on the use of location vs. time graphs to describe motion of car.

Drawing a Model: In his model representation (see Figure 6.2), a car moves along the street and the car's changing distance from a house located in the middle of the street with respect to time can be represented in the given graph.

He stated that "*I want to think like this; my house is located here which is far from the road about 3 meters. Then, this car [showing at the left side of the road] moves along this road. If we measure the distance between the house and the car going along the highway as time passes, we can get a location vs. time similar to this graph [showing the given question].*" At this point, I encouraged him to explain what he did when trying to construct a mathematical model representation after he presented it. I wanted him to analyze his location vs. time based mathematical model representation

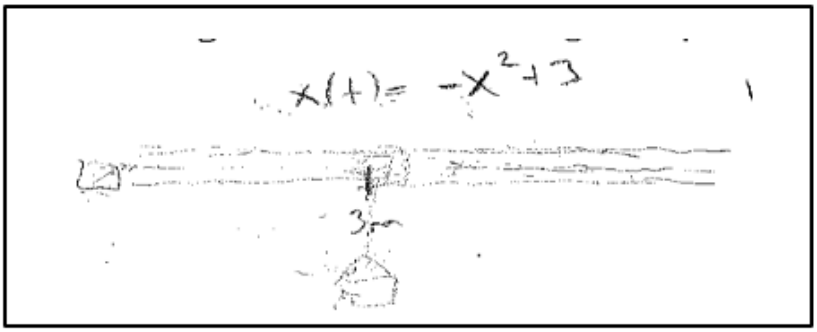


Figure 6.2. Tarık's model representation of house

on the given graph. He moved his pencil from positive to negative infinity on the x-coordinate in order to show car's distance from the house when it moves along the line that he drew. However, he realized his mistake of assigning variables that were inconsistent with the drawn graph when he came to the vertex of the parabola (shown in Figure 6.3.) since the distance of the car must be minimum on y-axis when it align to the house according to his drawing model representation. On the other hand, the y -axis of the vertex represents the maximum value of the parabola in the given graph.

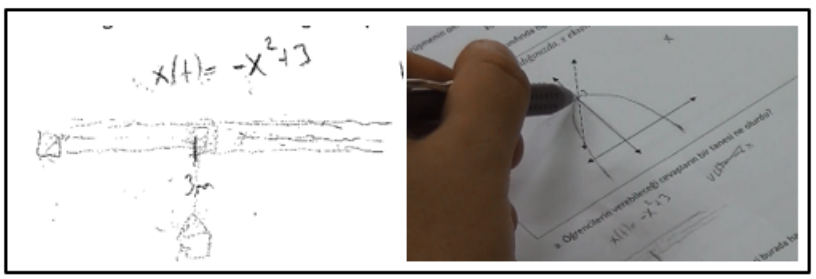


Figure 6.3. Tarık's drawing on quadratic functions

Writing Equation: After he failed in constructing an appropriate and valid mathematical model representation for the given representation, he turned back where he started and thought about the algebraic equation . He asserted that he would recognize a familiar situation, if he manages to make $y = -x^2 + 3$ a perfect square like $y = -(x-2)^2$ or multiplication of two factor like $x.(x-3)$. He said that he could find a familiar situation that can be represented by multiplication of two factors with respect to varying x values.

Interpretation of Functions: Later, Tarık changed his mind and gave up his equation-based approach and started to construct his model representation by directly focusing on real world situations of quadratic functions without thinking about equations. He argued that "OK... *I am trying to remind what kind of real world situations can be described by using parabolic curves... Where have I used quadratic functions in real world? I don't think about this '3' [pointing on the y-axis] at the moment. I am just focusing on where I can use quadratic functions in a real world situation?*"

As shown in the excerpt, he changed his focus from smaller aspects of creating a model representation onto the bigger picture. At that moment, he skipped incorporating with using symbolic algebra and algebraic manipulations. Instead, he preferred to find real life situations that can fit in the given graph and continued discussing: *"The concept of throwing motion in physics comes to my mind. When an object is thrown up, it rises and then falls down. I remember that the relationship between velocity and location of the ball is quadratic. For example, we used to have* $1/2at^2$*. This is a quadratic relationship and it may fit in my problem. I am thinking about throwing an object up in the air and it is coming down since it would be related to... I will say 'time' here, will I? Yes, I will. What if I try to draw location-time graph?"*

Excerpt above showed that in his second attempt, he tried to recall the concepts in the form of quadratic functions and create a story that would match the graphical data rather than establishing a brand new model representation. In this part, he developed an alternative way that would directly focus finding real world situation embedded in the graph instead of using symbolic manipulations. Besides, he suggested an equation $1/2at^2$ representing the laws of motion in physics and asserted that this formula $x = (1/2at^2)$ for the location of falling objects at any time can be applied in the given graph via precisely defining variables. By focusing on this theoretical formula, he started to think about appropriate model representation;

> T: I might think this as a projectile motion (Figure 6.4.) in which a ball was thrown off a certain cliff horizontally with a velocity and coming back. For example, if it comes from here [showing the ground level of platform] and I throw it up to the air and it falls. Because it will be related to time... I will say 'time'

yes. What if I drew location-time graph?

R: What do you mean by location-time?

T: I mean, what if I determine one of the variable here as 'time' and observe the location with respect to time.

R: What do you mean by "variable"?

T: My x-axis. At the very beginning, I tried to change my x values and reach something. In other words, my x values will alter my y values and I try to write a model according to these changing x and y v values...

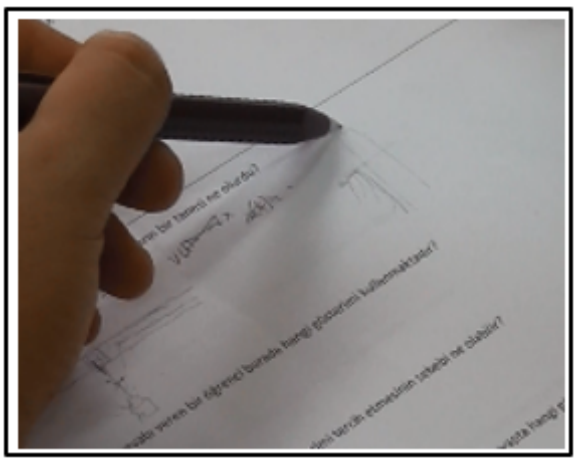


Figure 6.4. Tarık's drawing of projectile motion

As stated in the excerpt, he stated that graphical data could be modelled by projectile motion since he remembered that the objects' velocity and location is quadratic. This time, he controlled whether the variables in the function of motion ($1/2at^2$) can be relevant to given representation. Particularly, he proposed using the existing equation in the physics that is associated with relationship between location and time rather than creating a brand new system. It is important to note that by "ground level", he means the level of platform where the motion of ball starts (Figure 6.5.);

T: If I assume this 'time' [refers y-axis], its square will affect location. I mean, if I called ground level as zero, then there will be a platform like this. For example, this distance will be 3 [refers maximum height].

R: What is platform?

T: This is ground level. I am supposed to throw a ball up to the air from here [pointing ground level]. Here I called 0 [refers ground level] . Then it decreases and increases and fall to the ground.

R: Why do you call there '0'?

T: Since its maximum height is 3 m here [showing vertex], I called here 3 m. So this must be grand level so that it can come and go.

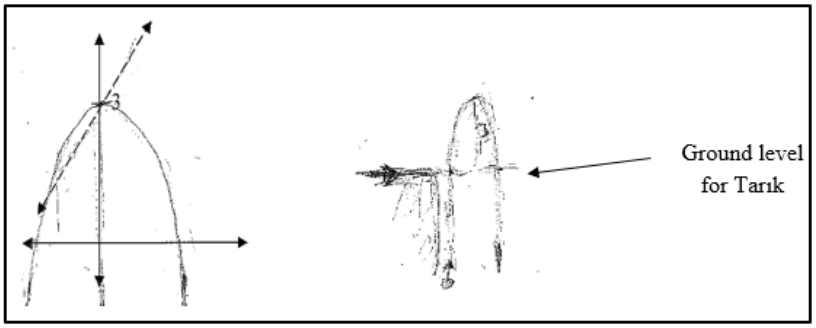


Figure 6.5. Tarık's final interpretation of projectile motion

He proposed projectile motion as mathematical model representation for the given graph. He developed his problem as; *"I modelled this graph as motion of ball thrown upward from a point 3 m above the ground with an initial velocity. In addition, this graph shows the change depending on the time. It will go upward and come down. There are two moments that are same as the ground level and those are two points that intersect x-axis on the given graph. The y values would be zero there."*

After focusing on graphical representation, he generated some real-life situations in which quadratic equations can serve as mathematical model representations and evaluate their suitability in projectile equation. From a physical standpoint, he looked for a pattern resembling one of the theme he knows and he was blindly in the hope getting somewhere on the problem. In other words, he thought a foreknown physics phenomenon including quadratics formula and manipulate its variables according to the graph value rather than formulae a more sophisticated real-world problem. It is important to note that in his both trying of creating a model representation, he needed to generate algebraic representation (as transitional stage) in the first place and focused on its variables even though the questions did not require him to write an equation.

6.2. Case of Çınar

Çınar was the other high achiever participant of this study since she has the second highest model translation score among participating preservice teachers. When looking from the questions related to translation between multiple representations, it

can be seen that Çınar got 2 points for the questions that required translations from symbolic to model, symbolic to tabular, symbolic to model, model to graph, graph to model. She did not get any point for the questions requiring translation from model to symbolic and graphical to symbolic representations.

Çınar's emergent codes, themes, explanation of each themes and example from the interviews were summarized in the following tables.

Table 6.3. Çınar's emergent codes and corresponding themes

RELEVANT CODES	THEME
Graph representation (false assumption) Graph completion Determining slope Determining values for horizontal and vertical axes	Interpretation of graph
Writing problem situation Solve conflicts on variable- graph interaction by creating a new model Writing reasonable real world situation	Making sense of model representation
IV → time, DV → velocity IV → time, DV → location IV → undetermined, DV → location IV → decide to change IV except "time" IV → temperature, DV → temperature IV → time, DV → location	Searching for variable

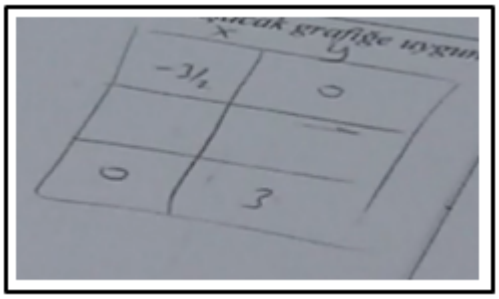
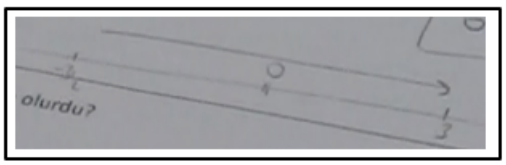
Table 6.3. Çınar's emergent codes and corresponding themes (cont.)

RELEVANT CODES	THEME
IV → time, DV → subjects completed during summer school	Searching for variable
IV → location, DV → time and looking their interaction Manipulation of graph properties to fit graph to determined variables Finding conflict when matching variables on graph	Alignment of variable-graph
Converting graph to table	Converting graph to table
Converting graph to model	Converting graph to model
Interpreting the number on the table Looking for correlation between x and y values Looking for variable that gives "y" value IV → time, DV → location confusion	Alignment of table and variables

Table 6.4. Çınar's theme categories and their definitions

THEME	EXPLANATION	EXAMPLE
Interpretation of graph	Determining the characterization of the graph based on information given in the explanation part	Ç: Well.. It intersects point 3 [on y-axis] at origin. Therefore, I need something at the beginning and then this will come point 0.
Making sense of model representation	Searching for meaning of model representation in mathematics	Ç: When I think about model representation, I suppose that I must a write a mathematical problem.
Searching for variables	Looking for different possible variable options based on the previous interpretation of graph	Ç: I am thinking about what those x'es can be? For example, if I thought them [refers values on y axis] as location, it is zero in the beginning. It moves to 3. Like 3 km [on x-axis]. So, I have to accept -3/2 as time here.
Alignment of variable and graph	Checking whether independent and dependent variables match with the given graph	Ç: Let's repoint here as -3 [referring point -5] in order to make slope 1. Then, something is travelling at 1 km/h. Who is this? A bicycle moves faster than 1 km/h so let's say this is a person, Ayşe. This is Ayşe's velocity time graph during one hour.

Table 6.4. Çınar's theme categories and their definitions (cont.)

THEME	EXPLANATION	EXAMPLE
Converting graph to table	Representing graph by making it a table representation	
Converting graph to model representation	Representing his model by a simple demonstration	
Alignment of table and variables	Checking whether independent and dependent variables match with the created table	R: Does it start at the point $\frac{-3}{2}$? Ç: Yes, it started from $\frac{-3}{2}$ and moves to point 3. It goes on like this. Actually it goes to point 0 not point 3 since it goes to point 0 here [showing y-axis on graph]. So, I need to find what x and y mean. What can $\frac{-3}{2}$ and 0 here?

6.2.1. Translation Process of Çınar

Interpretation of graph: The video recording transcripts of clinical interview indicated that Çınar began her work with interpreting graph by revisiting the problem required her to complete the graph, as intersecting two point on x-axis. However, she misinterpreted that two intersection points can be both on horizontal and vertical lines. Therefore she stretched out the given lines so that it crossed one point on x- axis and one point on y –axis and plotted x-axis as -5 shown in the Figure 6.6.

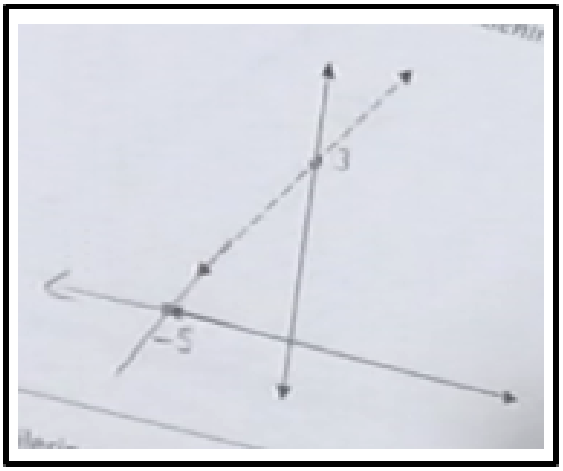


Figure 6.6. Çınar's completion of two intersection points on graph

Searching for variables: In her first attempt, Çınar started describing graph as a linear motion and searching for variables represented by horizontal and vertical lines. She preferred using velocity and time to write a mathematical model representation corresponding to given graph. At this point, the discussion followed: *"In the beginning, we must have something and that thing must increase. Because the slope of the line is constant, it must move with increasing velocity. If this is origin, then I pointed here -5 (Figure 6.5.). That thing started moving from point -5 and when it reaches point 0 [referring origin] its velocity becomes 3. So it is a velocity-time graph."*

Alignment of variable and graph: It is important to highlight that in her first attempt, Çınar verbally discussed what two variables can be in terms of the context of the graph and velocity-location graph occurred in her mind. Yet, when she proceeded to explain how velocity-location model representation works according to graphical representations, she concluded her words by stating y-axis as "time". In order to solve this conflict I asked her;

R: Whose velocity–time graph is this?

Ç: Whose? Let's repoint here as -3 [referring point -5] in order to make slope 1 (Figure 6.7.). Then, something is travelling at 1 km/h. Who is this? A bicycle moves faster than 1 km/h so let's say this is a person, Ayşe. This is Ayşe's velocity time graph during one hour. Her velocity is 1 km/h. She would start at the point -3... at the point 0...Yes after 3 hours later. I am confused. Let's start in the beginning.

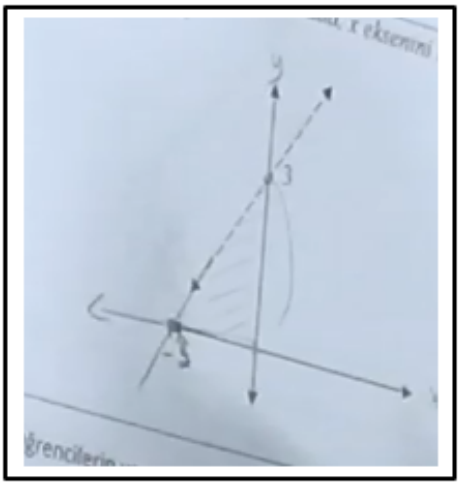


Figure 6.7. Çınar's construction of linear graph

Though she initially presented velocity-time graph belong to a person, she was unclear when focusing on graph-model matching. She explained her model representation as if someone started a motion at the point of -3 and moved on. In other words, she proposed horizontal axis as location, which a moving object travel along, and vertical axis as velocity of the object. However, on the other hand, she then indicated that the horizontal axis represents time, so -3 and 0 are time in hours while the vertical axis represents velocity. She could not complete her words when talking about the situation at the origin since she confused variables; location and velocity. Interestingly, she did not realize her mistake of confusion.

Searching meaning for model: Rather she wanted to try a different model representation since she started to question meaning of model and realized that she could not understand what writing a mathematical model representation exactly mean and she stated that *"When I think about model, I suppose that I must write a problem."* At this point, I reminded her that a mathematical model could be a real world problem that must be represented by the given graph in our scenario.

Alignment of variable and graph: She manipulated the value of x-axis to -3/2 since vertical axis seems longer than horizontal axis (Figure 6.8.). She stated that the distance of points on x, y to the origin do not seem equal in length, and she decided to change her value in order to make them appropriate with the ratio between two axes.

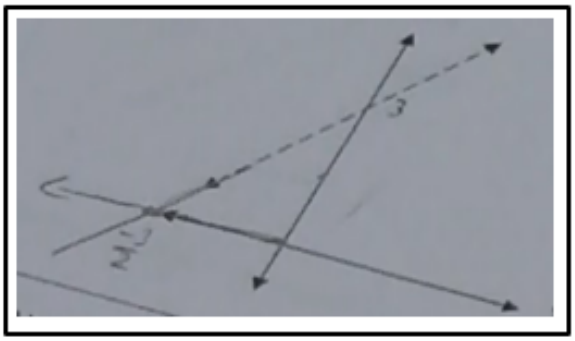


Figure 6.8. Çınar's plotting of x and y axes

Interpretation of graph: She then wanted to determine "the slope of the line". Yet, when she was encouraged to explain why she started to think about slope, she could not properly explain her thinking;

> Ç: This is supposed to be a velocity-time graph and its slope gives me the velocity. Then I will write my model something like that; what is the velocity of a thing that moves with this speed in an hour.
>
> R: Then what do -3/2 and 3 [on the x and y-axis] mean? What does this graph represent?
>
> Ç: This is velocity-time graph. Y-axis will be velocity and x-axis will be time. Ah! If this [referring x-axis] represents time, -3/2 cannot be time. It means it starts before zero point. Then I will not call here time, negative time may cause trouble. Let's think as input and output.

As stated in the above excerpt, Çınar was not able to think clearly when explaining how slope of the line helped her write a suitable model representation. She forgot that the slope of the line on a velocity versus time graph reveals an object's acceleration not its velocity. She maintained her previous misconceptions that refers horizontal axis as location and time. When Çınar explained the behavior of horizontal axis without matching it to graph, she refers it as the location of object. Nonetheless, she explained y-axis as if it represents time when working on graph.

Converting graph to table: Since she did not overcome this confusion, she decided to change her strategy and converted given data on graph to a table. She stated that *"What I expect here is to model this graph. For example, I can show it in this way [referring table shown in Figure 6.9.]*"

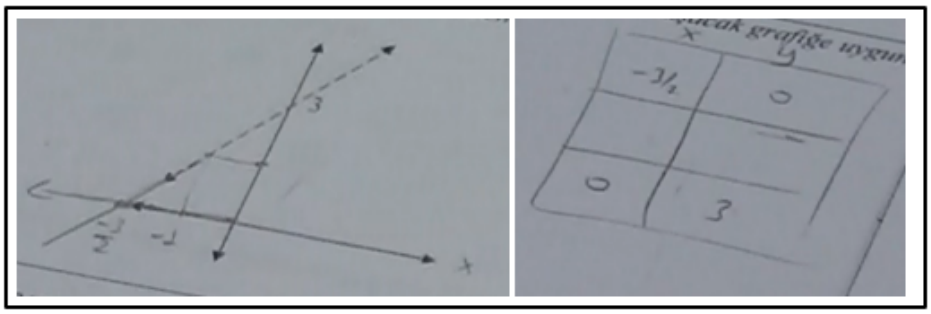


Figure 6.9. Çınar's graphical and tabular representations

Searching for variable: After plotting intersection points on both x and y-axis, Çınar wanted to find the y coordinate corresponds to -1 plotted on x-axis. Hence, she supposed that she could finally perceive the relationship between x values and y values and then reach rule of the function. She stated that *"I will write the function according to slope. If -3/2 goes 0 and 0 goes 3 then there is a relationship here, isn't it?"* At this point, I asked her why she wanted to find slope of the graph in order to write a mathematical model representation. She stated the following;

> Ç: Right now, I tried to model this in different ways. In other words, I try to understand where any point goes on the coordinate plane. I tried to understand the rule of this function. I think...-1 goes 2. So, what am I required to do now?
> R: I want you to write a model suitable with this graph.
> Ç: Like a real life example?
> R: Yes, like a real life example.

Converting graph to an equation: In performing translation between graph and model, Çınar introduced her idea of creating rule of the functions by using slope of the line and intersection points. However, she did not complete this idea.

Searching for variable: She again suddenly started to question process of writing model representation. Even though she could provide some examples such as a filling bottle that have possibility to fit this graph, she was unable to transform these variables into a mathematical model representation.

Converting graph to model: After once again being reminded that the problem wants her to write a real life example she gave up thinking about velocity and focusing

on location of a car starting from the point -3/2 and goes to point 3 as shown in Figure 6.10.

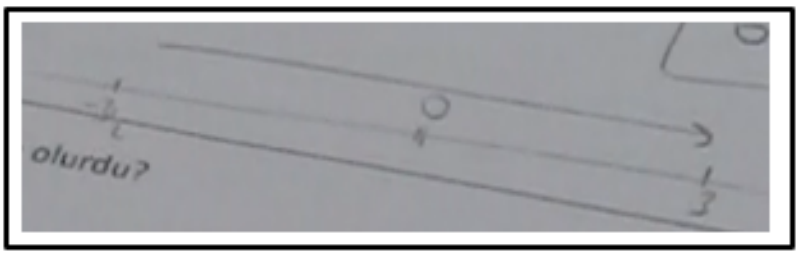


Figure 6.10. Çınar's model representation of a car moving from point $-3/2$ to point 3

However when she started to elaborate the process, she realized that negative values on the x-axis cannot stand for time in her representation. Then the discussion followed: *"Ok. I see... Then what can be represented by negative values? Inasmuch as I start with negative values, how can I associate this negativity in real life? If this is velocity time graph, then time cannot be negative. Maybe I can think about location. If I assume that 0 is the initial point, it may go along this way."*

Alignment between table and variables: When she explained her thinking, she decided to modify her variables and emphasized relationship between x and y values on the graph. In order to better observe covariation between x and y values, she checked her tabular representation and focuses on variables once again.

> R: Does it start at the point -3/2 ?
> Ç: Yes, it started from -3/2 and moves to 3. It goes on something like this.. Actually it goes to 0 not 3 since it goes to 0 here [showing y axis on graph]. So, I need to find what x and y mean. What is 3/2 and 0 here? -3/2 goes 0, what is 2? We have nothing at the time of -3/2.

As pointed above excerpt, she did not operate anything specific with regard to her table representation rather than reading off x and corresponding y values and find place for them on her model representation. Yet, it is important to state that this back and forth translation between tabular representations to model representation helped her correct mistake of putting 3 instead of 0 shown in Figure 6.11.



Figure 6.11. Çınar's model representation of a car moving from $-3/2$ to 0

Searching for variable: However, she was still unable to establish a valid model representation. I encouraged her to speak loudly.

> R: You are thinking about what -3/2 can be?
>
> Ç: Yes. I am thinking about what those x'es can be? For example, if I thought them [refers values on y-axis] as location, it is zero in the beginning. It moves to 3. Like 3 km. Therefore, I have to accept -3/2 as time here.
>
> R: If you accept -3/2 as time, how would it be?
>
> Ç: It may be like this; before the race, it gets far this and during the race it... No, it is not. What if I think a filling vessel? I always refer x –axis as time. Maybe because of this, I have confusion here. This may represent a relationship.

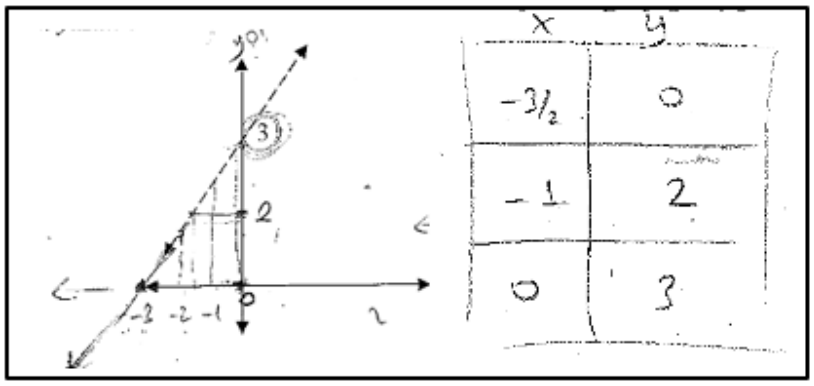


Figure 6.12. Çınar's graphical and tabular representations

According to conversation above, it is interesting to note that Çınar was persistent to maintain her preference of time as independent variable at first even if she stated that time cannot be expressed by negative values. She decided to change her focus from what dependent variable could be into what independent variable could be different from time. When she was finally accepted negative time blocked her creating a valid mathematical model representation, she started to look for a relationship represented by this graph and the discussion followed;

Ç: Let me think x-axis as temperature. At -3/2 degree... For example, this [refers y-axis] can be children' sense of temperature. They feel -3/2 degree as 0 and 0 degree as 3. What about this?

R: How?

Ç: This is the sensible temperature [refers y-axis] felt by children and this is the real temperature [refers x-axis]. They feel -3/2 degree as 0, 0 degree as 3. It seems to me that this may be logical.

R: You are talking about a general situation for children, right?

Ç: Yes. I thought that there might be this kind of sensation. Nevertheless, my problem must be realistic, right?

R: I want you to write a model.

Ç: Then, it is not suitable.

After giving up insisting on using time as independent variable, she changed her focus to find out a different variable that can be represented by values on x-axis. Consequently, Çınar then attempted to use the idea of using temperature and she tried to make it suitable in a model representation. However, Çınar knew that the problem situation that corresponds the values on x and y-axis must be appropriate for a real world situation, her tendency to neglect realistic consideration during mathematical modeling was explicitly revealed. I wanted to remind her that she is required to write a mathematical model representation since she was seemed that she did not know what she was doing. After this reminder, she tried to adopt a different scenario;

Ç: 0 goes 3. Well... Let it be this one; children are prepared for the exam. Here [refers origin] is the time where schools start. Here represents the time of summer school [refers negative values of x-axis]. At moment of -3/2...is the time where children start summer school. Let -1th time be first week. These can be weeks [refers y –axis]. I would change -3/2 to a whole number since these ratios [between x-axis and y-axis] seems to me not important. Therefore, -3 represents week here.

R: Then, why does 3rd week is negative?

Ç: This is the preparation term [refers negative values of x]. Term before the school starts. When the course starts, their knowledge about 12th grade. 1-week passes and it know these much [shows corresponded values of -2 on y axis] after 2 weeks, these much [shows corresponded values of -1 on y axis]. When the school starts [refers origin] they have already known 3 subjects.

R: What do your x-axis and y-axis represent?

Ç: Y-axis represents subjects learnt by students and x represents time in terms of week. Is it OK?

For her last attempt, she decided to correlate mathematics topics covered by students with time in terms of week. She clearly stated dependent and independent

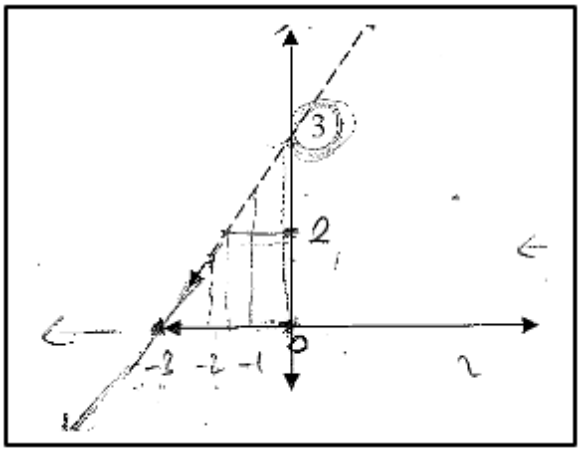


Figure 6.13. Çınar's graph of drawing

variable of the graphical representation. It is important to highlight that throughout her translation process she tried to adopt different variables in the hope that the restatement of variable supported by different scenario somehow conforms the given graph and she waited for approval at the end of her explanation. She generated a tabular representation as transitional stage to model representation, in order to see the behavior of the graphical representation. However, in the end, she focused on the direct reliance between graph and her model representation. As a whole, her process bear a resemblance to what Tarık was doing in terms of trying several real life contexts that might fit the graphical representation and creating an alternative representation to understand the graph.

6.3. Case of Deniz

Deniz was chosen for this study due to having one of the lowest total translation and model translation score among the participating teachers. According to her answers on questionnaire, she failed when interpreting and solving questions related to translation from graphical to model, model to symbolic and model to tabular representations. She could get 2 points when translating from tabular to symbolic, symbolic to model and graphical to symbolic representations.

Deniz's emergent codes, themes, explanation of each themes and example from the interviews were summarized in the following tables.

Table 6.5. Deniz's emergent codes and corresponding themes

RELEVANT CODES	THEME
Difficulty of finding intersection points Difficulty of obtaining parabola in graph	Interpretation of graph
Presenting parabola as a model representation Visualizing movement of roller coaster Mentioning vertex, line of symmetry, upward arms of parabola when explaining the movement of roller coaster	Giving example based on shape of the graph

Table 6.6. Deniz's theme categories and their definitions

THEME	EXPLANATION	EXAMPLE
Interpretation of graph	Determining the characterization of the graph based on information given in the explanation part	D: It will intersect two points on x-axis. It would not be a line. R: What would it be then? D: A parabola
Giving example based on the shape of the graph	Proposing parabola as a suitable function	D: So, the graph will be a parabola. It is appropriate for this graph. Parabola. Parabola is appropriate model for this graph. May I pass second question?

6.3.1. Translation Process of Deniz

Interpretation of graph: When she started to work on the first problem, Deniz pointed to the graph and said that it does not make sense since the shape of the given graph is a line. Though she showed serious effort to understand what questions required and attempted to intersect x-axis with one legs of the parabola, she could not give the meaning of completing this line, as it will be a parabola. She stated the following;

D: I don't understand how it zooms out.
R: This is zoomed image. When you zoom out, you will see that this line intersects two points on x axis.
D: I couldn't see it.
R: What would it be if this line intersects with two points?
D: It would not be a line.
R: What would it be?
D: A parabola

As stated above excerpt, her way of thinking was expected to entail a demonstration of the understanding that in order for a line to intersect two points on x axis, it might be parabola not a line. However, she had difficulty with interpreting the given problem and obtaining parabola by sketching out the given line. She stated that *"OK. It will intersect one point here. Then if I stretch it out ... Am I the one who do not see it? If I stretch it out, it will not intersect another point on x axis."* (see Figure 6.14)

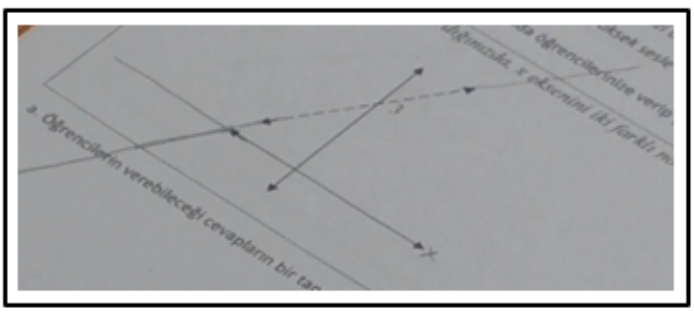


Figure 6.14. Deniz's drawing of a line crossing x-axis

It is interesting to note that, although she knew that the problem necessitated drawing parabola, she was unable to change the direction of the line and curl it up.

In this point, I insisted her to look graph from a different perspective and reminded her that this graph may be a part of larger display of another graph. After given this explanation, she could think that *"OK. Then it may be one of the legs of the parabola*" and make the other leg intersects on the negative side of the x-axis shown in Figure 6.15.

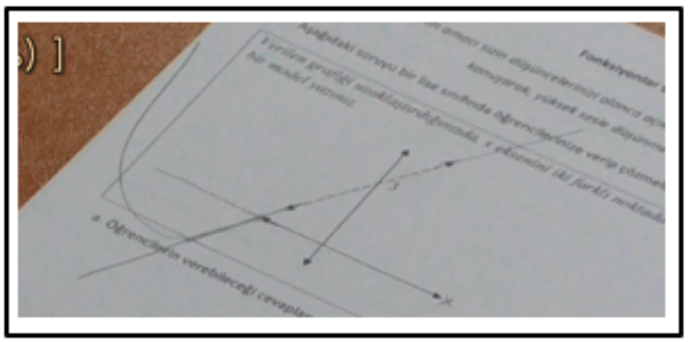


Figure 6.15. Deniz's drawing of a parabola

By doing this, Deniz could transform the given line into a parabola, which intersects two points on x-axis as problem required.

Giving example based on shape: When Deniz was seemed that she was able make sense of problem situations involving creating a model representation, I want to encourage her to keep talking about her thinking on modeling. However, our conversation rolled on into unexpected water:

> D: So, the graph will be a parabola (something like this). It is appropriate for this graph. Parabola. Parabola is appropriate model for this graph. May I pass second question?
> R: The question requires you to write a model like a real life example.
> D: OK. Then it is roller coaster.

Obviously, she first misinterpreted the idea of use of model representation and assumed that parabola can be mathematical model. To better understand what Deniz really knew about mathematical model representation, I briefly explained her what question required her by writing a mathematical model representation and hoped to trigger a discussion on the analyzing the process of modeling. She changed her focus to find a real life phenomenon that is roller coaster in our case. She supposed that

mathematical model representations can be a visual representations of graph in real life scenario.

Interpretation of function: In order to question Deniz regarding the details of construction of model representation, I encouraged her to explain thinking behind the idea of roller coaster the discussion followed;

> R: How do you think that it may be a roller coaster?
> D: Down and up. The shape of the graph reminds me roller coaster... axis of symmetry, upwards legs.

She is likely to think about a model representation based on her perception of the graphical shape. It was obvious that her understanding of concept of mathematical model representation rested on her reliance of finding physically similar real life example to the given graph. More interestingly, she tried to absorb parabola properties such as axis of symmetry and direction of legs in her context of roller coaster even though she never benefited from them.

I hoped that, at some point in our discussion, she changed her focus to find a model representation that might align with parabola properties, her tendency to think of graph shape maintained. It is important to highlight that her thinking is limited to static shape thinking that prohibited her to realize the role of dependent and independent variable as well as coordination between them and propose a valid model representation.

6.4. Case of Bilge

Bilge was the least successful preservice teacher according to her total translation total model translation. She failed to satisfy interpret questions which especially focused on translation of model representations. Her scores showed that she could not get any points in translating from symbolic to model, graph to model, model to symbolic and model to tabular. The two translation pair that Bilge received 2 points were translation from tabular to graph and graph to symbolic representations.

Bilge's emergent codes, themes, explanation of each themes and example from the interviews were summarized in tables.

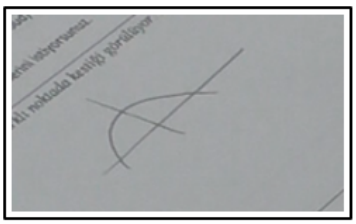
Table 6.7. Bilge's emergent codes and corresponding themes

RELEVANT CODES	THEME
Drawing a parabola independent to original question Pointing vertex	Interpretation of graph
Visualizing movement of roller coaster Visualizing movement of sprinkler	Giving example based on shape of graph
Searching a way to associate her variables and models	Alignment of graph and model

Table 6.8. Bilge's theme categories and their definitions

THEME	EXPLANATION	EXAMPLE
Interpretation of question	Determining the type of the graph (without completing it as it stated in the questions), she would get based on information given in the explanation part characterization of the graph based on information given in the explanation part	B: Since it intersects with two points... It intersects two points on the graph. This is the graph.. When I zoom it out... I think in this way; if x-axis was intersected by two points, it must be a parabola.

Table 6.8. Bilge's theme categories and their definitions (cont.)

THEME	EXPLANATION	EXAMPLE
Interpretation of graph	Determining the characterization of the graph based on information given in the explanation part	R: Can you draw what you would get after completing the graph? B: Something like this; 
Giving example based on the shape of the graph	Proposing parabola and sprinkler as suitable function	B: It can be a top of roller coaster
Alignment of graph and model	Seeking for association between independent and dependent variables and created model	B: I could not associate between roller coaster and this parabola

6.4.1. Translation Process of Bilge

Interpretation of problem: Bilge began with an overview of the problem and tried to understand what was required in the question. She reasoned that the graph would be parabola since two intersection points on x-axis pointed a parabola as question indicated.

> B: [reading question] So the graph most probably look alike a parabola.
> R: Why?
> B: Since it intersects with two points... It intersects two points on the graph. I have one graph here. When I zoom it out... I think in this way; if x-axis was intersected by two points, it must be a parabola.

In this point, it is valuable note that Bilge constructed her arguments only focusing on the written statements of the question. In other words, when Bilge decided to use parabola as an appropriate form for the given situation, she did not consider how a parabola can be formed from a line as presented in the question.

Interpretation of graph: Notwithstanding she concluded that she needed parabola, she was unable to show her work on the given graph. When the idea of a parabola was shown up, I encourage her to complete her words by drawing the parabola and she considered the following demonstration in Figure 6.16.

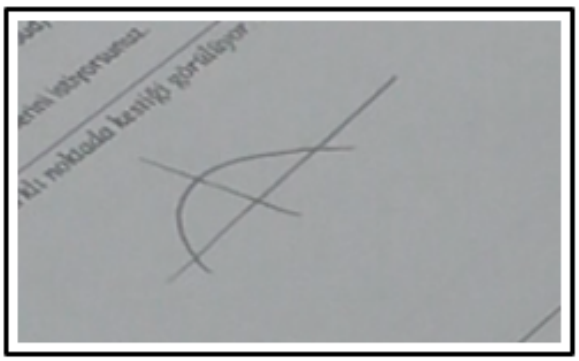


Figure 6.16. Bilge's drawing of parabola

Even though Bilge was able to make sense that the emerging graph must be parabola, she could not draw it on the given line. After creating her parabola, she read the problem once again and started to think about what she was going to do. Giving example based on the shape of the graph: After a long silence, I asked her to think loudly and explain what she was thinking.

R: How do you think?
B: I think about a real life example that resembles parabola.
R: Real life example that resembles parabola.
B: Yes. For example, roller coaster. It goes up. There is a parabola here.

As demonstrated in the above excerpts, Bilge followed a very similar path to Deniz's in modeling the graph and suggested the same concept as a mathematical model representation. Bilge knew that one of the most fundamental component of mathematical model representation is to find a real world situation. However, finding

a real life phenomenon, which looks like the given graph, is enough for her to suppose that she could write a model representation.

Interpretation of graph: She also pointed the vertex of the parabola and stated that the roller coaster move between legs of the parabola. Discussion becomes interesting, when I encourage her to explain her thinking in detail.

> R: OK. How do you decide parabola?
> B: I was thinking about the shape of the graph.. Well, it is not shape.. The parabola has a peak point. Here [referring peak point] there is no x values here. So how I think roller coaster here... When it reached the peak point, yes there is just y value there.[silence]
> R: So?
> B: No, it is not working.

Whereas she was required to explain how her parabola suits the given graph, she started to discuss characteristics of the parabola at the peak point of graph. While showing the movement of roller coaster with his hand, she realized that the corresponding values of x and y-axes on the graph did not make sense with her model representation.

Alignment between graph and model: It was obvious that her understanding of concept of mathematical model representation rested on her reliance of finding physically similar real life example to the given graph. When required to explain how roller coaster was an appropriate model representation for the given graph, Bilge understood that something went wrong with her example; however, she failed to find out what it is. The discussion followed:

> R: Why isn't it working?
> B: Yes. I could not match my model representation and this graph. I could not find attachment between parabola and this model representation.
> R: Why did you think that it could work in the beginning? In addition, how did you decide it is not going to work?
> B: Because it [referring roller coaster] resembles too much to parabola.
> R: So, why did you give up thinking about parabola?
> B: I do not understand what my purpose here is. In other words, I could not find a connection between graph and parabola.

The value of y-axis at the origin reminded her to assess mathematical practices of the question but she could not establish mathematical connections. In order to better understand her knowledge on modeling representations, I asked her is there another model representation that can fit the graph. Our discussion proceeded as follows;

R: Can you say a different model?
B: Another model?
R: Yes
B: (thinking) Sprinkler (shown Figure 6.17).
R: How?
B: They looks like parabola, too. They moved in parabolic path.
R: Can you show your work?

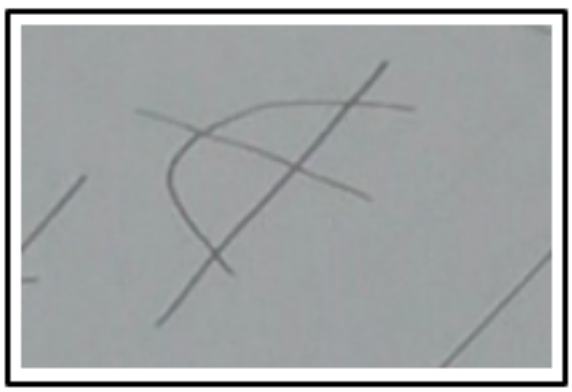


Figure 6.17. Bilge's drawing of model representation for water sprinkler

It is interesting to note that even though she showed awareness that the real life example must have mathematical underpinnings in order to be an appropriate model representation for her, she gave the similar example when I encourage her to create a new one. She was further asked why sprinkler was a suitable model representation and she once again hesitated and stated that she was unable to find a correlation between her model representation and given graph. As she proceeded, she remarked "*If this is roller coaster and if I start here [pointing starting point], I will come vertex. Then I will return where I start. It is same for the sprinkler. The water started to flow here [pointing starting point] and again reach the vertex.*"

Bilge typically embraced an approach that intended to find a physical phenomenon moving in a parabolic path as a mathematical model representation. At some point of interview, Bilge discussed the role of x and y-axis in our problem context

like Deniz yet, she could not touch the idea of variables because of her limited knowledge of mathematical representations. It was obvious that she was keen on focusing on the shape of the graph rather than describing the behavior of roller coaster or sprinkler with dependent and independent variables. In this sense, it is worth to note that Bilge placed a heavy emphasis on finding a real world mechanism or phenomenon that is similar to given representations in terms of their shapes.

6.5. Characteristics of Models

In this section, I will analyze preservice teachers' model representation according to aforementioned framework (see p.36) derived from several resources in order to assess the characteristics of models in this study.

6.5.1. Tarık's Model

After his detailed evaluation of graphical representation, Tarık proposed projectile motion as mathematical model for the given graph. He developed his model as follows;

> T: I modelled this graph as motion of ball thrown upward from a point 3 m above the ground with an initial velocity. Also, this graph shows the change depending on the time. It will go upward and come down. There are two moments that are the same as the ground level and those are two points that intersect x-axis on the given graph. The y values would b zero there.

According to research framework, the first criteria of realistic situation was successfully satisfied since projectile motion is a form of motion in physics. If an object is thrown, then it moves along a curved path. He described a real life situation involving a ball thrown up from an altitude and tried to adapt its changing location with respect to time into the graph. On the other hand, he could not realize that negative values on the x-axis cannot be explained by his model (x-axis was representing time). In his projectile motion, negative values in time stayed meaningless. In this sense, Tarık's model failed to satisfy criteria 3 of independent and dependent variables.

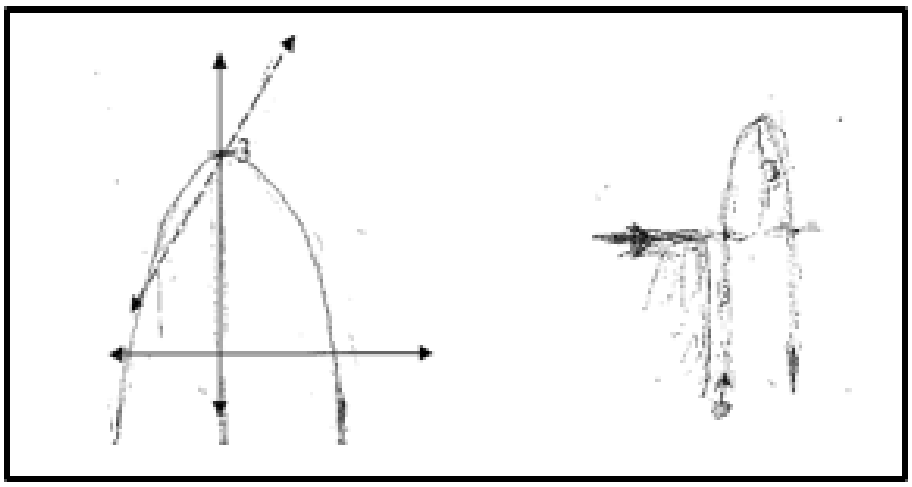


Figure 6.18. Tarık's model of projectile motion

Besides, his model did not fit the given graph. From the point of graph, the ball thrown vertically must reach its maximum height at the same time as the ball dropped and reached the ground. However, when approaching his projectile motion from his drawings, it was seen that duration of time for a ball to reach the bottom from the pick point was longer than the time elapsed to reach the pick point. Therefore, even if the negative values on time made sense, it is obvious that Tarık's projectile motion failed to satisfy criteria 2 of alignment to original representation, which is graphical representation in our case since his demonstration did not match the given graph.

6.5.2. Çınar's Model

The video recording transcripts of task-based interview indicated that when starting to complete the graph, she misunderstood the problem and completed the graph in which one points intersected with x-axis and the other was on the y-axis as shown on the Figure 6.19. She proposed the following model;

> D: Well.. Let it be this one; children are prepared for the exam. Here (refers origin) is the time where schools start. Here represents the weeks of summer school (negative values of x-axis) ... Let -1th time be first week ... so, -3 represents 3thweek before schools start... This is the preparation term [referring negative values of x]. Term before the school starts... When the course starts their knowledge about 12th grade. 1-week passes and they know these much [showing corresponded values of -2 on y-axis] after 2 weeks, these much [showing

corresponded values of -1 on y-axis]. When the school starts [referring origin], they have already known 3 subjects.

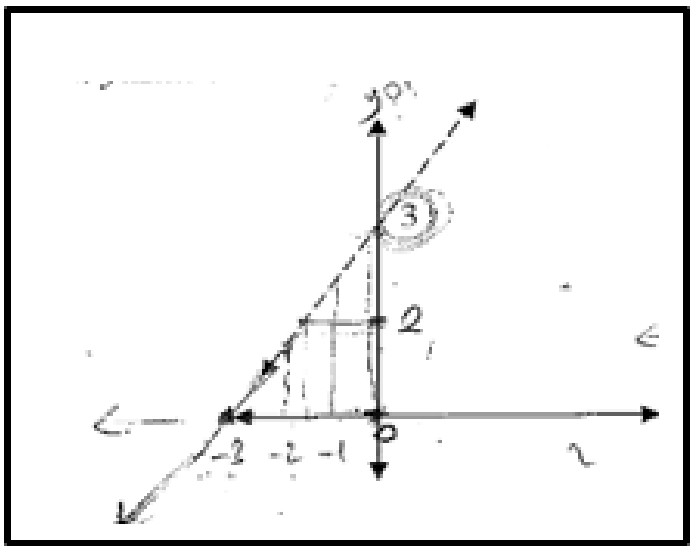


Figure 6.19. Deniz's linear graph for model representation

According to research framework assessing the quality of model, Çınar could find a real life example that can be observed daily life and successfully satisfied the first criteria, realistic situation. Also her example that models subjects learned by students with respect to time (passing weeks) can be observed through the given representation and she again satisfied criteria alignment to original representation. On the other hand, according to her linear function, the number of topics taught to students are the same for each hour, minute and even second interval. This contradicts the reality. In this sense, the third criteria, independent and dependent variables was not satisfied.

6.5.3. Deniz and Bilge's Model

These two participants struggled with both interpreting the given graph and formulating a mathematical model. Not only did they present the same idea as a mathematical model for given graph but also both substantially struggled to give meaning to mathematical model. In this respect, the most dramatic gap concerning translating to a model representation was revealed throughout Bilge and Deniz's interviews. According to their answers, both participants asserted that the movement of roller coaster could be a model for the given graph. They failed to understand the mathematical

thinking associated with real life example. They developed models as follows;

> B: For example, we go up in the roller coaster. There is a parabolic shape there.

Similarly, Deniz stated that;

> D: Down and up. The shape of the graph reminds me roller coaster... axis of symmetry, upwards legs



Figure 6.20. Bilge and Deniz's model representation of roller coaster

None of the criteria was met, both low-achieving preservice teacher indicated that the movement of roller coaster in real life inspired them. Even though it seems that their examples came from a daily life situation, it is important to note that they focused only on the shape of path that roller coaster followed rather than describing the behavior of roller coaster with dependent and independent variables. In this sense, their conception of mathematical model as a real world application is problematic.

7. DISCUSSION

The purpose of this study was to explore four secondary mathematics teachers' ways of algebraic thinking as they worked through a task required them to translate from graphical representation to modeling representation. To address two research questions "What are the thought processes of preservice mathematics teachers with regard to translating the idea of functions from graphical representation to model representation" and "What are the characteristics of model representations constructed by participating preservice teachers when translating from a graphical representation", a questionnaire was first administered to 24 preservice mathematics teachers. Then, interviews of four selected mathematics teachers were conducted to expand our knowledge on preservice teachers' thinking and reasoning on translation between representations. This chapter will provide an interpretation of the findings of this study in light of the literature. I want to discuss three main factors influencing preservice teachers' translation process when they worked through graphical representation to model representation: (1) Conceptual understanding of functions (2) Covariational thinking and (3) Perceptions of model representation.

7.1. Factors Influencing Translation from Graph Representations to Model Representations

7.1.1. Conceptual Understanding of Functions

According to Shulman (1986) the important thing in pedagogical content knowledge (PCK) is to blend subject knowledge and pedagogy rather than treat them as separate entities. In this sense, knowledge of subject matter and pedagogical knowledge alone is necessary but not sufficient for PCK since PCK occurred at the intersection of the sets of content knowledge and pedagogy. The role of teachers in terms of subject-matter knowledge influencing the way teachers provide instruction and overall quality of teaching which in turn will affect students' learning and understanding on this subject is highlighted (Ball, 1991, 2003). To aid students' mastery of mathematics,

teachers must provide students with appropriate representations of ideas, illustrations, analogies, demonstrations and explanations of concepts in order to make them comprehensible (Shulman, 1986).

In this respect, I wanted to investigate preservice teachers' comprehension of functions through their use of multiple representations in this study. Functions are one of the most essential topics in mathematics education since adequate understanding of functions serves a foundation for well-developed algebraic thinking. Preservice teachers are required to embrace profound understanding on language of functions especially that enables them develop flexible links between different forms of representations (Driscoll, 1999; Dreyfus and Eisenberg, 1983). In particular, Moschkovich, Schoenfeld and Arcavi (1993) presented two fundamental qualities that are implication of profound understanding of functions; (i) perceiving functions in multiple ways and (ii) demonstrating the ability to move flexibly among various representation (p. 20). In this context, I wanted to use a task created by Cooney *et al.* (2002) requiring writing a model representation from the graph that intersects with two points on x-axis when zooming out. This high-level task requires participants to think themselves and access relevant knowledge, not just follow the certain rules and guidelines to reach the solution.

Results for this study suggested that mathematically high achieving participants (determined by the pre-test on MRFQ) exhibited competency of recognizing and addressing functions in terms of dependent-independent relationships. Their primary focus were to find correspondence between x and y values as they observed progression of the output values with respect to input values. Both high achieving participants, Tarık and Çınar described and operated functions to create alternative ways to develop modeling representations. They were successful at plotting several points on both horizontal and vertical lines as well as coordinating between graphical representations to tabular representation and tabular representation to algebraic representation. Throughout their interviews, the graphical and algebraic perspective has strong dominance among other representations. It is important to note that both high achieving participants used tabular or algebraic representation as transitional stage to the mod-

eling representations. In other words, they could not establish the direct alignment between graphical representation and model representation without using alternative representations.

On the other hand, low achieving participants of this study, Bilge and Deniz demonstrated an action of function where the graph of parabola is viewed as a curve in the plane and they tried to generate a real life example that looks like the shape of parabola. Even though they were able to recognize that the function was parabola, they focused only on the appearance of the function on x-y plane. It can be concluded that they have some knowledge of functions, which is limited with recognizing shapes of the graph without using its properties. In this respect, I can conclude that they have procedural understanding of functions that allowed them determine what kind of functions they were dealing with but they failed to connect relationship between x and y values of graph and model.

Preservice teachers who participated in this study were recruited from the department of secondary school mathematics teaching program in one of the highest ranked public university of Turkey. According to data from Student Selection and Placement Center SSPC (Öğrenci Seçme ve Yerleştirme Merkezi – ÖSYM) in 2016, the students administering in this college have been from the top 1 among 1.072.396. In other words, the participants were all relatively well qualified in mathematics subjects. Furthermore, preservice teachers who participated in my study completed Calculus I-II- III courses and other advanced level mathematics courses such as Group Theory and Differential Equations. In this sense, it is considerable to investigate how preservice teachers reason and think about functions in multiple ways after years of use in advanced algebra courses. In my study, it is revealed that even the most top ranked preservice teachers' needs professional assistance in terms of building meaningful connections developed between different representations. Usiskin (2001) highlighted discrepancy between teachers' fields of training with respect to advanced mathematics courses taken during undergraduate studies and teachers' content knowledge by asserting "the more mathematics courses a teacher takes, the wider gap between the mathematics the teacher studies and the mathematics teacher teaches" (p. 86). In a similar vein, the

empirical literature suggested that reaching mathematical proficiency in undergraduate study do not guarantee conceptual underpinnings of ideas needed by teachers to teach (Kilpatrick, Swafford and Findell, 2001). When what kind of teachers' knowledge assists teaching and learning in mathematics is questioned, mathematical knowledge for teaching that was defined as a special kind of mathematical work encompassing explaining terms and concepts, interpreting students' solution strategy, using different representations of a concepts and providing examples and proofs for students (Hill, Rowan and Ball, 2005, p. 373) gained attention. Ertas (2014), for instance, reported no significant difference between mathematics pedagogical content knowledge scores of senior mathematics and secondary mathematics teaching education students. Due to the fact that preservice teachers were required to complete mathematical method and general pedagogy courses emphasizing how to teach mathematical concepts, it was expected to see considerable difference in favor of preservice teachers. Though it can be thought that content-focused advanced mathematics courses for preservice teachers during their undergraduate study is enough for mathematical knowledge for teaching, it is considerable to highlight that advanced mathematical knowledge is necessary but not sufficient condition for specialized knowledge for teaching high school mathematics (Zazkis and Leikin, 2010). In this study, by the time I conducted this research with preservice teachers, they had not been taught using multiple representation in their teaching method courses. In this respect, the findings of this study provides evidence of the need for more focused attention to content knowledge in method courses in addition to teaching methods in teaching education courses.

The participants of current study were potential teacher candidates who have chance to learn high school mathematics during their teaching method courses. The instructor of their method course claimed that she designed the course in way for preservice teachers to study high school mathematics by seeing connections between different mathematical concepts as well as pedagogical practices. For example, the main textbook of the course (Rubenstein, Beckmannn and Thompson, 2004) is also designed in that way. One of the goals of algebra unit was to allow preservice teachers to move flexibly between different representations. It is important to note that this study was conducted before participant teachers studied functions or multiple representations in

their method courses in order to prevent the intervention effect. By doing this, I would be able to investigate how preservice teachers reason and think about functions in multiple ways after years of use in advanced algebra courses. The findings of this study agree with the those of Ball and Wilson (1990), Livingston and Borko (1990), Usiskin (2001), Ball, Hill and Bass (2005) and Wilburne and Long (2010) who have drawn attention to preservice teachers' needs for professional assistance in terms of building necessary connections in their content knowledge. Ball (1990) criticized colleges of education with regard to providing pure content knowledge much of what preservice teachers need to know about mathematics. In this sense, Usiskin (2001) highlighted that preservice teachers are needed to acquire mathematical content knowledge different from how they normally receive in college level mathematics instruction since they are expected to teach mathematics with meaning in their student teaching.

Turning to the teacher preparation, Shulman (1986) highlighted the complexities of teacher understanding as well as transmission of content knowledge and advocated that knowing a subject for teaching is much beyond knowing its facts and concepts. In this case, in order to have a robust understanding of functions requires an elaborated comprehension of fundamental function components and the ability to coordinate each components properly. Therefore, college level mathematics instructions for preservice mathematics teachers must emphasize to connect mathematics taught in the high school curriculum.

7.1.2. Covariational Understanding of Functions

An important element of effective mathematics teaching at secondary school is using multiple representations and guiding students to translate between the representations (NCTM, 2000). In a similar context, using multiple embodiments for teaching different mathematical contexts have been dramatically penetrated into high school mathematics program in Turkey along with the curriculum changes in 2013. It is pointed in the curriculum that students' process of conceptual sense making must be supported by integration of multiple representations and educational materials (MoNE, 2013). Therefore, teachers are expected to be adaptive representation users, especially

in algebra. In the case of this research, I wanted to explore preservice teachers' perceptions of mathematical model representation when making translation from graphical representation. When interpreting the graphical representation, preservice teachers' understanding of covariation constituted the central part of our study. In the literature, Carlson *et al.* (2002) defined covariational reasoning as "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354).

In this study, a closer look at participating preservice teachers' covariational reasoning in the context of translating graphical representation to modeling representation revealed the main differentiation between low achieving and high achieving participants of this study. High achieving preservice teachers basically tried to select two varying quantities according to changes in x and y values and label horizontal and vertical axes. Even though, Tarık and Çınar could not state independent and dependent variables explicitly, they have been observed exhibiting specific behaviors that indicate some level of covariational thinking. Carlson *et al.* (2002) grouped covariational reasoning abilities of individuals into five distinct developmental levels (see p.11): Coordination (Level 1); Direction (Level 2); Quantitative Coordination (Level 3); Average Rate (Level 4); and Instantaneous Rate (Level 5). According to this framework, Tarık and Çınar demonstrated covariational reasoning ability at Level 3 (quantitative coordination), in which they could coordinate the amount of changes of one variable according to changes in the other variable by simultaneously tracking the values of two quantities.

Both high achieving participants plotted points and developed tabular charts in which changes in quantities were represented. Whereas Tarık demonstrated mental action of coordinating the non-proportional growth of the height of the objects with respect to the change of the time interval in his quadratic graph, Çınar expressed her awareness of how the amount of change in the number of topics that were covered by students varied according to change of the time in her linear graph. Besides, both students labeled x and y-axis, treated independent and dependent variables in appropriate ways and then explained the direction of the changes (e.g. as time goes by, the number of topics that are covered by student will be increased), that are mental

actions associated with Level 1 (coordination level) and Level 2 (direction level). Even though Çınar could discuss about the average rate of change for a certain time interval at some point, she did not provide necessary evidence that she possessed an understanding of coordinating rate of change that is the indicator of Level 4 (average rate). When Tarık's thought and actions take into consideration, there were nothing about interpreting varying rates of change over intervals of function's domain.

On the other hand, Deniz and Bilge, the low achieving participants of my study interpreted the graph of their functions as a picture of physical phenomenon rather than as a mapping from one domain of a set to a set of output values. In this sense, it is not possible to talk about any mental action associated with covariational reasoning. Therefore, I defined Level 0 (no covariational thinking) in which participants could only recognize parabola by its two intersection points on x-axis and try to find everyday phenomenon that looks like parabola. In this line, Deniz and Bilge will be in Level 0 that no mental action of covariational reasoning takes place. Neither did they seem to be aware of necessity of using properties of parabola, and in fact, they supposed that it has nothing to do with changes between variables of parabola. Their strict reliance of finding a real life phenomenon as visual equivalent of the shape of the given graph prevented them to embrace a holistic perspective in which they could plotted several points and coordinate amount of changes in dependent and independent variables. At the outset of her study, Frank (2017) reached the similar results with university students recruiting pre-calculus course and warned researchers to be mindful of Level 0, no image of coordination. As I expected that all preservice teachers were able to develop more sophisticated act of covariation than first year college students, my results implied that preservice teachers' ability to coordinate amount of change did not develop at all throughout college years.

An important element of effective mathematics teaching at secondary school is using multiple representations and guiding students to translate between the representations (NCTM, 2000). In that sense, teachers are expected to be adaptive representation users, especially in algebra. In my experience, the common behavior displayed by all participants has been construction of graph intersecting two points on the x-axis and

recognition of the type of graph as parabola. The high achieving participants of this study varied in their ability to determine the direction of the change of the dependent variable while monitoring the changes in the independent variable. On the other hand, low achieving participants of this study exhibited behaviors that suggested that they were unable to understand the necessity of coordinate changes in the dependent and independent variable. Actually this findings were in parallel with other studies reporting preservice teachers' trouble with covariation thinking of functions (Carlson, Jacobs, and Larsen, 2001; Carlson, Larsen and Lesh, 2003; Köklü, 2007; Zeytun, Çetinkaya and Erbas, 2010). In the outset of these studies, researchers revealed preservice and in-service teachers' weak covariational reasoning abilities and difficulties in interpreting variable of the graphs. They reported participants' weakness of interpreting continuously increasing and decreasing rate for a physical situation.

When Turkish mathematics curriculum imposed from 2013 is investigated, it can be seen that the basic idea of covariation is assessable from grade 9 objectives as it stated "students will be able to explain relationship between dependent and independent variables". Notwithstanding the curriculum approach which offers to cover covariational understanding in the subject of functions, my results suggested that none of the participants (even high achievers) stated independent and dependent variables. This illuminates participant preservice teachers' lack of flexible and robust covariational reasoning abilities. It is pointed in the literature that well developed understanding of function is possible by exploiting relationships between two variables (Carraher and Schliemann, 2002). In this sense, it can be concluded that preservice teachers' limited conceptual understanding on functions cause them to struggle interpreting the given functions and finding a model representation for the function graph based on the findings from participants.

7.1.3. Participants' Approach to Model Representation of Functions

Demonstrating model representation of functions is not only algorithmic process, it is also related to formulating problem situation, choosing appropriate variables according to given representations, determining the relationships between representations

and verifying the model and its implication. Within this context, integration of model representation as mathematical practice in teaching and learning mathematics is targeted an area of interest in revised mathematics curriculum (MoNE, 2018). Talim ve Terbiye Kurulu Başkanlığı (TTKB, 2017) identified symbolic, model, graphical and tabular representations as a way for developing appropriate solution strategies for real life problems. In this line, effective and fluent usage of model representation is an important practice associated within this goal.

NCTM (2000) underscores the role of using mathematical model in algebra learning and encourage instructional programs throughout Grade 9 to 12 to enable students use mathematical models to represent and understand mathematical relationships. More explicitly, NCTM Standards (2000) for using mathematical models to represent and understand quantitative relationships has been identified as follows; i. "Identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships; ii. Use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts; iii. Draw reasonable conclusions about a situation being modeled." (p. 296)

In this regard, teaching mathematical concepts to students using mathematical model representations of real-life situations has been offered by many scholars as an effective way of meaningful learning (Lesh and Doerr, 2003; Delice and Kertil, 2013). In parallel to this, preservice teachers are required to develop a rich sense of model representation of mathematics in which students create and analyze mathematical ideas and concepts embedded in real world context.

On the other hand, findings of this research demonstrated that participating preservice teachers had difficult time when translating the given representations into mathematical model representation, which is in accordance with the previous findings among preservice teachers (Delice and Kertil, 2013; Doerr, 2007; Eraslan, 2011). Besides, this study implies that even the top ranked students at university entrance exam who relatively higher understandings of mathematics had difficulty in developing and validating a proper model representation.

In this study, preservice teachers had the opportunity to be creative in their approach since the given problem was open-ended and can be solved in many ways. Tarık and Çınar tried to mathematize the graph by using tabular and algebraic forms of multiple representations and go through these representations cycle in order to verbalize a mathematical model. These two high achieving preservice teachers have the ability to change representations fluently. Çınar was the only participant who provided a model, which was aligned with the graph that she drew, even though she made a mistake in completing the graph, and she drew a linear function instead of a quadratic function. When other participants who completed the graph correctly are considered, none of those provided a model, which aligned with the values on the graph. In this point, it is important to highlight that Çınar considered mathematical model as real life scenario and connected her ideas to the real world. However, whenever Çınar came up with a model in her mind and explain it to the researcher, she waited for an approval before validating her solution. In other respects, Tarık did his best in terms of assigning variables and coordinating between variables when it is taught that the task do not fit any model representations. Tarık was the only participants who was able to provide a model, which was suitable for quadratic function. His model were directly taken from a well-known physics phenomenon and he tried to fit his model representation to graphical representation. Similarly, for both high achieving students, it was obvious that they struggled to construct the real world scenario so that they focused on numbers in the given graph, later tabular and symbolic representations that they created in order to do mathematical procedure (Blum, 2011; Crouch and Haines, 2004).

Low achiever participants could not even propose a model representation since their concept definition of mathematical model were not properly structured. They interpreted mathematical model representation as visual equivalent of graph shape in real world situation due to their limited knowledge of quadratic functions. Therefore, they only emphasized on correspondence between the shape of the graph and shape of the movement they imagined in real life.

These finding is in accordance with the findings among college students (Freeman, 2014), graduate mathematics students (Trelinski, 1983). In his studies, Trelinski (1983) observed graduate students' inability to analyze a non-standard problem embedded in chemistry context and construct a model representation for given situation. In a similar vein, Freeman (2014) studied with college students revealed their difficulties when developing a mathematical model representation in quadratic functions. Furthermore, this current study also revealed teachers' limited knowledge on quadratic functions as parallel to Huang (2014). Since not only have they difficulty when finding an appropriate mathematical model representation, but also they could not explain their model with respect to given graph.

In general, this study demonstrated that low achieving participating preservice teachers perceived model representation as visual correspondence of the movement of the graph. These preservice teachers were not aware that their understanding about model representation heavily relied on finding an objects moving in a particular shape rather than finding real-world relationships among quantities that may fit the graph values. On the other hand, high achieving participant preservice teachers have relatively higher awareness about model representations. For this purpose, Tarık focused on adopting a physics phenomenon to given graph whereas Çınar emphasized developing a specific real life situation which would fit the given graph. This different tendency between low-achieving and high-achieving preservice teachers is similar to Thomas *et al.* (2010) findings on expert and novice teachers in which experts emphasized the fundamental characteristics of given functions whereas novices embraced the graph as a whole without identifying the key features associated with the translation while translating between graphical and algebraic representations.

7.2. Limitations and Implications for Future Research

Qualitative method were highly appropriate in my research design since it allowed me to observe the way in which participant teachers interpret what they experience. On the other hand, it is fair to say that generalizability of findings, selected sample and data analysis may have some limitations.

First of all, I believe that the main limitation of this qualitative research is that my findings cannot be extended to wider populations. Because of the nature of qualitative research, findings cannot be generalize "to individuals, sites or places outside of the individuals, sites, or places under study" (Creswell, 2009, p. 193). Besides, participants of this study are not representative of entire preservice teacher since they are just 24 people who are at the top of 1 according to their scores on students' selection and placement test (ÖSS). Especially quantitative test results might have been different if a more homogeneous sample had been taken. Therefore, a further study including larger number of preservice teachers can be administered in order to get results that are more accurate.

Second, researchers' presence throughout the video recording sessions and quantitative data gathering might have affected the way in which participant teachers' responded questions. Because of the fact that preservice teachers have already administered multiple representations of functions questionnaire, they might have thought that they were required to use as many representations as possible when solving problems.

Third, I could spend more time with preservice teachers whom I studied in depth. Because of the fact that the researcher is the primary instrument of data collection and data analysis (Merriam, 2009), interpretations would be much detailed if I could make additional interviews. Concurring to this, future research may emphasize these preservice teachers' classroom practices in terms of their use of multiple representations when they become teachers.

Finally, my interview questions include a task in which a quadratic function embedded in graph, therefore, I could not observe participant teachers' approach to other functions families, which limited the depth of data collected. In this respect, a further study can be extended in use of multiple representations in other families of functions.

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APPENDIX A: SOURCES OF ITEMS OF QUESTIONNAIRE

Table A.1. Sources Table

Item Number	**Adopted From**
1	You, 2006
2	You, 2006
3	Researchers
4	Shell Centre for Mathematical Education, 1985
5	Leinhardt, Zavlavsky and Stein, 1990
6	Shell Centre for Mathematical Education, 1985
7	Researchers
8	Nilklad, 2004
9	Carlson et al., 2002
10	Researchers
11	Shell Centre for Mathematical Education, 1985
12	Bush and Leinward, 2002

APPENDIX B: DESCRIPTION OF ITEMS

Table B.1. Description of items and corresponding content areas

From \ To	Algebraic Rep.	Graphical Rep.	Tabular Rep.	Modeling Repr.
Algebraic Repr.		Item 8	Item 2	Item 7
Graphical Repr.	Item 1		Item 12	Item 5
Tabular Repr.	Item 4	Item 11		Item 9
Modeling Repr.	Item 3	Item 6	Item 10	

APPENDIX C: QUESTIONS OF QUESTIONNAIRE (MRFQ)

1. $f(x) = 6x^2 + 11x - 6$ ve $g(x) = x^2 - 4$ fonksiyonlarının kesiştikleri noktayı grafik çizerek gösteriniz.

2. Aşağıda verilen tablodaki değerler, bir araba modelinin fiyatının yıllar içindeki değişimini göstermektedir. Tabloda verilen bu ilişkiyi bir fonksiyon olarak yazmak mümkün mü? Mümkünse bu ilişkiyi gösteren fonksiyonun cebirsel ifadesini yazınız, mümkün değilse nedenini açıklayınız.

Zaman (yıl)	Fiyat ($)
0	$ 16.800
2	$13,600
4	$10,400
6	$7,200

Figure C.1. Questions relating to translation from symbolic to graphical and from tabular to symbolic representation

3. $y = -50 + \frac{1}{3}x$ fonksiyonuyla ifade edilebilecek bir günlük hayat modeli oluşturunuz. Oluşturduğunuz modelin doğru bir model olduğunu gerekçelendirerek açıklayınız.

4. Farklı firmaların satışa sürdüğü A, B ve C ilaçlarının vücuda girdikleri andan itibaren kandakikonsantrasyonlarının zamana bağlı değişimini ifade eden cebirsel ifadeler aşağıdaki şekilde gösterilmektedir.

İlacın ismi	Kandaki yaklaşık değeri
A	$y = A.(0.84)^x$
B	$y = A.(0.97)^x$
C	$y = A.(0.5)^x$

A: Vücuda giren doz miktarı
y: Kandaki ilaç miktarı
x: İlaç vücuda girdiği andan itibaren geçen zaman (saat)

A, B ve C ilaçlarının dozajı 4 mg olarak ayarlanmış şekilde üç farklı hastaya sunulmuştur. Bu üç hasta için, ilaçların alındığı andan itibaren 3 saat boyunca kandaki yoğunluklarını belirleyen bir tablo oluşturunuz. (hesap makinesi kullanabilirsiniz.)

Figure C.2. Questions relating to translation from symbolic to model and from symbolic to tabular representation

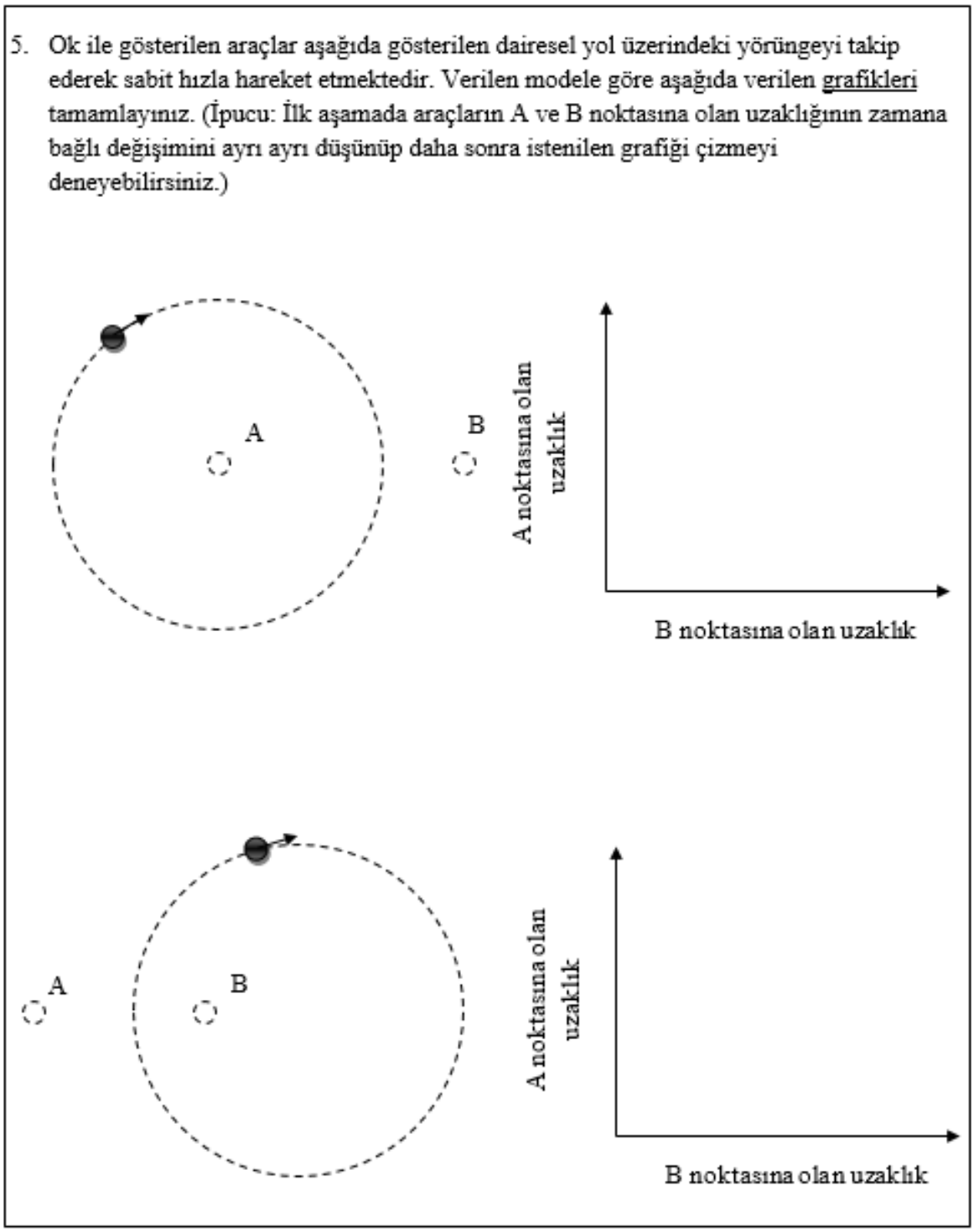
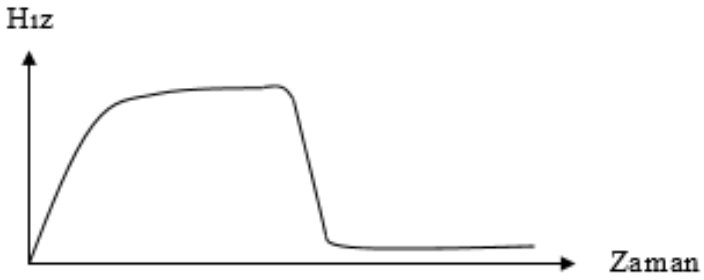


Figure C.3. Question relating to translation from model to graphical representation

6. Aşağıda verilen grafik hangi spor dalı ile ilgili olabilir?



Balık Tutma

Sırıkla Atlama

Paraşütle Atlama

Cirit Atma

7. Ülke gündemini etkileyen önemli bir olaya ait bilgi akışını sağlayan ve kaynak niteliği taşıyan bir gazeteci, kendi sosyal medya hesabında bu olay ile ilgili sürekli olarak bilgi paylaşmaktadır. Paylaşılan bu bilgilerin retweetlenme sayısı ve toplam retweetlenme sayısının zamana göre değişimini gösteren iki cebirsel ifadeler oluşturunuz.

Figure C.4. Questions relating to translation from graphical to model and from model to symbolic representations

8. Aşağıda koordinat sisteminde gösterilen grafik $y = a(x - h)^2 + k$ fonksiyonuna aittir. Fonksiyondaki bilinmeyen $a, h ve k$ değerleri için kabul edilebilir sayılar belirleyiniz.

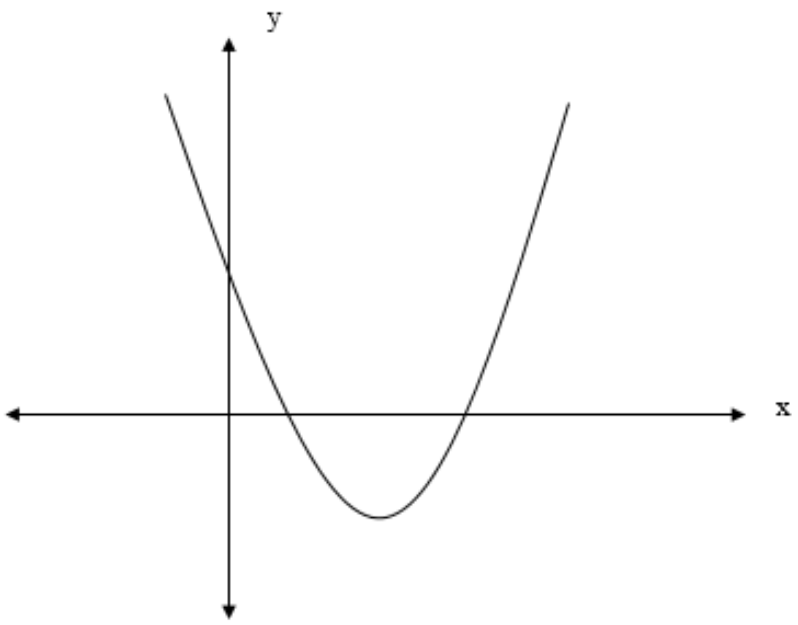
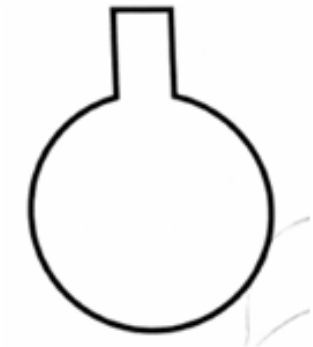


Figure C.5. Question relating to translation from graphical to algebraic representation

9. Aşağıda verilen şişenin su ile doldurulduğunu düşününüz. Şişenin 10 saniyede dolduğunu düşünürseniz, her bir saniye için suyun yüksekliğini tahmini olarak gösteren bir tablo oluşturunuz.



10. Aşağıdaki verilen tablolar iki farklı duruma ait ilişkiyi temsil etmektedir. Her bir tablo için uygun olabilecek günlük hayat modelleri oluşturunuz.

1880	1890	1900	1910	1920
0	1	5	17	30

6	8	10	12	14
$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$

Figure C.6. Questions relating to translation from model to tabular and from tabular to model representation

11. Verilen grafikbir otobandaki 24 saatlik araç sayısını göstermektedir. Bu grafikteki bilgileri bir tabloya aktarınız. Oluşturduğunuz tablonun fonksiyonun tüm hareketini anlamamızı sağlayacak bir tablo olması gerektiğini unutmayınız.

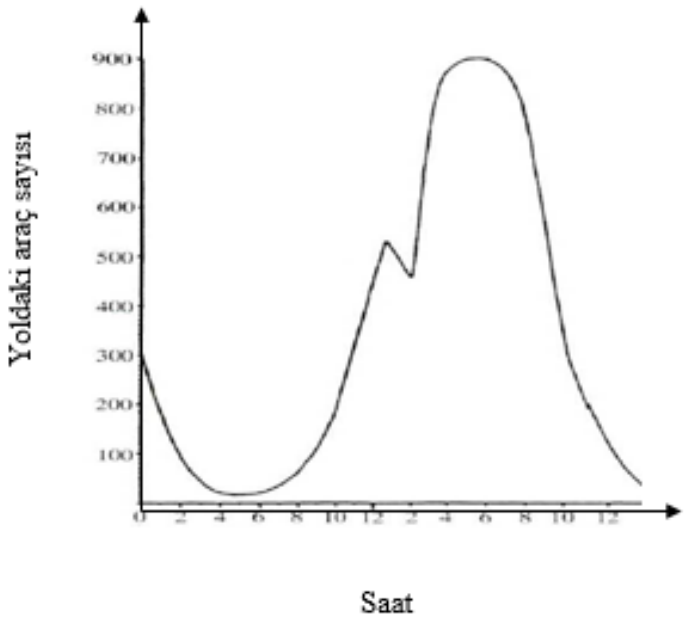


Figure C.7. Question relating to translation from graphical to tabular representation

12. Aşağıda verilen tabloları, en uygun grafiklerle eşleştiriniz.

i. Bir petri kabı içindeki bakteri popülasyonu

Zaman(saat)	2	3	6	10	14
Bakteri Sayısı	4	8	64	1024	16384

ii. Bir çiçeğin büyüme hızı

Yaş (ay)	2	3	4	5	6
Uzunluk (cm)	5	10	17	26	37

iii. Aşırı şeker tüketimi sonrası vücudun tepkisi

Zaman(saat)	1	2	3	4	5
Kandaki şeker miktarı (mg/100ml)	90	75	60	45	30

Figure C.8. Question relating to translation from tabular to graphical representation

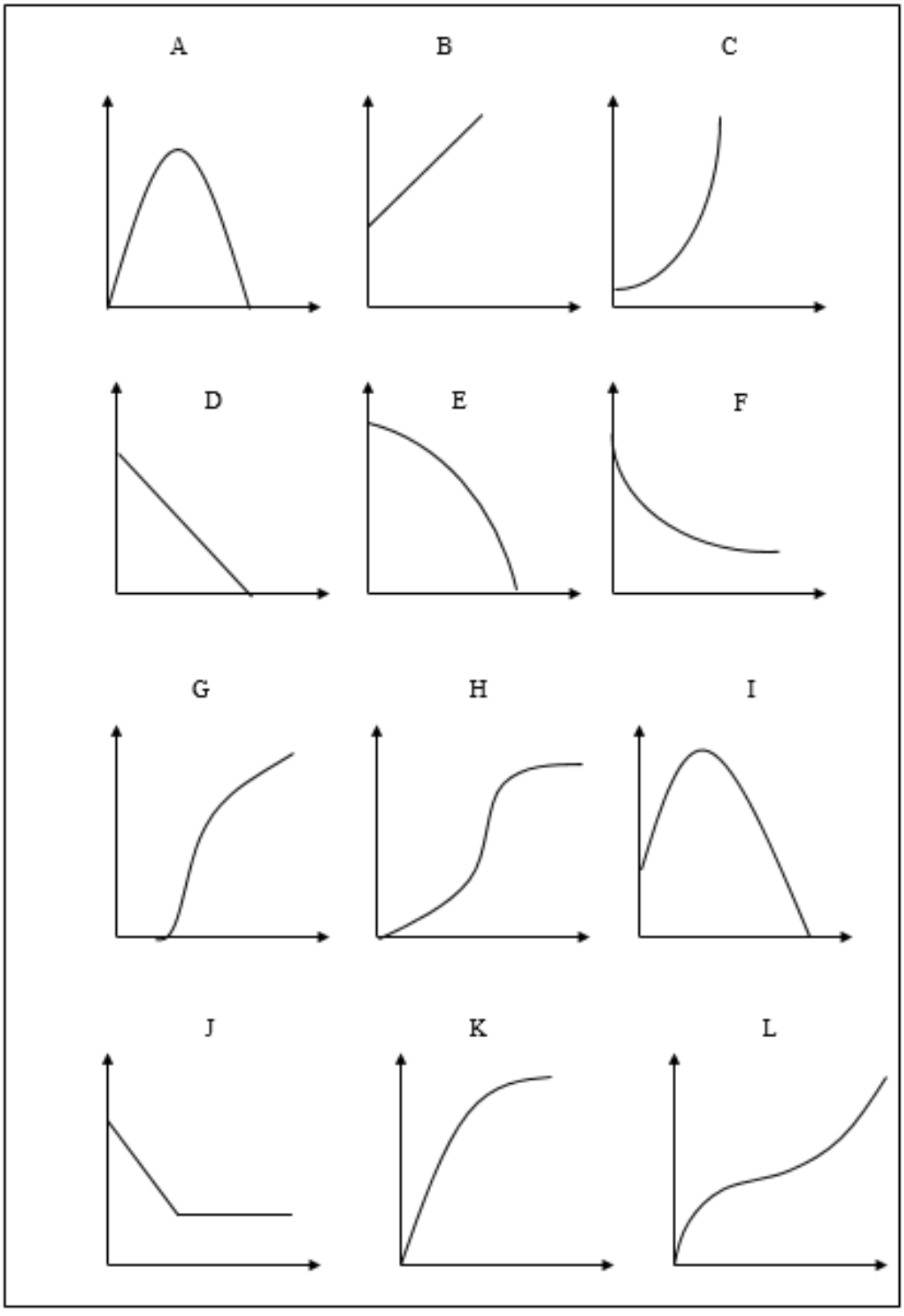
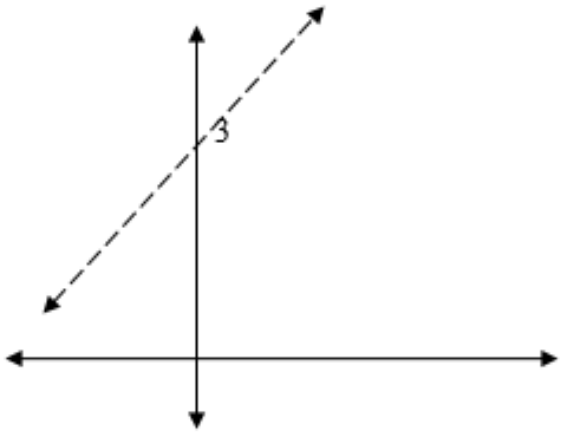


Figure C.9. Question relating to translation from tabular to graphical representation (cont.)

APPENDIX D: TASK-BASED INTERVIEW

Bu soruyu öğrencilerinize verdiğinizi düşünün.

Verilen grafiği uzaklaştırdığınızda, x eksenini iki farklı noktada kestiği görülüyor. Oluşacak grafiğe uygun bir model yazınız.



1. Sizce öğrenciler bu soruyu nasıl cevaplar?

2. Bu soruyu çözen bir öğrenci hangi gösterimi kullanmaktadır?

3. Bu modeli oluşturan öğrenciler nasıl bir yol izlemiş olabilirler?