CLUSTER CONSENSUS FOR NETWORKS WITH ANTAGONISTIC INTERACTIONS

by

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To my father...

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ABSTRACT

CLUSTER CONSENSUS FOR NETWORKS WITH ANTAGONISTIC INTERACTIONS

Consensus refers to the problem of distributed control protocols that enable all agents in a network to reach an agreement on a certain variable or quantity. Cluster consensus is a more generalized form of consensus where, instead of all agents converging to a single common value, they divide themselves into multiple clusters, and agents within each cluster reach an agreement on a common value. This thesis addresses the cluster consensus problem in multi-agent networks operating over arbitrary directed graphs, both unsigned and signed, without imposing restrictive structural assumptions common in existing literature. The research investigates cluster formation and stability across diverse system dynamics, including higher-order linear, time-delayed, inherent nonlinear, and matrix-weighted systems.

A major contribution is the development of a systematic framework, notably utilizing an extended graph representation, which enables the explicit determination of the number of clusters and their members for any given signed directed graph. The thesis provides necessary and sufficient conditions, stability analysis, and control design methods for these general networks and system types. Key findings include the explicit determination of clusters for arbitrary signed networks, handling delays and nonlinearities on general graphs, and analyzing matrix-weighted consensus without topological constraints. By overcoming restrictive structural assumptions and providing specific methods for cluster analysis on general directed networks, this research significantly enhances the understanding and solvability of cluster consensus problems for realistic multi-agent system applications. The theoretical results are supported by numerical examples.

ÖZET

ANTAGONİSTİK ETKİLEŞİMLİ AĞLAR İÇİN KÜME ONAYLAŞIMI

Onaylaşım, bir ağdaki tüm etmenlerin belirli bir değişken veya nicelik üzerinde anlaşmaya varmasını sağlayan dağıtılmış kontrol protokolleri problemini ifade eder. Küme onaylaşımı, tüm etmenlerin tek bir ortak değere yakınsaması yerine, kendilerini birden fazla kümeye ayırdıkları ve her küme içindeki etmenlerin ortak bir değer üzerinde anlaşmaya vardıkları daha genel bir onaylaşım biçimidir. Bu tez, mevcut literatürde yaygın olan kısıtlayıcı yapısal varsayımları empoze etmeden, hem işaretli hem de işaretsiz, rastgele seçilmiş yönlü çizgeler üzerinde çalışan çok etmenli ağlardaki küme onaylaşım problemini ele almaktadır. Araştırma, yüksek mertebeli doğrusal, zaman gecikmeli, içsel doğrusal olmayan ve matris ağırlıklı sistemler dahil olmak üzere çeşitli sistem dinamikleri arasında küme formasyonu ve kararlılığını araştırmaktadır.

Başlıca katkılarından biri, herhangi verilen bir işaretli yönlü çizge için kümelerin ve üyelerinin sayısının açıkça belirlenmesini sağlayan, özellikle genişletilmiş çizge gösterimini kullanan sistematik bir çerçevenin geliştirilmesidir. Tez, bu genel ağlar ve sistem tipleri için gerekli ve yeterli koşulları, kararlılık analizini ve kontrol tasarım yöntemlerini sağlar. Temel bulgular arasında, rastgele seçilmiş işaretli ağlar için kümelerin açıkça belirlenmesi, genel çizgelerdeki gecikmelerin ve doğrusal olmayanlıkların ele alınması ve topolojik kısıtlamalar olmadan matris ağırlıklı fikir birliğinin analiz edilmesi yer almaktadır. Kısıtlayıcı yapısal varsayımların üstesinden gelerek ve genel yönlü ağlarda küme analizi için özel yöntemler sağlayarak, bu araştırma gerçekçi çok etmenli sistem uygulamaları için küme onaylaşım sorunlarının anlaşılmasını ve çözülebilirliğini önemli ölçüde artırır. Teorik sonuçlar sayısal örneklerle desteklenmiştir.

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LIST OF SYMBOLS

$\mathcal{A}$	The unsigned or the signed adjacency matrix
$\bar{\mathcal{A}}$	The extended adjacency matrix of the signed adjacency matrix $\mathcal{A}$
$\mathbb{A}$	The matrix-weighted adjacency matrix
$\mathcal{E}$	The set of directed edges
$\bar{\mathcal{E}}$	The extended set of directed edges
$\mathcal{G}$	An unsigned or a signed directed graph
$\bar{\mathcal{G}}$	The extended directed graph of a signed directed graph $\mathcal{G}$
$\mathcal{G}_{p,i}$	The i-th PLS of an unsigned directed graph $\mathcal{G}$
$\mathcal{G}_{s,j}$	The j-th SLS of an unsigned directed graph $\mathcal{G}$
$\bar{\mathcal{G}}_{p,i}$	The i-th PLS of an extended directed graph $\bar{\mathcal{G}}$
$\bar{\mathcal{G}}_{s,j}$	The j-th SLS of an extended directed graph $\bar{\mathcal{G}}$
$\mathbb{G}$	A matrix-weighted directed graph
I_n	The $n \times n$ identity matrix
L	The unsigned or the signed Laplacian matrix
$\bar{L}$	The extended Laplacian matrix of the signed Laplacian matrix L
$\mathbb{L}$	The matrix-weighted Laplacian matrix
$\mathcal{L}_m$	The system matrix of m-th order continuous-time system
l_p	The number of PLSs for an unsigned directed graph $\mathcal{G}$
l_s	The number of SLSs for an unsigned directed graph $\mathcal{G}$
$\bar{l}_p$	The number of PLSs for an extended directed graph $\bar{\mathcal{G}}$
$\bar{l}_s$	The number of SLSs for an extended directed graph $\bar{\mathcal{G}}$
$\mathcal{N}_i$	The neighbor set of agent i
$\bar{\mathcal{N}}_i$	The extended neighbor set of agent i
$u_i(k)$	The control input of agent i at time step k
$u_i(t)$	The control input of agent i at time t
$\bar{u}_i(k)$	The extended control input of agent i at time step k
$\bar{u}_i(t)$	The extended control input of agent i at time t

$\mathcal{V}$	The set of agents
$\bar{\mathcal{V}}$	The extended set of agents
$\mathcal{V}_{p,i}$	The agent set of i-th PLS
$\mathcal{V}_{s,j}$	The agent set of j-th SLS
$\bar{\mathcal{V}}_{p,i}$	The extended agent set of i-th PLS
$\bar{\mathcal{V}}_{s,j}$	The extended agent set of j-th SLS
$\mathbb{W}$	The matrix-weighted system matrix
$\mathcal{W}_m$	The system matrix of m-th order discrete-time system
$x_i(k)$	The state of agent i at time step k
$x_i(t)$	The state of agent i at time t
$\mathbf{x}_p(k)$	The state vector of the PLSs at time step k
$\mathbf{x}_p(t)$	The state vector of the PLSs at time t
$\mathbf{x}_s(k)$	The state vector of the SLSs at time step k
$\mathbf{x}_s(t)$	The state vector of the PLSs at time t
γ_q	A control parameter
$\lambda_m(\cdot)$	The minimum eigenvalue of a matrix
$\lambda_M(\cdot)$	The maximum eigenvalue of a matrix
$\rho(\cdot)$	The spectral radius of a matrix
τ	Fixed time delay
$\mathbf{0}_n$	The $n \times 1$ column vector with all entries 0
$0_{n\times m}$	The $n \times m$ matrix with all entries 0
$\mathbf{1}_n$	The $n \times 1$ column vector with all entries 1
$\mathbb{R}\mathrm{e}\{\cdot\}$	The real part of a complex number
$\mathbb{I}\mathrm{m}\{\cdot\}$	The imaginary part of a complex number

LIST OF ACRONYMS/ABBREVIATIONS

OLHP	Open Left Half Complex Plane
ORHP	Open Right Half Complex Plane
PLS	Primary Layer Subgraph
SLS	Secondary Layer Subgraph

1. INTRODUCTION

In networked systems, the distributed coordination problem has received increasing attention of researchers due to its broad applicability in real world systems such as social networks, opinion dynamics, formation of unmanned vehicles, autonomous mobile robots, cooperative control of spacecrafts and wireless resource management [1–6]. There exists a vast amount of studies on the traditional coordination problem, namely *the complete consensus problem*, which is convergence of all members of a multi-agent network to a common final value over time. However, the agents may also attain separate groups, called *clusters*. This phenomenon is referred to as *cluster consensus*. The concept of cluster consensus has been also investigated using various terms, including group consensus, cluster synchronization, multi-equilibria consensus and multi-consensus [7–15].

As an illustration, consider the network depicted in Figure 1.1. This network features both cooperative (black arrows) and antagonistic (red arrows) interactions and is not bound by any structural constraints such as *strongly connectedness*, *spanning tree*, *structurally balance* and *digon sign-symmetry* (These terms will be explained in Chapter 2). For the sake of simplicity, communication between the agents is assumed to follow a first-order linear consensus algorithm. Based on this, the state of each agent $(x_i(t),\, i = 1, 2 \ldots, 6)$ is updated as described below [16]:

$$\dot{x}_1(t) = -1\big(x_4(t) + x_1(t)\big)$$
$$\dot{x}_2(t) = 1\big(x_1(t) - x_2(t)\big) - 2\big(x_3(t) + x_2(t)\big)$$

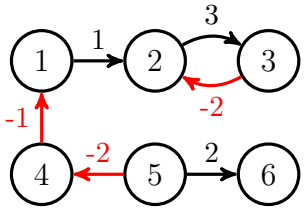


Figure 1.1. A network with cooperative and antagonistic interactions.

$$\dot{x}_3(t) = 3\big(x_2(t) - x_3(t)\big)$$
$$\dot{x}_4(t) = -2\big(x_5(t) + x_4(t)\big)$$
$$\dot{x}_5(t) = 0$$
$$\dot{x}_6(t) = 2\big(x_5(t) - x_6(t)\big)$$

Without explicitly solving the above differential equations, no existing study can determine the number of convergence points (clusters) for the agents in this network. When we simulate the network shown in Figure 1.1, states of the agents evolve as depicted in Figure 1.2. The network converges to 3 distinct equilibrium points. Even for such a small network, determining the number of clusters and their members is difficult; for larger networks, this determination becomes a distinct and challenging problem. In this thesis, our aim is to find a solution to this substantial problem. To this end, any given network is analyzed in different higher-order scenarios such that cooperative networks, cooperative-antagonistic networks, time delayed networks, inherently nonlinear networks and matrix-weighted networks. The following section provides a brief review of some related studies.

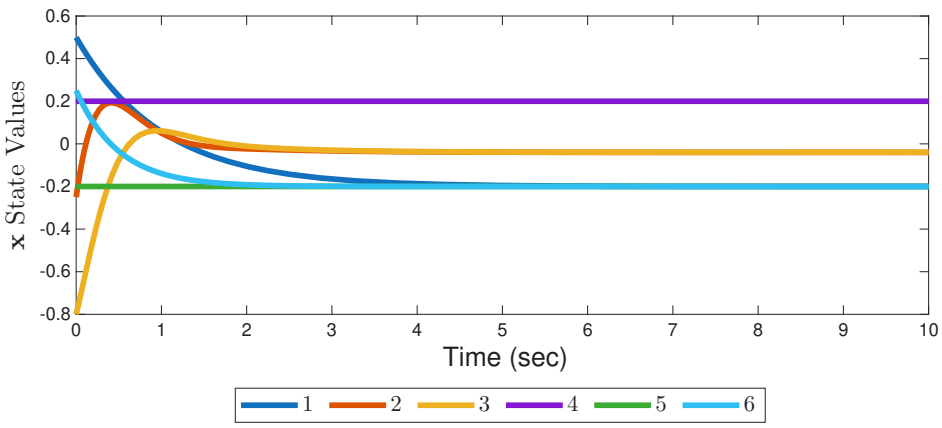


Figure 1.2. State trajectories of agents for the network in Figure 1.1.

1.1. Related Literature

1.1.1. Cooperative Networks

In the literature, most of the existing studies on cluster consensus focus on cooperative multi-agent systems with first-order dynamics [17–21]. A cooperative multi-agent network is a specialized system where agents work together to achieve a common goal or a cluster formation. In these networks, the connections between agents are always considered to be positive weighting. In [17], the authors have proposed a condition which assures that the number of separate clusters is equal to the period of the strongly connected system. Multi-agent systems that are modeled as interconnection of two pre-determined subsystems are examined in [18] for cluster consensus under the strict assumption that the total communication between clusters is limited to zero. The authors of [19] have studied the cluster consensus problem for systems with inter-cluster nonidentical inputs which are required to divide the system into desired clusters. In [20], a cluster consensus algorithm has been proposed that is valid only for two pre-determined clusters. Cluster consensus protocols of nonlinearly coupled systems have been investigated in [21] and a feedback controller is designed to determine the final values of artificially divided clusters.

There is also a substantial number of studies on cluster agreement of systems with second-order dynamics [22–25]. In [22], the authors have designed the weights of the network so that stable second-order cluster consensus is achieved. In nonlinear networks controlled via pinning control, the leader-following cluster agreement problem has been investigated in [23]. The authors of [24] have examined cluster consensus of delayed second-order networks by using Hopf bifurcation theory. In [25], model-based protocols have been proposed for so called scaled cluster consensus in the networks with pre-determined clusters.

Higher-order consensus has been investigated for networks with a spanning tree in [26–28]. However, in such cases, note that we have a single group due to the exis-

tence of a spanning tree. Observer-based cluster consensus has been examined in [29] and it has been found that intra-cluster balanced topologies with antisymmetric inter-cluster couplings promote cluster consensus, regardless of the magnitude of the coupling weights. The authors of [30] have investigated the cluster consensus problem for switching higher-order systems and have stated sufficient conditions to obtain pre-determined clusters under a restrictive assumption on the Laplacian matrix.

All of the cluster consensus research discussed above on first/higher-order systems are valid for specific conditions (e.g. *pre-determined clusters*) or under restrictive assumptions on the graph structure corresponding to the system. However, such conditions and assumptions may not be applicable in real world.

1.1.2. Cooperative-Antagonistic Networks

The complete and the cluster consensus problem are mostly discussed in the context of cooperative systems. In such systems, communication links between agents are assumed to be positive weighting. However, this assumption may not be realistic for some real-world systems such as opinion dynamics, decision making. In these systems, also called antagonistic systems, interaction flow from an agent to another may be weighted positive or negative based on agents' opinions. Positive links represent collaboration of agents while negative links represent antagonistic interaction of agents. Such a topology is represented by *signed directed graphs*. While the literature consists of studies that consider the consensus problem in signed networks, no research investigates the number of clusters and the agents in each cluster for a network of agents with antagonistic interactions. Research on cluster consensus has been mostly restricted to the bipartite consensus and interval bipartite consensus problems [31–35].

Altafini has investigated *bipartite consensus* where all members of a signed network converge to the same final value in modulus but not in sign [31]. Bipartite consensus is a special type of cluster consensus in which the number of clusters equals two. A necessary and sufficient condition to achieve bipartite consensus for strongly

connected networks is that the signed directed graph is structurally balanced and digon sign-symmetric. In [32], the authors have studied *interval bipartite consensus* where the root agents of networks reach bipartite consensus while the non-root agents converge to some value which is smaller than the bipartite consensus value in modulus. To achieve interval bipartite consensus, signed directed graphs associated with networks are required to have a spanning tree. However, in cases where interval bipartite consensus is achieved, the number of clusters and their members are unknown. Opinion dynamics in social networks have been investigated by constructing structurally balanced signed directed graphs in [33]. The authors have considered the case where the opinions are the same (consensus) or totally opposite (bipartite). Signed directed graphs containing a spanning tree have been used to achieve bipartite consensus. However, in real life, there may be more than two opposite opinions. This case has not been taken into account in [33]. Bipartite consensus is achieved for structurally balanced systems using binary cooperative-antagonistic interactions. In [34], the authors have argued that the level of collaboration or competition may be different from binary. It has been shown that multi-partite consensus can be achieved by using weighted cooperative-antagonistic interactions. Interactively balanced or sub-balanced connectivity of directed graph is a necessary condition to reach cluster consensus. In [35], two necessary and sufficient conditions have been derived for structurally balanced and unbalanced systems with weighted interactions. For structurally unbalanced systems, pinning control is required. In [36], the authors have shown that the solution of a signed network dynamics with n agents can be obtained from the solution of the corresponding extended unsigned network dynamics with $2n$ agents. This approach has been used in [37] for consensus analysis of the signed graph containing spanning tree.

As discussed above, research on the cluster consensus problem is mainly limited to bipartite consensus and interval bipartite consensus for networks evolving over signed directed graphs. Except for the bipartite consensus case (where there are only two clusters), there is no study that computes the number of clusters an arbitrary network converges to. Furthermore, structurally balanced directed graph structure is a necessary assumption to reach bipartite or interval bipartite consensus. The existing

results on networks characterized by signed directed graphs are restricted to limited directed graph structures i.e., structurally balanced graphs and graphs consisting of a spanning tree [31–35]. For signed directed graphs with no spanning tree, there is no available result in the literature. Moreover, the existing results [31–35] cannot be directly extended to cover such cases since the graph theoretical concepts used in [31–35] are restricted to directed graphs with spanning tree.

1.1.3. Time Delayed Networks

Most of the studies in the literature have focused on multi-agent systems without any delay. However, delays are unavoidable phenomenons in real-life networked systems due to the limited speed of information transfer. Furthermore, time delays may reduce the convergence performance of systems or even cause instability [38–42].

In [43], the authors have shown that bipartite consensus can be reached in a first-order signed network with arbitrary finite communication delay under the assumption of the directed graph being strongly connected. Second-order bipartite consensus problem with nonuniform input delay has been examined in [44] by using linear matrix inequalities and frequency domain approach. The authors of [45] have studied the first and second-order bipartite consensus problem for network with time varying delays by applying the delay partitioning approach. Two consensus protocols have been proposed in [46] to deal with the bipartite consensus problem for delayed systems with second-order time varying dynamics by transforming the given systems into reduced-order systems. In [47], the authors have proposed some event-based conditions to guarantee the bipartite consensus for balanced networks with first-order discrete-time dynamics under nonuniform communication delay. By the Lyapunov method, the problem of bipartite consensus has been investigated for second-order multi-agent systems with signed directed graphs and nonuniform communication delays under structurally balanced and unbalanced topologies in [48].

As discussed above, most of the cluster consensus studies for delayed systems with signed directed graphs are confined to bipartite consensus. In the case where systems converge to more than two clusters, no study in the literature discusses the number of clusters formed in a delayed network. For delayed networks described by signed directed graphs, restrictive directed graph structures (structurally balanced, strongly connected, spanning tree etc.) are required to obtain the existing results in the literature. Without any structural assumption on signed directed graphs, no results can be found in the existing literature for the cluster consensus problem in delayed networks.

1.1.4. Inherently Nonlinear Networks

While the cluster consensus problem is frequently formulated within a linear framework in the literature, real-world agents of networks demonstrate nonlinear dynamic properties. In [49], the single consensus problem has been investigated for first and second-order cooperative systems with inherent nonlinear dynamics that satisfy a Lipschitz condition. Sufficient conditions have been derived such that networks evolving over directed graphs with a spanning tree achieve consensus in [50]. However, the case where the directed graph does not contain a spanning tree has not been considered in [49, 50], which has been studied in [51–56] as the *containment control problem*. In containment control, the goal is to canalize followers to the dynamic convex hull spanned by multiple leaders. While the literature consists of studies that consider the containment control problem, no research investigates on grouping of followers and the number of groups. In [51], an algorithm based on high-frequency feedback robust control for second-order nonlinear networks has been proposed to ensure that all followers stay in the dynamic convex hull spanned by the leaders. The distributed containment control problem has also been investigated for second-order network with inherent nonlinear dynamics in [52]. Two control methods have been derived to guarantee that the followers converge to the dynamic convex hull space. However, the problem of how many clusters the followers converge to has not been investigated. In [53], the authors have established a distributed event-triggered control law for nonlinearly networked

cooperative systems to guarantee that the containment objective is achieved in fixed time without dependence of initial states of agents. In [54], a method on selection of control gain has been developed to realize containment control of the followers utilizing inherent nonlinear dynamics. While this method ensures that the followers stay in the area spanned by the leaders, it does not give any further information about the behaviors of the followers in that area. The authors in [55] have designed a centralized method to achieve the containment control objective and extended their results to the decentralized counterpart for second-order nonlinear systems. In [56], the same problem has been investigated for high-order networks with inherent nonlinear dynamics. The authors have proposed an adaptive method based on the relative state information to keep the followers in the dynamic convex hull spanned by the leaders. In [57], a distributed algorithm has been proposed to ensure that the followers converge to the region covered by multiple leaders. A control protocol has been presented to achieve containment control for networks with nonlinear dynamics in [58] and the finite convergence time has been analyzed. In [59], the authors have investigated the second-order consensus problem for systems having inherent nonlinear dynamics. For strongly connected networks, a new generalized algebraic connectivity has been characterized to derive some sufficient conditions for attaining consensus.

As discussed above, research on multi-agent systems [49–59] with inherent nonlinear dynamics is limited to ensure that the followers converge to the dynamic convex hull formed by the leaders. There is no study in the literature that computes the number of clusters of the followers and leaders converge to.

As previously delineated, most of the studies, addressing multi-agent systems with inherent nonlinear dynamics, is confined to cooperative network applications. However, real-world systems may not always be cooperative and may contain antagonistic communication links. Such systems are modeled with *signed directed graphs*.

In [60], the authors have proposed a distributed control algorithm for networks of inherently nonlinear agents characterized by signed directed graphs. The algorithm,

designed for agents with Lipschitz nonlinear dynamics, ensures that balanced signed networks reach bipartite consensus. The study [61] has investigated the overall behavior of an inherently nonlinear network with antagonistic interactions and switching topology. Assuming balanced signed directed graphs and one-sided Lipschitz condition for the nonlinear system, the authors have proved that bipartite consensus can be achieved by utilizing matrix measure and contradiction theory. In [62], the bipartite consensus problem has been examined of signed second-order systems with inherent nonlinear dynamics and delay. However, the result is valid with the assumption that the network can be structured into two distinct subgroups. The authors of [63] have explored the consensus problem for a group of first-order agents within a network that includes both cooperative and antagonistic interactions. To achieve consensus, they first separate the network into two parts (cooperative and antagonistic sub-networks), then develop time-delayed control protocol specifically for the antagonistic sub-network by using Lyapunov theory.

All the above mentioned studies (for cooperative or antagonistic networks) are limited to containment control or bipartite consensus. Current methods are unable to determine the composition of clusters or establish stability conditions for cluster consensus when dealing with inherently nonlinear dynamics on arbitrary signed directed graphs.

1.1.5. Matrix-Weighted Networks

Due to their practical use in diverse fields, consensus algorithms have received considerable research attention in the last two decades (see, [64] and the references therein). The majority of these studies address the case where the weights used in the state update laws of the agents are scalars. However, the weights in the network are represented by matrices in important application areas including synchronization of coupled oscillators [65–67], distributed moving horizon estimation [68], formation control [69–71] and coordinated movement [72] of multi robot systems.

In [73], the *matrix-weighted consensus algorithm* has been proposed for undirected networks with positive definite and positive semi-definite weight matrices and it has been shown that the multi-agent system admits average consensus provided that the network consists of a positive tree. Structurally balanced networks have been studied by the authors of [74] who stated that bipartite consensus is achieved as long as the null space of the weighted Laplacian matrix and the bipartite consensus subspace are the same. This algebraic condition is also shown to be equivalent to the existence a positive-negative spanning tree in the network. In [75], matrix-weighted consensus has been investigated in discrete-time switching networks where the weight matrices are assumed to be positive definite or positive semi-definite. When symmetric matrix weights are used by the agents, it has been shown that average consensus is established if the union of switching graphs over successive time intervals with the same length contains a positive spanning tree. The results have been also extended to the case where the matrix weights are not symmetric, but equal for all neighbors of an agent. The authors of [76] have exploited matrix-weighted consensus in unbalanced networks and stated the conditions which guarantee bipartite consensus. It has been shown in [77] that the structural balance is not necessary for bipartite consensus for matrix-weighted networks with symmetric weight matrices. The authors have used the non-trivial balancing set approach to provide a necessary and sufficient condition for bipartite consensus. In [78], the bipartite consensus problem has been explored within the context of second-order matrix-weighted networks having structurally unbalanced property. In [79], matrix-weighted consensus problem has been transformed into a multipartite consensus problem so that each state of the former problem is considered to be an agent in the latter. The authors have also proposed a method to unite multiple subgroups into a larger group where the agents in the final group achieve average consensus. Matrix-weighted consensus problem has further been investigated in undirected networks with varying topology [80, 81], event-triggered communication [82, 83], time-scaled graphs [84], sampled data [85] and second-order dynamics [86–90].

While matrix-weighted consensus has been extensively investigated for undirected networks in the recent years, relatively little research has been carried out for networks

represented by directed graphs containing spanning trees [81,91]. Although the studies in [81, 91] consider the matrix-weighted consensus algorithms evolving over *directed graphs*, they have investigated the conditions for which consensus on a single or two equilibria (bipartite consensus) is established. On the other hand, it is well known that agreement on a single equilibrium state may not be possible even when agents utilize scalar weights. The above results are limited to bipartite consensus for networks consisting of spanning trees.

1.2. The Motivation of the Thesis

Existing literature on multi-agent systems operating over various network structures, including cooperative and antagonistic networks, presents significant limitations, primarily focusing on containment control or bipartite consensus. A major gap exists in the ability to determine the composition of clusters and establish stability conditions for cluster consensus when dealing with diverse system dynamics, such as linear, delayed linear, inherently nonlinear or matrix-weighted behaviors. All of the cluster consensus research discussed above on first/higher-order systems often rely on specific, sometimes restrictive, conditions such as pre-determined clusters or strong assumptions about the network's graph structure. Furthermore, research into cluster consensus on cooperative-antagonistic networks is largely confined to bipartite and interval bipartite consensus. Critically, beyond the specific case of bipartite consensus (where there are only two clusters), there is a distinct lack of methods to compute the number of clusters that an arbitrary network will converge to. Again, existing results in this area are heavily dependent on restrictive graph structures such as structurally balanced or spanning tree. The challenges are similarly pronounced in the study of delayed cooperative-antagonistic systems. Most existing work is limited to bipartite consensus, and there is no literature addressing the number of clusters formed in delayed networks that converge to more than two clusters. The problem of cluster consensus in delayed networks without any such structural assumptions on graphs remains unaddressed in the existing literature. Similarly, research concerning multi-agent systems with inherent nonlinear dynamics has predominantly focused on ensuring containment control,

where followers converge within the dynamic convex hull of leaders. However, these studies do not provide methods to compute the number or composition of the clusters that the followers and leaders converge to. Similarly, the study of matrix-weighted consensus, while extensively investigated for undirected networks, has seen relatively little research for networks represented by directed graphs containing spanning trees. The motivation for this thesis stems directly from these identified shortcomings. This research aims to push beyond the boundaries of current knowledge by investigating the higher-order cluster consensus problem for multidimensional multi-agent systems under more general and realistic conditions. Specifically, this thesis seeks to develop novel methods applicable to networks with linear, delayed linear, inherently nonlinear and matrix-weighted dynamics operating over arbitrary unsigned or signed directed graphs, including those that do not satisfy traditional structural assumptions like the presence of a spanning tree, strong connectedness, structurally balance or digon sign-symmetry. A key focus is to provide the means to compute the number of clusters to which such systems converge and to identify the members belonging to each cluster. By relaxing restrictive structural assumptions, the conclusions of this thesis will be broadly applicable to a wider array of real-world multi-agent system applications.

1.3. The Contribution of the Thesis

The novelties of each chapter of this thesis can be summarized as follows:

1.3.1. Contributions of Chapter 2

In Chapter 2, we investigate the higher-order cluster consensus problem for multi-dimensional multi-agent systems whose results are published in [92]. The main contributions of this chapter can be summarized as follows:

1) Given a multi-agent network evolving over an arbitrary directed graph with any order of continuous-time dynamics, necessary and sufficient conditions are determined on the proposed distributed control law so that cluster consensus is

achieved.

2) For a given graph, the number of clusters and the cluster members are determined explicitly. It is also shown that the final values of the group members depend on the left and right eigenvectors of the Laplacian; and that it does not depend on the controller parameters.
3) Special cases of the general results are considered for second and third order systems. It is shown that a specific choice of controller parameters that ensures cluster consensus always exists.

1.3.2. Contributions of Chapter 3

Chapter 3 is the first study in the literature that addresses signed directed graphs that have no spanning tree whose results are published in [93]. We consider a general network characterized by positive and negative interactions to overcome the limitations in [31–35] to address the following fundamental questions:

- Given a multi-agent network evolving over a signed directed graph with no assumption on the graph structure, does the network achieve cluster consensus?
- If so, how many clusters are formed and what are the members of each cluster?

To address the above fundamental questions, we introduce the extended directed graph representation for the analysis of the corresponding signed directed graph. In this context, the contributions of Chapter 3 can be highlighted as follows:

1) By exploiting the lifting approach in [36], we first relate the dynamics of any signed directed graph (possibly with no spanning tree) to those of an unsigned directed graph and then study the convergence properties of the associated system that contains agents with higher-order dynamics in continuous-time.
2) For any given first or higher-order system with a signed directed graph, we compute the number of clusters and explicitly determine cluster members.
3) For higher-order systems, controller parameter selection for signed networks is

investigated and conditions that guarantee the cluster consensus are stated.

4) The above results are extended to discrete-time systems.

1.3.3. Contributions of Chapter 4

Chapter 4 investigates the cluster consensus problem for signed networks composed of agents with higher-order integrator dynamics, considering the effect of time delays. The related results are published in [94, 95]. We depart from prior works by considering networks with unrestricted structural properties, including those that may not exhibit characteristics such as spanning trees or strong connectivity. The main novelties of Chapter 4 can be summarized as follows:

1) The delayed third-order systems evolving over any given unsigned directed graph is considered and a non-empty set of control parameters is given.
2) The cluster analysis of the delayed third & higher-order is carried out and the number of clusters is explicitly determined.
3) Through the utilization of extended directed graph representation, the relation between signed networks with fixed input delay and their extended representations is examined. Furthermore, the convergence properties of these representations are analyzed within a continuous-time framework.
4) The stability criteria for unsigned higher-order continuous-time integrator networks, featuring fixed communication delays, are derived. In the case of a higher-order continuous-time integrator system represented by a signed directed graph without any special graph structures, the stability characteristics are obtained through the utilization of the extended directed graph representation.
5) In continuous-time delayed higher-order integrator systems, the maximum delay bound is identified to enable cluster consensus. This analysis includes computation of the number of clusters and the members within each cluster.
6) Analyzing higher-order discrete-time integrator systems with nonuniform delay displays their convergence properties, demonstrating consistent performance even in the presence of bounded delays.

1.3.4. Contributions of Chapter 5

In Chapter 5, we discuss the cluster consensus problem and investigate first and second-order cooperative and antagonistic networks with inherent nonlinear dynamics whose results are published in [96]. We develop methods that compute the number of clusters to which systems converge and the members of each cluster. The contributions of Chapter 5 can be summarized as follows:

1) In first-order networks with nonlinear inherent dynamics, a sufficient condition is obtained so that the systems can achieve cluster consensus.
2) Sufficient conditions on controller parameters are derived to attain cluster consensus in second-order inherently nonlinear systems.
3) For a given first or second-order inherently nonlinear network, the number of clusters and their members are explicitly computed.
4) Using the extended graph representation, we create techniques to determine both the number of stable clusters formed by the systems and the composition of each cluster.

1.3.5. Contributions of Chapter 6

In Chapter 6, we investigate matrix-weighted cluster consensus problem evolving over directed graphs. Contrary to the existing literature on matrix-weighted consensus, our work does not have any assumptions on the network topology such as existence of a spanning tree or graph being strongly connected, etc. The vast majority of studies in the literature investigate the problem for *unsigned* and *undirected* networks to enable the networks to reach matrix-weighted *single* consensus. Besides this, *signed* systems are analyzed under the assumption that the related graph is structurally balanced. The few existing studies address the problem for systems characterized by *directed* graphs. These studies are insufficient to determine the system's behavior when the system does not reach *single* consensus. Our study eliminates this inadequacy by calculating the number of clusters and determining the members of each cluster.

With this motivation in mind, the main novelties of Chapter 6 can be summarized as follows:

1) The cluster consensus problem is analyzed in *unsigned* directed networks where edge weights are positive definite. The number of clusters and the members of each cluster are theoretically determined.
2) Unlike many classical results that assume the presence of a directed spanning tree or strong connectivity, such restrictive assumptions are removed in this chapter. This allows us to explore cluster formation in weakly connected directed networks.
3) Our theoretical contribution provides a systematic method for selecting weight matrices to satisfy the eigenvalue condition of the Laplacian matrix.
4) These results are extended to *signed* network by including negative definite weights in addition to positive definite weights.
5) For discrete-time systems, we derive explicit conditions for step-size parameter selection to ensure stability and convergence to a matrix-weighted cluster consensus. We propose a distributed method for step-size computation, which requires only local information and an upper bound on the total number of agents.

1.4. The Organization of the Thesis

This thesis is organized as follows. Chapter 2 provides a study of the higher-order cluster agreement problem for networks evolving over arbitrary unsigned directed graphs. It derives necessary and sufficient conditions for achieving non-pre-determined clusters and demonstrates that appropriate control parameter choices do not influence the number or final values of the clusters.

Chapter 3 focuses on the cluster consensus problem for networks with antagonistic interactions modeled by signed directed graphs. Utilizing an extended graph representation with a lifting approach, it relates the system trajectories to an extended system on a positive directed graph. The chapter explicitly determines the number of clusters and their members for any signed directed graph, investigates the stability

relation between signed and extended systems for first and higher-order dynamics, and derives control parameters for first, second, and third-order systems. The results are extended to discrete-time networks.

Chapter 4 investigates the cluster consensus problem in higher-order integrator networks with delayed dynamics, represented by unsigned or signed directed graphs without requiring special graph structures. It analyzes input and communication delayed networks on unsigned directed graphs. For signed directed graphs with delays, it uses an extended directed graph approach to recover trajectories and investigates stability properties and upper bounds for continuous-time systems with fixed delays. The chapter also shows that discrete-time systems with non-uniform delay achieve cluster consensus if the undelayed system is stable.

Chapter 5 examines the cluster consensus challenge in networks of first and second-order systems with inherent nonlinear dynamics and both cooperative and competitive interactions. It derives sufficient conditions based on Lyapunov theory for cluster consensus in first-order systems on unsigned directed graphs (without necessarily a spanning tree) and extends these results to second-order nonlinear systems. The stability analysis is further expanded to include first and second-order nonlinear systems on signed directed graphs using the extended graph representation.

Chapter 6 investigates the matrix-weighted cluster consensus problem in directed networks, removing restrictive topological assumptions. It develops a theoretical framework for systems with positive definite or negative definite matrix edge weights, providing conditions to determine the number and structure of clusters. The analysis includes an extension to discrete-time systems with a distributed step-size selection method for stability and convergence to multiple clusters using only local information.

Chapter 7 delineates the concluding remarks.

2. CLUSTER CONSENSUS IN UNSIGNED NETWORKS

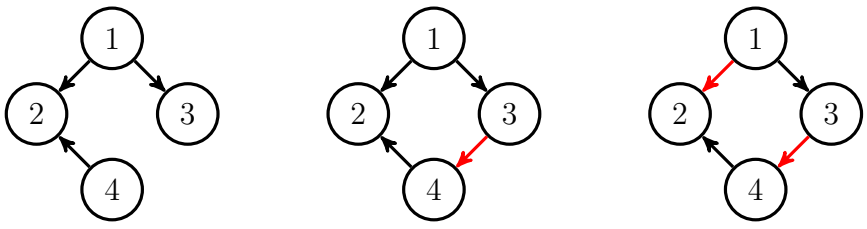
This chapter studies the higher-order cluster agreement problem for continuous-time networks evolving over any given unsigned directed graph. Necessary and sufficient conditions are derived to reach clusters which are not pre-determined as opposed to existing literature.

2.1. Graph Theoretical Concepts

Given a multi-agent network consisting of n agents, the connections between agents are modeled via a directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. A directed edge, (j, i), denotes information flow from agent j to agent i. The adjacency element, a_{ij}, is the weight of the corresponding edge, (j, i), is defined such that $a_{ij} \neq 0 \iff (j, i) \in \mathcal{E}$. $\mathcal{G}$ is called an *unsigned* directed graph, if all adjacency elements are non-negative, i.e., $a_{ij} \geq 0$. Otherwise, $\mathcal{G}$ is named as *signed* directed graph. A signed directed graph, $\mathcal{G}$, is called digon sign-symmetric if $a_{ij}a_{ji} \geq 0$. The neighbor set of agent i is the set of agents that are adjacent to agent i and defined by $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$.

A directed path $\mathcal{P}$ between two agents (from agent i to agent j) is defined as a finite sequence of agents such that $(l_1, l_2, \ldots, l_{m-1}, l_m)$ where $l_1 = i$, $l_m = j$ and $(l_k, l_{k+1}) \in \mathcal{E}$ for $k = 1, 2, \ldots, m-1$. A directed graph $\mathcal{G}$ is called *strongly connected* if a path exists between every pair of its agents in both directions. A directed graph contains a *spanning tree* if there is at least one agent, known as the root, that has a directed path available to all other agents in the graph.

Signed directed graphs have particular properties defined for them, such as *structural balance* and *digon sign-symmetry*. If the weights of the edge pairs $\{(i, j), (j, i)\}$ in a signed directed graph cannot have opposite signs, i.e., $a_{ij}a_{ji} \geq 0$, the graph is digon sign-symmetric. A signed directed graph is structurally balanced if there exists



(a) (b) (c)

Figure 2.1. Graph theory visualization.

a bipartition of agents $\mathcal{V}_1$ and $\mathcal{V}_2$ such that $\mathcal{V}_1 \bigcup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \bigcap \mathcal{V}_2 = \emptyset$, $a_{ij} \geq 0$ for $i, j \in \mathcal{V}_q (q = 1, 2)$ and $a_{ij} \leq 0$ for $i \in \mathcal{V}_q, j \in \mathcal{V}_p, q \neq p \quad (q, p = 1, 2)$. The following example illustrates the general concepts related the graph theory:

Example 2.1. *Consider the network with 4 agents communicating in 3 different ways as shown in Figure 2.1. Edges with positive weights are shown as black arrows, whereas those with negative weights are depicted as red arrows. In Figure 2.1(a), there is no any agent that can reach all other agents via a directed path so that the network does not contain a spanning tree. When an edge from agent 3 to 4 is added to the network as shown in Figure 2.1(b), agent 1 has directed paths to the remaining agents in the network. Therefore, the network has a spanning tree, and agent 1 is the root. Although the network in Figure 2.1(b) is represented by a signed directed graph, it is not available a bipartition that makes the network structurally balanced. When the edge from agent 1 to agent 2 is negatively weighted, the network is obtained in Figure 2.1(c). In this scenario, it is possible to have a bipartition that ensures the network is structurally balanced for* $\mathcal{V}_1 = \{1, 3\}$ *and* $\mathcal{V}_2 = \{2, 4\}$.

In the analysis of the cluster consensus problem, PLS (Primary Layer Subgraph) and SLS (Secondary Layer Subgraph) concepts related to unsigned directed graphs are used to determine the number of clusters. The definitions are given below:

Definition 2.1. ([13])Primary and Secondary Layer Subgraphs (PLS and SLS): *An unsigned directed graph,* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$*, can be uniquely parted into* l_p *primary and* l_s *secondary layer subgraphs with the following properties:*

- *a* PLS ($\mathcal{G}_{p,i}$, $i = 1, 2, \ldots, l_p$)
  - *(i) has an agent set, $\mathcal{V}_{p,i}$, containing a spanning tree and consisting of the maximum possible number of agents that do not receive information from $\mathcal{V} \setminus \mathcal{V}_{p,i}$.*
- *a* SLS ($\mathcal{G}_{s,j}$, $j = 1, 2, \ldots, l_s$)
  - *(i) has an agent set, $\mathcal{V}_{s,j}$, containing a spanning tree and a root agent that receives information from agents of at least two different subgraphs in $\mathcal{V} \setminus \mathcal{V}_{s,j}$ and*
  - *(ii) except for its root agent, does not receive information from $\mathcal{V} \setminus \mathcal{V}_{s,j}$.*

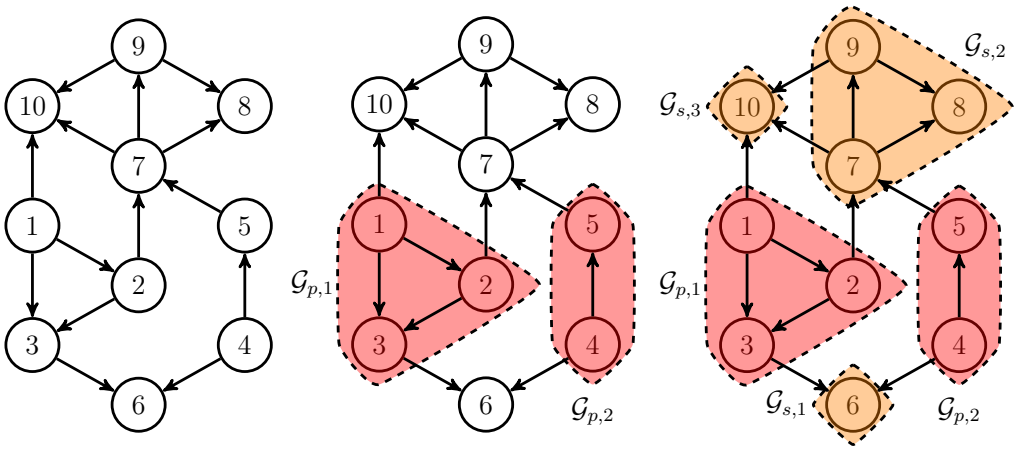
The PLS and SLS concepts are functional tools in analysis of the cluster consensus problem. The following remark gives detailed information on PLS and SLS notions:

Remark 2.1. The PLSs and SLSs in Definition 2.1 can be applied to an arbitrary unsigned directed graph without any structural limitations to determine clusters by using the detection algorithms in [13]. In order to determine PLSs and SLSs, the following procedure can be followed:

- Firstly, PLS detection algorithm in [13] is utilized to determine PLS agent sets $\mathcal{V}_{p,i}$, $i = 1, 2, \ldots, l_p$. Each PLS should contain a spanning tree and cannot receive information from out of the group.
- Secondly, SLS agent sets $\mathcal{V}_{s,j}$, $j = 1, 2, \ldots, l_s$ is obtained from the remaining agent sets $\mathcal{V} \setminus \bigcup_{i=1,2,\ldots,l_p} \mathcal{V}_{p,i}$ by using SLS detection algorithm in [13]. Each SLS should contain a spanning tree, its root agent should receive information from at least two different group and other agents cannot receive information from out of the group.

In order to illustrate determination of PLS and SLS, an example is given below.

Example 2.2. *Consider the network represented by unsigned directed graph with 10 agents in Figure 2.2(a). By following the procedure in Remark 2.1, two PLS agent sets ($\mathcal{V}_{p,1}$ and $\mathcal{V}_{p,2}$) are determined in accordance with the properties of PLS in Definition 2.1. The corresponding subgraphs ($\mathcal{G}_{p,1}$ and $\mathcal{G}_{p,2}$) are shown in Figure 2.2(b). In light*



(a) (b) (c)

Figure 2.2. An unsigned directed graph with 10 agents and 14 edges.

of Definition 2.1 note that two PLSs ($\mathcal{G}_{p,1}$ and $\mathcal{G}_{p,2}$) contain a spanning tree and do not have a link coming from other groups. Subsequently, three SLS agent sets ($\mathcal{V}_{s,1}$, $\mathcal{V}_{s,2}$ and $\mathcal{V}_{s,3}$) are obtained from the remaining agent set. The corresponding subgraphs ($\mathcal{G}_{s,1}$, $\mathcal{G}_{s,2}$ and $\mathcal{G}_{s,3}$) are shown in Figure 2.2(c). Each SLS ($\mathcal{G}_{s,1}$, $\mathcal{G}_{s,2}$ and $\mathcal{G}_{s,3}$) consists of a spanning tree whose root agent receives information from at least two different agents in other clusters. By using PLS and SLS detection algorithms in [13], PLSs and SLSs and their members can be obtained as $\mathcal{V}_{p,1} = \{1, 2, 3\}$, $\mathcal{V}_{p,2} = \{4, 5\}$ and $\mathcal{V}_{s,1} = \{6\}$, $\mathcal{V}_{s,2} = \{7, 8, 9\}$, $\mathcal{V}_{s,3} = \{10\}$, respectively. PLS and SLS decompositions can be applied to any given unsigned directed graph without any topological structure constraints. PLSs and SLSs can be uniquely determined by the algorithms proposed in [13].

2.2. Cooperative Higher-Order System Model

Consider the m-th order dynamics of agent i described by

$$\begin{aligned} \dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\ &\vdots \\ \dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\ \dot{x}_i^{(m)}(t) &= u_i(t), \quad i = 1, 2, \ldots, n \end{aligned} \tag{2.1}$$

where $x_i^{(q)}(t)$ $(q = 1, 2, \ldots, m)$ are the states of agent i at time t, $x_i^{(q)}$ denotes the $(q-1)$st derivative of $x_i^{(1)}$ and $u_i(t)$ is the control input described as follows:

$$u_i(t) = \sum_{q=1}^{m} \gamma_q \sum_{j \in \mathcal{N}_i} a_{ij}(x_j^{(q)}(t) - x_i^{(q)}(t)) \tag{2.2}$$

where $\gamma_q > 0$ $(q = 1, 2, \ldots, m)$ are the control parameters.

The dynamics of system (2.1) with the control input (2.2) are equivalent to

$$\begin{aligned} \dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\ &\vdots \\ \dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\ \dot{x}_i^{(m)}(t) &= -\sum_{q=1}^{m} \gamma_q \sum_{j \in \mathcal{N}_i} l_{ij} x_j^{(q)}(t), \quad i = 1, 2, \ldots, n \end{aligned} \tag{2.3}$$

where l_{ij} is the element of the unsigned Laplacian matrix, L, which is defined by

$$l_{ij} = \begin{cases} \sum\limits_{k=1, k \neq i}^{n} a_{ik}, & \text{if } i = j \\ -a_{ij}, & \text{otherwise.} \end{cases}$$

Remark 2.2. For a given network represented by an unsigned directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, it is well-known in the literature [6, 97] that the network with continuous-time first-order dynamics achieves complete consensus if and only if the directed graph contains a spanning tree. When the network does not contain a spanning tree, the network with first-order dynamics converges to multiple groups determined by the graph structure [9, 98]. For first-order systems, the network always converges to the clusters so long as each agent uses non-negative edge weights. For second-order systems [98], there are two scaling parameters γ_1 and γ_2 which affect the converge properties. The problem becomes more challenging when we consider continuous-time dynamics beyond second-order as not only the stability analysis but also the theoretical analysis regarding the formation clusters becomes more involved. To illustrate some of these

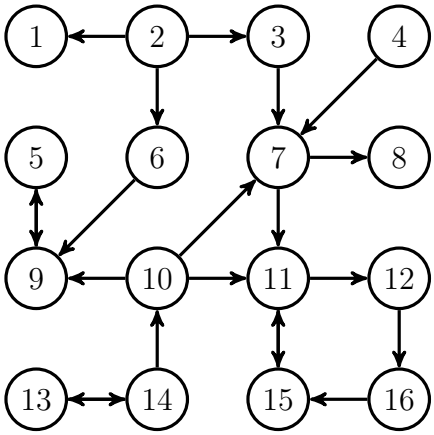


Figure 2.3. An unsigned directed graph with 16 agents and 21 edges.

challenges, consider the following example.

Example 2.3. *Consider the network depicted in Figure 2.3. Note that the graph does not contain a spanning tree. Therefore, the analysis in [26, 99] is not applicable. The objective of the present chapter is to determine whether there exist a set of controller parameters* $\gamma_1, \gamma_2, \ldots, \gamma_m$ *which guarantee the stability of* (2.3) *as well determine the final group formations when convergence is achieved. To further illustrate the concept, suppose that each agent has third-order continuous-time dynamics* ($m = 3$) *and utilizes the controller coefficients* $\gamma_1 = 3$, $\gamma_2 = 0.5, \gamma_3 = 1.5$. *As depicted in Figure 2.4, the network with these settings diverges. Note once again that this would not be an issue if each agent were to use first-order dynamics. The analysis of this network cannot be carried out using the existing techniques in the literature.*

From the above example, we conclude that appropriate choice of control parameters plays an important role in the stability analysis. Moreover, even when stability is ensured, it is a challenging task to determine the resulting number of clusters as well as the members of each cluster for a given graph.

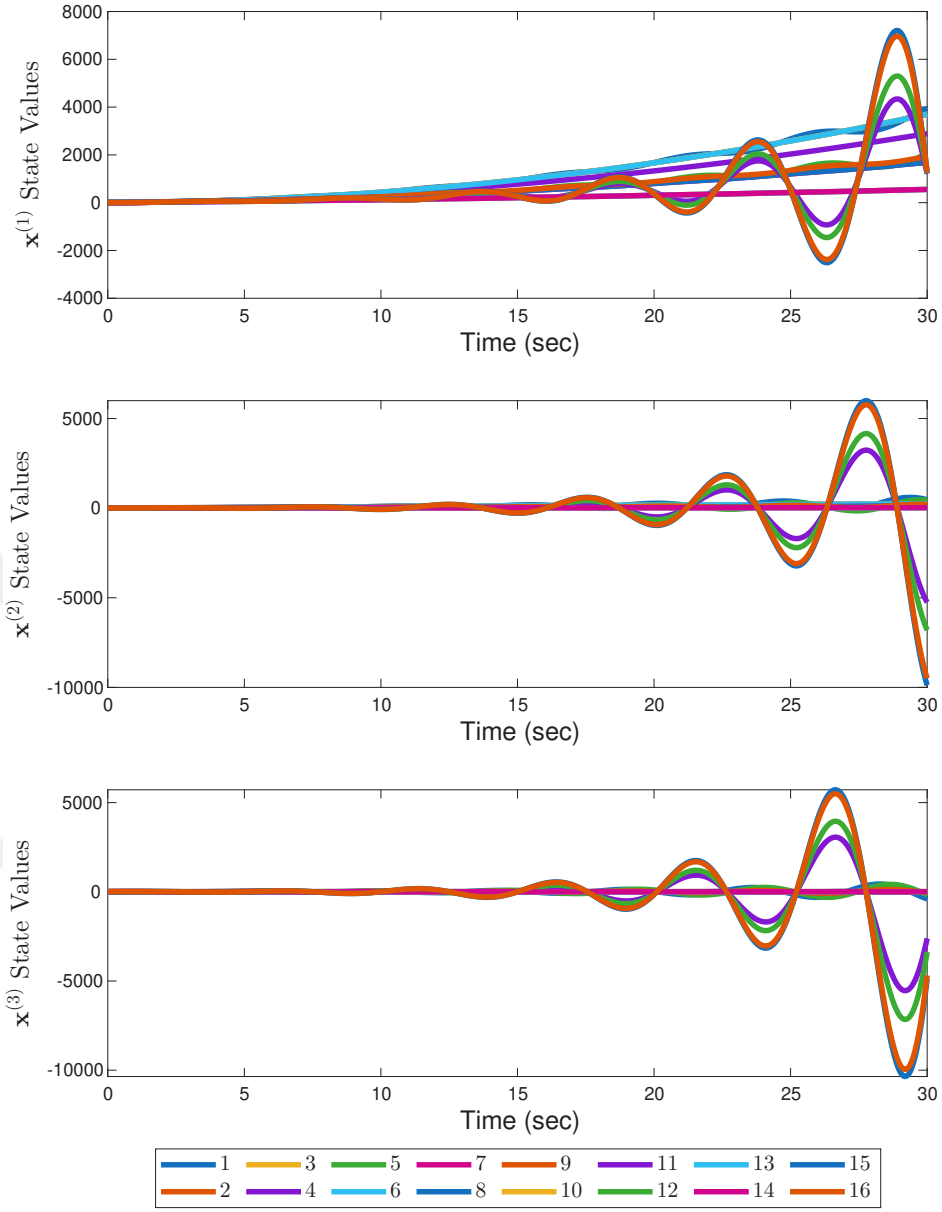


Figure 2.4. State trajectories of agents associated with unsigned directed graph in Figure 2.3 with $\gamma_1 = 3$, $\gamma_2 = 0.5$ and $\gamma_3 = 1.5$.

Before analyzing the cluster consensus problem for higher-order systems, we define higher-order cluster consensus.

Definition 2.2. *(Cluster Consensus) The network* (2.3) *is said to achieve cluster consensus if there exist* $\mathcal{K}$ *disjoint non-empty clusters,* $\{\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_\mathcal{K}\}$*, such that*

- $\bigcup_{p=1}^{\mathcal{K}} \mathcal{C}_p = \mathcal{V}$,

- $\lim_{t\to\infty} \|x_i^{(q)}(t) - x_j^{(q)}(t)\| = 0$, *for all* $i, j \in \mathcal{C}_p$, $p = 1, 2, \ldots, \mathcal{K}$ *and* $q = 1, 2, \ldots, m$.

The objective of this chapter is to determine clusters of the system (2.3) with any order dynamics and to state necessary and sufficient conditions on control parameters so that the network dynamics is stable.

2.3. Analysis of Cluster Consensus in Unsigned Directed Graphs

In this section, we investigate the cluster consensus properties of the higher-order network in (2.3) evolving over an arbitrary unsigned directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$. We can rewrite (2.3) as

$$\dot{\mathbf{x}}(t) = \mathcal{L}_m \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{2.4}$$

where $\mathbf{x}(t) = \left[\mathbf{x}^{(1)}(t)^\mathsf{T}, \mathbf{x}^{(2)}(t)^\mathsf{T}, \ldots, \mathbf{x}^{(m)}(t)^\mathsf{T}\right]^\mathsf{T}$ is the state vector with components $\mathbf{x}^{(q)}(t) = \left[x_1^{(q)}(t), x_2^{(q)}(t), \ldots, x_n^{(q)}(t)\right]^\mathsf{T}$ for $q = 1, 2, \ldots, m$, $\mathbf{x}_0$ is the initial condition and $\mathcal{L}_m \in \mathbb{R}^{mn\times mn}$

$$\mathcal{L}_m = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ -\gamma_1 L & -\gamma_2 L & -\gamma_3 L & \ldots & -\gamma_m L \end{bmatrix}. \tag{2.5}$$

Given an unsigned directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, it can be divided into l_p PLSs and l_s SLSs using the decomposition algorithm proposed in [13] and the unsigned Laplacian matrix in (2.5) can be decomposed by using row and column operations as

$$L = \begin{bmatrix} L_p & 0_{n_p\times n_s} \\ L_{sp} & L_s \end{bmatrix} \tag{2.6}$$

where

$$L_p = \text{blkdiag}\{L_{1,1}, \ldots, L_{l_p,l_p}\}, \quad L_{sp} = \begin{bmatrix} L_{l_p+1,1} & \cdots & L_{l_p+1,l_p} \\ \vdots & \ddots & \vdots \\ L_{l_p+l_s,1} & \cdots & L_{l_p+l_s,l_p} \end{bmatrix},$$

$$L_s = \begin{bmatrix} L_{l_p+1,l_p+1} & \cdots & L_{l_p+1,l_p+l_s} \\ \vdots & \ddots & \vdots \\ L_{l_p+l_s,l_p+1} & \cdots & L_{l_p+l_s,l_p+l_s} \end{bmatrix}$$

Note that the first l_p rows of L are structured such that only the diagonal blocks contain non-zero elements and each of these diagonal blocks $L_{i,i}$ for $i = 1, 2, \ldots, l_p$ is an unsigned Laplacian matrix. Each row of $-L_s^{-1}L_{sp}$ adds up to 1 and all members of $-L_s^{-1}L_{sp}$ are non-negative [98].

Let $\mathcal{I}_p = \{1, 2, \ldots, l_p\}$ and $\mathcal{I}_s = \{1, 2, \ldots, l_s\}$ be index sets of the PLSs and SLSs, respectively. $n_{p,i}$, $i \in \mathcal{I}_p$ denotes the number of agents in the i-th PLS and $n_{s,j}$, $j \in \mathcal{I}_s$ denotes the j-th SLS. Let $n_p = \sum_{i=1}^{l_p} n_{p,i}$ and $n_s = \sum_{j=1}^{l_s} n_{s,j}$ be the total number of agents in the PLSs and the SLSs, respectively. Then, we can rewrite (2.4) as

$$\begin{bmatrix} \dot{\mathbf{x}}_p^{(1)}(t) \\ \dot{\mathbf{x}}_s^{(1)}(t) \\ \vdots \\ \dot{\mathbf{x}}_p^{(m)}(t) \\ \dot{\mathbf{x}}_s^{(m)}(t) \end{bmatrix} = \mathcal{L}_m \begin{bmatrix} \mathbf{x}_p^{(1)}(t) \\ \mathbf{x}_s^{(1)}(t) \\ \vdots \\ \mathbf{x}_p^{(m)}(t) \\ \mathbf{x}_s^{(m)}(t) \end{bmatrix}$$

where $\mathbf{x}_p^{(q)}(t) \in \mathbb{R}^{n_p}$, $\mathbf{x}_s^{(q)}(t) \in \mathbb{R}^{n_s}$ $(q = 1, 2, \ldots, m)$ are the state vectors for the PLSs and SLSs, respectively.

The eigenvalues of $\mathcal{L}_m$ can be computed by solving the characteristic equation $\det(\mu I_{mn} - \mathcal{L}_m) = 0$ where

$$\det(\mu I_{mn} - \mathcal{L}_m) = \det\left(\begin{bmatrix} \mu I_n & -I_n & 0_{n\times n} & \dots & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \mu I_n & -I_n & \dots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \dots & \mu I_n & -I_n \\ \gamma_1 L & \gamma_2 L & \gamma_3 L & \dots & \gamma_{m-1} L & \mu I_n + \gamma_m L \end{bmatrix}\right)$$
$$= \det(\mu^m I_n + (\gamma_1 + \gamma_2\mu + \cdots + \gamma_m\mu^{m-1})L).$$

The above expression can be rewritten with respect to the eigenvalues of the unsigned Laplacian matrix, L, as

$$\det(\mu I_{mn} - \mathcal{L}_m) = \prod_{i=1}^{n} \left(\mu^m + (\gamma_1 + \gamma_2\mu + \cdots + \gamma_m\mu^{m-1})\lambda_i\right) = 0 \tag{2.7}$$

where λ_i is the i-th eigenvalue of L. From (2.7), the eigenvalues of $\mathcal{L}_m$ can be obtained from

$$\mu^m + \gamma_m\lambda_i\mu^{m-1} + \cdots + \gamma_2\lambda_i\mu + \gamma_1\lambda_i = 0 \tag{2.8}$$

for $i = 1, 2, \dots, n$. Each eigenvalue of L corresponds to m eigenvalues of $\mathcal{L}_m$.

The first result concerning higher-order cluster consensus is stated in Theorem 2.1 below.

Theorem 2.1. *The number of clusters for the m-th order multi-agent system is given by $\mathcal{K} = l_p + l_s$ if and only if the roots of* (2.8) *are in the open left half complex plane (OLHP) except ml_p zero eigenvalues where l_p and l_s are the number of PLSs and SLSs, respectively.*

Proof. Suppose that the roots of (2.8) are in the OLHP except ml_p zero eigenvalues for some constants γ_1, $\gamma_2, \dots, \gamma_m$. Note that $\mathcal{L}_m$ has ml_p exactly zero eigenvalues for

$\gamma_1, \gamma_2, \ldots, \gamma_m \neq 0$. From $\mathcal{L}_m \xi_{i,1} = \mathbf{0}_{mn}$, l_p right eigenvectors of $\mathcal{L}_m$ associated with the eigenvalue 0 can be obtained as $\xi_{i,1} = \begin{bmatrix} \kappa_i^\mathsf{T} & -(L_s^{-1} L_{sp} \kappa_i)^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \cdots & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} \end{bmatrix}^\mathsf{T}$ for $i = 1, 2, \ldots, l_p$, where $\kappa_i \in \mathbb{R}^{n_p}$ is a right eigenvector of L_p that can be computed as follows:

$$
\begin{aligned}
\kappa_1 &= \begin{bmatrix} \mathbf{1}_{n_{p,1}}^\mathsf{T} & \mathbf{0}_{n_{p,2}}^\mathsf{T} & \mathbf{0}_{n_{p,3}}^\mathsf{T} & \cdots & \mathbf{0}_{n_{p,l_p-1}}^\mathsf{T} & \mathbf{0}_{n_{p,l_p}}^\mathsf{T} \end{bmatrix}^\mathsf{T}, \\
\kappa_2 &= \begin{bmatrix} \mathbf{0}_{n_{p,1}}^\mathsf{T} & \mathbf{1}_{n_{p,2}}^\mathsf{T} & \mathbf{0}_{n_{p,3}}^\mathsf{T} & \cdots & \mathbf{0}_{n_{p,l_p-1}}^\mathsf{T} & \mathbf{0}_{n_{p,l_p}}^\mathsf{T} \end{bmatrix}^\mathsf{T}, \\
&\vdots \\
\kappa_{l_p} &= \begin{bmatrix} \mathbf{0}_{n_{p,1}}^\mathsf{T} & \mathbf{0}_{n_{p,2}}^\mathsf{T} & \mathbf{0}_{n_{p,3}}^\mathsf{T} & \cdots & \mathbf{0}_{n_{p,l_p-1}}^\mathsf{T} & \mathbf{1}_{n_{p,l_p}}^\mathsf{T} \end{bmatrix}^\mathsf{T}.
\end{aligned}
$$

Note that $\mathcal{L}_m$ cannot be diagonal since there are only l_p right eigenvectors of $\mathcal{L}_m$ associated with the eigenvalue 0. Therefore, a Jordan form is used here. The nonzero eigenvalues of $\mathcal{L}_m$ can be calculated from the characteristic equation $\Delta(\mu) = \mu^m + \gamma_m \lambda_i \mu^{m-1} + \cdots + \gamma_2 \lambda_i \mu + \gamma_1 \lambda_i = 0$ with $\lambda_i \neq 0$. Therefore, $\mathcal{L}_m$ has exactly a zero eigenvalue of algebraic multiplicity ml_p and all the other eigenvalues have negative real parts. Then there exists a nonsingular matrix $P \in \mathbb{R}^{mn \times mn}$, such that $P^{-1} \mathcal{L}_m P = J$, where J is the Jordan canonical form associated with $\mathcal{L}_m$. Thus, one has

$$
\begin{aligned}
\mathcal{L}_m &= PJP^{-1} \\
&= \begin{bmatrix} \xi_{1,1} & \cdots & \xi_{1,m} & \cdots & \xi_{l_p,1} & \cdots & \xi_{l_p,m} & \xi_{ml_p+1} & \cdots & \xi_{mn} \end{bmatrix} \\
&\quad \times \begin{bmatrix} J_0 & & & \\ & \ddots & & \\ & & J_0 & \\ & & & \tilde{J} \end{bmatrix} \begin{bmatrix} \eta_{1,1}^\mathsf{T} \\ \vdots \\ \eta_{1,m}^\mathsf{T} \\ \vdots \\ \eta_{l_p,1}^\mathsf{T} \\ \vdots \\ \eta_{l_p,m}^\mathsf{T} \\ \eta_{ml_p+1}^\mathsf{T} \\ \vdots \\ \eta_{mn}^\mathsf{T} \end{bmatrix}
\end{aligned} \tag{2.9}
$$

where $\xi_{i,1}, \ldots, \xi_{i,m}$ and $\eta_{i,1}, \ldots, \eta_{i,m}$, $i = 1, 2, \ldots, l_p$, are right and left eigenvectors of $\mathcal{L}_m$ associated with the eigenvalue 0; ξ_j and η_j, $j = ml_p + 1, ml_p + 2, \ldots, mn$, are the rest of the right and left eigenvectors of $\mathcal{L}_m$, respectively; $J_0 \in \mathbb{R}^{m \times m}$ is given by

$$J_0 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix};$$

and $\tilde{J} \in \mathbb{R}^{mn_s \times mn_s}$ is the diagonal Jordan block matrix with the nonzero eigenvalues.

$\mathcal{L}_m P = PJ$ yields

$$\begin{aligned} \mathcal{L}_m \xi_{i,1} &= \mathbf{0}_{mn}, \\ \mathcal{L}_m \xi_{i,2} &= \xi_{i,1}, \\ &\vdots \\ \mathcal{L}_m \xi_{i,m} &= \xi_{i,m-1}, \quad i = 1, 2, \ldots, l_p. \end{aligned}$$

Hence, one can obtain

$$\begin{aligned} \xi_{i,2} &= \begin{bmatrix} \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \kappa_i^\mathsf{T} & -(L_s^{-1} L_{sp} \kappa_i)^\mathsf{T} & \cdots & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} \end{bmatrix}^\mathsf{T}, \\ &\vdots \\ \xi_{i,m-1} &= \begin{bmatrix} \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \cdots & \kappa_i^\mathsf{T} & -(L_s^{-1} L_{sp} \kappa_i)^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} \end{bmatrix}^\mathsf{T}, \\ \xi_{i,m} &= \begin{bmatrix} \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \cdots & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \kappa_i^\mathsf{T} & -(L_s^{-1} L_{sp} \kappa_i)^\mathsf{T} \end{bmatrix}^\mathsf{T} \end{aligned}$$

as $(m-1)l_p$ generalized right eigenvectors of $\mathcal{L}_m$ associated with the eigenvalue 0 for $i = 1, 2, \ldots, l_p$. Accordingly,

$$\eta_{i,1} = \begin{bmatrix} \varsigma_i^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \cdots & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} \end{bmatrix}^\mathsf{T},$$

$$\eta_{i,2} = \begin{bmatrix} \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \varsigma_i^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \cdots & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} \end{bmatrix}^\mathsf{T},$$
$$\vdots$$
$$\eta_{i,m-1} = \begin{bmatrix} \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \cdots & \varsigma_i^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} \end{bmatrix}^\mathsf{T},$$

and

$$\eta_{i,m} = \begin{bmatrix} \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \cdots & \mathbf{0}_{n_p}^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} & \varsigma_i^\mathsf{T} & \mathbf{0}_{n_s}^\mathsf{T} \end{bmatrix}^\mathsf{T}$$

are $(m-1)l_p$ generalized left eigenvectors and l_p left eigenvector of $\mathcal{L}_m$ associated with the eigenvalue 0, respectively where $\varsigma_i^\mathsf{T}\kappa_i = 1$. From properties of the Kronecker product, one has

$$\begin{aligned} e^{\mathcal{L}_m t} &= e^{P(Jt)P^{-1}} = Pe^{(Jt)}P^{-1} \\ &= P \begin{bmatrix} e^{J_0 t} & & & \\ & \ddots & & \\ & & e^{J_0 t} & \\ & & & e^{\tilde{J}t} \end{bmatrix} P^{-1} \end{aligned} \tag{2.10}$$

where

$$e^{J_0 t} = \begin{bmatrix} 1 & t & \frac{1}{2!}t^2 & \cdots & \frac{1}{(m-1)!}t^{m-1} \\ 0 & 1 & t & \cdots & \frac{1}{(m-2)!}t^{m-2} \\ 0 & 0 & 1 & \cdots & \frac{1}{(m-3)!}t^{m-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

and $e^{\tilde{J}t} \to 0_{mn_s \times mn_s}$ as $t \to \infty$. Let $\mathbf{y}(t)$ be defined as

$$\mathbf{y}(t) = \left(\Gamma\left(I_m \otimes \boldsymbol{\kappa}\boldsymbol{\varsigma}^\mathsf{T}\right)\right) \begin{bmatrix} \mathbf{x}_p^{(1)}(0) \\ \vdots \\ \mathbf{x}_p^{(m)}(0) \end{bmatrix}$$

where $\mathbf{x}_p^{(q)}(t)$, $\mathbf{x}_s^{(q)}(t)$ $(q = 1, 2, \ldots, m)$ are the state vectors for the PLSs and SLSs; and $\boldsymbol{\kappa} = [\kappa_1, \kappa_2, \ldots, \kappa_{l_p}]$, $\boldsymbol{\varsigma} = [\varsigma_1, \varsigma_2, \ldots, \varsigma_{l_p}]$ are $n_p \times l_p$ matrices containing the right and left eigenvectors associated with the eigenvalue 0 of L_p, respectively; and

$$\Gamma = \begin{bmatrix} I_{n_p} & tI_{n_p} & \cdots & \frac{t^{(m-1)}}{(m-1)!}I_{n_p} \\ 0_{n_p \times n_p} & I_{n_p} & \cdots & \frac{t^{(m-2)}}{(m-2)!}I_{n_p} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n_p \times n_p} & 0_{n_p \times n_p} & \cdots & I_{n_p} \\ -L_s^{-1}L_{sp} & -tL_s^{-1}L_{sp} & \cdots & -\frac{t^{(m-1)}}{(m-1)!}L_s^{-1}L_{sp} \\ 0_{n_s \times n_p} & -L_s^{-1}L_{sp} & \cdots & -\frac{t^{(m-2)}}{(m-2)!}L_s^{-1}L_{sp} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{n_s \times n_p} & 0_{n_s \times n_p} & \cdots & -L_s^{-1}L_{sp} \end{bmatrix}.$$

Then, $\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = \lim_{t\to\infty} \|e^{\mathcal{L}_m t}\mathbf{x}(0) - \mathbf{y}(t)\|$ can be computed as

$$\lim_{t\to\infty} \|\mathbf{x}(t) - \mathbf{y}(t)\| = \lim_{t\to\infty} \left\| \begin{bmatrix} \mathbf{x}_p^{(1)}(t) \\ \vdots \\ \mathbf{x}_p^{(m)}(t) \\ \mathbf{x}_s^{(1)}(t) \\ \vdots \\ \mathbf{x}_s^{(m)}(t) \end{bmatrix} - \mathbf{y}(t) \right\| = \lim_{t\to\infty} \left\| e^{\mathcal{L}_m t} \begin{bmatrix} \mathbf{x}_p^{(1)}(0) \\ \vdots \\ \mathbf{x}_p^{(m)}(0) \\ \mathbf{x}_s^{(1)}(0) \\ \vdots \\ \mathbf{x}_s^{(m)}(0) \end{bmatrix} - \mathbf{y}(t) \right\| = 0.$$

From [100], the number of clusters for system (2.1) converges to

$$\mathcal{K} = l_p + l_s.$$

□

2.4. Control Parameter Selection

In this section, conditions on the choice of control parameters (γ_q) are investigated to achieve cluster consensus for systems with any order of dynamics. In deriving the general stability condition for m-th order systems, recall from the proof of Theorem 2.1 that the following polynomial with complex coefficients

$$\Delta(\mu) = \mu^m + \gamma_m\lambda_i\mu^{m-1} + \cdots + \gamma_2\lambda_i\mu + \gamma_1\lambda_i \tag{2.11}$$

has to have its roots in the OLHP. Associated with (2.11), define Υ as

$$\Upsilon = \begin{bmatrix} \alpha'_1 & -\alpha''_2 & -\alpha'_3 & \alpha''_4 & \alpha'_5 & \cdots & (-1)^{m+1}\alpha'_{2m-1} \\ 1 & -\alpha''_1 & -\alpha'_2 & \alpha''_3 & \alpha'_4 & \cdots & (-1)^{m+1}\alpha'_{2m-2} \\ 0 & \alpha'_1 & -\alpha''_2 & -\alpha'_3 & \alpha''_4 & \cdots & (-1)^{m+1}\alpha'_{2m-2} \\ 0 & 1 & -\alpha''_1 & -\alpha'_2 & \alpha''_3 & \cdots & (-1)^{m+1}\alpha'_{2m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{2.12}$$

where $\alpha'_r = \mathbb{R}\mathrm{e}\{\gamma_{m-r+1}\lambda_i\}$, $\alpha''_r = \mathbb{I}\mathrm{m}\{\gamma_{m-r+1}\lambda_i\}$ and $\alpha_r = 0$ for $r > m$. Let Υ_{ii}, $i = 1, 2, \ldots, 2m-1$ denote the leading principal minors of Υ. The general stability condition for m-th order systems is given below.

Theorem 2.2. *The roots of* (2.11) *are in the OLHP if and only if all members of the sequence of* $1, \Upsilon_{11}, \Upsilon_{33}, \ldots, \Upsilon_{2m-1,2m-1}$ *are positive.*

Proof. The proof follows from [101] by setting $\alpha'_r = \mathbb{R}\mathrm{e}\{\gamma_{m-r+1}\lambda_i\}$, $\alpha''_r = \mathbb{I}\mathrm{m}\{\gamma_{m-r+1}\lambda_i\}$ and $\alpha_r = 0$ for $r > m$. □

Remark 2.3. From Theorem 2.1 and 2.2, m-th order systems reach cluster consensus if and only if all members of the sequence of $1, \Upsilon_{11}, \Upsilon_{33}, \ldots, \Upsilon_{2m-1,2m-1}$ are positive.

In the sequel, we consider special cases for which stability regions are derived explicitly.

2.4.1. Second-Order Systems

Consider second-order systems, i.e., $m = 2$. In this case, the matrix Υ becomes

$$\Upsilon = \begin{bmatrix} \mathbb{R}\mathrm{e}\{\gamma_2\lambda_i\} & -\mathbb{I}\mathrm{m}\{\gamma_1\lambda_i\} & 0 \\ 1 & -\mathbb{I}\mathrm{m}\{\gamma_2\lambda_i\} & -\mathbb{R}\mathrm{e}\{\gamma_1\lambda_i\} \\ 0 & \mathbb{R}\mathrm{e}\{\gamma_2\lambda_i\} & -\mathbb{I}\mathrm{m}\{\gamma_1\lambda_i\} \end{bmatrix}.$$

From Theorem 2.2, all members of the sequence of $1, \Upsilon_{11}, \Upsilon_{33}$ have to be positive to achieve cluster consensus. Hence, one can obtain the following necessary and sufficient conditions which agree with [98]:

$$(i) \quad \gamma_2 > 0, \tag{2.13a}$$

$$(ii) \quad \frac{\gamma_2^2}{\gamma_1} > \max_i \frac{\mathbb{I}\mathrm{m}^2\{\lambda_i\}}{\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2} \tag{2.13b}$$

where λ_i are the non-zero eigenvalues of L.

2.4.2. Third-Order Systems

Consider third-order systems, i.e., $m = 3$. In this case, the matrix Υ becomes

$$\Upsilon = \begin{bmatrix} \mathbb{R}\mathrm{e}\{\gamma_3\lambda_i\} & -\mathbb{I}\mathrm{m}\{\gamma_2\lambda_i\} & -\mathbb{R}\mathrm{e}\{\gamma_1\lambda_i\} & 0 & 0 \\ 1 & -\mathbb{I}\mathrm{m}\{\gamma_3\lambda_i\} & -\mathbb{R}\mathrm{e}\{\gamma_2\lambda_i\} & \mathbb{I}\mathrm{m}\{\gamma_1\lambda_i\} & 0 \\ 0 & \mathbb{R}\mathrm{e}\{\gamma_3\lambda_i\} & -\mathbb{I}\mathrm{m}\{\gamma_2\lambda_i\} & -\mathbb{R}\mathrm{e}\{\gamma_1\lambda_i\} & 0 \\ 0 & 1 & -\mathbb{I}\mathrm{m}\{\gamma_3\lambda_i\} & -\mathbb{R}\mathrm{e}\{\gamma_2\lambda_i\} & \mathbb{I}\mathrm{m}\{\gamma_1\lambda_i\} \\ 0 & 0 & \mathbb{R}\mathrm{e}\{\gamma_3\lambda_i\} & -\mathbb{I}\mathrm{m}\{\gamma_2\lambda_i\} & -\mathbb{R}\mathrm{e}\{\gamma_1\lambda_i\} \end{bmatrix}$$

By applying Theorem 2.2, one can obtain the following necessary and sufficient conditions:

$$(i) \quad \gamma_3 > 0, \tag{2.14a}$$

$$(ii) \quad \gamma_3^2\gamma_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2 - \gamma_3\gamma_1\,\mathbb{R}\mathrm{e}^2\{\lambda_i\} - \gamma_2^2\,\mathbb{I}\mathrm{m}^2\{\lambda_i\} > 0, \tag{2.14b}$$

$$(iii) \quad \gamma_2^2\gamma_3^2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^4 - 2\gamma_1\gamma_2\gamma_3\,\mathbb{R}\mathrm{e}^2\{\lambda_i\}|\lambda_i|^2 - \gamma_2^3\,\mathbb{I}\mathrm{m}^2\{\lambda_i\}|\lambda_i|^2 + \gamma_1^2\,\mathbb{R}\mathrm{e}^3\{\lambda_i\} > 0. \tag{2.14c}$$

For third-order systems, a non-empty set of control parameters $\{\gamma_1, \gamma_2, \gamma_3\}$ is given by the following corollary.

Corollary 2.1. *Let $m = 3$ and $\gamma_1, \gamma_2, \gamma_3$ be chosen such that the following conditions are satisfied:*

$$(i) \quad \gamma_1 > 0, \gamma_2 > 0, \gamma_3 > 0, \tag{2.15a}$$

$$(ii) \quad \gamma_1\gamma_3 = \gamma_2^2, \tag{2.15b}$$

$$(iii) \quad \frac{\gamma_3^2}{\gamma_2} \geq \max_i \frac{2}{\mathbb{R}\mathrm{e}\{\lambda_i\}} \tag{2.15c}$$

where λ_i are the non-zero eigenvalues of L. Then,

$$\Delta(\mu) = \mu^3 + \gamma_3\lambda_i\mu^2 + \gamma_2\lambda_i\mu + \gamma_1\lambda_i$$

has its roots in the OLHP.

Proof. Suppose that the conditions in (2.15) hold. We will show that the necessary and sufficient conditions in (2.14) are satisfied.

(i) It is straightforward to note that (2.14a) follows from (2.15a).

(ii) To verify that (2.14b) is satisfied, substitute (2.15b) in (2.14b) to obtain

$$\frac{\gamma_3^2}{\gamma_2} > \frac{1}{\mathbb{R}\mathrm{e}\{\lambda_i\}}.$$

which certainly holds with (2.15c) in place.

(iii) To verify (2.14c), substitute (2.15b) in (2.14c) to obtain

$$\frac{\gamma_3^2}{\gamma_2} + \frac{\gamma_2}{\gamma_3^2}\frac{\mathbb{R}\mathrm{e}^2\{\lambda_i\}}{|\lambda_i|^4} > \frac{2\,\mathbb{R}\mathrm{e}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^2\{\lambda_i\}}{\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2}.$$

Let σ denote $\frac{\gamma_3^2}{\gamma_2}$. The above inequality becomes the following quadratic inequality with respect to σ:

$$\sigma^2 - \frac{2\,\mathbb{R}\mathrm{e}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^2\{\lambda_i\}}{\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2}\sigma + \frac{\mathbb{R}\mathrm{e}^2\{\lambda_i\}}{|\lambda_i|^4} > 0$$

From solving the above inequality, one can obtain 2 critical values of σ as

$$\sigma_{1,2} = \frac{2\,\mathbb{R}\mathrm{e}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^2\{\lambda_i\}}{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2} \pm \frac{1}{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2}\sqrt{4\,\mathbb{R}\mathrm{e}^2\{\lambda_i\}\,\mathbb{I}\mathrm{m}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^4\{\lambda_i\}}$$

where σ_2 is taken to be the larger value. From the above equation, we always have two real roots since $4\,\mathbb{R}\mathrm{e}^2\{\lambda_i\}\,\mathbb{I}\mathrm{m}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^4\{\lambda_i\} \geq 0$. Then, the following inequalities always hold:

$$\begin{aligned}\sigma_2 &\geq \frac{2\,\mathbb{R}\mathrm{e}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^2\{\lambda_i\}}{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2} + \frac{1}{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2}\sqrt{\mathbb{I}\mathrm{m}^4\{\lambda_i\}} \\ &\geq \frac{1}{\mathbb{R}\mathrm{e}\{\lambda_i\}}\end{aligned}$$

and

$$\begin{aligned}\sigma_1 &\leq \frac{2\,\mathbb{R}\mathrm{e}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^2\{\lambda_i\}}{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2} - \frac{1}{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2}\sqrt{\mathbb{I}\mathrm{m}^4\{\lambda_i\}} \\ &\leq \frac{\mathbb{R}\mathrm{e}\{\lambda_i\}}{|\lambda_i|^2} \leq \frac{1}{\mathbb{R}\mathrm{e}\{\lambda_i\}}.\end{aligned}$$

Combining the above inequalities and $\frac{1}{\mathbb{R}\mathrm{e}\{\lambda_i\}}$ found in *(ii)*, we can obtain the lower bound of $\frac{\gamma_3^2}{\gamma_2}$ as σ_2. To simplify the choice of parameters, we will determine a new bound which does not require the imaginary part of λ_i. Note that the following inequality

always holds:

$$\begin{aligned}\sigma_2 &< \frac{2\,\mathbb{R}\mathrm{e}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^2\{\lambda_i\}}{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2} + \frac{1}{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2}\sqrt{(2\,\mathbb{R}\mathrm{e}^2\{\lambda_i\} + 3\,\mathbb{I}\mathrm{m}^2\{\lambda_i\})^2} \\ &< \frac{2}{\mathbb{R}\mathrm{e}\{\lambda_i\}}.\end{aligned}$$

Hence by choosing the ratio $\frac{\gamma_3^2}{\gamma_2}$ greater than or equal to $\max_i \frac{2}{\mathbb{R}\mathrm{e}\{\lambda_i\}}$ as in (2.15c) implies that (2.14c) is satisfied. This completes the proof. □

Remark 2.4. From Corollary 2.1, it is easy to choose control parameters that guarantee cluster consensus. When γ_2 is taken as 1, γ_3 can be determined as $\gamma_3 = \alpha > 0$ that satisfies $\alpha^2 \geq \max_i \frac{2}{\mathbb{R}\mathrm{e}\{\lambda_i\}}$. In such a case, γ_1 becomes $\frac{1}{\alpha} > 0$ which meets *(i)* and *(ii)* conditions in Corollary 2.1.

2.5. Numerical Analysis

Consider the multi-agent system illustrated in Figure 2.3. For the simulations, the following randomly generated system matrices are used:

$$L = \begin{bmatrix}
2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -5 & 0 & 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & -3 & 0 & 0 & 9 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -8 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -3 & 0 & 0 & 9 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & -6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 5 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -7 & 7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 7 & -5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 4
\end{bmatrix}$$

The eigenvalues of the given unsigned Laplacian matrix can be calculated as 17, 14, 12, 11, 10.8833, 10, 9, 8, 7, 6, $4.9425 + j2.3258$, $4.9425 - j2.3258$, 1.2318, 0, 0, 0. From Corollary 2.1, a sufficient stability condition of the system is given by $\frac{\gamma_3^2}{\gamma_2} > \max_i \frac{2}{\mathbb{R}\mathrm{e}\{\lambda_i\}} = 1.6236$. The state values are shown in Figure 2.5 for the control parameters $\gamma_1 = 2.25$, $\gamma_2 = 3$ and $\gamma_3 = 4$. The $\mathbf{x}^{(3)}$ states converge to six clusters such that each cluster achieves a final steady state. The $\mathbf{x}^{(2)}$ and $\mathbf{x}^{(1)}$ states reach six clusters whose values are increasing.

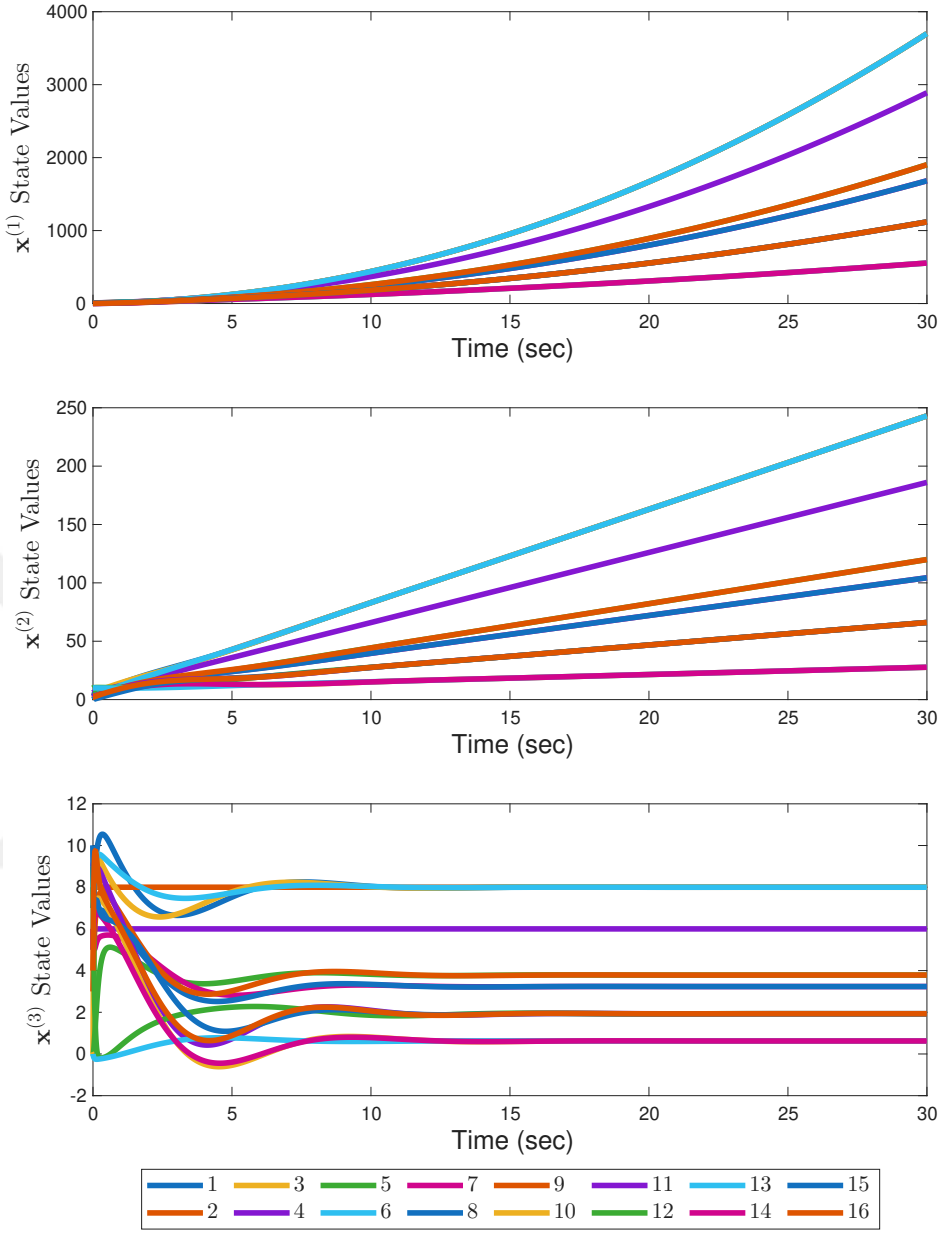


Figure 2.5. State trajectories of agents associated with unsigned directed graph in Figure 2.3 with $\gamma_1 = 2.25$, $\gamma_2 = 3$ and $\gamma_3 = 4$.

Apart from the simulation results, the number of PLSs and SLSs are calculated as $l_p = 3$ and $l_s = 3$ in the network shown in Figure 2.3 by the decomposition algorithm proposed in [13]. These clusters are given in Figure 2.6. The PLSs and SLSs of the network are determined as $\mathcal{V}_{p,1} = \{1, 2, 3, 6\}$, $\mathcal{V}_{p,2} = \{4\}$, $\mathcal{V}_{p,3} = \{10, 13, 14\}$ and $\mathcal{V}_{s,1} = \{5, 9\}$, $\mathcal{V}_{s,2} = \{7, 8\}$, $\mathcal{V}_{s,3} = \{11, 12, 15, 16\}$, respectively. As expected from Theorem 2.1, these clusters are the same clusters observed in the simulation results depicted in Figure 2.5.

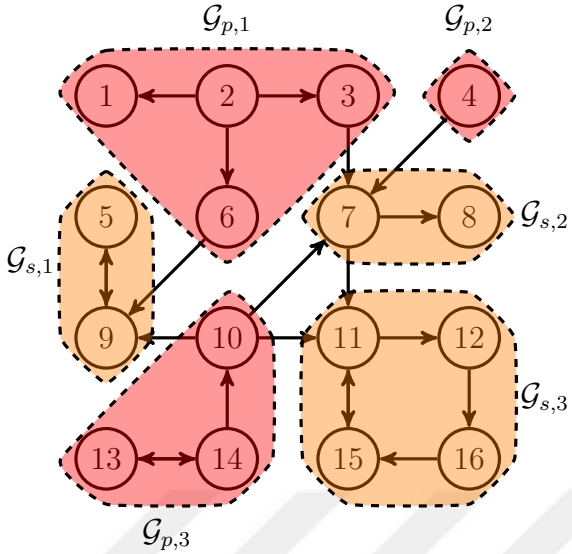


Figure 2.6. PLSs and SLSs of the network in Figure 2.3.

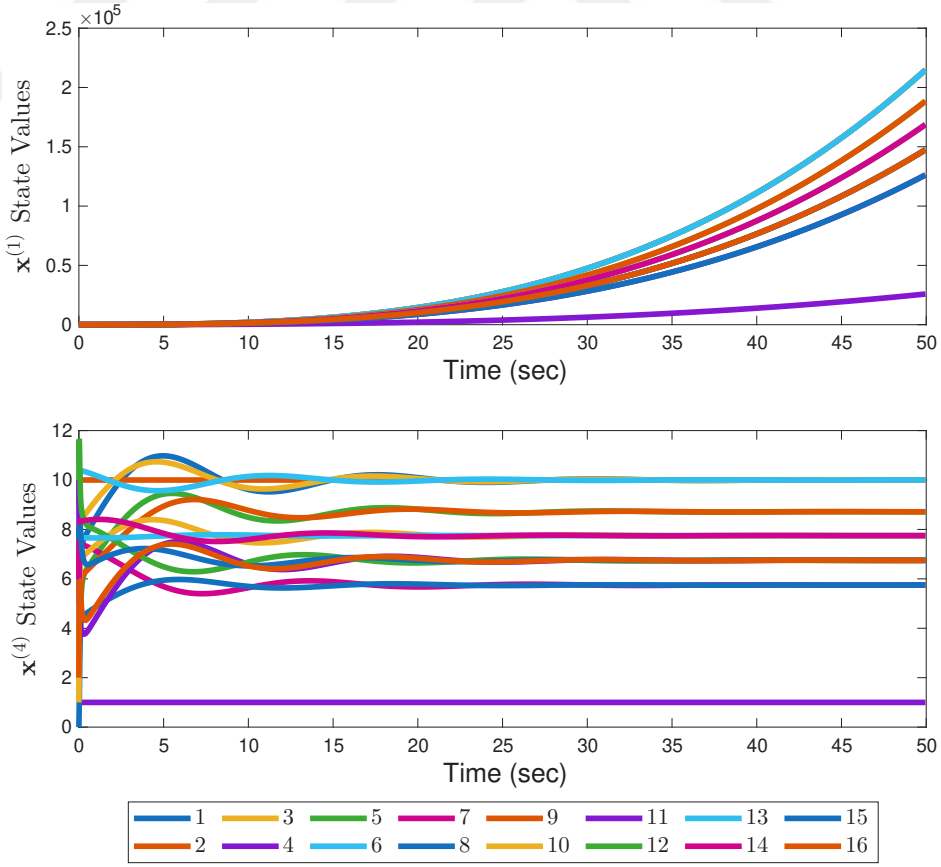


Figure 2.7. State trajectories of agents associated with unsigned directed graph in Figure 2.3 with $\gamma_1 = 2.25$, $\gamma_2 = 3$ and $\gamma_3 = 4$.

Now let us consider the same network with 4-th order dynamics. The choice of the controller parameters, $\gamma_1 = 1$, $\gamma_2 = 3$, $\gamma_3 = 6$, $\gamma_4 = 8$ lead to $4l_p$ ($l_p = 3$) zero eigenvalues and the rest of eigenvalues are in the OLHP. The states changes are shown in Figure 2.7. As expected, the network converges to $\mathcal{K} = 3 + 3 = 6$ clusters.

2.6. Chapter Summary & Concluding Remarks

In this chapter, we have investigated the any-order cluster consensus problem for cooperative multi-agent networks whose agents evolve over any given directed graph. We state necessary and sufficient conditions that ensure cluster agreement by choosing the control parameters based on Theorem 2.1. Moreover, we show that there is always a non-empty set of coupling parameters that guarantees cluster consensus for third-order systems. The results are illustrated by numerical examples for third/fourth-order systems. While this chapter presents stability conditions for higher-order cooperative networks to reach cluster consensus, achieving stability conditions for antagonistic systems is considered in Chapter 3.

3. CLUSTER CONSENSUS IN SIGNED NETWORKS

This chapter studies the cluster consensus problem for networks with antagonistic interactions modeled by adjacency matrices with negative weights. The behaviors of agents in a signed network are extracted from its extended directed graph by introducing an extended graph representation consisting of purely cooperative interactions with lifting approach. Consequently, the number of clusters and the cluster members are explicitly determined for any signed directed graph by using PLS and SLS concepts. The relation between the dynamics of a signed system and its extended representation is investigated for first and higher-order systems and the conditions that make both systems stable are stated. The obtained results for continuous-time networks are subsequently extended to discrete-time networks.

3.1. Antagonistic Higher-Order System Model

Consider a multi-agent system consisting of n agents with the following linear consensus protocol

$$\begin{aligned} \dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\ &\vdots \\ \dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\ \dot{x}_i^{(m)}(t) &= u_i(t), \quad i = 1, 2, \ldots, n \end{aligned} \tag{3.1}$$

where $x_i^{(q)}(t) \in \mathbb{R}$ $(q = 1, 2, \ldots, m)$ are the states of agent i at time t, $x_i^{(q)}$ denotes the $(q-1)$st derivative of $x_i^{(1)}$ and $u_i(t)$ is the control input described as follows:

$$u_i(t) = \sum_{q=1}^{m} \gamma_q \sum_{j \in \mathcal{N}_i} a_{ij}(x_j^{(q)}(t) - \operatorname{sgn}(a_{ij}) x_i^{(q)}(t)) \tag{3.2}$$

where $\gamma_q > 0$ are the control parameters.

The dynamics of system (3.1) are equivalent to

$$\begin{aligned}
\dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\
&\vdots \\
\dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\
\dot{x}_i^{(m)}(t) &= -\sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} l_{ij} x_j^{(q)}(t), \quad i = 1, 2, \ldots, n
\end{aligned} \tag{3.3}$$

where l_{ij} is the element of the signed Laplacian matrix, L, which is defined by

$$l_{ij} = \begin{cases} \sum\limits_{k=1, k\neq i}^{n} |a_{ik}|, & \text{if } i = j \\ -a_{ij}, & \text{otherwise.} \end{cases} \tag{3.4}$$

System (3.1) will be investigated under first and higher-order ($m \geq 2$) dynamics separately. In this context, system (3.1) can be expressed for first-order dynamics as

$$\dot{\mathbf{x}}(t) = -L\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{3.5}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathsf{T}}$ with $x_i(t) \in \mathbb{R}$ $(i = 1, 2, \ldots, n)$ is the state vector of the first-order system at time t and $\mathbf{x}_0$ is the initial condition. For $m \geq 2$, system (3.1) can be rewritten in the following form:

$$\dot{\mathbf{x}}(t) = \mathcal{L}_m \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{x}(t) = \left[\mathbf{x}^{(1)}(t)^{\mathsf{T}}, \mathbf{x}^{(2)}(t)^{\mathsf{T}}, \ldots, \mathbf{x}^{(m)}(t)^{\mathsf{T}}\right]^{\mathsf{T}}$ is the state vector with components $\mathbf{x}^{(q)}(t) = \left[x_1^{(q)}(t), x_2^{(q)}(t), \ldots, x_n^{(q)}(t)\right]^{\mathsf{T}}$ for $q = 1, 2, \ldots, m$, $\mathbf{x}_0$ is the initial condition

and

$$\mathcal{L}_m = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \dots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \dots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \dots & I_n \\ -\gamma_1 L & -\gamma_2 L & -\gamma_3 L & \dots & -\gamma_m L \end{bmatrix}. \tag{3.6}$$

The eigenvalues of $\mathcal{L}_m$ can be computed by solving the characteristic equation $\det(\mu I_{mn} - \mathcal{L}_m) = 0$ where

$$\begin{aligned} \det(\mu I_{mn} - \mathcal{L}_m) &= \det\left(\begin{bmatrix} \mu I_n & -I_n & 0_{n\times n} & \dots & 0_{n\times n} & 0_{n\times n} \\ 0_{n\times n} & \mu I_n & -I_n & \dots & 0_{n\times n} & 0_{n\times n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \dots & \mu I_n & -I_n \\ \gamma_1 L & \gamma_2 L & \gamma_3 L & \dots & \gamma_{m-1} L & \mu I_n + \gamma_m L \end{bmatrix}\right) \\ &= \det(\mu^m I_n + (\gamma_1 + \gamma_2\mu + \cdots + \gamma_m \mu^{m-1})L). \end{aligned}$$

The above expression can be rewritten with respect to the eigenvalues of the signed Laplacian matrix, L, as

$$\det(\mu I_{mn} - \mathcal{L}_m) = \prod_{i=1}^{n} \left(\mu^m + (\gamma_1 + \gamma_2\mu + \cdots + \gamma_m\mu^{m-1})\lambda_i\right) = 0 \tag{3.7}$$

where λ_i is the i-th eigenvalue of L. From (3.7), the eigenvalues of $\mathcal{L}_m$ can be obtained from

$$\mu^m + \gamma_m\lambda_i\mu^{m-1} + \cdots + \gamma_2\lambda_i\mu + \gamma_1\lambda_i = 0 \tag{3.8}$$

for $i = 1, 2, \dots, n$. Each eigenvalue of L corresponds to m eigenvalues of $\mathcal{L}_m$.

3.2. Extended Directed Graph

The evolution of an any-order system with a signed directed graph $\mathcal{G}$ is investigated by using its extended directed graph representation introduced in [36]. We define this representation as follows:

Definition 3.1. *For a given signed directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the agent set, $\mathcal{E}$ is the edge set and $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix; the corresponding extended directed graph is $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ with the agent set $\bar{\mathcal{V}} = \{1, 2, \ldots, n, 1', 2', \ldots, n'\}$, the edge set $\bar{\mathcal{E}}$ and the adjacency matrix $\bar{\mathcal{A}} = [\bar{a}_{ij}] \in \mathbb{R}^{2n\times 2n}$. The edge set, $\bar{\mathcal{E}}$, and the adjacency matrix, $\bar{\mathcal{A}}$, are defined as follows:*

1) If $(j, i) \in \mathcal{E}$ and $a_{ij} > 0$, then
   a) $(j, i) \in \bar{\mathcal{E}}$ and $\bar{a}_{ij} = a_{ij}$, and
   b) $(j', i') \in \bar{\mathcal{E}}$ and $\bar{a}_{i+n,j+n} = a_{ij}$ for $i, j = 1, 2, \ldots, n$.

2) If $(j, i) \in \mathcal{E}$ and $a_{ij} < 0$, then
   a) $(j, i') \in \bar{\mathcal{E}}$ and $\bar{a}_{i+n,j} = -a_{ij}$, and
   b) $(j', i) \in \bar{\mathcal{E}}$ and $\bar{a}_{i,j+n} = -a_{ij}$ for $i, j = 1, 2, \ldots, n$.

Remark 3.1. The extended adjacency elements $\bar{a}_{ij}$ in Definition 3.1 can be also formulated as

$$\bar{a}_{ij} = \bar{a}_{i+n,j+n} = \frac{|a_{ij}| + a_{ij}}{2} \qquad \text{and} \qquad \bar{a}_{i+n,j} = \bar{a}_{i,j+n} = \frac{|a_{ij}| - a_{ij}}{2},$$

for $i, j = 1, 2, \ldots, n$. From the adjacency elements $\bar{a}_{ij}$, the extended adjacency matrix $\bar{\mathcal{A}}$ is obtained as

$$\bar{\mathcal{A}} = \frac{1}{2}\begin{bmatrix} \mathtt{A} + \mathcal{A} & \mathtt{A} - \mathcal{A} \\ \mathtt{A} - \mathcal{A} & \mathtt{A} + \mathcal{A} \end{bmatrix}. \tag{3.9}$$

where $\mathtt{A} = [|a_{ij}|] \in \mathbb{R}^{n\times n}$. Note that an extended directed graph has only non-negative adjacency elements, i.e., $\bar{\mathcal{G}}$ is an unsigned directed graph consisting of $2n$ agents.

The extended directed graph concept is illustrated by the following example.

Example 3.1. *Consider the signed directed graph in Figure 3.1. Black and red arrows denote positive and negative edges, respectively. Note that the directed graph has no spanning tree. In order to construct the extended directed graph of the given network, the following procedure can be used:*

1) A new agent, i', is created for each agent, i, $i = 1, 2, \ldots, 5$. In this way, the number of agents is doubled.

2) While the positive edges in Figure 3.1 are preserved in Figure 3.2, the corresponding new positive edges are created according to Definition 3.1. For example, the positive edge $(1, 3)$ is maintained and the corresponding new edge $(1', 3')$ is created with the same weight i.e., $a_{31} = 0.7$.

3) The negative edges in Figure 3.1 are removed in Figure 3.2 and new positive edges are generated in its place based on Definition 3.1. For example, the negative edge $(1, 2)$ is removed and the corresponding new edges are created as $(1, 2')$ and $(1', 2)$ with the weight of $|a_{21}| = 0.1$.

By following the steps above, the corresponding extended directed graph is obtained as shown in Figure 3.2. Notice that the weights of all edges in the extended directed graph are positive.

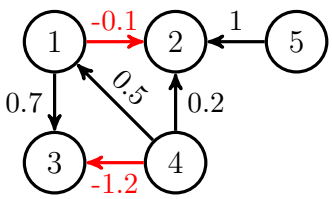


Figure 3.1. Signed directed graph in Example 3.1

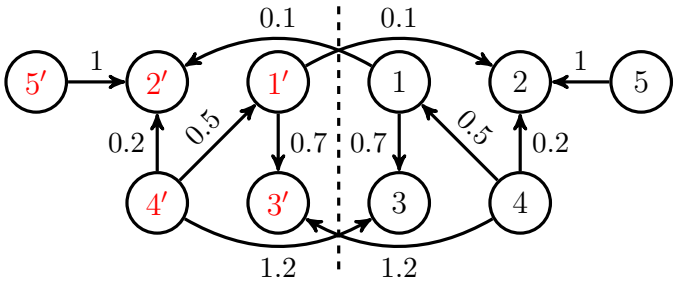


Figure 3.2. Extended directed graph in Example 3.1

3.3. Analysis of Cluster Consensus in Signed Directed Graphs

Given system (3.1) evolving over signed directed graph $\mathcal{G}$, the extended system associated with extended directed graph $\bar{\mathcal{G}}$ is defined as

$$\begin{aligned} \dot{\bar{x}}_i^{(1)}(t) &= \bar{x}_i^{(2)}(t), \\ &\vdots \\ \dot{\bar{x}}_i^{(m-1)}(t) &= \bar{x}_i^{(m)}(t), \\ \dot{\bar{x}}_i^{(m)}(t) &= \bar{u}_i(t), \quad i = 1, 2, \ldots, 2n \end{aligned} \tag{3.10}$$

where $\bar{x}_i^{(q)}(t)$ $(q = 1, 2, \ldots, m)$ are the states at time t, $\bar{x}_i^{(q)}$ denotes the $(q-1)$st derivative of $\bar{x}_i^{(1)}$ and $\bar{u}_i(t)$ is the control input described as follows:

$$\bar{u}_i(t) = \sum_{q=1}^{m} \gamma_q \sum_{j \in \bar{\mathcal{N}}_i} \bar{a}_{ij}(\bar{x}_j^{(q)}(t) - \bar{x}_i^{(q)}(t)), \quad i = 1, 2, \ldots, 2n. \tag{3.11}$$

The dynamics of system (3.10) are equivalent to

$$\begin{aligned} \dot{\bar{x}}_i^{(1)}(t) &= \bar{x}_i^{(2)}(t), \\ &\vdots \\ \dot{\bar{x}}_i^{(m-1)}(t) &= \bar{x}_i^{(m)}(t), \\ \dot{\bar{x}}_i^{(m)}(t) &= -\sum_{q=1}^{m} \gamma_q \sum_{j \in \bar{\mathcal{N}}_i} \bar{l}_{ij} \bar{x}_j^{(q)}(t), \quad i = 1, 2, \ldots, 2n \end{aligned} \tag{3.12}$$

where $\bar{l}_{ij}$ is the element of the unsigned Laplacian matrix, $\bar{L}$, which is defined by

$$\bar{l}_{ij} = \begin{cases} \sum_{k=1,k\neq i}^{2n} \bar{a}_{ik}, & \text{if } i = j \\ -\bar{a}_{ij}, & \text{otherwise.} \end{cases}$$

System (3.12) can be rewritten as in the following form:

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathcal{L}}_m \bar{\mathbf{x}}(t), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0$$

where $\bar{\mathbf{x}}(t) = \left[\bar{\mathbf{x}}^{(1)}(t)^\mathsf{T}, \bar{\mathbf{x}}^{(2)}(t)^\mathsf{T}, \ldots, \bar{\mathbf{x}}^{(m)}(t)^\mathsf{T}\right]^\mathsf{T}$ is the state vector with components $\bar{\mathbf{x}}^{(q)}(t) = \left[\bar{x}_1^{(q)}(t), \bar{x}_2^{(q)}(t), \ldots, \bar{x}_{2n}^{(q)}(t)\right]^\mathsf{T}$ for $q = 1, 2, \ldots, m$, $\bar{\mathbf{x}}_0$ is the initial condition and

$$\bar{\mathcal{L}}_m = \begin{bmatrix} 0_{2n\times 2n} & I_{2n} & 0_{2n\times 2n} & \ldots & 0_{2n\times 2n} \\ 0_{2n\times 2n} & 0_{2n\times 2n} & I_{2n} & \ldots & 0_{2n\times 2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \ldots & I_{2n} \\ -\gamma_1\bar{L} & -\gamma_2\bar{L} & -\gamma_3\bar{L} & \ldots & -\gamma_m\bar{L} \end{bmatrix}. \tag{3.13}$$

The eigenvalues of $\bar{\mathcal{L}}_m$ can be computed by solving the characteristic equation $\det(\bar{\mu} I_{2mn} - \bar{\mathcal{L}}_m) = 0$ where

$$\begin{aligned} \det(\bar{\mu} I_{2mn} - \bar{\mathcal{L}}_m) &= \det\left(\begin{bmatrix} \bar{\mu} I_{2n} & -I_{2n} & 0_{2n\times 2n} & \ldots & 0_{2n\times 2n} & 0_{2n\times 2n} \\ 0_{2n\times 2n} & \bar{\mu} I_{2n} & -I_{2n} & \ldots & 0_{2n\times 2n} & 0_{2n\times 2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \ldots & \bar{\mu} I_{2n} & -I_{2n} \\ \gamma_1\bar{L} & \gamma_2\bar{L} & \gamma_3\bar{L} & \ldots & \gamma_{m-1}\bar{L} & \bar{\mu} I_{2n} + \gamma_m\bar{L} \end{bmatrix}\right) \\ &= \det(\bar{\mu}^m I_{2n} + (\gamma_1 + \gamma_2\bar{\mu} + \cdots + \gamma_m\bar{\mu}^{m-1})\bar{L}). \end{aligned}$$

The above expression can be rewritten with respect to the eigenvalues of the extended unsigned Laplacian matrix, $\bar{L}$, as

$$\det(\bar{\mu} I_{2mn} - \bar{\mathcal{L}}_m) = \prod_{i=1}^{2n} \left(\bar{\mu}^m + (\gamma_1 + \gamma_2 \bar{\mu} + \cdots + \gamma_m \bar{\mu}^{m-1}) \bar{\lambda}_i\right) = 0 \tag{3.14}$$

where $\bar{\lambda}_i$ is the i-th eigenvalue of $\bar{L}$. From (3.14), the eigenvalues of $\bar{\mathcal{L}}_m$ can be obtained from

$$\bar{\mu}^m + \gamma_m \bar{\lambda}_i \bar{\mu}^{m-1} + \cdots + \gamma_2 \bar{\lambda}_i \bar{\mu} + \gamma_1 \bar{\lambda}_i = 0, \quad i = 1, 2, \ldots, 2n. \tag{3.15}$$

Each eigenvalue of $\bar{L}$ corresponds to m eigenvalues of $\bar{\mathcal{L}}_m$.

The following result relates the trajectories of system (3.1) described by a signed directed graph to that of system (3.10) which is described by an unsigned directed graph.

Theorem 3.1. *The trajectory* $\mathbf{x}(t)$ *of system* (3.1) *can be recovered from that of system* (3.10) *as* $x_i^{(q)}(t) = \bar{x}_i^{(q)}(t)$, $i = 1, 2, \ldots, n$ *for all* $t \geq 0$ *by setting* $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ *and* $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ *for* $i = 1, 2, \ldots, n$ *and* $q = 1, 2, \ldots, m$.

Proof. The control input (3.11) can be rewritten as follows:

$$\bar{u}_i(t) = \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{ij} (\bar{x}_j^{(q)}(t) - \bar{x}_i^{(q)}(t)) + \sum_{j=1}^{n} \bar{a}_{i,j+n} (\bar{x}_{j+n}^{(q)}(t) - \bar{x}_i^{(q)}(t)) \right]$$

for $i = 1, 2, \ldots, 2n$. By decomposing the sum into two parts, the extended system's control input is obtained as

$$\begin{aligned}
\bar{u}_i(t) &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(t) - \bar{x}_i^{(q)}(t) \right) + \sum_{j=1}^{n} \bar{a}_{i,j+n} \left(\bar{x}_{j+n}^{(q)}(t) - \bar{x}_i^{(q)}(t) \right) \right], \\
\bar{u}_{i+n}(t) &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{i+n,j+n} \left(\bar{x}_{j+n}^{(q)}(t) - \bar{x}_{i+n}^{(q)}(t) \right) + \sum_{j=1}^{n} \bar{a}_{i+n,j} \left(\bar{x}_j^{(q)}(t) - \bar{x}_{i+n}^{(q)}(t) \right) \right]
\end{aligned} \tag{3.16}$$

for $i = 1, 2, \ldots, n$. Let $\hat{x}_i^{(q)}(t) = \bar{x}_i^{(q)}(t) + \bar{x}_{i+n}^{(q)}(t)$ for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, m$. Then, we have

$$\begin{aligned}\dot{\hat{x}}_i^{(q)}(t) &= \dot{\bar{x}}_i^{(q)}(t) + \dot{\bar{x}}_{i+n}^{(q)}(t) \\ &= \bar{x}_i^{(q+1)}(t) + \bar{x}_{i+n}^{(q+1)}(t) \\ &= \hat{x}_i^{(q+1)}(t)\end{aligned}$$

for $q = 1, 2, \ldots, m-1$ and

$$\begin{aligned}\dot{\hat{x}}_i^{(m)}(t) &= \dot{\bar{x}}_i^{(m)}(t) + \dot{\bar{x}}_{i+n}^{(m)}(t) \\ &= \bar{u}_i(t) + \bar{u}_{i+n}(t) \\ &= \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} (\bar{a}_{ij} + \bar{a}_{i+n,j}) \bar{x}_j^{(q)}(t) + (\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n}) \bar{x}_{j+n}^{(q)}(t) \right) \\ &\quad - \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} (\bar{a}_{ij} + \bar{a}_{i,j+n}) \bar{x}_i^{(q)}(t) + (\bar{a}_{i+n,j} + \bar{a}_{i+n,j+n}) \bar{x}_{i+n}^{(q)}(t) \right).\end{aligned}$$

From Definition 3.1, we obtain

$$\bar{a}_{i+n,j} = \bar{a}_{i,j+n}, \quad \bar{a}_{ij} = \bar{a}_{i+n,j+n} \tag{3.17}$$

for $i = 1, 2, \ldots, n$ and $|a_{ij}| = \bar{a}_{ij} + \bar{a}_{i+n,j}$. By using (3.17), we also have $\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n} = \bar{a}_{ij} + \bar{a}_{i,j+n} = \bar{a}_{i+n,j} + \bar{a}_{i+n,j+n} = |a_{ij}|$. Then, we have

$$\begin{aligned}\dot{\hat{x}}_i^{(m)}(t) &= \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} |a_{ij}| \left(\bar{x}_j^{(q)}(t) + \bar{x}_{j+n}^{(q)}(t) \right) \right) - \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} |a_{ij}| \left(\bar{x}_i^{(q)}(t) + \bar{x}_{i+n}^{(q)}(t) \right) \right) \\ &= \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} |a_{ij}| \left(\hat{x}_j^{(q)}(t) - \hat{x}_i^{(q)}(t) \right), \quad i = 1, 2, \ldots, n. \end{aligned} \tag{3.18}$$

The dynamics of the above system is equivalent to

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathcal{L}}_m \hat{\mathbf{x}}(t), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \tag{3.19}$$

where $\hat{\mathbf{x}}(t) = \left[\hat{\mathbf{x}}^{(1)}(t)^{\mathsf{T}}, \hat{\mathbf{x}}^{(2)}(t)^{\mathsf{T}}, \ldots, \hat{\mathbf{x}}^{(m)}(t)^{\mathsf{T}}\right]^{\mathsf{T}}$ is the state vector with components $\hat{\mathbf{x}}^{(q)}(t) = \left[\hat{x}_1^{(q)}(t), \hat{x}_2^{(q)}(t), \ldots, \hat{x}_n^{(q)}(t)\right]^{\mathsf{T}}$ for $q = 1, 2, \ldots, m$, $\hat{\mathbf{x}}_0$ is the initial condition and $\hat{\mathcal{L}}_m \in \mathbb{R}^{mn \times mn}$ is given by

$$\hat{\mathcal{L}}_m = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ -\gamma_1\hat{L} & -\gamma_2\hat{L} & -\gamma_3\hat{L} & \ldots & -\gamma_m\hat{L} \end{bmatrix} \tag{3.20}$$

where $\hat{L}$ is the unsigned Laplacian matrix whose elements are defined by

$$\hat{l}_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{n} |a_{ik}|, & \text{if } i = j \\ -|a_{ij}|, & \text{otherwise.} \end{cases} \tag{3.21}$$

The solution of (3.19) is

$$\hat{\mathbf{x}}(t) = e^{\hat{\mathcal{L}}_m t}\hat{\mathbf{x}}_0 \tag{3.22}$$

Note that $\hat{\mathbf{x}}_0 = [0, \ldots, 0]^{\mathsf{T}}$ implies $\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_0 = [0, \ldots, 0]^{\mathsf{T}}$ for all t, i.e., $\bar{x}_i^{(q)}(t) = -\bar{x}_{i+n}^{(q)}(t)$ for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, m$.

Substituting $\bar{x}_{j+n}^{(q)}(t) = -\bar{x}_j^{(q)}(t)$ into (3.16), we obtain

$$\dot{\bar{u}}_i(t) = \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} (\bar{a}_{ij} - \bar{a}_{i,j+n})\bar{x}_j^{(q)}(t) - \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{a}_{i,j+n})\bar{x}_i^{(q)}(t)\right]$$

for $i = 1, 2, \ldots, n$. From Definition 3.1, we get

$$\dot{\bar{u}}_i(t) = \sum_{q=1}^{m} \left[\sum_{j=1}^{n} \left(\frac{|a_{ij}| + a_{ij}}{2} - \frac{|a_{ij}| - a_{ij}}{2}\right)\bar{x}_j^{(q)}(t) - \left(\frac{|a_{ij}| + a_{ij}}{2} + \frac{|a_{ij}| - a_{ij}}{2}\right)\bar{x}_i^{(q)}(t)\right]$$

$$= \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} a_{ij}(x_j^{(q)}(t) - \mathrm{sgn}(a_{ij})x_i^{(q)}(t)) \tag{3.23}$$

for $1 = 1, 2, \ldots, n$. Note that (3.23) is equal to (3.1). By setting $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ and $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, m$ the trajectory $\mathbf{x}^{(q)}(t)$ can be recovered. This completes the proof. □

While Theorem 3.1 focuses on the relation of the trajectories of systems (3.1) and (3.10), it does not address the relationship between their stability properties. By using the relation of L and $\bar{L}$, below we first study the stability relation for first-order dynamics and then extend the results to higher-order systems.

Theorem 3.2. *For $m = 1$, the extended system* (3.10) *is stable if and only if the system* (3.1) *is stable.*

Proof. ($\Longrightarrow$) Suppose system (3.10) is stable. From (3.9), the unsigned Laplacian system matrix, $\bar{L}$, for system (3.10) is expressed as

$$\bar{L} = \frac{1}{2}\begin{bmatrix} \hat{L} + L & \hat{L} - L \\ \hat{L} - L & \hat{L} + L \end{bmatrix}$$

where L and $\hat{L}$ are both Laplacian matrices defined as in (3.4) and (3.21), respectively. By using the similarity transformation matrix

$$T = \begin{bmatrix} I_n & 0_{n\times n} \\ I_n & I_n \end{bmatrix}, \tag{3.24}$$

we have

$$T\bar{L}T^{-1} = \begin{bmatrix} L & \frac{1}{2}(\hat{L} - L) \\ 0_{n\times n} & \hat{L} \end{bmatrix} \tag{3.25}$$

from which we conclude that the eigenvalues of $\bar{L}$ are the combined eigenvalues of L and $\hat{L}$. Since system (3.10) is stable and L in (3.4) is the system matrix for system (3.1), we conclude that system (3.1) is stable.

($\Longleftarrow$) Suppose system (3.1) is stable, i.e., L is a Laplacian matrix. Note from (3.25) that $\bar{L}$ has eigenvalues of L and $\hat{L}$. As $\hat{L}$ is a Laplacian matrix by definition in (3.21) and system (3.1) is stable, it follows that system (3.10) is stable. □

Theorem 3.3. *For $m \geq 2$, if the extended system* (3.10) *is stable, then the system* (3.1) *is stable.*

Proof. Suppose the extended system (3.10) is stable. From (3.25), recall that the eigenvalues of $\bar{L}$ for the extended system (3.10) are the union of the eigenvalues of L and $\hat{L}$. By using this result, the eigenvalues of $\bar{\mathcal{L}}_m$ can be obtained with respect to the eigenvalues of L and $\hat{L}$ as

$$\bar{\mu}^m + \gamma_m \lambda_i \bar{\mu}^{m-1} + \cdots + \gamma_2 \lambda_i \bar{\mu} + \gamma_1 \lambda_i = 0, \quad i = 1, 2, \ldots, n, \tag{3.26a}$$

$$\bar{\mu}^m + \gamma_m \hat{\lambda}_i \bar{\mu}^{m-1} + \cdots + \gamma_2 \hat{\lambda}_i \bar{\mu} + \gamma_1 \hat{\lambda}_i = 0, \quad i = 1, 2, \ldots, n \tag{3.26b}$$

where $\bar{\mu}$ is an eigenvalue of $\bar{\mathcal{L}}_m$, λ_i and $\hat{\lambda}_i$ are the eigenvalues of L and $\hat{L}$, respectively. Note that (3.26a) and (3.26b) together are the characteristic polynomial of the system the extended system (3.10) and only (3.26a) is the characteristic polynomial of the system (3.1). This implies that the characteristic polynomial of the extended system (3.10) includes that of the system (3.1). Given that the extended system (3.10) is stable, so is the system (3.1). □

From Theorem 3.3, we note that the stability of the extended system (3.10) implies the stability of the system (3.1) for $m \geq 2$. On the other hand, a stable system (3.1) does not necessarily imply a stable extended system (3.10) as illustrated in the following example.

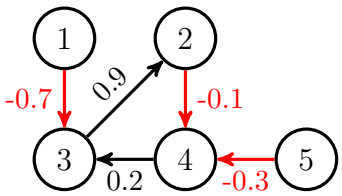


Figure 3.3. Signed directed graph in Example 3.2

Example 3.2. *Consider the signed network in Figure 3.3. For the given network, the signed Laplacian matrix, L, and its unsigned version, $\hat{L}$, are obtained as follows:*

$$L = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & -0.9 & 0 & 0 \\ 0.7 & 0 & 0.9 & -0.2 & 0 \\ 0 & 0.1 & 0 & 0.4 & 0.3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \hat{L} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & -0.9 & 0 & 0 \\ -0.7 & 0 & 0.9 & -0.2 & 0 \\ 0 & -0.1 & 0 & 0.4 & -0.3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of $-L$ and $-\hat{L}$ are calculated as $\{-1.0646,\ -0.6,\ -0.5354,\ 0,\ 0\}$ and $\{-0.9289 \pm j0.1773,\ -0.3422,\ 0,\ 0\}$, respectively. Note that the spectra of $-L$ and $-\hat{L}$ are in the OLHP except zero eigenvalues. From Theorem 3.2, the eigenvalues of $-\bar{L}$ is the union of the eigenvalues of $-L$ and $-\hat{L}$, i.e., $\{-1.0646,\ -0.6,\ -0.5354,\ 0,\ 0\} \bigcup \{-0.9289 \pm j0.1773,\ -0.3422,\ 0,\ 0\}$. In the light of this fact, the following results are obtained:

- *For $m = 1$, system matrices for systems (3.1) and (3.10) are $-L$ and $-\bar{L}$, respectively. Therefore, both systems are stable.*
- *For $m = 2$ and $\gamma_1 = 10, \gamma_2 = 0.5$, system matrices for systems (3.1) and (3.10) are $\mathcal{L}_m$ and $\bar{\mathcal{L}}_m$, respectively. All nonzero eigenvalues of $\mathcal{L}_m$ have negative real parts. In the corresponding extended system matrix $\bar{\mathcal{L}}_m$, there is a pair of complex conjugate eigenvalues whose real parts are positive ($0.0548 \pm j3.0971$). In this case, system (3.1) is stable whereas extended system (3.10) is unstable.*

In the case of structurally balanced systems, the stability of systems (3.1) and (3.10) are equivalent regardless of the degree of the dynamics as delineated in the next result.

Corollary 3.1. *Consider a structurally balanced system* (3.1)*. For* $m \geq 1$*, the extended system* (3.10) *is stable if and only if the system* (3.1) *is stable.*

Proof. Given a structurally balanced system (3.1), its signed Laplacian matrix, L, in (3.4) can be expressed as

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

where the elements of $L_{11} \in \mathbb{R}^{n_1 \times n_1}$ and $L_{22} \in \mathbb{R}^{n_2 \times n_2}$ contains positive edges and the elements of $L_{12} \in \mathbb{R}^{n_1 \times n_2}$ and $L_{21} \in \mathbb{R}^{n_2 \times n_1}$ contain negative edges. Therefore, one can obtain $\hat{L}$ in (3.21) by using the similarity transformation as

$$\hat{L} = \begin{bmatrix} L_{11} & -L_{12} \\ -L_{21} & L_{22} \end{bmatrix} = \begin{bmatrix} -I_{n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & I_{n_2} \end{bmatrix} \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix} \begin{bmatrix} -I_{n_1} & 0_{n_1 \times n_2} \\ 0_{n_2 \times n_1} & I_{n_2} \end{bmatrix}$$

This implies that L and $\hat{L}$ have the same eigenvalues. From (3.25), $\bar{L}$ also has the same eigenvalues with L. In the structurally balanced case, the stabilities of systems (3.10) and (3.1) are equivalent for $m \geq 1$. □

For any given higher-order system with negative links, its extended higher-order system with positive links can be obtained by using Definition 3.1. For the reader's convenience, we restate Theorem 2.1 below as Lemma 3.1, adapted to the notation and equations introduced in this chapter:

Lemma 3.1. *The number of clusters for the* m*-th order system* (3.10) *associated with an unsigned directed graph is equal to* $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ *if and only if the roots of* (3.14) *are in the OLHP except* $m\bar{l}_p$ *zero eigenvalues where* $\bar{l}_p$ *and* $\bar{l}_s$ *are the number of PLSs and SLSs, respectively.*

Note that Lemma 3.1 is applicable for networks whose communication links are positive weighted. By using Definition 3.1, a signed directed graph can be augmented

to its extended directed graph whose edges are positive. In this context, the following result is obtained to determine the clusters for a general higher-order multi-agent system that has no restrictive structure properties.

Theorem 3.4. *The number of clusters for the m-th order system* (3.1) *associated with an signed directed graph is equal to* $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ *if the roots of* (3.15) *are in the OLHP where* $\bar{l}_p$ *and* $\bar{l}_s$ *are the number of PLSs and SLSs of the extended system* (3.10)*, respectively; and* l' *is the number of subgraphs whose agent set consists of only a subset of* $\bar{\mathcal{V}} \setminus \mathcal{V} = \{1', 2', \ldots, n'\}$.

Proof. From Definition 3.1, it follows that the extended graph $\bar{\mathcal{G}}$ contains non-negative edge weights. Hence, Lemma 3.1 is applicable. For a given system (3.1) with n agents, its extended representation (3.10) with $2n$ agents is obtained based on Definition 3.1. By using Lemma 3.1, the extended system (3.10) that is extended representation of the system (3.1) converges to $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ clusters. Since l' subgraphs in $\bar{l}_p + \bar{l}_s$ subgraphs do not contain any agent of the system (3.1), the number of clusters for system (3.1) is found as

$$\mathcal{K} = \bar{l}_p + \bar{l}_s - l'.$$

□

Remark 3.2. Theorem 3.4 delineates the condition to determine the number of clusters for any directed graph with positively or negatively weighted edges from its extended directed graph. If the original system (3.1) is completely given by non-negative weights with l_p PLSs and l_s SLSs, then the corresponding extended system (3.10) has $\bar{l}_p = 2l_p$ PLSs and $\bar{l}_s = 2l_s$ SLSs with $l' = l_p + l_s$. From Theorem 3.4, the number of clusters for system (3.1) is $\mathcal{K} = \bar{l}_p + \bar{l}_s - l' = l_p + l_s$ which agrees with the result in [13].

Remark 3.3. It can also be shown that the existing results in the literature on the special class of structurally balanced signed directed graphs can be recovered from Theorem 3.4. Systems with structurally balanced directed graphs always converge to 2 clusters [31–35]. Extended directed graph of a structurally balanced has 2 independent

subgraphs which contain a spanning tree. Therefore, it can be found that $\bar{l}_p = 2$ PLSs and $\bar{l}_s = 0$ SLSs for the extended system. Since each PLS always has at least one member of the original system, it can be obtained as $l' = 0$. Consequently, a structurally balanced system attains $\bar{l}_p + \bar{l}_s - l' = 2$ equilibria that agrees with the existing results in the literature.

The above result brings a novel contribution to the study of the cluster consensus problem for higher-order multi-agent systems. In the literature, there is no any result on determining the number of clusters to which a given higher-order system with negative links converges.

Given a multi-agent system with a signed directed graph, the application of Theorem 3.4 can be summarized as in Procedure 1.

Procedure 1

Given a multi-agent system of n agents with signed directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, 2, \ldots, n\}$

Step 1: Construct the multi-agent system of $2n$ agents with $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ where $\mathcal{V} = \{1, 2, \ldots, n, 1', 2', \ldots, n'\}$, $\mathcal{E}$ and $\mathcal{A}$ satisfy Definition 3.1.

Step 2: Determine the PLSs $\bar{\mathcal{G}}_{p,i}$ for $i = 1, 2, \ldots, \bar{l}_p$, in $\bar{\mathcal{G}}$ by using the detection algorithm in [13].

Step 3: Determine the SLSs $\bar{\mathcal{G}}_{s,i}$ for $i = 1, 2, \ldots, \bar{l}_s$, in $\bar{\mathcal{G}}$ by using the detection algorithm in [13].

Step 4: Determine the subgraphs among $\bar{\mathcal{G}}_{p,i}$ and $\bar{\mathcal{G}}_{s,j}$ for $i = 1, 2, \ldots, \bar{l}_p$, $j = 1, 2, \ldots, \bar{l}_s$ whose agent set consists of only a subset of $\bar{\mathcal{V}} \backslash \mathcal{V} = \{1', 2', \ldots, n'\}$. Let the total number of such subgraphs be l'

Step 5: Calculate the number of clusters in $\mathcal{G}$ as $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$.

Figure 3.4. Algorithm to calculate the number of clusters in a given signed directed graph.

Given a signed directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the time complexity of Algorithm 1 can be obtained as follows:

- **Step 1:** The extended unsigned directed graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathcal{A}})$ is constructed based on Definition 3.1 where the agent set $\bar{\mathcal{V}}$ is duplicated that requires $2n$ operations and the edge set $\bar{\mathcal{E}}$ is extended based on the connections that requires $4n^2 - 2n$ operations. Hence, the time complexity of Step 1 is $O(n^2)$.
- **Step 2:** The time complexity of the PLS detection algorithm is studied in [13] and found to be $O(n^4)$.
- **Step 3:** In [13], the time complexity of the SLS detection algorithm is found to be $O(n^6)$.
- **Step 4:** Searching n elements $(1', 2', \ldots, n')$ in set of $2n$ elements in the worst case requires at most $2n^2$ comparisons. Therefore, the time complexity of Step 4 is $O(n^2)$.
- **Step 5:** As this step involves a scalar calculation, its time complexity is $O(1)$.

Based on the above discussion, Algorithm 1 is polynomial time and its time complexity is given by $O(n^6)$.

3.3.1. Control Parameter Selection

In this section, conditions on the choice of control parameters (γ_q) are investigated to achieve cluster consensus for systems with any order of dynamics. In deriving the general stability condition for m-th order systems and their extended dynamics, consider the following characteristic polynomials with complex coefficients

$$\Delta(\mu) = \mu^m + \gamma_m \lambda_i \mu^{m-1} + \cdots + \gamma_2 \lambda_i \mu + \gamma_1 \lambda_i, \tag{3.27a}$$

$$\hat{\Delta}(\mu) = \mu^m + \gamma_m \hat{\lambda}_i \mu^{m-1} + \cdots + \gamma_2 \hat{\lambda}_i \mu + \gamma_1 \hat{\lambda}_i \tag{3.27b}$$

that should have its roots in the OLHP. Associated with (3.27a) or (3.27b), define Υ as

$$\Upsilon = \begin{bmatrix} \alpha'_1 & -\alpha''_2 & -\alpha'_3 & \alpha''_4 & \alpha'_5 & \cdots & (-1)^{m+1}\alpha'_{2m-1} \\ 1 & -\alpha''_1 & -\alpha'_2 & \alpha''_3 & \alpha'_4 & \cdots & (-1)^{m+1}\alpha'_{2m-2} \\ 0 & \alpha'_1 & -\alpha''_2 & -\alpha'_3 & \alpha''_4 & \cdots & (-1)^{m+1}\alpha'_{2m-2} \\ 0 & 1 & -\alpha''_1 & -\alpha'_2 & \alpha''_3 & \cdots & (-1)^{m+1}\alpha'_{2m-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \tag{3.28}$$

where $\alpha'_r = \mathbb{R}\mathrm{e}\{\gamma_{m-r+1}\lambda_i\}$ or $\alpha'_r = \mathbb{R}\mathrm{e}\{\gamma_{m-r+1}\hat{\lambda}_i\}$, $\alpha''_r = \mathbb{I}\mathrm{m}\{\gamma_{m-r+1}\lambda_i\}$ or $\alpha''_r = \mathbb{I}\mathrm{m}\{\gamma_{m-r+1}\hat{\lambda}_i\}$ and $\alpha_r = 0$ for $r > m$. Let Υ_{ii}, $i = 1, 2, \ldots, 2m-1$ denote the leading principal minors of Υ. From Chapter 2, it follows that the roots of (3.27a) and (3.27b) are in the OLHP, i.e., the $m-$th order systems are stable if and only if all members of the sequence of $1, \Upsilon_{11}, \Upsilon_{33}, \ldots, \Upsilon_{2m-1,2m-1}$ are positive. These conditions can be simplified for second and third-order systems.

For second-order systems:

(i) $\gamma_2 > 0$,

(ii) $\dfrac{\gamma_2^2}{\gamma_1} > \max\limits_i \dfrac{\mathbb{I}\mathrm{m}^2\{\lambda_i\}}{\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2}$.

(λ_i are non-zero eigenvalues of L or $\hat{L}$)

For third-order systems:

(i) $\gamma_3 > 0$,

(ii) $\gamma_3^2\gamma_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^2 - \gamma_3\gamma_1\,\mathbb{R}\mathrm{e}^2\{\lambda_i\} - \gamma_2^2\,\mathbb{I}\mathrm{m}^2\{\lambda_i\} > 0$,

(iii) $\gamma_2^2\gamma_3^2\,\mathbb{R}\mathrm{e}\{\lambda_i\}|\lambda_i|^4 - 2\gamma_1\gamma_2\gamma_3\,\mathbb{R}\mathrm{e}^2\{\lambda_i\}|\lambda_i|^2 - \gamma_2^3\,\mathbb{I}\mathrm{m}^2\{\lambda_i\}|\lambda_i|^2 + \gamma_1^2\,\mathbb{R}\mathrm{e}^3\{\lambda_i\} > 0$.

(λ_i are non-zero eigenvalues of L or $\hat{L}$)

Under the conditions given above for second and third-order systems, the extended representation of any signed network is stable from Chapter 2. In this case, the number of clusters for the extended system can be calculated as $\bar{l}_p + \bar{l}_s$ based on Theorem 3.4 where $\bar{l}_p$ and $\bar{l}_s$ are the number of PLSs and SLSs of the extended system, respectively. Since l' is the number of subgraphs containing only agents from $\{1', 2', \ldots, n'\}$, it can be concluded that the number of clusters for a signed network is equal to $\bar{l}_p + \bar{l}_s - l'$ provided that the following respective conditions are satisfied.

3.4. Extension to Discrete-Time Networks

Consider a multi-agent system consisting of n agents with m-th order integrator dynamics in discrete time.

$$\begin{aligned} x_i^{(1)}(k+1) &= x_i^{(1)}(k) + x_i^{(2)}(k), \\ &\vdots \\ x_i^{(m-1)}(k+1) &= x_i^{(m-1)}(k) + x_i^{(m)}(k), \\ x_i^{(m)}(k+1) &= x_i^{(m)}(k) + u_i(k), \quad i = 1, 2, \ldots, n \end{aligned} \tag{3.29}$$

with the initial condition $x_i^{(q)}(0) = x_{i0}^{(q)}$ where $x_i^{(q)}(k)$ $(q = 1, 2, \ldots, m)$, are the states of agent i at time step k. We introduce the control input of agent i as follows

$$u_i(k) = \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} a_{ij} \left(x_j^{(q)}(k) - \operatorname{sgn}(a_{ij}) x_i^{(q)}(k) \right) \tag{3.30}$$

where a_{ij} are the elements of the adjacency matrix and $\gamma_q > 0$ $(q = 1, 2, \ldots, m)$ are the controller parameters to be designed.

System (3.29) will be investigated under first and higher-order ($m \geq 2$) dynamics separately. In this context, system (3.29) can be expressed for first-order dynamics as

$$\begin{aligned} \mathbf{x}(k+1) &= (I_n - L)\mathbf{x}(k) \\ \mathbf{x}(k+1) &= W\mathbf{x}(k), \quad \mathbf{x}(0) = \mathbf{x}_0 \end{aligned} \tag{3.31}$$

where $\mathbf{x}(k) = [x_1(k), x_2(k), \ldots, x_n(k)]^\mathsf{T}$ is the state vector of the first-order system at time step k, $W = [w_{ij}]$ is the system matrix and $\mathbf{x}_0$ is the initial condition. The following conditions on the system matrix are assumed to be satisfied.

Assumption 3.1. *The elements of W, w_{ij}, satisfy the followings:*

(i) $w_{ii} > 0$ *for* $i = 1, 2, \ldots, n$,
(ii) $w_{ij} \neq 0$ *for all* $(j, i) \in \mathcal{E}$,
(iii) $w_{ij} = 0$ *for all* $(j, i) \notin \mathcal{E}$,
(iv) $\sum_{j=1}^{n} |w_{ij}| = 1$ *for* $i = 1, 2, \ldots, n$

Assumption 3.1(i) ensures that each agent uses its local information by positive weighting. Assumptions 3.1(ii) and (iii) guarantee that the weighting coefficients are nonzero if there is a direct connection between the agents and zero if there is none. The absolute values of the weighting coefficients used by an agent sum up to 1 as stated in Assumption 3.1(iv). Assumption 3.1 assures that the matrix $\mathtt{W} = [|w_{ij}|] \in \mathbb{R}^{n \times n}$ is a row-stochastic matrix.

For $m \geq 2$, the state dynamics given in (3.29) and (3.30) can be expressed in matrix form as

$$\mathbf{x}(k+1) = \mathcal{W}_m \mathbf{x}(k) \tag{3.32}$$

where the system matrix $\mathcal{W}_m$ is defined as

$$\mathcal{W}_m = \begin{bmatrix} I_n & I_n & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & I_n & I_n & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} & I_n & I_n \\ -\gamma_1 L & -\gamma_2 L & \cdots & \cdots & -\gamma_{m-1}L & I_n - \gamma_m L \end{bmatrix}_{mn\times mn}. \tag{3.33}$$

Given system (3.29) evolving over signed directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, the extended system is defined as

$$\begin{aligned} \bar{x}_i^{(1)}(k+1) &= \bar{x}_i^{(1)}(k) + \bar{x}_i^{(2)}(k), \\ &\vdots \\ \bar{x}_i^{(m-1)}(k+1) &= \bar{x}_i^{(m-1)}(k) + \bar{x}_i^{(m)}(k), \\ \bar{x}_i^{(m)}(k+1) &= \bar{x}_i^{(m)}(k) + \bar{u}_i(k), \quad i = 1, 2, \ldots, 2n \end{aligned} \tag{3.34}$$

with the initial condition $\bar{x}_i^{(q)}(0) = \bar{x}_{i0}^{(q)}$ for $q = 1, 2, \ldots, m$. The control input is given as

$$\bar{u}_i(k) = \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{2n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k) \right) \tag{3.35}$$

where $\bar{a}_{ij}$ are the non-negative weighting coefficients. The state dynamics given in (3.34) and (3.35) can be expressed in matrix form as

$$\bar{\mathbf{x}}(k+1) = \bar{\mathcal{W}}_m \bar{\mathbf{x}}(k), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0. \tag{3.36}$$

Here the system matrix $\bar{\mathcal{W}}_m$ is defined as

$$\bar{\mathcal{W}}_m = \begin{bmatrix} I_n & I_n & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & I_n & I_n & 0_{n\times n} & \cdots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} & I_n & I_n \\ -\gamma_1\bar{L} & -\gamma_2\bar{L} & \cdots & \cdots & -\gamma_{m-1}\bar{L} & I_n - \gamma_m\bar{L} \end{bmatrix}_{2mn\times 2mn} \tag{3.37}$$

and $\bar{\mathbf{x}}(k) = [\bar{\mathbf{x}}^{(1)}(k)^\mathsf{T}, \bar{\mathbf{x}}^{(2)}(k)^\mathsf{T}, \ldots, \bar{\mathbf{x}}^{(m)}(k)^\mathsf{T}]^\mathsf{T}$ is the state vector with components $\bar{\mathbf{x}}^{(q)}(k) = [\bar{x}_1^{(q)}(k), \bar{x}_2^{(q)}(k), \ldots, \bar{x}_{2n}^{(q)}(k)]^\mathsf{T}$ for $q = 1, 2, \ldots, m$, $\bar{\mathbf{x}}_0$ is the initial condition and $\bar{L}$ is the unsigned Laplacian matrix of the extended graph $\bar{\mathcal{G}}$.

The state trajectory of the original system (3.29) can be recovered from that of the extended system (3.34) as delineated by the following result:

Theorem 3.5. *The trajectory* $\mathbf{x}(k)$ *of system* (3.29) *can be recovered from that of system* (3.34) *as* $x_i^{(q)}(k) = \bar{x}_i^{(q)}(k)$, $i = 1, 2, \ldots, n$ *and* $q = 1, 2, \ldots, m$ *for all* $k \geq 0$ *by setting* $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ *and* $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ *for* $i = 1, 2, \ldots, n$ *and* $q = 1, 2, \ldots, m$.

Proof. The input of the extended system (3.34) can be rewritten as follows

$$\bar{u}_i(k) = \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k) \right) + \sum_{j=1}^{n} \bar{a}_{i,j+n} \left(\bar{x}_{j+n}^{(q)}(k) - \bar{x}_i^{(q)}(k) \right) \right), \tag{3.38}$$

$$\bar{u}_{i+n}(k) = \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} \bar{a}_{i+n,j+n} \left(\bar{x}_{j+n}^{(q)}(k) - \bar{x}_{i+n}^{(q)}(k) \right) + \sum_{j=1}^{n} \bar{a}_{i+n,j} \left(\bar{x}_j^{(q)}(k) - \bar{x}_{i+n}^{(q)}(k) \right) \right) \tag{3.39}$$

for $i = 1, 2, \ldots, n$. Let $\hat{x}_i^{(q)}(k) = \bar{x}_i^{(q)}(k) + \bar{x}_{i+n}^{(q)}(k)$ for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, m$.

Then we have

$$\begin{aligned}\hat{x}_i^{(q)}(k+1) &= \bar{x}_i^{(q)}(k+1) + \bar{x}_{i+n}^{(q)}(k+1) \\ &= \bar{x}_i^{(q)}(k) + \bar{x}_i^{(q+1)}(k) + \bar{x}_{i+n}^{(q)}(k) + \bar{x}_{i+n}^{(q+1)}(k) \\ &= \hat{x}_i^{(q)}(k) + \hat{x}_i^{(q+1)}(k)\end{aligned} \tag{3.40}$$

for $q = 1, 2, \ldots, m-1$ and

$$\begin{aligned}\hat{x}_i^{(m)}(k+1) &= \bar{x}_i^{(m)}(k+1) + \bar{x}_{i+n}^{(m)}(k+1) \\ &= \bar{x}_i^{(m)}(k) + \bar{u}_i(k) + \bar{x}_{i+n}^{(m)}(k) + \bar{u}_{i+n}(k) \\ &= \hat{x}_i^{(m)}(k) + \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k) \right) + \sum_{j=1}^{n} \bar{a}_{i,j+n} \left(\bar{x}_{j+n}^{(q)}(k) - \bar{x}_i^{(q)}(k) \right) \right. \\ &\quad \left. + \sum_{j=1}^{n} \bar{a}_{i+n,j+n} \left(\bar{x}_{j+n}^{(q)}(k) - \bar{x}_{i+n}^{(q)}(k) \right) + \sum_{j=1}^{n} \bar{a}_{i+n,j} \left(\bar{x}_j^{(q)}(k) - \bar{x}_{i+n}^{(q)}(k) \right) \right).\end{aligned}$$

From Definition 3.1, we obtain

$$\bar{a}_{i+n,j} = \bar{a}_{i,j+n}, \quad \bar{a}_{ij} = \bar{a}_{i+n,j+n} \tag{3.41}$$

for $i = 1, 2, \ldots, n$ and $|a_{ij}| = \bar{a}_{ij} + \bar{a}_{i+nj}$. By using (3.41), we also have $\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n} = \bar{a}_{ij} + \bar{a}_{i,j+n} = \bar{a}_{i+n,j} + \bar{a}_{i+n,j+n} = |a_{ij}|$. Then we have

$$\begin{aligned}\hat{x}_i^{(m)}(k+1) &= \hat{x}_i^{(m)}(k) + \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} |a_{ij}| \left(\bar{x}_j^{(q)}(k) + \bar{x}_{j+n}^{(q)}(k) - \bar{x}_i^{(q)}(k) - \bar{x}_{i+n}^{(q)}(k) \right) \right) \\ &= \hat{x}_i^{(m)}(k) + \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} |a_{ij}| \left(\hat{x}_j^{(q)}(k) - \hat{x}_i^{(q)}(k) \right).\end{aligned} \tag{3.42}$$

The dynamics of the system represented by (3.40) and (3.42) is equivalent to

$$\begin{aligned}
\hat{x}_i^{(1)}(k+1) &= \hat{x}_i^{(1)}(k) + \hat{x}_i^{(2)}(k), \\
&\vdots \\
\hat{x}_i^{(m-1)}(k+1) &= \hat{x}_i^{(m-1)}(k) + \hat{x}_i^{(m)}(k), \\
\hat{x}_i^{(m)}(k+1) &= \hat{x}_i^{(m)}(k) + \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} |a_{ij}| \left(\hat{x}_j^{(q)}(k) - \hat{x}_i^{(q)}(k) \right)
\end{aligned} \tag{3.43}$$

which can be expressed in matrix form as

$$\hat{\mathbf{x}}(k+1) = \hat{\mathcal{W}}_m \hat{\mathbf{x}}(k), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \tag{3.44}$$

where $\hat{\mathbf{x}}(k) = [\hat{\mathbf{x}}^{(1)}(k)^\mathsf{T}, \hat{\mathbf{x}}^{(2)}(k)^\mathsf{T}, \ldots, \hat{\mathbf{x}}^{(m)}(k)^\mathsf{T}]^\mathsf{T}$ is the state vector with components $\hat{\mathbf{x}}^{(q)}(k) = [\hat{x}_1^{(q)}(k), \hat{x}_2^{(q)}(k), \ldots, \hat{x}_n^{(q)}(k)]^\mathsf{T}$ $(q = 1, 2, \ldots, m)$, $\hat{\mathbf{x}}_0$ is the initial condition and

$$\hat{\mathcal{W}}_m = \begin{bmatrix}
I_n & I_n & 0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} \\
0_{n\times n} & I_n & I_n & 0_{n\times n} & \cdots & 0_{n\times n} \\
0_{n\times n} & 0_{n\times n} & \ddots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0_{n\times n} \\
0_{n\times n} & 0_{n\times n} & \cdots & 0_{n\times n} & I_n & I_n \\
-\gamma_1 \hat{L} & -\gamma_2 \hat{L} & \cdots & \cdots & -\gamma_{m-1}\hat{L} & I_n - \gamma_m \hat{L}
\end{bmatrix}_{mn\times mn}. \tag{3.45}$$

Since the solution of (3.44) is $\hat{\mathbf{x}}(k) = \hat{\mathcal{W}}_m^k \hat{\mathbf{x}}_0$, the initial condition $\hat{\mathbf{x}}_0 = [0, \ldots, 0]^\mathsf{T}$ implies $\hat{\mathbf{x}}(k) = \hat{\mathbf{x}}_0 = [0, \ldots, 0]^\mathsf{T}$ for all k, i.e., $\bar{x}_i^{(q)}(k) = -\bar{x}_{i+n}^{(q)}(k)$ for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, m$.

Substituting $\bar{x}_{j+n}^{(q)}(k) = -\bar{x}_j^{(q)}(k)$ into (3.38), we obtain

$$\begin{aligned}
\bar{u}_i(k) &= \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k) \right) + \sum_{j=1}^{n} \bar{a}_{i,j+n} \left(-\bar{x}_j^{(q)}(k) - \bar{x}_i^{(q)}(k) \right) \right) \\
&= \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} (\bar{a}_{ij} - \bar{a}_{i,j+n}) \bar{x}_j^{(q)}(k) - (\bar{a}_{ij} + \bar{a}_{i,j+n}) \bar{x}_i^{(q)}(k) \right)
\end{aligned} \tag{3.46}$$

for $i = 1, 2, \ldots, n$. From Definition 3.1, we get

$$\begin{aligned}\bar{u}_i(k) &= \sum_{q=1}^{m} \gamma_q \left(\sum_{j=1}^{n} \left(\frac{|a_{ij}|+a_{ij}}{2} - \frac{|a_{ij}|-a_{ij}}{2} \right) \bar{x}_j^{(q)}(k) - \left(\frac{|a_{ij}|+a_{ij}}{2} + \frac{|a_{ij}|-a_{ij}}{2} \right) \bar{x}_i^{(q)}(k) \right) \\ &= \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} a_{ij} \left(x_j^{(q)}(k) - \operatorname{sgn}(a_{ij}) x_i^{(q)}(k) \right) \end{aligned} \tag{3.47}$$

for $i = 1, 2, \ldots, n$. Note that (3.47) is equal to (3.30). By setting $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ and $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, m$, the trajectory $\mathbf{x}(k)$ can be recovered. This completes the proof. □

While Theorem 3.5 focuses on the relation of the trajectories of systems (3.29) and (3.34), it does not address the relationship between their stability properties.

Theorem 3.6. *For $m = 1$, the extended system* (3.34) *is stable if and only if the system* (3.29) *is stable.*

Proof. ($\Longrightarrow$) Suppose extended system (3.34) is stable. From (3.9), the system matrix $\bar{W}$ of the extended system (3.34) is expressed as

$$\bar{W} = \frac{1}{2} \begin{bmatrix} \mathtt{W} + W & \mathtt{W} - W \\ \mathtt{W} - W & \mathtt{W} + W \end{bmatrix}$$

where $\mathtt{W} = [|w_{ij}|] \in \mathbb{R}^{n \times n}$. W and $\bar{W}$ are both system matrices satisfying Assumption 3.1. By using the similarity transformation matrix defined as in (3.24), we have

$$T \bar{L} T^{-1} = \begin{bmatrix} W & \frac{1}{2}(\mathtt{W} - W) \\ 0_{n \times n} & \mathtt{W} \end{bmatrix} \tag{3.48}$$

from which we conclude that the eigenvalues of $\bar{W}$ are the combined eigenvalues of W and $\mathbb{W}$. Since extended system (3.34) is stable and W is the system matrix for system (3.29), we conclude that system (3.29) is stable.

($\Longleftarrow$) Suppose system (3.29) is stable, i.e., W is a system matrix satisfying Assumption 3.1. Note from (3.48) that $\bar{W}$ has eigenvalues of W and $\mathtt{W}$. As $\mathtt{W}$ is a system matrix satisfying Assumption 3.1 and system (3.29) is stable, it follows that extended system (3.34) is stable. □

The following theorem states the stability relationship between the original and the extended system.

Theorem 3.7. *For $m \geq 2$, if the extended system* (3.34) *is stable, then the system* (3.29) *is stable.*

Proof. In order to prove the theorem, we show that the spectrum of $\bar{\mathcal{W}}_m$ is the union of the spectra of $\mathcal{W}_m$ and $\hat{\mathcal{W}}_m$.

The eigenvalues of $\mathcal{W}_m$ are the roots of the polynomial

$$\Delta(\lambda) = \prod_{i=1}^{n}(\lambda-1)^m + \left(\sum_{d=1}^{m}\gamma_d(\lambda-1)^{d-1}\right)\mu_i$$

where μ_i are the eigenvalues of L. The eigenvalues of $\hat{\mathcal{W}}_m$ are the roots of the polynomial

$$\hat{\Delta}(\lambda) = \prod_{i=1}^{n}(\lambda-1)^m + \left(\sum_{d=1}^{m}\gamma_d(\lambda-1)^{d-1}\right)\hat{\mu}_i$$

where $\hat{\mu}_i$ are the eigenvalues of $\hat{L}$. The eigenvalues of $\bar{\mathcal{W}}_m$ are the roots of the polynomial

$$\bar{\Delta}(\lambda) = \prod_{i=1}^{n}(\lambda-1)^m + \left(\sum_{d=1}^{m}\gamma_d(\lambda-1)^{d-1}\right)\bar{\mu}_i$$

where $\bar{\mu}_i$ are the eigenvalues of $\bar{L}$. Recall from (3.25) that the spectrum of $\bar{L}$ is the union of the spectra of L and $\hat{L}$. Therefore, for all λ such that $\Delta(\lambda) = 0$ or $\hat{\Delta}(\lambda) = 0$, we have $\bar{\Delta}(\lambda) = 0$ which concludes the proof. □

Remark 3.4. From Theorem 3.7, we note that the stability of extended system (3.34) implies the stability of system (3.29) for $m \geq 2$. On the other hand, a stable system (3.29) does not necessarily imply a stable extended system (3.34).

In the case of structurally balanced systems, the stability of systems (3.29) and (3.34) are equivalent regardless of the degree of the dynamics as delineated in the next result.

Corollary 3.2. *Consider a structurally balanced system* (3.29). *For* $m \geq 1$, *the extended system* (3.34) *is stable if and only if the system* (3.29) *is stable.*

Proof. Suppose that the system (3.29) is structurally balanced. From Corollary 3.1, the signed Laplacian matrix L and the unsigned Laplacian matrix $\hat{L}$ have the same eigenvalues. From (3.25), $\bar{L}$ also has the same eigenvalues as L. Since the eigenvalues of $\bar{\mathcal{W}}_m$ and $\mathcal{W}_m$ are functions of the eigenvalues of $\bar{L}$ and L, we conclude that the stabilities of systems (3.34) and (3.29) are equivalent for structurally balanced graphs. □

The stability conditions for cluster consensus of (3.29) evolving over an unsigned directed graph are stated in the following lemma.

Lemma 3.2. ([102]) *Suppose that* $\gamma_1 \neq 0$ *and the underlying unsigned directed graph of the unsigned network consists of* $\bar{l}_p$ *PLSs and* $\bar{l}_s$ *SLSs. Then the system* (3.34) *achieves* $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ *cluster consensus if and only if the roots of the equation*

$$\sum_{q=0}^{m-1} 2^q (\sigma - 1)^{m-q} \gamma_{q+1} \bar{\lambda}_i + 2^m \tag{3.49}$$

are located in the OLHP where $\bar{\lambda}_i$ *are the nonzero eigenvalues of* $\bar{L}$ *and the multiplicity of the eigenvalue* 0 *of* $\bar{L}$ *is* $m\bar{l}_p$.

The above lemma is only valid for multi-agent systems evolving over unsigned directed graphs. By using Definition 3.1, this lemma can be generalized for signed networks without any structural restriction. Therefore, the following theorem is obtained:

Theorem 3.8. *The system* (3.29) *achieves* $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ *cluster consensus if the roots of the equation*

$$\sum_{q=0}^{m-1} 2^q (\sigma - 1)^{m-q} \gamma_{q+1} \bar{\lambda}_i + 2^m \tag{3.50}$$

are located in the OLHP where $\bar{l}_p$ *and* $\bar{l}_s$ *are the number of PLSs and SLSs of the extended system* (3.34)*, respectively;* l' *is the number of subgraphs whose agent set consists of only a subset of* $\bar{\mathcal{V}} \setminus \mathcal{V} = \{1', 2', \ldots, n'\}$*,* $\bar{\lambda}_i$ *are the nonzero eigenvalues of* $\bar{L}$ *and the multiplicity of the eigenvalue* 0 *of* $\bar{L}$ *is* $m\bar{l}_p$*.*

Proof. From Lemma 3.2, the total number of clusters of extended system (3.34) is $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$. Finally, the number of clusters for system (3.29) is found as

$$\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$$

since l' clusters have no member from the original system (3.29). □

Remark 3.5. Lemma 3.2 states the stability conditions for cluster consensus of the extended system. Note that when the extended system achieves $\bar{\mathcal{K}}$-cluster consensus, the original system achieves $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ cluster consensus where l' is the number of subgraphs whose agent set consists of only a subset of $\bar{\mathcal{V}} \setminus \mathcal{V} = \{1', 2', \ldots, n'\}$.

3.4.1. Controller Parameter Selection

The choice of controller parameters γ_i plays a crucial role for the stability of the system (3.29). In order to ensure stability, the controller parameters should be chosen such that the roots of (3.50) are located in the OLHP and the conditions obtained in

Section 4 can be applied. By using the results in [103] and [100], these conditions can be simplified for second and third-order systems.

For second-order systems:

(i) $\gamma_1 = (\epsilon_l/\delta)^2$,

(ii) $\gamma_2 = \epsilon_l$.

($\delta = \sqrt{\frac{2}{r_l}}$ and ϵ_l are two positive numbers with $0 < r_l \leq \min \mathbb{R}\mathrm{e}\{\mu_i\}$, $0 < \epsilon_l \leq \min \frac{\mathbb{R}\mathrm{e}\{\mu_i\}}{|\mu_i|^2}$, and μ_i are the nonzero eigenvalues of $\bar{L}$)

For third-order systems:

(i) $\gamma_1 = \dfrac{\alpha\epsilon_{\mathbb{R}\mathrm{e}} - \epsilon_{\mathbb{R}\mathrm{e}} - 2\Delta_{\mathbb{R}\mathrm{e}} - 0.5\Delta_{\mathbb{I}\mathrm{m}}}{(\alpha - 1.5)(\alpha - 1)(\alpha^2 - 3\alpha + 2.5)}$,

(ii) $\gamma_2 = \alpha\gamma_1$,

(iii) $\gamma_3 = \dfrac{(\Delta_{\mathbb{I}\mathrm{m}} + 4\Delta_{\mathbb{R}\mathrm{e}})(\alpha^3 - 4.5\alpha^2 + 6.5\alpha - 3.25) + \epsilon_{\mathbb{R}\mathrm{e}}(\alpha - 1)^2}{(\alpha - 1.5)(\alpha - 1)(\alpha^2 - 3\alpha + 2.5)}$.

($\Delta_{\mathbb{R}\mathrm{e}}, \epsilon_{\mathbb{R}\mathrm{e}}$, $\Delta_{\mathbb{I}\mathrm{m}}$ and α are positive parameters such that $\Delta_{\mathbb{R}\mathrm{e}} \geq \max_i \frac{\mathbb{R}\mathrm{e}\{\mu_i\}}{|\mu_i|^2}$, $0 < \epsilon_{\mathbb{R}\mathrm{e}} \leq \min_i \frac{\mathbb{R}\mathrm{e}\{\mu_i\}}{|\mu_i|^2}$, $\Delta_{\mathbb{I}\mathrm{m}} \geq \max_i \frac{\mathbb{I}\mathrm{m}\{\mu_i\})}{|\mu_i|^2}$, $\alpha = \alpha_0 \dfrac{2\epsilon_{\mathbb{R}\mathrm{e}} + 4\Delta_{\mathbb{R}\mathrm{e}} + \Delta_{\mathbb{I}\mathrm{m}}}{2\epsilon_{\mathbb{R}\mathrm{e}}}$ with $\alpha_0 > 1$ hold where μ_i are the nonzero eigenvalues of $\bar{L}$)

Under the conditions given above for second and third-order systems, the extended representation of any signed network is stable from [103] and [100]. In this case, the number of clusters for the extended system can be calculated as $\bar{l}_p + \bar{l}_s$ based on Lemma 3.2 where $\bar{l}_p$ and $\bar{l}_s$ are the number of PLSs and SLSs of the extended system, respectively. Since l' is the number of subgraphs containing only agents from $\{1', 2', \ldots, n'\}$, it can be concluded that the number of clusters for a signed network is equal to $\bar{l}_p + \bar{l}_s - l'$ provided that the following respective conditions are satisfied.

3.5. Numerical Analysis

Consider the network in Figure 3.5 where the red arrows represent negative information flows and the black arrows represent positive information flows. *Note that the network is not structurally balanced and has no spanning tree. As such, the existing results in [31–35] are not applicable.*

The system matrix is generated as

$$L = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -1 & 3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 4 & 5 & -1 \\ 0 & 0 & 0 & 0 & -2 & 2 \end{bmatrix}.$$

The corresponding extended system matrix can be obtained as follows:

$$\bar{L} = \left[\begin{array}{cccccc:cccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ -1 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 5 & -1 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & -1 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 2 \end{array}\right].$$

The eigenvalues of $\bar{L}$ are $7, 7, 3, 3, 2, 2, 1, 1, 0, 0, 0, 0$. In the case where the network has second-order dynamics with $\gamma_1 = \gamma_2 = 1$, the stability conditions are satisfied. The signed directed graph given in Figure 3.5 can be extended into the directed graph in Figure 3.6. Notice that all connections between agents of the extended network are positively weighted in Figure 3.6. The extended directed graph can also be redrawn as shown in Figure 3.7 depicting also PLSs and SLSs. In this context, PLSs and SLSs of the extended network can be extracted as shown in Figure 3.7 by using the PLS and SLS detection algorithms from [13]. Subgraphs are labeled as $\bar{\mathcal{G}}_{p,1}, \bar{\mathcal{G}}_{p,2}, \bar{\mathcal{G}}_{p,3}, \bar{\mathcal{G}}_{p,4}$ for the PLSs and $\bar{\mathcal{G}}_{s,1}, \bar{\mathcal{G}}_{s,2}$ for the SLSs. The members of the clusters in the extended directed graph are determined as $\bar{\mathcal{V}}_{p,1} = \{3\}$, $\bar{\mathcal{V}}_{p,2} = \{4, 1', 5', 6'\}$, $\bar{\mathcal{V}}_{p,3} = \{1, 5, 6, 4'\}$, $\bar{\mathcal{V}}_{p,4} = \{3'\}$, $\bar{\mathcal{V}}_{s,1} = \{2\}$ and $\bar{\mathcal{V}}_{s,2} = \{2'\}$. The number of PLSs and SLSs of the extended network are $\bar{l}_p = 4$ and $\bar{l}_s = 2$, respectively. The number of subgraphs whose agent set consists of only a subset of $\bar{\mathcal{V}} \setminus \mathcal{V} = \{1', 2', \ldots, n'\}$ is $l' = 2$ ($\bar{\mathcal{V}}_{p,4}$ and $\bar{\mathcal{V}}_{s,2}$). From Theorem 3.4, the original system in Figure 3.5 reaches $\bar{l}_p + \bar{l}_s - l' = 4$ clusters.

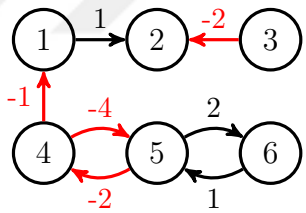


Figure 3.5. A signed directed graph not containing a spanning tree.

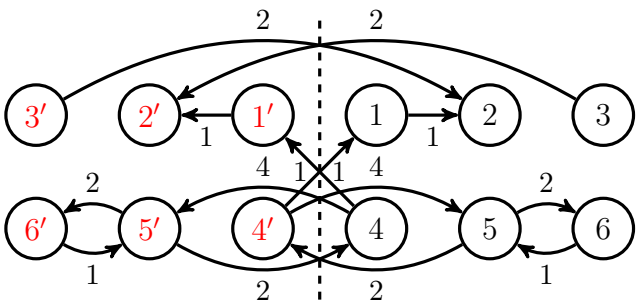


Figure 3.6. Extended directed graph for the network in Figure 3.5.

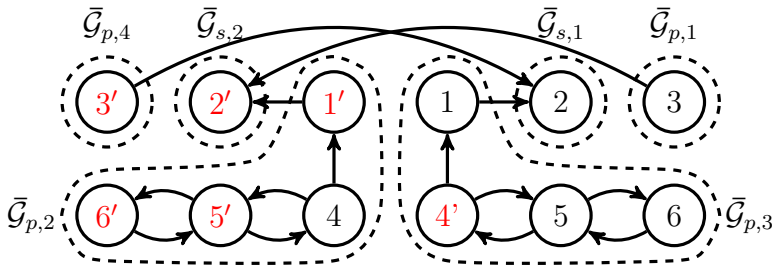


Figure 3.7. Redrawing of the extended network in Figure 3.6.

For the signed directed graph, shown in Figure 3.5, with $\mathbf{x}(0) = \mathbf{x}_0 = [0.8,\ 0.25,\ 0.95,\ 0.35,\ 0.2,\ 0.25]^\mathsf{T}$, the second-order simulation result is depicted in Figure 3.8. The system converges to 4 clusters which are $\{1, 5, 6\}$, $\{2\}$, $\{3\}$ and $\{4\}$.

To recover the states of original agents from its extended representation, initial states of the extended system are set as $\bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 = [\mathbf{x}_0^\mathsf{T},\ -\mathbf{x}_0^\mathsf{T}]^\mathsf{T}$. With this setting, the related second-order simulation result is given in Figure 3.9. Note that the trajectories of the original agents are the same in both figures.

From this numerical example, the following comparative results can be obtained:

(i) The main result in [31] allows to determine whether the network converges to bipartite consensus or not. If given network is strongly connected, digon-sign symmetric and structurally balanced, it can be said that bipartite consensus is achieved. The studies [33–35] also focus on networks that have special graph structures such as structurally balanced, spanning tree. However, there are no results for networks that do not containing special graph structures. When structurally balanced systems are considered numerically, its extended representations have always only 2 PLSs, i.e., $\bar{l}_p = 2, \bar{l}_s = 0$. Since the PLSs always contain a member of the structurally balanced signed network, l' can be calculated as 0. Hence, one can conclude that a structurally balanced signed network always converges to $\bar{l}_p + \bar{l}_s - l' = 2$ clusters in perspective of numerical analysis.

(ii) In [32], results are obtained on networks having a spanning tree. In addition to the related result in [31], if given network has a spanning tree, interval bipartite

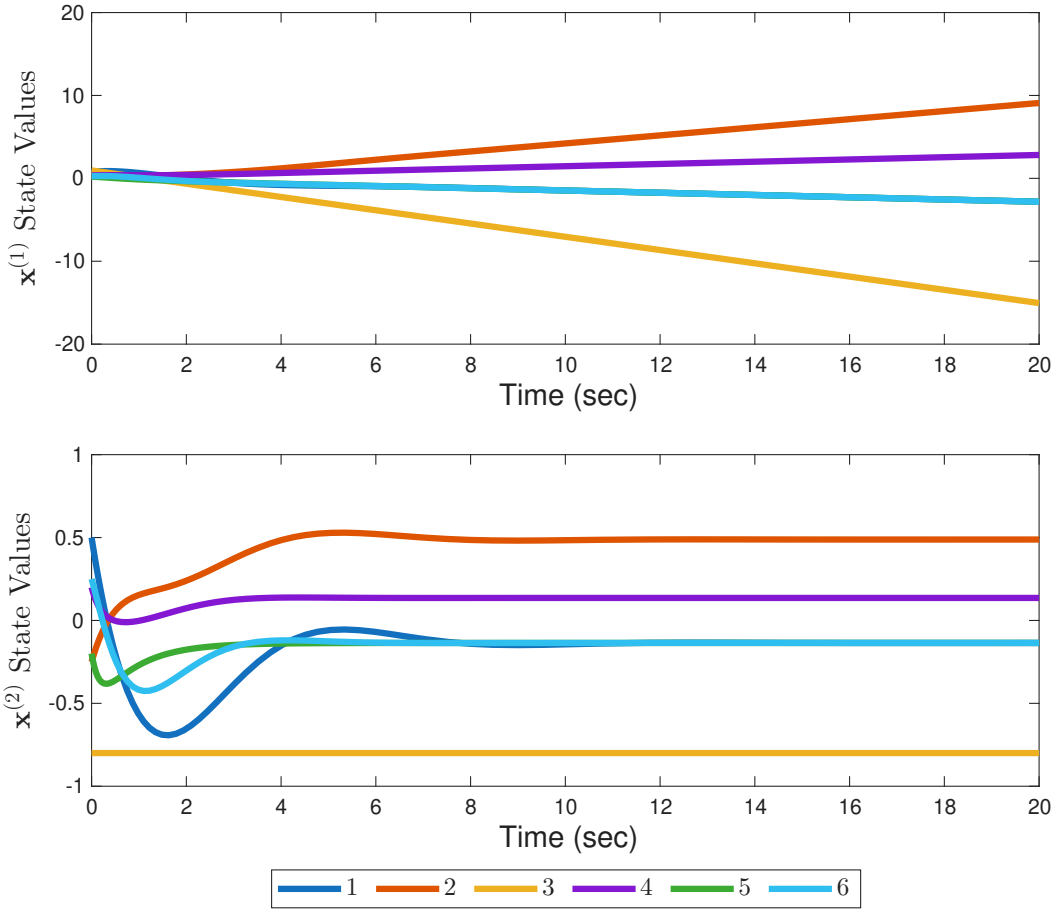


Figure 3.8. State trajectories of agents associated with signed directed graph in Figure 3.5.

consensus is attained. However, the number of clusters and their members cannot be determined from this work.

(iii) The network in Figure 3.5 is not structurally balanced and has no spanning tree. All papers mentioned above and others in the literature are insufficient to define how the given network behaves. In this chapter, we determine the number of clusters and their members for a given network without any restrictive structural properties by using PLS/SLS concepts and lifting approach. In numerical manner, $\bar{l}_p, \bar{l}_s$ and l' can be calculated as any value under PLS and SLS conditions.

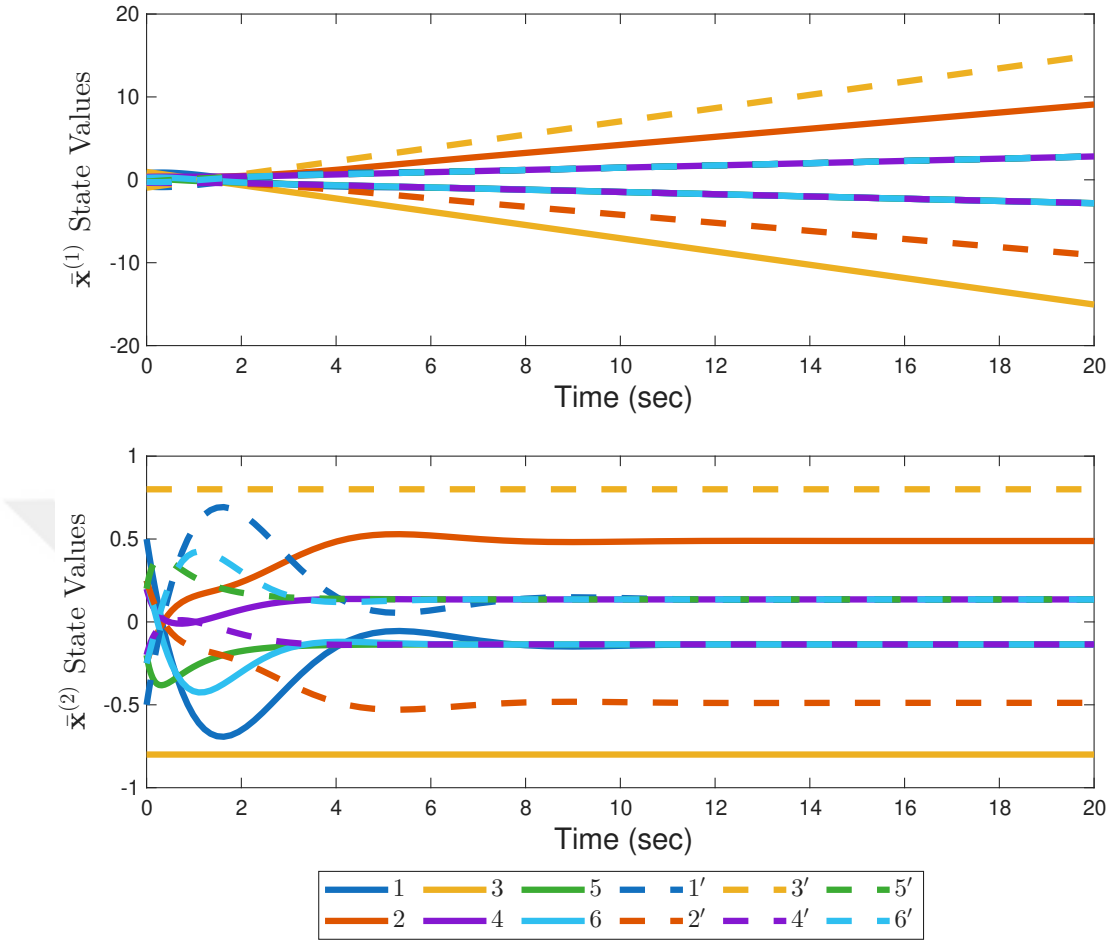


Figure 3.9. State trajectories of agents associated with extended directed graph in Figure 3.6.

3.6. Chapter Summary & Concluding Remarks

This chapter has discussed the cluster agreement problem for a higher-order network of agents evolving over any given signed directed graph $\mathcal{G}$. We have introduced an extended graph representation to analyze the convergence properties of any given signed directed graph. This extended directed graph structure has enabled us to compute the number of clusters and the agents in each cluster for a multi-agent network without any restriction on the graph structure. Systematic controller parameter selection methods are also provided for both continuous-time and discrete-time networks. Up to this point, the cluster consensus problem has been studied in cooperative or antagonistic networks without any delay. The effects of delay on the cluster consensus property of a given system are discussed in Chapter 4.

4. EFFECT OF DELAY ON CLUSTER CONSENSUS

In this chapter, we study the cluster consensus problem in higher-order integrator networks represented by unsigned or signed directed graphs with delayed dynamics. First, we investigate input or communication delayed networks modeled via unsigned directed graphs. Second, we consider networks evolving over signed directed graphs with input or communication delays. By utilizing an extended digraph approach, the trajectories of a signed delayed network are recovered from its corresponding extended unsigned digraph. For higher-order integrator systems with fixed input or communication delay under continuous-time topology, the stability properties are investigated and the upper bounds are determined so that the systems achieve cluster consensus without any change in cluster members. Furthermore, we investigate the discrete-time higher-order integrator systems with nonuniform delay.

4.1. Delayed Networks with Unsigned Directed Graphs

In multi-agent networks, delays can be broadly classified into input and communication delays. In this section, we examine the effect of the delay on the stability of cluster consensus of unsigned networks in continuous-time.

4.1.1. Input Delay Effect

Consider the $m-$th order linear delayed dynamics of agent i described by

$$\begin{aligned} \dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\ &\vdots \\ \dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\ \dot{x}_i^{(m)}(t) &= u_i(t-\tau), \quad i = 1, 2, \ldots, n \end{aligned} \tag{4.1}$$

where $x_i^{(q)}(t) \in \mathbb{R}$ $(q = 1, 2, \ldots, m)$ are the states of agent i at time t, $x_i^{(q)}$ denotes the $(q-1)$st derivative of $x_i^{(1)}$, τ is the fixed input delay and $u_i(t-\tau)$ is the delayed input described as follows:

$$u_i(t) = \sum_{q=1}^{m} \gamma_q \sum_{j \in \mathcal{N}_i} a_{ij}(x_j^{(q)}(t) - x_i^{(q)}(t)) \tag{4.2}$$

where γ_q are control parameters.

Necessary and sufficient conditions for cluster consensus of agents with the above dynamics are discussed in Chapter 2. Cluster consensus analysis of higher-order systems is complicated in comparison to that of first/second-order systems. Adding time delay to the system makes the cluster consensus problem even more challenging for higher-order systems. In this chapter, we examine the effect of time delay on cluster consensus for m-th order multi-agent systems.

The dynamics of the delayed higher-order system can be written as

$$\dot{\mathbf{x}}(t) = \mathcal{L}_0 \mathbf{x}(t) + \mathcal{L}_d \mathbf{x}(t-\tau), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{4.3}$$

where $\mathbf{x}(t) = \left[\mathbf{x}^{(1)}(t)^\mathsf{T}, \mathbf{x}^{(2)}(t)^\mathsf{T}, \ldots, \mathbf{x}^{(m)}(t)^\mathsf{T}\right]^\mathsf{T}$ is the state vector with components $\mathbf{x}^{(q)}(t) = \left[x_1^{(q)}(t), x_2^{(q)}(t), \ldots, x_n^{(q)}(t)\right]^\mathsf{T}$ for $q = 1, 2, \ldots, m$, $\mathbf{x}_0$ is the initial condition and

$$\mathcal{L}_0 = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \end{bmatrix}, \mathcal{L}_d = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ -\gamma_1 L & -\gamma_2 L & -\gamma_3 L & \ldots & -\gamma_m L \end{bmatrix}. \tag{4.4}$$

4.1.2. Analysis of Delayed Third-Order Cluster Consensus

In this section, we present the results on the cluster consensus problem in third-order systems with input delay. Its generalization to the systems with higher-order dimensions will be discussed in the next section.

Reconsider the system matrices $\mathcal{L}_0$ and $\mathcal{L}_d$ in (4.4). The eigenvalues of the given third-order system with input delay can be computed by solving the characteristic equation $\det(\mu I_{3n} - \mathcal{L}_0 - e^{-\mu\tau}\mathcal{L}_d) = 0$ where

$$\begin{aligned}\det(\mu I_{3n} - \mathcal{L}_0 - e^{-\mu\tau}\mathcal{L}_d) &= \det\left(\begin{bmatrix} \mu I_n & -I_n & 0_{n\times n} \\ 0_{n\times n} & \mu I_n & -I_n \\ \gamma_1 e^{-\mu\tau}L & \gamma_2 e^{-\mu\tau}L & \mu I_n + \gamma_3 e^{-\mu\tau}L \end{bmatrix}\right) \\ &= \det(\mu^3 I_n + (\gamma_1 + \gamma_2\mu + \gamma_3\mu^2)e^{-\mu\tau}L).\end{aligned}$$

The above expression can be written with respect to eigenvalues of the unsigned Laplacian matrix, L, as

$$\det(\mu I_{3n} - \mathcal{L}_0 - e^{-\mu\tau}\mathcal{L}_d) = \prod_{i=1}^{n}\left[\mu^3 + (\gamma_1 + \gamma_2\mu + \gamma_3\mu^2)e^{-\mu\tau}\lambda_i\right] = 0 \qquad (4.5)$$

where λ_i is the i-th eigenvalue of L. From the above expression, it is easy to see that the characteristic equation (4.5) has a zero eigenvalue of algebraic multiplicity $3l_p$ if and only if L has a zero eigenvalue of algebraic multiplicity l_p.

Firstly, we provide the following lemma that is to be utilized in the stability analysis. Then, we present the theorem on control parameters and input delay.

Lemma 4.1. (Cardano, 1545, [104]) *Let P be the cubic equation:*

$$P(r) = ar^3 + br^2 + cr + d = 0$$

with $a \neq 0$. Then, P has solutions as follows:

$$\begin{aligned} r_1 &= S_0 + S_1 - \frac{b}{3a}, \\ r_2 &= -\frac{S_0 + S_1}{2} - \frac{b}{3a} + \frac{j\sqrt{3}}{2}(S_0 - S_1), \\ r_3 &= -\frac{S_0 + S_1}{2} - \frac{b}{3a} - \frac{j\sqrt{3}}{2}(S_0 - S_1) \end{aligned}$$

where

$$S_0 = \sqrt[3]{R + \sqrt{Q^3 + R^2}}, \quad S_1 = \sqrt[3]{R - \sqrt{Q^3 + R^2}}$$

with

$$Q = \frac{3ac - b^2}{9a^2}, \quad R = \frac{9abc - 27a^2d - 2b^3}{54a^3}.$$

The expression $D = Q^3 + R^2$ is called the discriminant of the equation. Then,

(1) If $D > 0$, then one root is real and two are complex conjugates.

(2) If $D = 0$, then all roots are real, and at least two are equal.

(3) If $D < 0$, then all roots are real and unequal.

For input delayed third-order systems, a non-empty set of control parameters $\{\gamma_1, \gamma_2, \gamma_3\}$ is given by the following theorem.

Theorem 4.1. *Let $\gamma_1, \gamma_2, \gamma_3$ be chosen such that the following conditions are satisfied:*

(i) $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$,

(ii) $\gamma_1\gamma_3 = \gamma_2^2$,

(iii) $\frac{\gamma_3^2}{\gamma_2} \geq \max_i \frac{2}{\mathbb{R}\mathrm{e}\{\lambda_i\}}$,

(iv) $\tau < \tau^* = \min_i \tau_i$

where τ_i is calculated for nonzero eigenvalue λ_i as

$$\tau_i = \frac{1}{\omega_i} \arctan\left(\frac{-(\gamma_1 - \gamma_3\omega_i^2)\,\mathbb{R}\mathrm{e}\{\lambda_i\} + \gamma_2\omega_i\,\mathbb{I}\mathrm{m}\{\lambda_i\}}{(\gamma_1 - \gamma_3\omega_i^2)\,\mathbb{I}\mathrm{m}\{\lambda_i\} + \gamma_2\omega_i\,\mathbb{R}\mathrm{e}\{\lambda_i\}}\right).$$

Then, the third-order system with input delay results in $\mathcal{K} = l_p + l_s$ clusters where l_p and l_s are the number of PLSs and SLSs, respectively.

Proof. Conditions $(i), (ii)$ and (iii) are stated for third-order cluster consensus without delay in Chapter 2. These three conditions guarantee cluster consensus for any given third-order system for $\tau = 0$.

To find the maximum tolerable upper bound of the input delay for any given third-order system, let $\mu = j\omega_i$. One has

$$j\omega_i^3 = (\gamma_1 + j\gamma_2\omega_i - \gamma_3\omega_i^2)e^{-j\omega_i\tau_i}\lambda_i. \tag{4.6}$$

Taking modulus on both sides of (4.6), one obtains

$$\begin{aligned}\omega_i^6 &= \left[(\gamma_1 - \gamma_3\omega_i^2)^2 + (\gamma_2\omega_i)^2\right]\left[\mathbb{R}\mathrm{e}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^2\{\lambda_i\}\right] \\ &= \left[(\gamma_1 - \gamma_3\omega_i^2)\,\mathbb{R}\mathrm{e}\{\lambda_i\} - \gamma_2\omega_i\,\mathbb{I}\mathrm{m}\{\lambda_i\}\right]^2 + \left[(\gamma_1 - \gamma_3\omega_i^2)\,\mathbb{I}\mathrm{m}\{\lambda_i\} + \gamma_2\omega_i\,\mathbb{R}\mathrm{e}\{\lambda_i\}\right]^2.\end{aligned} \tag{4.7}$$

From (4.7), the equation, $P(\omega_i^2)$, can be written as

$$P(\omega_i^2) = \omega_i^6 - \gamma_3^2|\lambda_i|^2\omega_i^4 - (\gamma_2^2 - 2\gamma_1\gamma_3)|\lambda_i|^2\omega_i^2 - \gamma_1^2|\lambda_i|^2 = 0.$$

By imposing the condition (ii), one has

$$P(\omega_i^2) = \omega_i^6 - \gamma_3^2|\lambda_i|^2\omega_i^4 + \gamma_1\gamma_3|\lambda_i|^2\omega_i^2 - \gamma_1^2|\lambda_i|^2 = 0 \tag{4.8}$$

The discriminant of $P(\omega_i^2)$ can be calculated as

$$\begin{aligned} D &= Q^3 + R^2 \\ &= \frac{9\gamma_1^2|\lambda_i|^2}{54^2}\Big(9(\gamma_3|\lambda_i|)^6 - 42(\gamma_3|\lambda_i|)^3(\gamma_1|\lambda_i|) + 81(\gamma_1|\lambda_i|)^2\Big) \\ &= \frac{9\gamma_1^2|\lambda_i|^2}{54^2}\Big(\left[3(\gamma_3|\lambda_i|)^3 - 9(\gamma_1|\lambda_i|)\right]^2 + 12(\gamma_3|\lambda_i|)^3(\gamma_1|\lambda_i|)\Big) \\ &> 0 \end{aligned}$$

since $\gamma_1 > 0$, $\gamma_2 > 0$, $\gamma_3 > 0$. From Lemma 4.1, we have one real root and two complex conjugate roots. The sign of the real root can be determined by checking the sign of R. The real root of $P(\omega_i^2)$ is positive since $-\frac{b}{3a}$ is greater than zero and we have

$$\begin{aligned} R &= \frac{1}{54}\left[2(\gamma_3|\lambda_i|)^6 - 9(\gamma_3|\lambda_i|)^3(\gamma_1|\lambda_i|) + 27(\gamma_1|\lambda_i|)^2\right] \\ &= \frac{1}{54}\left(\left[\sqrt{2}(\gamma_3|\lambda_i|)^3 - \sqrt{27}(\gamma_1|\lambda_i|)\right]^2 + (2\sqrt{54} - 9)(\gamma_3|\lambda_i|)^3(\gamma_1|\lambda_i|)\right) \\ &> 0. \end{aligned}$$

This means that we have one real positive root of $P(\omega_i^2)$. Therefore, we obtain one positive ω_i that satisfies (4.8).

To find the upper bound of the input delay, separating the real and imaginary parts of (4.6) yields

$$\begin{aligned} 0 &= \cos(\omega_i\tau_i)\left[(\gamma_1 - \gamma_3\omega_i^2)\,\mathbb{R}\mathrm{e}\{\lambda_i\} - \gamma_2\omega_i\,\mathbb{I}\mathrm{m}\{\lambda_i\}\right] \\ &\quad + \sin(\omega_i\tau_i)\left[(\gamma_1 - \gamma_3\omega_i^2)\,\mathbb{I}\mathrm{m}\{\lambda_i\} + \gamma_2\omega_i\,\mathbb{R}\mathrm{e}\{\lambda_i\}\right], \\ \omega_i^3 &= \cos(\omega_i\tau_i)\left[(\gamma_1 - \gamma_3\omega_i^2)\,\mathbb{I}\mathrm{m}\{\lambda_i\} + \gamma_2\omega_i\,\mathbb{R}\mathrm{e}\{\lambda_i\}\right] \\ &\quad + \sin(\omega_i\tau_i)\left[-(\gamma_1 - \gamma_3\omega_i^2)\,\mathbb{R}\mathrm{e}\{\lambda_i\} + \gamma_2\omega_i\,\mathbb{I}\mathrm{m}\{\lambda_i\}\right]. \end{aligned} \tag{4.9}$$

By simple calculations from (4.7) and (4.9), one obtains

$$\begin{aligned}\cos(\omega_i\tau_i) &= \frac{(\gamma_1-\gamma_3\omega_i^2)\,\mathbb{I}\mathrm{m}\{\lambda_i\}+\gamma_2\omega_i\,\mathbb{R}\mathrm{e}\{\lambda_i\}}{\omega_i^3},\\ \sin(\omega_i\tau_i) &= \frac{-(\gamma_1-\gamma_3\omega_i^2)\,\mathbb{R}\mathrm{e}\{\lambda_i\}+\gamma_2\omega_i\,\mathbb{I}\mathrm{m}\{\lambda_i\}}{\omega_i^3}.\end{aligned} \tag{4.10}$$

From the above equation, τ_i can be calculated as

$$\tau_i = \frac{1}{\omega_i}\arctan\left(\frac{-(\gamma_1-\gamma_3\omega_i^2)\,\mathbb{R}\mathrm{e}\{\lambda_i\}+\gamma_2\omega_i\,\mathbb{I}\mathrm{m}\{\lambda_i\}}{(\gamma_1-\gamma_3\omega_i^2)\,\mathbb{I}\mathrm{m}\{\lambda_i\}+\gamma_2\omega_i\,\mathbb{R}\mathrm{e}\{\lambda_i\}}\right). \tag{4.11}$$

For each nonzero eigenvalue of λ_i, there will be a τ_i. The upper bound of the input delay that can be tolerated by the whole system can be found as

$$\tau^* = \min_i \tau_i.$$

This completes the proof. □

4.1.3. Analysis of Delayed Higher-Order Cluster Consensus

In this section, we provide necessary and sufficient conditions for cluster consensus in higher-order systems with input delay.

The eigenvalues of higher-order ($m \geq 3$) system (4.3) can be computed by solving the characteristic equation $\det(\mu I_{mn} - \mathcal{L}_0 - e^{-\mu\tau}\mathcal{L}_d) = 0$ where

$$\begin{aligned}\det(\mu I_{mn} - \mathcal{L}_0 - e^{-\mu\tau}\mathcal{L}_d) &= \det\left(\begin{bmatrix} \mu I_n & -I_n & \cdots & 0_{n\times n} \\ 0_{n\times n} & \mu I_n & \cdots & 0_{n\times n} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1 e^{-\mu\tau}L & \gamma_2 e^{-\mu\tau}L & \cdots & \mu I_n + \gamma_m e^{-\mu\tau}L \end{bmatrix}\right)\\ &= \det(\mu^m I_n + (\gamma_1 + \gamma_2\mu + \cdots + \gamma_m\mu^{m-1})e^{-\mu\tau}L).\end{aligned}$$

The above expression can be written with respect to eigenvalues of the unsigned Laplacian matrix, L, as

$$\det(\mu I_{mn} - \mathcal{L}_0 - e^{-\mu\tau}\mathcal{L}_d) = \prod_{i=1}^{n} \left[\mu^m + (\gamma_1 + \gamma_2\mu + \cdots + \gamma_m\mu^{m-1})e^{-\mu\tau}\lambda_i\right] = 0 \quad (4.12)$$

where λ_i is the i-th eigenvalue of L. From the above expression, it is easy to deduce that the characteristic equation (4.12) has a zero eigenvalue of algebraic multiplicity ml_p if and only if L has a zero eigenvalue of algebraic multiplicity l_p.

In Chapter 2, necessary and sufficient conditions for higher-order cluster consensus in networks without delay are stated. In deriving the general stability condition for m-th order systems, the characteristic equation (4.12) with complex coefficients and $\tau = 0$

$$\Delta(\mu) = \mu^m + \gamma_m\lambda_i\mu^{m-1} + \cdots + \gamma_2\lambda_i\mu + \gamma_1\lambda_i \quad (4.13)$$

has to have its roots in the OLHP. Associated with (4.13), define Υ as

$$\Upsilon = \begin{bmatrix} \alpha_1' & -\alpha_2'' & -\alpha_3' & \alpha_4'' & \alpha_5' & \cdots & (-1)^{m+1}\alpha_{2m-1}' \\ 1 & -\alpha_1'' & -\alpha_2' & \alpha_3'' & \alpha_4' & \cdots & (-1)^{m+1}\alpha_{2m-2}' \\ 0 & \alpha_1' & -\alpha_2'' & -\alpha_3' & \alpha_4'' & \cdots & (-1)^{m+1}\alpha_{2m-2}' \\ 0 & 1 & -\alpha_1'' & -\alpha_2' & \alpha_3'' & \cdots & (-1)^{m+1}\alpha_{2m-3}' \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \quad (4.14)$$

where $\alpha_r' = \mathbb{R}\mathrm{e}\{\gamma_{m-r+1}\lambda_i\}$, $\alpha_r'' = \mathbb{I}\mathrm{m}\{\gamma_{m-r+1}\lambda_i\}$ and $\alpha_r = 0$ for $r > m$. Let Υ_{ii}, $i = 1, 2, \ldots, 2m-1$ denote the leading principal minors of Υ. We restate Theorem 2.2 as Lemma 4.2, using the notation introduced in this chapter:

Lemma 4.2. *The roots of* (4.13) *are in the OLHP if and only if all members of the sequence of* $1, \Upsilon_{11}, \Upsilon_{33}, \ldots, \Upsilon_{2m-1,2m-1}$ *are positive.*

In the case where input delay exists, a delay condition is needed in addition to the conditions in Chapter 2. For some large τ, at least one root of the characteristic equation (4.12) will move the right half plane and (4.12) will become unstable. Thus, there exists a positive upper bound τ^* such that the characteristic equation (4.12) has a zero on the imaginary axis and (4.12) remains stable for $\tau < \tau^*$. To this end, let $\mu = j\omega_i$. One has

$$-(j\omega_i)^m = (C_1 + jC_2)e^{-j\omega_i\tau_i}\lambda_i \tag{4.15}$$

where

$$C_1 = \mathbb{R}\mathrm{e}\left\{\sum_{q=1}^{m}\gamma_q(j\omega_i)^{q-1}\right\}, \quad C_2 = \mathbb{I}\mathrm{m}\left\{\sum_{q=1}^{m}\gamma_q(j\omega_i)^{q-1}\right\}.$$

Taking modulus on both sides of (4.15), one obtains

$$\begin{aligned}\omega_i^{2m} &= \left(C_1^2 + C_2^2\right)\left(\mathbb{R}\mathrm{e}^2\{\lambda_i\} + \mathbb{I}\mathrm{m}^2\{\lambda_i\}\right)\\ &= \left(C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\}\right)^2 + \left(C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}\right)^2.\end{aligned} \tag{4.16}$$

If m is odd, we can separate the real and imaginary parts of (4.15) as follows:

$$\begin{aligned}0 &= \cos(\omega_i\tau_i)\left(C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\}\right) + \sin(\omega_i\tau_i)\left(C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}\right),\\ (-1)^{\frac{m+1}{2}}\omega_i^m &= \cos(\omega_i\tau_i)\left(C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}\right) - \sin(\omega_i\tau_i)\left(C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\}\right).\end{aligned} \tag{4.17}$$

By simple calculations from (4.16) and (4.17), one obtains

$$\begin{aligned}\cos(\omega_i\tau) &= \frac{(-1)^{\frac{m+1}{2}}\left(C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}\right)}{\omega_i^m},\\ \sin(\omega_i\tau) &= \frac{-(-1)^{\frac{m+1}{2}}\left(C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\}\right)}{\omega_i^m}.\end{aligned} \tag{4.18}$$

From the above equation, the upper bound of the input delay can be calculated as

$$\tau_i = \frac{1}{\omega_i} \arctan\left(\frac{-(C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\})}{C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}}\right). \tag{4.19}$$

If m is even, we can separate the real and imaginary parts of (4.15) as follows:

$$\begin{aligned}(-1)^{\frac{m+2}{2}}\omega_i^m &= \cos(\omega_i\tau_i)(C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\}) + \sin(\omega_i\tau_i)(C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}),\\ 0 &= \cos(\omega_i\tau_i)(C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}) - \sin(\omega_i\tau_i)(C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\}).\end{aligned} \tag{4.20}$$

By simple calculations from (4.16) and (4.20), one obtains

$$\begin{aligned}\cos(\omega_i\tau_i) &= \frac{(-1)^{\frac{m+2}{2}}(C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\})}{\omega_i^m},\\ \sin(\omega_i\tau_i) &= \frac{(-1)^{\frac{m+2}{2}}(C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\})}{\omega_i^m}.\end{aligned} \tag{4.21}$$

From the above equation, the upper bound of the input delay can be calculated as

$$\tau_i = \frac{1}{\omega_i} \arctan\left(\frac{C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}}{C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\}}\right). \tag{4.22}$$

Combining of (4.19) and (4.22), the upper bound of τ can be calculated as

$$\tau^* = \min_i \tau_i \tag{4.23}$$

where

$$\tau_i = \begin{cases} \dfrac{1}{\omega_i} \arctan\left(\dfrac{-(C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\})}{C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}}\right), & \text{if } m \text{ is odd} \\ \dfrac{1}{\omega_i} \arctan\left(\dfrac{C_1\,\mathbb{I}\mathrm{m}\{\lambda_i\} + C_2\,\mathbb{R}\mathrm{e}\{\lambda_i\}}{C_1\,\mathbb{R}\mathrm{e}\{\lambda_i\} - C_2\,\mathbb{I}\mathrm{m}\{\lambda_i\}}\right), & \text{if } m \text{ is even.} \end{cases}$$

The outline of the procedure for the calculation of the upper bound τ^* is provided as follows:

Procedure 2

Step 1: By solving (4.16), obtain the positive solution ω_i for each nonzero λ_i.

Step 2: By using ω_i, calculate C_1 and C_2 as in (4.15) and then find τ_i as in (4.23).

Step 3: Determine the upper bound of the input delay as $\tau^* = \min_i \tau_i$.

Figure 4.1. Algorithm to calculate the upper bound of input delay τ^*.

The above derivation, together with Lemma 4.2 and the proposed procedure lead us to the following result.

Theorem 4.2. *Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be chosen such that all members of the sequence of $1, \Upsilon_{11}, \Upsilon_{33}, \ldots, \Upsilon_{2m-1,2m-1}$ in (4.14) are positive. The number of clusters for the input delayed higher-order ($m \geq 3$) system (4.3) is given by $\mathcal{K} = l_p + l_s$ if and only if $\tau < \tau^* = \min_i \tau_i$ where τ_i is calculated delay from (4.23) for nonzero eigenvalue λ_i; l_p and l_s are the number of PLSs and SLSs, respectively.*

4.1.4. Communication Delay Effect

Consider a network of agents with m-th order linear delayed dynamics described by

$$\begin{aligned} \dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\ &\vdots \\ \dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\ \dot{x}_i^{(m)}(t) &= u_i(t), \quad i = 1, 2, \ldots, n \end{aligned} \tag{4.24}$$

where $x_i^{(q)}(t)$ $(q = 1, 2, \ldots, m)$ represents the states of agent i at time t, $x_i^{(q)}$ is the $(q-1)$th derivative of $x_i^{(1)}$, τ is the communication delay and $u_i(t)$ is the delayed input

defined as

$$u_i(t) = \sum_{q=1}^{m} \gamma_q \sum_{j \in \mathcal{N}_i} a_{ij} \left(x_j^{(q)}(t-\tau) - x_i^{(q)}(t) \right), \quad i = 1, 2, \ldots, n. \tag{4.25}$$

In this section we assume that the elements of the adjacency matrix satisfies the following conditions:

Assumption 4.1. *The adjacency elements, a_{ij}, hold the followings:*

- $a_{ii} > 0$ *for* $i = 1, 2, \ldots, n$.
- $a_{ij} \neq 0$ *for all* $(j, i) \in \mathcal{E}$.
- $a_{ij} = 0$ *for all* $(j, i) \notin \mathcal{E}$.
- $\sum_{j=1}^{n} |a_{ij}| = 1$ *for* $i = 1, 2, \ldots, n$.

The dynamics of the unsigned higher-order integrator system with communication delay can be written as

$$\dot{\mathbf{x}}(t) = \mathcal{D}_0 \mathbf{x}(t) + \mathcal{A}_d \mathbf{x}(t-\tau), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{4.26}$$

where

$$\mathcal{D}_0 = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ -\gamma_1 I_n & -\gamma_2 I_n & -\gamma_3 I_n & \ldots & -\gamma_m I_n \end{bmatrix}, \mathcal{A}_d = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ \gamma_1 \mathcal{A} & \gamma_2 \mathcal{A} & \gamma_3 \mathcal{A} & \ldots & \gamma_m \mathcal{A} \end{bmatrix}.$$

Theorem 4.3. *Let $\gamma_1, \gamma_2, \ldots, \gamma_m$ be chosen such that all members of the sequence of $1, \Upsilon_{11}, \Upsilon_{33}, \ldots, \Upsilon_{2m-1,2m-1}$ in (4.14) are positive and its directed graph contains l_p PLSs and l_s SLSs. Under Assumption 4.1, there exists an upper bound τ^* such that the system (4.24)-(4.25) represented by an unsigned directed graph forms $\mathcal{K} = l_p + l_s$ clusters for $\tau < \tau^*$.*

Proof. The system (4.26) can be written in terms of PLS and SLS dynamics by using detection algorithms in [13]. For this case, let $n_{p,i}$ and $n_{s,j}$ be the number of agents in the $i-$th PLS and $j-$th SLS, respectively for $i = 1, 2, \ldots, l_p$ and $j = 1, 2, \ldots, l_s$. The total number of agents in the PLSs and SLSs can be expressed as $n_p = \sum_i n_{p,i}$ and $n_s = \sum_j n_{s,j}$, respectively. The dynamics of the agents in PLSs and SLSs can be written as follows:

$$\begin{bmatrix} \dot{\mathbf{x}}_p^{(1)}(t) \\ \vdots \\ \dot{\mathbf{x}}_p^{(m)}(t) \end{bmatrix} = \left[\begin{array}{c|c} 0_{(m-1)n_p \times n_p} & I_{(m-1)n_p} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes I_{n_p} & \end{array}\right] \begin{bmatrix} \mathbf{x}_p^{(1)}(t) \\ \vdots \\ \mathbf{x}_p^{(m)}(t) \end{bmatrix} + \left[\begin{array}{c} 0_{(m-1)n_p \times mn_p} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \otimes \mathcal{A}_p \end{array}\right] \begin{bmatrix} \mathbf{x}_p^{(1)}(t-\tau) \\ \vdots \\ \mathbf{x}_p^{(m)}(t-\tau) \end{bmatrix}, \tag{4.27}$$

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{x}}_s^{(1)}(t) \\ \vdots \\ \dot{\mathbf{x}}_s^{(m)}(t) \end{bmatrix} = & \left[\begin{array}{c|c} 0_{(m-1)n_s \times n_s} & I_{(m-1)n_s} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes I_{n_s} & \end{array}\right] \begin{bmatrix} \mathbf{x}_s^{(1)}(t) \\ \vdots \\ \mathbf{x}_s^{(m)}(t) \end{bmatrix} + \left[\begin{array}{c} 0_{(m-1)n_s \times mn_s} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \otimes \mathcal{A}_s \end{array}\right] \begin{bmatrix} \mathbf{x}_s^{(1)}(t-\tau) \\ \vdots \\ \mathbf{x}_s^{(m)}(t-\tau) \end{bmatrix} \\ & + \left[\begin{array}{c} 0_{(m-1)n_s \times mn_p} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \otimes \mathcal{A}_{sp} \end{array}\right] \begin{bmatrix} \mathbf{x}_p^{(1)}(t-\tau) \\ \vdots \\ \mathbf{x}_p^{(m)}(t-\tau) \end{bmatrix}. \end{aligned} \tag{4.28}$$

where $\mathbf{x}_p^{(q)}(t) = \left[\mathbf{x}_{p,1}^{(q)}(t)^{\mathsf{T}}, \mathbf{x}_{p,2}^{(q)}(t)^{\mathsf{T}}, \ldots, \mathbf{x}_{p,l_p}^{(q)}(t)^{\mathsf{T}}\right]^{\mathsf{T}}$, $\mathbf{x}_s^{(q)}(t) = \left[\mathbf{x}_{s,1}^{(q)}(t)^{\mathsf{T}}, \mathbf{x}_{s,2}^{(q)}(t)^{\mathsf{T}}, \ldots, \mathbf{x}_{s,l_s}^{(q)}(t)^{\mathsf{T}}\right]^{\mathsf{T}}$ $(q = 1, 2, \ldots, m)$ are the states vectors of the agents in PLSs and SLSs, respectively; $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, \ldots, \gamma_m]^{\mathsf{T}}$ is the controller parameter vector, $\mathcal{A}_p = I_{n_p} - L_p$, $\mathcal{A}_{sp} = -L_{sp}$ and $\mathcal{A}_s = I_{n_s} - L_s$. From [98], L_p, L_{sp} and L_s are

$$L_p = \text{blkdiag}\{L_{1,1}, \ldots, L_{l_p,l_p}\}, \quad L_{sp} = \begin{bmatrix} L_{l_p+1,1} & \cdots & L_{l_p+1,l_p} \\ \vdots & \ddots & \vdots \\ L_{l_p+l_s,1} & \cdots & L_{l_p+l_s,l_p} \end{bmatrix},$$

$$L_s = \begin{bmatrix} L_{l_p+1,l_p+1} & \cdots & L_{l_p+1,l_p+l_s} \\ \vdots & \ddots & \vdots \\ L_{l_p+l_s,l_p+1} & \cdots & L_{l_p+l_s,l_p+l_s} \end{bmatrix}.$$

Step 1 (Stability and the Number of Clusters for PLS Dynamics): For each PLS, the following singular value decomposition can be obtained:

$$S_i^{-1} L_{i,i} S_i = \begin{bmatrix} \tilde{L}_{i,i} & \mathbf{0}_{n_{p,i}-1} \\ \mathbf{0}_{n_{p,i}-1}^{\mathsf{T}} & 0 \end{bmatrix}, \quad i = 1, 2, \ldots, l_p. \tag{4.29}$$

The last column of S_i is chosen as $\mathbf{1}_{n_{p,i}}$ which is the right eigenvector of $L_{i,i}$ associated with the zero eigenvalue. All eigenvalues of $\tilde{L}_{i,i}$ have positive real parts since each PLS contains a spanning tree [105].

Note that $\mathcal{A}_{i,i} = I_{n_{p,i}} - L_{i,i}$ for $i = 1, 2, \ldots, l_p$. From (4.29), it can be obtained as $S^{-1}\mathcal{A}_{i,i}S = \begin{bmatrix} I_{n_{p,i}-1} - \tilde{L}_{i,i} & \mathbf{0}_{n_{p,i}-1} \\ \mathbf{0}_{n_{p,i}-1}^{\mathsf{T}} & 1 \end{bmatrix}$. Define $\tilde{\mathcal{A}}_{i,i} = I_{n_{p,i}-1} - \tilde{L}_{i,i}$ and

$$\begin{aligned} \tilde{\mathbf{y}}_{p,i}^{(q)}(t) &= S_i^{-1} \mathbf{x}_{p,i}^{(q)}(t) \\ &= \begin{bmatrix} y_{p,i,1}^{(q)}(t) & \cdots & y_{p,i,n_{p,i}}^{(q)}(t) \end{bmatrix}^{\mathsf{T}} \\ &= \begin{bmatrix} \mathbf{y}_{p,i}^{(q)}(t)^{\mathsf{T}} & y_{p,i,n_{p,i}}^{(q)}(t) \end{bmatrix}^{\mathsf{T}} \end{aligned}$$

for $q = 1, 2, \ldots, m$. One can obtain the following equations:

$$\begin{bmatrix} \dot{\mathbf{y}}_{p,i}^{(1)}(t) \\ \vdots \\ \dot{\mathbf{y}}_{p,i}^{(m-1)}(t) \\ \dot{\mathbf{y}}_{p,i}^{(m)}(t) \end{bmatrix} = \left[\begin{array}{c|c} 0_{(m-1)(n_{p,i}-1)\times n_{p,i}-1} & I_{(m-1)(n_{p,i}-1)} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes I_{n_{p,i}-1} & \end{array}\right] \begin{bmatrix} \mathbf{y}_{p,i}^{(1)}(t) \\ \vdots \\ \mathbf{y}_{p,i}^{(m-1)}(t) \\ \mathbf{y}_{p,i}^{(m)}(t) \end{bmatrix},$$

$$+\begin{bmatrix} 0_{(m-1)(n_{p,i}-1)\times m(n_{p,i}-1)} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \otimes \tilde{\mathcal{A}}_{i,i} \end{bmatrix} \begin{bmatrix} \mathbf{y}_{p,i}^{(1)}(t-\tau) \\ \vdots \\ \mathbf{y}_{p,i}^{(m-1)}(t-\tau) \\ \mathbf{y}_{p,i}^{(m)}(t-\tau) \end{bmatrix} \tag{4.30a}$$

$$\begin{bmatrix} \dot{y}_{p,i,n_{p,i}}^{(1)}(t) \\ \vdots \\ \dot{y}_{p,i,n_{p,i}}^{(m-1)}(t) \\ \dot{y}_{p,i,n_{p,i}}^{(m)}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{(m-1)} \mid I_{m-1} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} y_{p,i,n_{p,i}}^{(1)}(t) \\ \vdots \\ y_{p,i,n_{p,i}}^{(m-1)}(t) \\ y_{p,i,n_{p,i}}^{(m)}(t) \end{bmatrix} + \begin{bmatrix} 0_{(m-1)\times m} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \end{bmatrix} \begin{bmatrix} y_{p,i,n_{p,i}}^{(1)}(t-\tau) \\ \vdots \\ y_{p,i,n_{p,i}}^{(m-1)}(t-\tau) \\ y_{p,i,n_{p,i}}^{(m)}(t-\tau) \end{bmatrix}. \tag{4.30b}$$

From Theorem 3 and Corollary 4 in [106], the stability of system (4.30a) depends on the eigenvalues of $\gamma_m \tilde{\mathcal{A}}_{i,i}$. The upper bound of τ for system (4.30a) can be calculated as

$$\tau_1^* = \min_i \tau_i \tag{4.31}$$

where

$$\tau_i = \begin{cases} \dfrac{1}{\tilde{\lambda}_i}, & \text{if } \tilde{\lambda}_i \text{ is real and positive} \\ \dfrac{k\pi + \arctan\left(\frac{\mathbb{Im}\{\tilde{\lambda}_i\}}{\mathbb{Re}\{\tilde{\lambda}_i\}}\right)}{\mathbb{Im}\{\tilde{\lambda}_i\}} > 0, \quad k = 0, \pm 1, \ldots, & \text{if } \tilde{\lambda}_i \text{ is complex and } \mathbb{Im}\{\tilde{\lambda}_i\} \neq 0 \end{cases}$$

and $\tilde{\lambda}_i$ is a nonzero eigenvalue of $\gamma_m \tilde{\mathcal{A}}_{i,i}$. The dynamics of (4.30a) are stable for $\tau < \tau_1^*$.

In order to analyze the stability of system (4.30b), consider its characteristic equation:

$$s^m + (\gamma_m s^{m-1} + \gamma_{m-1} s^{m-2} + \cdots + \gamma_1)(1 - e^{-\tau s}) = 0. \tag{4.32}$$

As the communication delay gets larger, at least one solution to the characteristic equation (4.32) will have positive real part, resulting in an unstable system. Therefore, an upper bound $\tau^* > 0$ exists such that (4.32) has a root on the imaginary axis and (4.32) is stable for $\tau < \tau^*$. In order to calculate τ^*, let $s = j\omega_i$ and we have

$$-(j\omega_i)^m = (C_1 + jC_2)(1 - e^{-j\omega_i\tau_i}) \tag{4.33}$$

where

$$C_1 = \mathbb{R}\mathrm{e}\left\{\sum_{q=1}^{m} \gamma_q (j\omega_i)^{q-1}\right\}, \quad C_2 = \mathbb{I}\mathrm{m}\left\{\sum_{q=1}^{m} \gamma_q (j\omega_i)^{q-1}\right\}.$$

If m is odd, from (4.33) one can obtain

$$\begin{aligned} 0 &= C_1\left(1 - \cos(\omega_i\tau_i)\right) - C_2 \sin(\omega_i\tau_i), \\ (-1)^{\frac{m+1}{2}}\omega_i^m &= C_2\left(1 - \cos(\omega_i\tau_i)\right) + C_1 \sin(\omega_i\tau_i) \end{aligned} \tag{4.34}$$

which yields

$$\begin{aligned} \cos(\omega_i\tau_i) &= \frac{C_1^2 + C_2^2 - (-1)^{\frac{m+1}{2}}\omega_i^m C_2}{C_1^2 + C_2^2}, \\ \sin(\omega_i\tau_i) &= \frac{(-1)^{\frac{m+1}{2}}\omega_i^m C_1}{C_1^2 + C_2^2}. \end{aligned} \tag{4.35}$$

If m is even, from (4.33), one can obtain

$$\begin{aligned} (-1)^{\frac{m+2}{2}}\omega_i^m &= C_1\left(1 - \cos(\omega_i\tau_i)\right) - C_2 \sin(\omega_i\tau_i), \\ 0 &= C_2\left(1 - \cos(\omega_i\tau_i)\right) + C_1 \sin(\omega_i\tau_i) \end{aligned} \tag{4.36}$$

which yields

$$\begin{aligned} \cos(\omega_i\tau_i) &= \frac{C_1^2 + C_2^2 - (-1)^{\frac{m+2}{2}}\omega_i^m C_1}{C_1^2 + C_2^2}, \\ \sin(\omega_i\tau_i) &= \frac{-(-1)^{\frac{m+2}{2}}\omega_i^m C_2}{C_1^2 + C_2^2}. \end{aligned} \tag{4.37}$$

By solving $\cos^2(\omega_i\tau_i)+\sin^2(\omega_i\tau_i)=1$, a positive solution ω_i can be obtained from one of the following polynomials:

$$\begin{aligned}\omega_i^m - 2(-1)^{\frac{m+1}{2}}C_2 = 0 \text{ if } m \text{ is odd},\\ \omega_i^m - 2(-1)^{\frac{m+2}{2}}C_1 = 0 \text{ if } m \text{ is even}.\end{aligned} \tag{4.38}$$

Consequently, an upper bound for τ is computed as

$$\tau_2^* = \min_i \tau_i \tag{4.39}$$

where

$$\tau_i = \begin{cases} \dfrac{1}{\omega_i}\left[\arctan\left(\dfrac{(-1)^{\frac{m+1}{2}}\omega_i^m C_1}{C_1^2+C_2^2-(-1)^{\frac{m+1}{2}}\omega_i^m C_2}\right)\pm 2k\pi\right] > 0, & \text{if } m \text{ is odd} \\ \dfrac{1}{\omega_i}\left[\arctan\left(\dfrac{-(-1)^{\frac{m+2}{2}}\omega_i^m C_2}{C_1^2+C_2^2-(-1)^{\frac{m+2}{2}}\omega_i^m C_1}\right)\pm 2k\pi\right] > 0, & \text{if } m \text{ is even} \end{cases} \tag{4.40}$$

for $k=1,2,\ldots$. For $\tau<\tau_2^*$, the dynamics of (4.30b) with communication delay is stable. Together with (4.31) and (4.39), PLSs converge to l_p clusters for $\tau<\min\{\tau_1^*,\tau_2^*\}$.

Step 2 (Stability and the Number of Clusters for SLS Dynamics): In this part, we prove that the following equation always holds:

$$\lim_{t\to\infty}\|\mathbf{x}_s^{(q)}(t)+L_s^{-1}L_{sp}\mathbf{x}_p^{(q)}(t)\|=0, \quad q=1,2,\ldots,m. \tag{4.41}$$

Let $\mathbf{e}_q(t)=\mathbf{x}_s^{(q)}(t)+L_s^{-1}L_{sp}\mathbf{x}_p^{(q)}(t)$ for $q=1,2,\ldots,m$. Taking derivatives yields the following matrix equation:

$$\begin{bmatrix} \dot{\mathbf{e}}_1(t) \\ \vdots \\ \dot{\mathbf{e}}_{m-1}(t) \\ \dot{\mathbf{e}}_m(t) \end{bmatrix} = \left[\begin{array}{c|c} 0_{(m-1)n_s \times n_s} & I_{(m-1)n_s} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes I_{n_s} & \end{array}\right] \begin{bmatrix} \mathbf{e}_1(t) \\ \vdots \\ \mathbf{e}_{m-1}(t) \\ \mathbf{e}_m(t) \end{bmatrix} + \left[\begin{array}{c} 0_{(m-1)n_s \times mn_s} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \otimes \mathcal{A}_s \end{array}\right] \begin{bmatrix} \mathbf{e}_1(t-\tau) \\ \vdots \\ \mathbf{e}_{m-1}(t-\tau) \\ \mathbf{e}_m(t-\tau) \end{bmatrix} + \left[\begin{array}{c} 0_{(m-1)n_s \times mn_s} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes (L_s^{-1} L_{sp} L_p) \end{array}\right] \begin{bmatrix} \mathbf{x}_p^{(1)}(t-\tau) \\ \vdots \\ \mathbf{x}_p^{(m-1)}(t-\tau) \\ \mathbf{x}_p^{(m)}(t-\tau) \end{bmatrix}. \tag{4.42}$$

Note that the term of $-\gamma_1 L_s^{-1} L_{sp} L_p \mathbf{x}_p^{(1)}(t-\tau) - \gamma_2 L_s^{-1} L_{sp} L_p \mathbf{x}_p^{(2)}(t-\tau) - \cdots - \gamma_m L_s^{-1} L_{sp} L_p \mathbf{x}_p^{(m)}(t-\tau)$ converges to zero as $t \to \infty$ since each PLS achieves consensus. Therefore the stability of (4.42) depends on the stability of the following dynamics:

$$\begin{bmatrix} \dot{\mathbf{e}}_1(t) \\ \vdots \\ \dot{\mathbf{e}}_{m-1}(t) \\ \dot{\mathbf{e}}_m(t) \end{bmatrix} = \left[\begin{array}{c|c} 0_{(m-1)n_s \times n_s} & I_{(m-1)n_s} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes I_{n_s} & \end{array}\right] \begin{bmatrix} \mathbf{e}_1(t) \\ \vdots \\ \mathbf{e}_{m-1}(t) \\ \mathbf{e}_m(t) \end{bmatrix} + \left[\begin{array}{c} 0_{(m-1)n_s \times mn_s} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \otimes \mathcal{A}_s \end{array}\right] \begin{bmatrix} \mathbf{e}_1(t-\tau) \\ \vdots \\ \mathbf{e}_{m-1}(t-\tau) \\ \mathbf{e}_m(t-\tau) \end{bmatrix}. \tag{4.43}$$

From Theorem 3 and Corollary 4 in [106], the stability of error dynamics (4.43) for SLS depends on the eigenvalues of $\gamma_m \mathcal{A}_s$. The upper bound of τ for system (4.43) can be calculated as

$$\tau_3^* = \min_i \tau_i \tag{4.44}$$

where

$$\tau_i = \begin{cases} \dfrac{1}{\lambda_{si}}, & \text{if } \lambda_{si} \text{ is real and positive} \\ \dfrac{k\pi + \arctan\left(\frac{\mathbb{I}\mathrm{m}\{\lambda_{si}\}}{\mathbb{R}\mathrm{e}\{\lambda_{si}\}}\right)}{\mathbb{I}\mathrm{m}\{\lambda_{si}\}} > 0, \quad k = 0, \pm 1, \ldots, & \text{if } \lambda_{si} \text{ is complex and } \mathbb{I}\mathrm{m}\{\lambda_{si}\} \neq 0 \end{cases}$$

where λ_{si} is a nonzero eigenvalue of $\gamma_m \mathcal{A}_s$. The dynamics of (4.43) are stable for $\tau < \tau_3^*$ i.e., $\lim_{t\to\infty} \|\mathbf{e}_q(t)\| = 0$ for $q = 1, 2, \ldots, m$. This implies that the following expression holds:

$$\mathbf{x}_s^{(q)}(\infty) = -L_s^{-1} L_{sp} \mathbf{x}_p^{(q)}(\infty)$$

for $q = 1, 2, \ldots, m$ under $\tau < \tau_3^*$. From [98], it is known that the above dynamics converge to l_s clusters.

Step 3 (Total Number of Clusters): One can conclude that the system (4.26) evolving over an unsigned directed graph converges to

$$\mathcal{K} = l_p + l_s$$

clusters for $\tau < \tau^* = \min\{\tau_1^*, \tau_2^*, \tau_3^*\}$ where τ_1^*, τ_2^* and τ_3^* are calculated from (4.31), (4.39) and (4.44), respectively. □

4.2. Delayed Networks with Signed Directed Graphs

4.2.1. Input Delay Effect

Consider the $m-$th order linear delayed dynamics of agent i described by

$$\begin{aligned} \dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\ &\vdots \\ \dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\ \dot{x}_i^{(m)}(t) &= u_i(t-\tau), \quad i = 1, 2, \ldots, n \end{aligned} \tag{4.45}$$

with initial conditions $x_i^{(q)}(0) = x_{i0}^{(q)}$ where $x_i^{(q)}(t)$ $(q = 1, 2, \ldots, m)$ represents the states of agent i, $x_i^{(q)}$ is the $(q-1)$th derivative of $x_i^{(1)}$, τ is the input delay and $u_i(t-\tau)$ is the delayed input defined as

$$u_i(t) = \sum_{q}^{m} \gamma_q \sum_{j \in \mathcal{N}_i} a_{ij} \left(x_j^{(q)}(t) - \text{sgn}(a_{ij}) x_i^{(q)}(t) \right), \quad i = 1, 2, \ldots, n \tag{4.46}$$

with control parameters $\gamma_q > 0$ and the signum function $\text{sgn}(\cdot)$. For analysis purposes, we define the augmented state vector $\mathbf{x}(t) = [\mathbf{x}^{(1)}(t)^\mathsf{T}, \mathbf{x}^{(2)}(t)^\mathsf{T}, \ldots, \mathbf{x}^{(m)}(t)^\mathsf{T}]^\mathsf{T}$ where $\mathbf{x}^{(q)}(t) = [x_1^{(q)}(t), x_2^{(q)}(t), \ldots, x_n^{(q)}(t)]^\mathsf{T}$ for $q = 1, 2, \ldots, m$ and $\mathbf{x}_0$ is the initial condition. Then, the dynamics of the signed system (4.45)-(4.46) can be written as

$$\dot{\mathbf{x}}(t) = \mathcal{L}_0 \mathbf{x}(t) + \mathcal{L}_d \mathbf{x}(t-\tau), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{4.47}$$

where

$$\mathcal{L}_0 = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \end{bmatrix}, \quad \mathcal{L}_d = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ -\gamma_1 L & -\gamma_2 L & -\gamma_3 L & \ldots & -\gamma_m L \end{bmatrix}$$

with L is the signed Laplacian matrix whose elements are defined by

$$l_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{n} |a_{ik}|, & \text{if } i = j \\ -a_{ij}, & \text{otherwise.} \end{cases} \tag{4.48}$$

Given the delayed system (4.45)-(4.46) modeled via $\mathcal{G}$, the corresponding extended delayed dynamics of $\bar{\mathcal{G}}$ are defined as

$$\begin{aligned} \dot{\bar{x}}_i^{(1)}(t) &= \bar{x}_i^{(2)}(t), \\ &\vdots \\ \dot{\bar{x}}_i^{(m-1)}(t) &= \bar{x}_i^{(m)}(t), \\ \dot{\bar{x}}_i^{(m)}(t) &= \bar{u}_i(t-\tau), \quad i = 1, 2, \ldots, 2n \end{aligned} \tag{4.49}$$

where $\bar{x}_i^{(q)}(t)$ $(q = 1, 2, \ldots, m)$ represents the states of agents, $\bar{x}_i^{(q)}$ is $(q-1)$th derivative of $\bar{x}_i^{(1)}(t)$ and $\bar{u}_i(t-\tau)$ is control input defined as

$$\bar{u}_i(t) = \sum_{q=1}^{m} \gamma_q \sum_{j\in\bar{\mathcal{N}}_i} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(t) - \bar{x}_i^{(q)}(t) \right), \quad i = 1, 2, \ldots, 2n. \tag{4.50}$$

Note from Definition 3.1 that $\bar{a}_{ij}$ are non-negative which implies that the extended delayed dynamics (4.49)-(4.50) correspond to an *unsigned* network. Our approach is to derive stability and clustering properties of (4.45)-(4.46) through those of (4.49)-(4.50).

For analysis purposes, we define the augmented state vector $\bar{\mathbf{x}}(t) = \left[\bar{\mathbf{x}}^{(1)}(t)^\mathsf{T}, \bar{\mathbf{x}}^{(2)}(t)^\mathsf{T}, \ldots, \bar{\mathbf{x}}^{(m)}(t)^\mathsf{T}\right]^\mathsf{T}$ where $\bar{\mathbf{x}}^{(q)}(t) = \left[\bar{x}_1^{(q)}(t), \bar{x}_2^{(q)}(t), \ldots, \bar{x}_{2n}^{(q)}(t)\right]^\mathsf{T}$ for $q = 1, 2, \ldots, m$ and $\bar{\mathbf{x}}_0$ is the initial condition. Then, the dynamics of the extended input delayed system (4.49) can be expressed in matrix form as

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathcal{L}}_0 \bar{\mathbf{x}}(t) + \bar{\mathcal{L}}_d \bar{\mathbf{x}}(t-\tau), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \tag{4.51}$$

where

$$\bar{\mathcal{L}}_0 = \begin{bmatrix} 0_{2n\times 2n} & I_{2n} & 0_{2n\times 2n} & \dots & 0_{2n\times 2n} \\ 0_{2n\times 2n} & 0_{2n\times 2n} & I_{2n} & \dots & 0_{2n\times 2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \dots & I_{2n} \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \dots & 0_{2n\times 2n} \end{bmatrix}, \bar{\mathcal{L}}_d = \begin{bmatrix} 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \dots & 0_{2n\times 2n} \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \dots & 0_{2n\times 2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \dots & 0_{2n\times 2n} \\ -\gamma_1\bar{L} & -\gamma_2\bar{L} & -\gamma_3\bar{L} & \dots & -\gamma_m\bar{L} \end{bmatrix}$$

with $\bar{L}$ is the unsigned Laplacian matrix whose elements are defined by

$$\bar{l}_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{2n} \bar{a}_{ik}, & \text{if } i = j \\ -\bar{a}_{ij}, & \text{otherwise.} \end{cases} \tag{4.52}$$

The following lemma gives the relationship between trajectories of the signed system (4.45)-(4.46) and its extended counterpart (4.49)-(4.50).

Lemma 4.3. *The states* $\mathbf{x}(t)$ *of the delayed system* (4.45)-(4.46) *can be obtained from its extended delayed system* (4.49)-(4.50) *as* $x_i^{(q)}(t) = \bar{x}_i^{(q)}(t)$*, for all* $t \geq 0$ *by placing* $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ *and* $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ *for* $i = 1, 2, \dots, n$ *and* $q = 1, 2, \dots, m$.

Proof. Equation (4.50) can be re-expressed as follows:

$$\bar{u}_i(t) = \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(t) - \bar{x}_i^{(q)}(t) \right) + \sum_{j=1}^{n} \bar{a}_{i,j+n} \left(\bar{x}_{j+n}^{(q)}(t) - \bar{x}_i^{(q)}(t) \right) \right]$$

for $i = 1, 2, \dots, 2n$. By substituting $t - \tau$ for t and decomposing the sum into two parts, the extended system's delayed input is obtained as

$$\begin{aligned}
\bar{u}_i(t-\tau) &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(t-\tau) - \bar{x}_i^{(q)}(t-\tau) \right) \right. \\
&\quad \left. + \sum_{j=1}^{n} \bar{a}_{i,j+n} \left(\bar{x}_{j+n}^{(q)}(t-\tau) - \bar{x}_i^{(q)}(t-\tau) \right) \right], \\
\bar{u}_{i+n}(t-\tau) &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{i+n,j+n} \left(\bar{x}_{j+n}^{(q)}(t-\tau) - \bar{x}_{i+n}^{(q)}(t-\tau) \right) \right. \\
&\quad \left. + \sum_{j=1}^{n} \bar{a}_{i+n,j} \left(\bar{x}_j^{(q)}(t-\tau) - \bar{x}_{i+n}^{(q)}(t-\tau) \right) \right]
\end{aligned} \tag{4.53}$$

for $i = 1, 2, \ldots, n$. Let $\hat{x}_i^{(q)}(t) = \bar{x}_i^{(q)}(t) + \bar{x}_{i+n}^{(q)}(t)$ $(i = 1, 2, \ldots, n,\ q = 1, 2, \ldots, m)$. Then, the dynamics of $\hat{x}_i^{(q)}(t)$ can be expressed as

$$\begin{aligned}
\dot{\hat{x}}_i^{(q)}(t) &= \dot{\bar{x}}_i^{(q)}(t) + \dot{\bar{x}}_{i+n}^{(q)}(t) \\
&= \bar{x}_i^{(q+1)}(t) + \bar{x}_{i+n}^{(q+1)}(t) \\
&= \hat{x}_i^{(q+1)}(t)
\end{aligned}$$

for $q = 1, 2, \ldots, m-1$ and

$$\begin{aligned}
\dot{\hat{x}}_i^{(m)}(t) &= \dot{\bar{x}}_i^{(m)}(t) + \dot{\bar{x}}_{i+n}^{(m)}(t) \\
&= \bar{u}_i(t-\tau) + \bar{u}_{i+n}(t-\tau) \\
&= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} (\bar{a}_{ij} + \bar{a}_{i+n,j}) \bar{x}_j^{(q)}(t-\tau) + (\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n}) \bar{x}_{j+n}^{(k)}(t-\tau) \right] \\
&\quad - \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} (\bar{a}_{ij} + \bar{a}_{i,j+n}) \bar{x}_i^{(q)}(t-\tau) + (\bar{a}_{i+n,j} + \bar{a}_{i+n,j+n}) \bar{x}_{i+n}^{(q)}(t-\tau) \right].
\end{aligned}$$

From Definition 3.1, we get

$$\bar{a}_{i+n,j} = \bar{a}_{i,j+n}, \quad \bar{a}_{ij} = \bar{a}_{i+n,j+n} \tag{4.54}$$

for $i = 1, 2, \ldots, n$. By using (4.54) together with Definition 3.1, we also have $\bar{a}_{ij} +$

$\bar{a}_{i+n,j} = \bar{a}_{i,j+n} + \bar{a}_{i+n,j+n} = \bar{a}_{ij} + \bar{a}_{i,j+n} = \bar{a}_{i+n,j} + \bar{a}_{i+n,j+n} = |a_{ij}|$. Then, we have

$$\begin{aligned}\dot{\hat{x}}_i^{(m)}(t) &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} |a_{ij}| \left(\bar{x}_j^{(q)}(t-\tau) + \bar{x}_{j+n}^{(q)}(t-\tau) \right) \right] \\ &\quad - \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} |a_{ij}| \left(\bar{x}_i^{(q)}(t-\tau) + \bar{x}_{i+n}^{(q)}(t-\tau) \right) \right] \\ &= \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} |a_{ij}| \left(\hat{x}_j^{(q)}(t-\tau) - \hat{x}_i^{(q)}(t-\tau) \right) \end{aligned} \tag{4.55}$$

for $i = 1, 2, \ldots, n$. In matrix form, (4.55) can be re-expressed as

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathcal{L}}_0 \hat{\mathbf{x}}(t) + \hat{\mathcal{L}}_d \hat{\mathbf{x}}(t-\tau), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \tag{4.56}$$

where $\hat{\mathbf{x}}(t) = \left[\hat{\mathbf{x}}^{(1)}(t)^\mathsf{T}, \hat{\mathbf{x}}^{(2)}(t)^\mathsf{T}, \ldots, \hat{\mathbf{x}}^{(m)}(t)^\mathsf{T} \right]^\mathsf{T}$ is the state vector with components $\hat{\mathbf{x}}^{(q)}(t) = \left[\hat{x}_1^{(q)}(t), \hat{x}_2^{(q)}(t) \ldots, \hat{x}_n^{(q)}(t) \right]^\mathsf{T}$ for $q = 1, 2, \ldots, m$, $\hat{\mathbf{x}}_0$ is the initial condition and

$$\hat{\mathcal{L}}_0 = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \end{bmatrix}, \quad \hat{\mathcal{L}}_d = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ -\gamma_1 \hat{L} & -\gamma_2 \hat{L} & -\gamma_3 \hat{L} & \ldots & -\gamma_m \hat{L} \end{bmatrix}.$$

Here $\hat{L} = [\hat{l}_{ij}]$ is an unsigned Laplacian matrix with the components

$$\hat{l}_{ij} = \begin{cases} \sum\limits_{k=1, k\neq i}^{n} |a_{ik}|, & \text{if } i = j \\ -|a_{ik}|, & \text{otherwise.} \end{cases} \tag{4.57}$$

Since the solution of (4.56) is $\hat{\mathbf{x}}(t) = \mathcal{L}^{-1} \left\{ \left(sI_{mn} - \hat{\mathcal{L}}_0 - e^{-\tau s} \hat{\mathcal{L}}_d \right)^{-1} \right\} \hat{\mathbf{x}}_0$ with the initial condition $\hat{\mathbf{x}}_0 = [0, \ldots, 0]^\mathsf{T}$, we conclude that $\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_0 = [0, \ldots, 0]^\mathsf{T}$ for all t, that is, $\bar{x}_i^{(q)}(t) = -\bar{x}_{i+n}^{(q)}(t)$ $(i = 1, 2, \ldots, n,\ q = 1, 2, \ldots, m)$. We substitute $\bar{x}_{j+n}^{(q)}(t) = -\bar{x}_j^{(q)}(t)$

into (4.53) to obtain

$$\begin{aligned}\bar{u}_i(t-\tau) &= \sum_{q=1}^{m}\gamma_q\left[\sum_{j=1}^{n}\bar{a}_{ij}\left(\bar{x}_j^{(q)}(t-\tau)-\bar{x}_i^{(q)}(t-\tau)\right)-\sum_{j=1}^{n}\bar{a}_{i,j+n}\left(\bar{x}_j^{(q)}(t-\tau)+\bar{x}_i^{(q)}(t-\tau)\right)\right] \\ &= \sum_{q=1}^{m}\gamma_q\left[\sum_{j=1}^{n}(\bar{a}_{ij}-\bar{a}_{i,j+n})\bar{x}_j^{(q)}(t-\tau)-(\bar{a}_{ij}+\bar{a}_{i,j+n})\bar{x}_i^{(q)}(t-\tau)\right] \qquad (4.58)\end{aligned}$$

for $i = 1, 2, \ldots, n$. From Definition 3.1, we get

$$\begin{aligned}\bar{u}_i(t-\tau) &= \sum_{q=1}^{m}\gamma_q\left[\sum_{j=1}^{n}\left(\frac{|a_{ij}|+a_{ij}}{2}-\frac{|a_{ij}|-a_{ij}}{2}\right)\bar{x}_j^{(q)}(t-\tau)\right. \\ &\quad \left.-\left(\frac{|a_{ij}|+a_{ij}}{2}+\frac{|a_{ij}|-a_{ij}}{2}\right)\bar{x}_i^{(q)}(t-\tau)\right] \\ &= \sum_{q=1}^{m}\gamma_q\sum_{j=1}^{n}a_{ij}\left(x_j^{(q)}(t-\tau)-\mathrm{sgn}(a_{ij})x_i^{(q)}(t-\tau)\right) \qquad (4.59)\end{aligned}$$

for $i = 1, 2, \ldots, n$. Note here that (4.59) and (4.46) are equivalent. By the choice of initial states $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ and $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ $(i = 1, 2, \ldots, n,\ q = 1, 2, \ldots, m)$ one can recover the trajectories $\mathbf{x}^{(q)}(t)$ which completes the proof. □

Lemma 4.1 relates the trajectories of the signed system (4.45)-(4.46) to that of the extended system (4.49)-(4.50) obtained by utilizing Definition 3.1. The objective in the sequel is to investigate the clustering properties of the signed system (4.45)-(4.46) by examining the stability and the clusters of the related extended system (4.49)-(4.50). To this end, we first relate the stability of the signed and extended systems as below.

The characteristic polynomial for the signed system (4.45)-(4.46) is obtained from (4.47) as $\Delta(s) = \det(sI_{mn} - \mathcal{L}_0 - e^{-s\tau}\mathcal{L}_d) = 0$ which can further be represented in terms of the eigenvalues of L

$$\Delta(s) = \prod_{i=1}^{n}\left(s^m + (\gamma_1 + \gamma_2 s + \cdots + \gamma_m s^{m-1})e^{-s\tau}\lambda_i\right) \qquad (4.60)$$

where λ_i are eigenvalues of L in (4.48). Similarly, from (4.51) the characteristic polynomial for the extended system (4.49)-(4.50) is

$$\bar{\Delta}(s) = \prod_{i=1}^{2n} \left(s^m + (\gamma_1 + \gamma_2 s + \cdots + \gamma_m s^{m-1}) e^{-s\tau} \bar{\lambda}_i \right) \tag{4.61}$$

where $\bar{\lambda}_i$ are the eigenvalues of $\bar{L}$ in (4.52). Employing the characteristic polynomials from equations (4.60) and (4.61) in conjunction with Lemma 4.1 leads to the derivation of the subsequent theorem:

Theorem 4.4. *Suppose that the extended system* (4.49)-(4.50) *is stable for* $\tau = 0$ *and its directed graph contains* $\bar{l}_p$ *PLSs and* $\bar{l}_s$ *SLSs. Let* l' *be the number of subgraphs of the extended system that does not contain any agent of the original system* (4.45)-(4.46). *Then there exists an upper bound* τ^* *such that the signed system* (4.45)-(4.46) *forms* $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ *clusters for* $\tau < \tau^*$.

Proof. The eigenvalues of $\bar{L}$ in (4.52) include those of L in (4.48) from Chapter 3. Therefore, the characteristic polynomial (4.60) is a divisor of (4.61). Hence, the stability of the system (4.51) implies the stability of (4.47).

We determine an upper bound for the extended input delayed system (4.51) which will also be effective for the signed input delayed system (4.47). Suppose that the non-delayed system is stable. An upper bound $\tau^* > 0$ exists for which (4.61) has a root on the imaginary axis and the system with characteristic polynomial (4.61) remains stable for $\tau < \tau^*$. In order to calculate τ^*, let $s = j\bar{\omega}_i$ and we have

$$-(j\bar{\omega}_i)^m = (C_1 + jC_2) e^{-j\bar{\omega}_i \tau_i} \bar{\lambda}_i \tag{4.62}$$

where

$$C_1 = \mathbb{R}\mathrm{e} \left\{ \sum_{q=1}^{m} \gamma_q (j\bar{\omega}_i)^{q-1} \right\}, \quad C_2 = \mathbb{I}\mathrm{m} \left\{ \sum_{q=1}^{m} \gamma_q (j\bar{\omega}_i)^{q-1} \right\}. \tag{4.63}$$

One obtains the following equation by taking modulus on both sides of (4.62):

$$\bar{\omega}_i^{2m} = \left(C_1^2 + C_2^2\right)\left(\mathbb{R}\mathrm{e}^2\{\bar{\lambda}_i\} + \mathbb{I}\mathrm{m}^2\{\bar{\lambda}_i\}\right). \tag{4.64}$$

From Theorem 4.2, τ^* can be calculated as

$$\tau^* = \min_i \tau_i \tag{4.65}$$

where

$$\tau_i = \begin{cases} \dfrac{1}{\bar{\omega}_i} \arctan\left(\dfrac{-\left(C_1\,\mathbb{R}\mathrm{e}\{\bar{\lambda}_i\} - C_2\,\mathbb{I}\mathrm{m}\{\bar{\lambda}_i\}\right)}{C_1\,\mathbb{I}\mathrm{m}\{\bar{\lambda}_i\} + C_2\,\mathbb{R}\mathrm{e}\{\bar{\lambda}_i\}}\right), & \text{if } m \text{ is odd} \\ \dfrac{1}{\bar{\omega}_i} \arctan\left(\dfrac{C_1\,\mathbb{I}\mathrm{m}\{\bar{\lambda}_i\} + C_2\,\mathbb{R}\mathrm{e}\{\bar{\lambda}_i\}}{C_1\,\mathbb{R}\mathrm{e}\{\bar{\lambda}_i\} - C_2\,\mathbb{I}\mathrm{m}\{\bar{\lambda}_i\}}\right), & \text{if } m \text{ is even.} \end{cases} \tag{4.66}$$

For $\tau < \tau^*$, the extended system is stable. Furthermore, by using the result in Theorem 4.2, we can determine the number of clusters of (4.51) as $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ if and only if $\tau < \tau^*$. From Lemma 4.1, recall that the trajectories of (4.47) are obtained from those of (4.51). Therefore, among the $\bar{\mathcal{K}}$ clusters, we need to remove l' clusters whose are not in the original set. Hence, the original network consists of

$$\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$$

clusters. □

The initial step in the stability analysis of a system with time delays is to assess the stability of its non-delayed counterpart. In Chapter 2, the stability conditions of the extended system (4.49)-(4.50) are presented for $\tau = 0$.

For a chain of m integrators, the characteristic polynomial of the system utilizing (4.49)-(4.50) with $\tau = 0$ can be computed as

$$\bar{\Delta}(s) = \prod_{i=1}^{2n} \left(s^m + (\gamma_1 + \gamma_2 s + \cdots + \gamma_m s^{m-1})\bar{\lambda}_i\right) \tag{4.67}$$

where $\bar{\lambda}_i$ are the eigenvalues of $\bar{L}$ in (4.52). For a first-order system ($m = 1$), the characteristic polynomial (4.67) is reduced to

$$\bar{\Delta}(s) = \prod_{i=1}^{2n} \left(s + \gamma_1 \bar{\lambda}_i\right) \tag{4.68}$$

where $\gamma_1 > 0$ is the only control parameter. Since the real parts of the nonzero Laplacian eigenvalues are positive ($\mathbb{R}\text{e}\{\bar{\lambda}_i\} > 0$), we have $\mathbb{R}\text{e}\{\gamma_1 \bar{\lambda}_i\} \geq 0$ for all i. Therefore, the first-order system utilizing (4.49)-(4.50) with $\tau = 0$ is always stable. For a second-order system ($m = 2$), the following characteristic equation is reduced to

$$\bar{\Delta}(s) = \prod_{i=1}^{2n} \left(s^2 + (\gamma_1 + \gamma_2 s)\bar{\lambda}_i\right) \tag{4.69}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are the control parameters. From Chapter 2, a second-order system utilizing (4.49)-(4.50) with $\tau = 0$ is stable if and only if the positive control parameters γ_1 and γ_2 hold the following condition:

- $\dfrac{\gamma_2^2}{\gamma_1} > \max\limits_i \dfrac{\mathbb{I}\text{m}\{\bar{\lambda}_i\}}{\mathbb{R}\text{e}\{\bar{\lambda}_i\}|\bar{\lambda}_i|^2}$.

For a third-order system ($m = 3$), the characteristic polynomial can be expressed as

$$\bar{\Delta}(s) = \prod_{i=1}^{2n} \left(s^3 + (\gamma_1 + \gamma_2 s + \gamma_3 s^2)\bar{\lambda}_i\right) \tag{4.70}$$

where $\gamma_1 > 0, \gamma_2 > 0$ and $\gamma_3 > 0$ are the control parameters. A third-order system utilizing (4.49)-(4.50) with $\tau = 0$ is stable if and only if the positive control parameters γ_1, γ_2 and γ_3 satisfy the following conditions derived from Chapter 2

- $\gamma_3^2\gamma_2\,\mathbb{R}\mathrm{e}\{\bar{\lambda}_i\}|\bar{\lambda}_i|^2 - \gamma_3\gamma_1\,\mathbb{R}\mathrm{e}^2\{\bar{\lambda}_i\} - \gamma_2^2\,\mathbb{I}\mathrm{m}^2\{\bar{\lambda}_i\} > 0,$
- $\gamma_2^2\gamma_3^2\,\mathbb{R}\mathrm{e}\{\bar{\lambda}_i\}|\bar{\lambda}_i|^4 - 2\gamma_1\gamma_2\gamma_3\,\mathbb{R}\mathrm{e}^2\{\bar{\lambda}_i\}|\bar{\lambda}_i|^2 - \gamma_3^2\,\mathbb{I}\mathrm{m}^2\{\bar{\lambda}_i\}|\bar{\lambda}_i|^2 + \gamma_1^2\,\mathbb{R}\mathrm{e}^3\{\bar{\lambda}_i\} > 0.$

As the degree of the system increases, the stability analysis becomes more complicated. For systems of order higher than three, the stability conditions are given in Chapter 2.

In summary, as evident from the derivation above, the stability analysis becomes increasingly complex as the order of the system increases. With a higher system order, the number of controller parameters involved in the stability analysis also rises. Each additional controller parameter contributes to the growing complexity of the stability evaluation.

Remark 4.1. Although the system's degree may impact its stability, Theorem 4.4 demonstrates that cluster composition remains unchanged when the system is stable. In other words, the clusters and their members identified by the PLS and SLS detection algorithms are identical across all system degrees.

Remark 4.2. Theorem 4.4 establishes the delay condition enabling the system (4.45)-(4.46) to achieve cluster consensus with $\overline{l}_p+\overline{l}_s-l'$ clusters. For the system (4.45)-(4.46), the delay condition on the upper bound τ^* is deduced from its extended dynamics employing Lemma 4.3. To ascertain τ^*, the following procedure provides a systematic approach.

Procedure 3

Step 1: For each nonzero $\bar{\lambda}_i$, obtain the positive solution $\bar{\omega}_i$ by solving (4.64).

Step 2: For each $\bar{\omega}_i$, compute C_1 and C_2 by using (4.63) and find τ_i from (4.66).

Step 3: Calculate an upper bound for the input delay as $\tau^* = \min_i \tau_i$.

Figure 4.2. Algorithm to calculate an upper bound of input delay τ^*.

An important distinction arises when considering the allowable delay tolerances within the signed system and its extended counterpart. While Theorem 4.4 establishes a crucial connection between their state trajectories, it is essential to acknowledge that

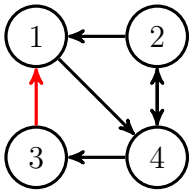


Figure 4.3. A signed directed graph.

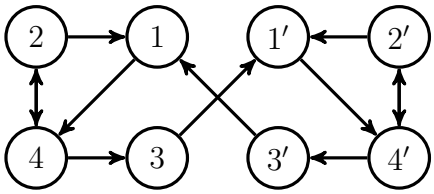


Figure 4.4. Extended directed graph for the network in Figure 4.3.

the precise upper bound on the input delay tolerable by the original signed system might deviate from the delay bound applicable to its extended dynamics. This difference can stem from the additional eigenvalues extended system has. This situation is discussed in the example below:

Example 4.1. *Consider the signed network in Figure 4.3. As illustrated in the figure, the network under consideration employs red arrows to depict negative communication flows, whereas positive communication flows are represented by black arrows. Without loss of generality, the system has first-order dynamics with input delay. Assume that the system utilizes the following system matrix:*

$$L = \begin{bmatrix} 6 & -1 & 5 & 0 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 4 & -4 \\ -1 & -2 & 0 & 3 \end{bmatrix}. \tag{4.71}$$

The eigenvalues of L can be calculated as $\{7.5449,\ 1.0166,\ 3.7193 \pm j1.3462\}$. *Leveraging the methodology established in the proof of Theorem 4.4, we can determine that the upper bound on the input delay for the signed system is equal to* 0.2082. *An extended counterpart of the signed network can be constructed by applying Definition 3.1, as illustrated in Figure 4.4. The eigenvalues of $\bar{L}$ are* $\{7.5449,\ 1.0166,\ 3.7193 \pm j1.3462,\ 6.2657 \pm j2.0675,\ 3.4686,\ 0\}$. *According to Theorem 4.1, the upper bound τ^**

is calculated as 0.1898. *The original system demonstrates stable behavior for delays between* 0.1898 *and* 0.2082. *In contrast, the extended system becomes unstable when the input delay surpasses* 0.1898 *for arbitrary initial values. The state trajectories of a stable signed system (*$0.1898 < \tau \leq 0.2082$*) can still be recovered from its unstable extended system by imposing initial conditions as in Lemma 4.1. However, we are unable to definitively determine the number of clusters formed in the original system upon convergence based on the present investigation for* $0.1898 < \tau \leq 0.2082$. *Therefore, we rely on the upper bound* τ^* *obtained for extended system.*

Within this section, we have delved into examining the stability characteristics of input delayed networks exhibiting antagonistic dynamics in continuous-time. The subsequent section will revisit the cluster consensus problem, this time addressing alterations in delay dynamics within networks. Specifically, the focus will shift to examining delay from the perspective of communication delay.

4.2.2. Communication Delay Effect

Consider a network of agents with m-th order linear delayed dynamics described by

$$\begin{aligned} \dot{x}_i^{(1)}(t) &= x_i^{(2)}(t), \\ &\vdots \\ \dot{x}_i^{(m-1)}(t) &= x_i^{(m)}(t), \\ \dot{x}_i^{(m)}(t) &= u_i(t), \quad i = 1, 2, \ldots, n \end{aligned} \tag{4.72}$$

where $x_i^{(q)}(t)$ $(q = 1, 2, \ldots, m)$ represents the states of agent i at time t, $x_i^{(q)}$ is the $(q-1)$th derivative of $x_i^{(1)}$, τ is the fixed communication delay and $u_i(t)$ is the delayed input defined as

$$u_i(t) = \sum_{q=1}^{m} \gamma_q \sum_{j \in \mathcal{N}_i} a_{ij} \left(x_j^{(q)}(t-\tau) - \operatorname{sgn}(a_{ij}) x_i^{(q)}(t) \right), \quad i = 1, 2, \ldots, n. \tag{4.73}$$

In this section we assume that the elements of the adjacency matrix satisfies the following conditions:

Assumption 4.2. *The adjacency elements, a_{ij}, hold the followings:*

- $a_{ii} > 0$ *for* $i = 1, 2, \ldots, n$.
- $a_{ij} \neq 0$ *for all* $(j, i) \in \mathcal{E}$.
- $a_{ij} = 0$ *for all* $(j, i) \notin \mathcal{E}$.
- $\sum_{j=1}^{n} |a_{ij}| = 1$ *for* $i = 1, 2, \ldots, n$.

The dynamics of the signed higher-order integrator system with communication delay can be written as

$$\dot{\mathbf{x}}(t) = \mathcal{D}_0 \mathbf{x}(t) + \mathcal{A}_d \mathbf{x}(t - \tau), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{4.74}$$

where

$$\mathcal{D}_0 = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ -\gamma_1 I_n & -\gamma_2 I_n & -\gamma_3 I_n & \ldots & -\gamma_m I_n \end{bmatrix}, \mathcal{A}_d = \begin{bmatrix} 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & 0_{n\times n} \\ \gamma_1 \mathcal{A} & \gamma_2 \mathcal{A} & \gamma_3 \mathcal{A} & \ldots & \gamma_m \mathcal{A} \end{bmatrix}.$$

Given the signed delayed system in (4.72)-(4.73) evolving over $\mathcal{G}$, dynamics of the the extended delayed system is given as

$$\begin{aligned} \dot{\bar{x}}_i^{(1)}(t) &= \bar{x}_i^{(2)}(t), \\ &\vdots \\ \dot{\bar{x}}_i^{(m-1)}(t) &= \bar{x}_i^{(m)}(t), \\ \dot{\bar{x}}_i^{(m)}(t) &= \bar{u}_i(t), \quad i = 1, 2, \ldots, 2n \end{aligned} \tag{4.75}$$

where $\bar{x}_i^{(q)}(t)$ $(q = 1, 2, \ldots, m)$ represents the state of agent i at time t, $\bar{x}_i^{(q)}$ is $(q-1)$th derivative of $\bar{x}_i^{(1)}(t)$ and $\bar{u}_i(t-\tau)$ is control input defined as

$$\bar{u}_i(t) = \sum_{q=1}^{m} \gamma_q \sum_{j \in \bar{\mathcal{N}}_i} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(t-\tau) - \bar{x}_i^{(q)}(t) \right), \quad i = 1, 2, \ldots, 2n. \tag{4.76}$$

Note once again that $\bar{a}_{ij} \geq 0$ by Definition 3.1 and the system in (4.75)-(4.76) evolves as an unsigned network.

For analysis purposes, we define the augmented state vector $\bar{\mathbf{x}}(t) = \left[\bar{\mathbf{x}}^{(1)}(t)^\mathsf{T}, \bar{\mathbf{x}}^{(2)}(t)^\mathsf{T}, \ldots, \bar{\mathbf{x}}^{(m)}(t)^\mathsf{T}\right]^\mathsf{T}$ where $\bar{\mathbf{x}}^{(q)}(t) = \left[\bar{x}_1^{(q)}(t), \bar{x}_2^{(q)}(t), \ldots, \bar{x}_{2n}^{(q)}(t)\right]^\mathsf{T}$ for $q = 1, 2, \ldots, m$ and $\bar{\mathbf{x}}_0$ is the initial condition. Then, the dynamics of the extended delayed system (4.75)-(4.76) are equivalent to

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathcal{D}}_0 \bar{\mathbf{x}}(t) + \bar{\mathcal{A}}_d \bar{\mathbf{x}}(t-\tau), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \tag{4.77}$$

where

$$\bar{\mathcal{D}}_0 = \begin{bmatrix} 0_{2n\times 2n} & I_{2n} & 0_{2n\times 2n} & \ldots & 0_{2n\times 2n} \\ 0_{2n\times 2n} & 0_{2n\times 2n} & I_{2n} & \ldots & 0_{2n\times 2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \ldots & I_{2n} \\ -\gamma_1\bar{\mathcal{D}} & -\gamma_2\bar{\mathcal{D}} & -\gamma_3\bar{\mathcal{D}} & \ldots & -\gamma_m\bar{\mathcal{D}} \end{bmatrix}, \bar{\mathcal{A}}_d = \begin{bmatrix} 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \ldots & 0_{2n\times 2n} \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \ldots & 0_{2n\times 2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{2n\times 2n} & 0_{2n\times 2n} & 0_{2n\times 2n} & \ldots & 0_{2n\times 2n} \\ \gamma_1\bar{\mathcal{A}} & \gamma_2\bar{\mathcal{A}} & \gamma_3\bar{\mathcal{A}} & \ldots & \gamma_m\bar{\mathcal{A}} \end{bmatrix}.$$

Through an analysis of the stability property of the extended system (4.77), we can ascertain the necessary upper bound for the system to achieve cluster consensus.

Next we relate the trajectories of the signed system (4.72)-(4.73) to those of the associated extended network (4.75)-(4.76).

Lemma 4.4. *The states* $\mathbf{x}(t)$ *of the delayed system* (4.72)-(4.73) *can be obtained from its extended delayed system* (4.75)-(4.76) *as* $x_i^{(q)}(t) = \bar{x}_i^{(q)}(t)$ *for all* $t \geq 0$ *by placing* $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ *and* $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ *for* $i = 1, 2, \ldots, n$ *and* $q = 1, 2, \ldots, m$.

Proof. We can rewrite $\bar{u}_i(t)$ in (4.76) of the extended delayed system as

$$\begin{aligned}
\bar{u}_i(t) &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(t-\tau) - \bar{x}_i^{(q)}(t) \right) + \sum_{j=1}^{n} \bar{a}_{i,j+n} \left(\bar{x}_{j+n}^{(q)}(t-\tau) - \bar{x}_i^{(q)}(t) \right) \right], \\
\bar{u}_{i+n}(t) &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{i+n,j+n} \left(\bar{x}_{j+n}^{(q)}(t-\tau) - \bar{x}_{i+n}^{(q)}(t) \right) + \sum_{j=1}^{n} \bar{a}_{i+n,j} \left(\bar{x}_j^{(q)}(t-\tau) - \bar{x}_{i+n}^{(q)}(t) \right) \right]
\end{aligned} \tag{4.78}$$

for $i = 1, 2, \ldots, n$. Let $\hat{x}_i^{(q)}(t) = \bar{x}_i^{(q)}(t) + \bar{x}_{i+n}^{(q)}(t)$ $(i = 1, 2, \ldots, n,\ q = 1, 2, \ldots, m)$. Then, the dynamics of $\hat{x}_i^{(q)}(t)$ can be expressed as

$$\begin{aligned}
\dot{\hat{x}}_i^{(q)}(t) &= \dot{\bar{x}}_i^{(q)}(t) + \dot{\bar{x}}_{i+n}^{(q)}(t) \\
&= \bar{x}_i^{(q+1)}(t) + \bar{x}_{i+n}^{(q+1)}(t) \\
&= \hat{x}_i^{(q+1)}(t)
\end{aligned}$$

for $q = 1, 2, \ldots, m-1$ and

$$\begin{aligned}
\dot{\hat{x}}_i^{(m)}(t) &= \dot{\bar{x}}_i^{(m)}(t) + \dot{\bar{x}}_{i+n}^{(m)}(t) \\
&= \bar{u}_i(t-\tau) + \bar{u}_{i+n}(t-\tau) \\
&= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} (\bar{a}_{ij} + \bar{a}_{i+n,j}) \bar{x}_j^{(q)}(t-\tau) + (\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n}) \bar{x}_{j+n}^{(k)}(t-\tau) \right] \\
&\quad - \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} (\bar{a}_{ij} + \bar{a}_{i,j+n}) \bar{x}_i^{(q)}(t) + (\bar{a}_{i+n,j} + \bar{a}_{i+n,j+n}) \bar{x}_{i+n}^{(q)}(t) \right].
\end{aligned}$$

We can utilize Definition 3.1 to obtain

$$\bar{a}_{i+n,j} = \bar{a}_{i,j+n}, \quad \bar{a}_{ij} = \bar{a}_{i+n,j+n} \tag{4.79}$$

for $i = 1, 2, \ldots, n$. By using (4.79) together with Definition 3.1, we also have $\bar{a}_{ij} +$

$\bar{a}_{i+n,j} = \bar{a}_{i,j+n} + \bar{a}_{i+n,j+n} = \bar{a}_{ij} + \bar{a}_{i,j+n} = \bar{a}_{i+n,j} + \bar{a}_{i+n,j+n} = |a_{ij}|$. Then, we have

$$\begin{aligned}\dot{\hat{x}}_i^{(m)}(t) &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} |a_{ij}| \left(\bar{x}_j^{(q)}(t-\tau) + \bar{x}_{j+n}^{(q)}(t-\tau) \right) \right] \\ &\quad - \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} |a_{ij}| \left(\bar{x}_i^{(q)}(t) + \bar{x}_{i+n}^{(q)}(t) \right) \right] \\ &= \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} |a_{ij}| \left(\hat{x}_j^{(q)}(t-\tau) - \hat{x}_i^{(q)}(t) \right) \end{aligned} \tag{4.80}$$

for $i = 1, 2, \ldots, n$ which is equivalent to

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathcal{D}}_0 \hat{\mathbf{x}}(t) + \hat{\mathcal{A}}_d \hat{\mathbf{x}}(t-\tau), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \tag{4.81}$$

where $\hat{\mathbf{x}}(t) = \left[\hat{\mathbf{x}}^{(1)}(t)^\mathsf{T}, \hat{\mathbf{x}}^{(2)}(t)^\mathsf{T}, \ldots, \hat{\mathbf{x}}^{(m)}(t)^\mathsf{T} \right]^\mathsf{T}$ is the state vector with components $\hat{\mathbf{x}}^{(q)}(t) = \left[\hat{x}_1^{(q)}(t), \hat{x}_2^{(q)}(t), \ldots, \hat{x}_n^{(q)}(t) \right]^\mathsf{T}$ for $q = 1, 2, \ldots, m$, $\hat{\mathbf{x}}_0$ is the initial condition and

$$\hat{\mathcal{D}}_0 = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ -\gamma_1 I_n & -\gamma_2 I_n & -\gamma_3 I_n & \ldots & -\gamma_m I_n \end{bmatrix}, \hat{\mathcal{A}}_d = \begin{bmatrix} 0_{n\times n} & I_n & 0_{n\times n} & \ldots & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} & I_n & \ldots & 0_{n\times n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0_{n\times n} & 0_{n\times n} & 0_{n\times n} & \ldots & I_n \\ \gamma_1 \hat{A} & \gamma_2 \hat{A} & \gamma_3 \hat{A} & \ldots & \gamma_m \hat{A} \end{bmatrix}$$

where $\hat{L} = I_n - \hat{A}$ is an unsigned Laplacian matrix defined as in (4.57). Since the solution of the system (4.80) is $\hat{\mathbf{x}}(t) = \mathcal{L}^{-1} \left\{ \left(sI_{mn} - \hat{\mathcal{D}}_0 - e^{-\tau s} \hat{\mathcal{A}}_d \right)^{-1} \right\} \hat{\mathbf{x}}_0$ with the initial condition $\hat{\mathbf{x}}_0 = [0, \ldots, 0]^\mathsf{T}$, it implies $\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_0 = [0, \ldots, 0]^\mathsf{T}$ for all t, i.e., $\bar{x}_i^{(q)}(t) = -\bar{x}_{i+n}^{(q)}(t)$ for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, m$. We substitute $\bar{x}_{j+n}^{(q)}(t) = -\bar{x}_j^{(q)}(t)$ into (4.78) to obtain

$$\begin{aligned}\bar{u}_i(t) &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(t-\tau) - \bar{x}_i^{(q)}(t) \right) - \sum_{j=1}^{n} \bar{a}_{i,j+n} \left(\bar{x}_j^{(q)}(t-\tau) + \bar{x}_i^{(q)}(t) \right) \right] \\ &= \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} (\bar{a}_{ij} - \bar{a}_{i,j+n}) \bar{x}_j^{(q)}(t-\tau) - (\bar{a}_{ij} + \bar{a}_{i,j+n}) \bar{x}_i^{(q)}(t) \right] \end{aligned} \tag{4.82}$$

for $i = 1, 2, \ldots, n$. By utilizing Definition 3.1, one has

$$\begin{aligned}\bar{u}_i(t) =& \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \left(\frac{|a_{ij}| + a_{ij}}{2} - \frac{|a_{ij}| - a_{ij}}{2} \right) \bar{x}_j^{(q)}(t-\tau) - \left(\frac{|a_{ij}| + a_{ij}}{2} + \frac{|a_{ij}| - a_{ij}}{2} \right) \bar{x}_i^{(q)}(t) \right] \\ =& \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{n} a_{ij} \left(x_j^{(q)}(t-\tau) - \mathrm{sgn}(a_{ij}) x_i^{(q)}(t) \right) \end{aligned} \tag{4.83}$$

for $i = 1, 2, \ldots, n$. Note here that (4.83) and (4.73) are equivalent. By the choice of initial states $\bar{x}_{i0}^{(q)} = x_{i0}^{(q)}$ and $\bar{x}_{i+n,0}^{(q)} = -x_{i0}^{(q)}$ (for $i = 1, 2, \ldots, n$, $q = 1, 2, \ldots, m$), one can recover the trajectories $\mathbf{x}^{(q)}(t)$ which completes the proof. □

Lemma 4.4 details how the trajectories of the signed network (4.72)-(4.73) can be recovered from those of the extended network (4.75)-(4.76) for which Theorem 4.2 is applicable. In other words, from Theorem 4.2, (4.75)-(4.76) achieves $\bar{l}_p + \bar{l}_s$ clusters for $\tau < \tau^*$. By eliminating those clusters of (4.75)-(4.76) that are unrelated to the original signed network (4.72)-(4.73), the following result is obtained.

Theorem 4.5. *Suppose that the extended system* (4.75)-(4.76) *is stable for* $\tau = 0$ *and its directed graph contains* $\bar{l}_p$ *PLSs and* $\bar{l}_s$ *SLSs. Let* l' *be the number of subgraphs of the extended delayed system that does not contain any agent of the original system* (4.72)-(4.73). *Under Assumption 4.2, there exists an upper bound* τ^* *such that the delayed system* (4.72)-(4.73) *represented by a signed directed graph forms* $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ *clusters for* $\tau < \tau^*$.

Proof. The system (4.74) can be written in terms of PLS and SLS dynamics by using detection algorithms in [13]. For this case, let $\bar{n}_{p,i}$ and $\bar{n}_{s,j}$ be the number of agents in the $i-$th PLS and $j-$th SLS, respectively for $i = 1, 2, \ldots, \bar{l}_p$ and $j = 1, 2, \ldots, \bar{l}_s$. The total number of agents in the PLSs and SLSs can be expressed as $\bar{n}_p = \sum_i \bar{n}_{p,i}$ and $\bar{n}_s = \sum_j \bar{n}_{s,j}$, respectively. The dynamics of the agents in PLSs and SLSs can be written as follows:

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_p^{(1)}(t) \\ \vdots \\ \dot{\bar{\mathbf{x}}}_p^{(m)}(t) \end{bmatrix} = \left[\begin{array}{c|c} 0_{(m-1)\bar{n}_p\times\bar{n}_p} & I_{(m-1)\bar{n}_p} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}}\otimes I_{\bar{n}_p} \end{array}\right] \begin{bmatrix} \bar{\mathbf{x}}_p^{(1)}(t) \\ \vdots \\ \bar{\mathbf{x}}_p^{(m)}(t) \end{bmatrix} + \begin{bmatrix} 0_{(m-1)\bar{n}_p\times m\bar{n}_p} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}}\otimes \bar{\mathcal{A}}_p \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_p^{(1)}(t-\tau) \\ \vdots \\ \bar{\mathbf{x}}_p^{(m)}(t-\tau) \end{bmatrix}, \tag{4.84}$$

$$\begin{aligned} \begin{bmatrix} \dot{\bar{\mathbf{x}}}_s^{(1)}(t) \\ \vdots \\ \dot{\bar{\mathbf{x}}}_s^{(m)}(t) \end{bmatrix} &= \left[\begin{array}{c|c} 0_{(m-1)\bar{n}_s\times\bar{n}_s} & I_{(m-1)\bar{n}_s} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}}\otimes I_{\bar{n}_s} \end{array}\right] \begin{bmatrix} \bar{\mathbf{x}}_s^{(1)}(t) \\ \vdots \\ \bar{\mathbf{x}}_s^{(m)}(t) \end{bmatrix} + \begin{bmatrix} 0_{(m-1)\bar{n}_s\times m\bar{n}_s} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}}\otimes \bar{\mathcal{A}}_s \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_s^{(1)}(t-\tau) \\ \vdots \\ \bar{\mathbf{x}}_s^{(m)}(t-\tau) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0_{(m-1)\bar{n}_s\times m\bar{n}_p} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}}\otimes \bar{\mathcal{A}}_{sp} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_p^{(1)}(t-\tau) \\ \vdots \\ \bar{\mathbf{x}}_p^{(m)}(t-\tau) \end{bmatrix}. \end{aligned} \tag{4.85}$$

where $\bar{\mathbf{x}}_p^{(q)}(t) = \left[\bar{\mathbf{x}}_{p,1}^{(q)}(t)^{\mathsf{T}}, \bar{\mathbf{x}}_{p,2}^{(q)}(t)^{\mathsf{T}}, \ldots, \bar{\mathbf{x}}_{p,\bar{l}_p}^{(q)}(t)^{\mathsf{T}}\right]^{\mathsf{T}}$, $\bar{\mathbf{x}}_s^{(q)}(t) = \left[\bar{\mathbf{x}}_{s,1}^{(q)}(t)^{\mathsf{T}}, \bar{\mathbf{x}}_{s,2}^{(q)}(t)^{\mathsf{T}}, \ldots, \bar{\mathbf{x}}_{s,\bar{l}_s}^{(q)}(t)^{\mathsf{T}}\right]^{\mathsf{T}}$, $q = 1, 2, \ldots, m$, are the states vectors of the agents in PLSs and SLSs, respectively; $\boldsymbol{\gamma} = [\gamma_1, \gamma_2, \ldots, \gamma_m]^{\mathsf{T}}$ is the controller parameter vector, $\bar{\mathcal{A}}_p = I_{\bar{n}_p} - \bar{L}_p$, $\bar{\mathcal{A}}_{sp} = -\bar{L}_{sp}$ and $\bar{\mathcal{A}}_s = I_{\bar{n}_s} - \bar{L}_s$. $\bar{L}_p, \bar{L}_{sp}$ and $\bar{L}_s$ are

$$\bar{L}_p = \text{blkdiag}\{\bar{L}_{1,1}, \ldots, \bar{L}_{\bar{l}_p,\bar{l}_p}\}, \quad \bar{L}_{sp} = \begin{bmatrix} \bar{L}_{\bar{l}_p+1,1} & \cdots & \bar{L}_{\bar{l}_p+1,\bar{l}_p} \\ \vdots & \ddots & \vdots \\ \bar{L}_{\bar{l}_p+\bar{l}_s,1} & \cdots & \bar{L}_{\bar{l}_p+\bar{l}_s,\bar{l}_p} \end{bmatrix},$$

$$\bar{L}_s = \begin{bmatrix} \bar{L}_{\bar{l}_p+1,\bar{l}_p+1} & \cdots & \bar{L}_{\bar{l}_p+1,\bar{l}_p+\bar{l}_s} \\ \vdots & \ddots & \vdots \\ \bar{L}_{\bar{l}_p+\bar{l}_s,\bar{l}_p+1} & \cdots & \bar{L}_{\bar{l}_p+\bar{l}_s,\bar{l}_p+\bar{l}_s} \end{bmatrix}.$$

Step 1 (Stability and the Number of Clusters for PLS Dynamics): For each PLS, the following singular value decomposition can be obtained:

$$S_i^{-1}\bar{L}_{i,i}S_i = \begin{bmatrix} \tilde{L}_{i,i} & \mathbf{0}_{\bar{n}_{p,i}-1} \\ \mathbf{0}_{\bar{n}_{p,i}-1}^{\mathsf{T}} & 0 \end{bmatrix}, \quad i = 1, 2, \ldots, \bar{l}_p. \tag{4.86}$$

The last column of S_i is chosen as $\mathbf{1}_{\bar{n}_{p,i}}$ which is the right eigenvector of $\bar{L}_{i,i}$ associated with the zero eigenvalue. All eigenvalues of $\tilde{L}_{i,i}$ have positive real parts since each PLS contains a spanning tree [105].

Note that $\bar{\mathcal{A}}_{i,i} = I_{\bar{n}_{p,i}} - \bar{L}_{i,i}$ for $i = 1, 2, \ldots, \bar{l}_p$. From (4.86), it can be obtained as $S^{-1}\bar{\mathcal{A}}_{i,i}S = \begin{bmatrix} I_{\bar{n}_{p,i}-1} - \tilde{L}_{i,i} & \mathbf{0}_{\bar{n}_{p,i}-1} \\ \mathbf{0}_{\bar{n}_{p,i}-1}^{\mathsf{T}} & 1 \end{bmatrix}$. Define $\tilde{\mathcal{A}}_{i,i} = I_{\bar{n}_{p,i}-1} - \tilde{L}_{i,i}$ and

$$\begin{aligned} \tilde{\mathbf{y}}_{p,i}^{(q)}(t) &= S_i^{-1}\bar{\mathbf{x}}_{p,i}^{(q)}(t) \\ &= \begin{bmatrix} \bar{y}_{p,i,1}^{(q)}(t) & \cdots & \bar{y}_{p,i,\bar{n}_{p,i}}^{(q)}(t) \end{bmatrix}^{\mathsf{T}} \\ &= \begin{bmatrix} \bar{\mathbf{y}}_{p,i}^{(q)}(t)^{\mathsf{T}} & \bar{y}_{p,i,\bar{n}_{p,i}}^{(q)}(t) \end{bmatrix}^{\mathsf{T}} \end{aligned}$$

for $q = 1, 2, \ldots, m$. One can obtain the following equations:

$$\begin{aligned} \begin{bmatrix} \dot{\bar{\mathbf{y}}}_{p,i}^{(1)}(t) \\ \vdots \\ \dot{\bar{\mathbf{y}}}_{p,i}^{(m-1)}(t) \\ \dot{\bar{\mathbf{y}}}_{p,i}^{(m)}(t) \end{bmatrix} &= \left[\begin{array}{c|c} 0_{(m-1)(\bar{n}_{p,i}-1)\times\bar{n}_{p,i}-1} & I_{(m-1)(\bar{n}_{p,i}-1)} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes I_{\bar{n}_{p,i}-1} & \end{array}\right] \begin{bmatrix} \bar{\mathbf{y}}_{p,i}^{(1)}(t) \\ \vdots \\ \bar{\mathbf{y}}_{p,i}^{(m-1)}(t) \\ \bar{\mathbf{y}}_{p,i}^{(m)}(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} 0_{(m-1)(\bar{n}_{p,i}-1)\times m(\bar{n}_{p,i}-1)} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \otimes \tilde{\mathcal{A}}_{i,i} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{y}}_{p,i}^{(1)}(t-\tau) \\ \vdots \\ \bar{\mathbf{y}}_{p,i}^{(m-1)}(t-\tau) \\ \bar{\mathbf{y}}_{p,i}^{(m)}(t-\tau) \end{bmatrix}, \end{aligned} \tag{4.87a}$$

$$\begin{bmatrix} \dot{\bar{y}}^{(1)}_{p,i,\bar{n}_{p,i}}(t) \\ \vdots \\ \dot{\bar{y}}^{(m-1)}_{p,i,\bar{n}_{p,i}}(t) \\ \dot{\bar{y}}^{(m)}_{p,i,\bar{n}_{p,i}}(t) \end{bmatrix} = \left[\begin{array}{c|c} \mathbf{0}_{(m-1)} & I_{m-1} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} & \end{array}\right] \begin{bmatrix} \bar{y}^{(1)}_{p,i,\bar{n}_{p,i}}(t) \\ \vdots \\ \bar{y}^{(m-1)}_{p,i,\bar{n}_{p,i}}(t) \\ \bar{y}^{(m)}_{p,i,\bar{n}_{p,i}}(t) \end{bmatrix} + \left[\begin{array}{c} 0_{(m-1)\times m} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \end{array}\right] \begin{bmatrix} \bar{y}^{(1)}_{p,i,\bar{n}_{p,i}}(t-\tau) \\ \vdots \\ \bar{y}^{(m-1)}_{p,i,\bar{n}_{p,i}}(t-\tau) \\ \bar{y}^{(m)}_{p,i,\bar{n}_{p,i}}(t-\tau) \end{bmatrix}. \tag{4.87b}$$

From Theorem 3 and Corollary 4 in [106], the stability of system (4.87a) depends on the eigenvalues of $\gamma_m \tilde{\mathcal{A}}_{i,i}$. The upper bound of τ for system (4.87a) can be calculated as

$$\tau_1^* = \min_i \tau_i \tag{4.88}$$

where

$$\tau_i = \begin{cases} \dfrac{1}{\tilde{\lambda}_i}, & \text{if } \tilde{\lambda}_i \text{ is real and positive} \\ \dfrac{k\pi + \arctan\left(\frac{\mathbb{Im}\{\tilde{\lambda}_i\}}{\mathbb{Re}\{\tilde{\lambda}_i\}}\right)}{\mathbb{Im}\{\tilde{\lambda}_i\}} > 0, \quad k = 0, \pm 1, \ldots, & \text{if } \tilde{\lambda}_i \text{ is complex and } \mathbb{Im}\{\tilde{\lambda}_i\} \neq 0 \end{cases}$$

and $\tilde{\lambda}_i$ is a nonzero eigenvalue of $\gamma_m \tilde{\mathcal{A}}_{i,i}$. The dynamics of (4.87a) are stable for $\tau < \tau_1^*$.

In order to analyze the stability of system (4.87b), consider its characteristic equation:

$$s^m + (\gamma_m s^{m-1} + \gamma_{m-1} s^{m-2} + \cdots + \gamma_1)(1 - e^{-\tau s}) = 0. \tag{4.89}$$

As the communication delay gets larger, at least one solution to the characteristic equation (4.89) will have positive real part, resulting in an unstable system. Therefore, an upper bound $\tau^* > 0$ exists such that (4.89) has a root on the imaginary axis and

(4.89) is stable for $\tau < \tau^*$. In order to calculate τ^*, let $s = j\bar{\omega}_i$ and we have

$$-(j\bar{\omega}_i)^m = (C_1 + jC_2)(1 - e^{-j\bar{\omega}_i\tau_i}) \tag{4.90}$$

where

$$C_1 = \mathbb{R}\mathrm{e}\left\{\sum_{q=1}^{m}\gamma_q(j\bar{\omega}_i)^{q-1}\right\}, \quad C_2 = \mathbb{I}\mathrm{m}\left\{\sum_{q=1}^{m}\gamma_q(j\bar{\omega}_i)^{q-1}\right\}.$$

From Theorem 4.3, a positive solution $\bar{\omega}_i$ can be obtained from one of the following polynomials:

$$\begin{aligned}\bar{\omega}_i^m - 2(-1)^{\frac{m+1}{2}}C_2 = 0 \text{ if } m \text{ is odd},\\ \bar{\omega}_i^m - 2(-1)^{\frac{m+2}{2}}C_1 = 0 \text{ if } m \text{ is even}.\end{aligned} \tag{4.91}$$

Consequently, an upper bound for τ is computed as

$$\tau_2^* = \min_i \tau_i \tag{4.92}$$

where

$$\tau_i = \begin{cases}\dfrac{1}{\bar{\omega}_i}\left[\arctan\left(\dfrac{(-1)^{\frac{m+1}{2}}\bar{\omega}_i^m C_1}{C_1^2 + C_2^2 - (-1)^{\frac{m+1}{2}}\bar{\omega}_i^m C_2}\right) \pm 2k\pi\right] > 0, & \text{if } m \text{ is odd}\\ \dfrac{1}{\bar{\omega}_i}\left[\arctan\left(\dfrac{-(-1)^{\frac{m+2}{2}}\bar{\omega}_i^m C_2}{C_1^2 + C_2^2 - (-1)^{\frac{m+2}{2}}\bar{\omega}_i^m C_1}\right) \pm 2k\pi\right] > 0, & \text{if } m \text{ is even}\end{cases} \tag{4.93}$$

for $k = 1, 2, \ldots$. For $\tau < \tau_2^*$, the dynamics of (4.87b) with communication delay is stable. Together with (4.88) and (4.92), PLSs converge to $\bar{l}_p$ clusters for $\tau < \min\{\tau_1^*, \tau_2^*\}$.

Step 2 (Stability and the Number of Clusters for SLS Dynamics): In this part, we prove that the following equation always holds:

$$\lim_{t\to\infty}\|\bar{\mathbf{x}}_s^{(q)}(t) + \bar{L}_s^{-1}\bar{L}_{sp}\bar{\mathbf{x}}_p^{(q)}(t)\| = 0, \quad q = 1, 2, \ldots, m. \tag{4.94}$$

Let $\mathbf{e}_q(t) = \bar{\mathbf{x}}_s^{(q)}(t) + \bar{L}_s^{-1}\bar{L}_{sp}\bar{\mathbf{x}}_p^{(q)}(t)$ for $q = 1, 2, \ldots, m$. Taking derivatives yields the following matrix equation:

$$\begin{bmatrix} \dot{\mathbf{e}}_1(t) \\ \vdots \\ \dot{\mathbf{e}}_{m-1}(t) \\ \dot{\mathbf{e}}_m(t) \end{bmatrix} = \left[\begin{array}{c|c} 0_{(m-1)\bar{n}_s \times \bar{n}_s} & I_{(m-1)\bar{n}_s} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes I_{\bar{n}_s} & \end{array}\right] \begin{bmatrix} \mathbf{e}_1(t) \\ \vdots \\ \mathbf{e}_{m-1}(t) \\ \mathbf{e}_m(t) \end{bmatrix} + \left[\begin{array}{c} 0_{(m-1)\bar{n}_s \times m\bar{n}_s} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \otimes \bar{\mathcal{A}}_s \end{array}\right] \begin{bmatrix} \mathbf{e}_1(t-\tau) \\ \vdots \\ \mathbf{e}_{m-1}(t-\tau) \\ \mathbf{e}_m(t-\tau) \end{bmatrix}$$
$$+ \left[\begin{array}{c} 0_{(m-1)\bar{n}_s \times m\bar{n}_s} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes (\bar{L}_s^{-1}\bar{L}_{sp}\bar{L}_p) \end{array}\right] \begin{bmatrix} \bar{\mathbf{x}}_p^{(1)}(t-\tau) \\ \vdots \\ \bar{\mathbf{x}}_p^{(m-1)}(t-\tau) \\ \bar{\mathbf{x}}_p^{(m)}(t-\tau) \end{bmatrix}. \tag{4.95}$$

Note that the term of $-\gamma_1\bar{L}_s^{-1}\bar{L}_{sp}\bar{L}_p\bar{\mathbf{x}}_p^{(1)}(t-\tau) - \gamma_2\bar{L}_s^{-1}\bar{L}_{sp}\bar{L}_p\bar{\mathbf{x}}_p^{(2)}(t-\tau) - \cdots - \gamma_m\bar{L}_s^{-1}\bar{L}_{sp}\bar{L}_p\bar{\mathbf{x}}_p^{(m)}(t-\tau)$ converges to zero as $t \to \infty$ since each PLS achieves consensus. Therefore the stability of (4.95) depends on the stability of the following dynamics:

$$\begin{bmatrix} \dot{\mathbf{e}}_1(t) \\ \vdots \\ \dot{\mathbf{e}}_{m-1}(t) \\ \dot{\mathbf{e}}_m(t) \end{bmatrix} = \left[\begin{array}{c|c} 0_{(m-1)\bar{n}_s \times \bar{n}_s} & I_{(m-1)\bar{n}_s} \\ \hline -\boldsymbol{\gamma}^{\mathsf{T}} \otimes I_{\bar{n}_s} & \end{array}\right] \begin{bmatrix} \mathbf{e}_1(t) \\ \vdots \\ \mathbf{e}_{m-1}(t) \\ \mathbf{e}_m(t) \end{bmatrix} + \left[\begin{array}{c} 0_{(m-1)\bar{n}_s \times m\bar{n}_s} \\ \hline \boldsymbol{\gamma}^{\mathsf{T}} \otimes \bar{\mathcal{A}}_s \end{array}\right] \begin{bmatrix} \mathbf{e}_1(t-\tau) \\ \vdots \\ \mathbf{e}_{m-1}(t-\tau) \\ \mathbf{e}_m(t-\tau) \end{bmatrix}. \tag{4.96}$$

From Theorem 3 and Corollary 4 in [106], the stability of error dynamics (4.96) for SLS depends on the eigenvalues of $\gamma_m\bar{\mathcal{A}}_s$. The upper bound of τ for system (4.96) can be calculated as

$$\tau_3^* = \min_i \tau_i \tag{4.97}$$

where

$$\tau_i = \begin{cases} \dfrac{1}{\lambda_{si}}, & \text{if } \lambda_{si} \text{ is real and positive} \\ \dfrac{k\pi + \arctan\left(\frac{\mathbb{I}\mathrm{m}\{\lambda_{si}\}}{\mathbb{R}\mathrm{e}\{\lambda_{si}\}}\right)}{\mathbb{I}\mathrm{m}\{\lambda_{si}\}} > 0, \quad k = 0, \pm 1, \ldots, & \text{if } \lambda_{si} \text{ is complex and } \mathbb{I}\mathrm{m}\{\lambda_{si}\} \neq 0 \end{cases}$$

where λ_{si} is a nonzero eigenvalue of $\gamma_m \bar{\mathcal{A}}_s$. The dynamics of (4.96) are stable for $\tau < \tau_3^*$ i.e., $\lim_{t\to\infty} \|\mathbf{e}_q(t)\| = 0$ for $q = 1, 2, \ldots, m$. This implies that the following expression holds:

$$\bar{\mathbf{x}}_s^{(q)}(\infty) = -\bar{L}_s^{-1} \bar{L}_{sp} \bar{\mathbf{x}}_p^{(q)}(\infty)$$

for $q = 1, 2, \ldots, m$ under $\tau < \tau_3^*$. From [98], it is known that the above dynamics converge to $\bar{l}_s$ clusters.

Step 3 (Total Number of Clusters): One can conclude that the system (4.74) evolving over an unsigned directed graph converges to $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ clusters for $\tau < \tau^* = \min\{\tau_1^*, \tau_2^*, \tau_3^*\}$ where τ_1^*, τ_2^* and τ_3^* are calculated from (4.88), (4.92) and (4.97), respectively.

In light of Theorem 4.2, the total number of clusters for extended delayed system (4.75)-(4.76), which is constructed from signed delayed system (4.72)-(4.73) by using Definition 3.1, is $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ if and only if $\tau < \tau^*$. By Lemma 4.4 together with Theorem 4.2, the signed delayed system forms

$$\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$$

clusters if $\tau < \tau^*$ since the extended delayed system has l' clusters without any member from the original delayed system (4.72)-(4.73). □

Remark 4.3. Theorem 4.5 establishes the delay condition enabling the system (4.72)-(4.73) to achieve cluster consensus with $\bar{l}_p + \bar{l}_s - l'$ clusters. For the system (4.75)-(4.76),

the delay condition on the upper bound τ^* is deduced from its extended dynamics employing Theorem 4.2. To ascertain τ^*, the following procedure provides a systematic approach.

Procedure 4

Step 1: Calculate τ_1^* from (4.88).

Step 2: Obtain the positive solution ω_i, by solving (4.91),

Step 3: For each ω_i, calculate C_1 and C_2 and then find τ_i from (4.93).

Step 4: Determine τ_2^* as $\tau_2^* = \min_i \tau_i$.

Step 5: Calculate τ_3^* from (4.97).

Step 6: Find an upper bound $\tau^* = \min\{\tau_1^*, \tau_2^*, \tau_3^*\}$.

Figure 4.5. Algorithm to calculate an upper bound of communication delay τ^*.

Up to here, the delayed dynamics of any given system (input or communication) have been examined within the realm of continuous-time. However, in the upcoming section, we will delve into the cluster consensus problem within the context of discrete-time networks.

4.3. Extension to Delayed Discrete-Time Networks

Consider a signed network where the m-th order discrete-time agent dynamics are given as

$$\begin{aligned} x_i^{(1)}(k+1) &= x_i^{(1)}(k) + x_i^{(2)}(k), \\ &\vdots \\ x_i^{(m-1)}(k+1) &= x_i^{(m-1)}(k) + x_i^{(m)}(k), \\ x_i^{(m)}(k+1) &= x_i^{(m)}(k) + u_i(k), \quad i = 1, 2, \ldots, n \end{aligned} \tag{4.98}$$

with $x_i^{(q)}(0) = x_{i0}^{(q)}$ where $x_i^{(q)}(k) \in R$ $(q = 1, 2, \ldots, m)$, represents the states of agent i at time k and the control signal $u_i(k)$ is chosen as

$$u_i(k) = \sum_{q=1}^{m} \gamma_q \sum_{j \in \mathcal{N}_i} a_{ij} \left(x_j^{(q)}(k - \tau_{ij}) - \text{sgn}(a_{ij}) x_i^{(q)}(k) \right), \quad i = 1, 2, \ldots, n \tag{4.99}$$

where a_{ij} denote the components of the adjacency matrix and $\gamma_q > 0$ are the controller parameters. Note that as a_{ij} can be negative, (4.98)-(4.99) model a signed network. On the other hand, there is no assumption that the network contains a spanning tree, structurally balanced.

For analysis purposes, we define the augmented state vector $\mathbf{y}(k) = \big[\mathbf{x}(k)^\mathsf{T}, \mathbf{x}(k+1)^\mathsf{T}, \ldots, \mathbf{x}(k+\tau_{max})^\mathsf{T}\big]^\mathsf{T}$ where $\mathbf{x}(k) = \big[\mathbf{x}^{(1)}(k)^\mathsf{T}, \mathbf{x}^{(2)}(k)^\mathsf{T}, \ldots, \mathbf{x}^{(m)}(k)^\mathsf{T}\big]^\mathsf{T}$ with $\mathbf{x}^{(q)}(k) = \left[x_1^{(q)}(k), x_2^{(q)}(k), \ldots, x_n^{(q)}(k)\right]^\mathsf{T}$ for $q = 1, 2, \ldots, m$ and $\mathbf{y}_0$ is the initial condition. Then (4.98)-(4.99) can be written in matrix form as

$$\mathbf{y}(k+1) = W\mathbf{y}(k), \quad \mathbf{y}(0) = \mathbf{y}_0 \tag{4.100}$$

where $W \in \mathbb{R}^{(\tau_{max}+1)mn \times (\tau_{max}+1)mn}$ is given by

$$W = \begin{bmatrix}
0_{n\times n} & \cdots & 0_{n\times n} & I_n & 0_{n\times n} & \cdots & \cdots & 0_{n\times n} \\
\vdots & \cdots & \cdots & 0_{n\times n} & \ddots & 0_{n\times n} & \cdots & \vdots \\
\vdots & \cdots & \cdots & \cdots & 0_{n\times n} & \ddots & \ddots & \vdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & 0_{n\times n} \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0_{n\times n} & I_n \\
\vdots & \cdots & \cdots & 0_{n\times n} & I_n & I_n & 0_{n\times n} & 0_{n\times n} \\
\vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & 0_n \\
0_{n\times n} & \cdots & \cdots & \cdots & \cdots & 0_{n\times n} & I_n & I_n \\
-\gamma_1 L_{\tau_{max}} & \cdots & -\gamma_m L_{\tau_{max}} & \cdots & -\gamma_1 L_0 & \cdots & -\gamma_{m-1} L_0 & I_n - \gamma_m L_0
\end{bmatrix} \tag{4.101}$$

with

$$[L_r]_{ij} = \begin{cases} -a_{ij}, & i \neq j \text{ and } \tau_{ij} = r, \\ \sum_{k=1,k\neq i}^{n} a_{ik}, & i = j \text{ and } \tau_{ij} = r \\ 0 & \text{otherwise} \end{cases}$$

for $r = 1, 2, \ldots, \tau_{max}$.

Given the delayed system in (4.98)-(4.99), its extended system is defined as

$$\begin{aligned} \bar{x}_i^{(1)}(k+1) &= \bar{x}_i^{(1)}(k) + \bar{x}_i^{(2)}(k), \\ &\vdots \\ \bar{x}_i^{(m-1)}(k+1) &= \bar{x}_i^{(m-1)}(k) + \bar{x}_i^{(m)}(k), \\ \bar{x}_i^{(m)}(k+1) &= \bar{x}_i^{(m)}(k) + \bar{u}_i(k), \quad i = 1, 2, \ldots, 2n \end{aligned} \tag{4.102}$$

where the weighting coefficients $\bar{w}_{ij}$ are non-negative and the control input is given as

$$\bar{u}_i(k) = \sum_{q=1}^{m} \gamma_q \sum_{j=1}^{2n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(k - \tau_{ij}) - \bar{x}_i^{(q)}(k) \right), \quad i = 1, 2, \ldots, 2n. \tag{4.103}$$

For analysis purposes, we define the augmented state vector $\bar{\mathbf{y}}(k) = \left[\bar{\mathbf{x}}(k)^\mathsf{T}, \bar{\mathbf{x}}(k+1)^\mathsf{T}, \ldots, \bar{\mathbf{x}}(k+\tau_{max})^\mathsf{T}\right]^\mathsf{T}$ where $\bar{\mathbf{x}}(k) = \left[\bar{\mathbf{x}}^{(1)}(k)^\mathsf{T}, \bar{\mathbf{x}}^{(2)}(k)^\mathsf{T}, \ldots, \bar{\mathbf{x}}^{(m)}(k)^\mathsf{T}\right]^\mathsf{T}$ with $\bar{\mathbf{x}}^{(q)}(k) = \left[\bar{x}_1^{(q)}(k), \bar{x}_2^{(q)}(k), \ldots, \bar{x}_{2n}^{(q)}(k)\right]^\mathsf{T}$ for $q = 1, 2, \ldots, m$ and $\bar{\mathbf{y}}_0$ is the initial condition. Then (4.102)-(4.103) can be written in matrix form as

$$\bar{\mathbf{y}}(k+1) = \bar{W}\bar{\mathbf{y}}(k), \quad \bar{\mathbf{y}}(0) = \bar{\mathbf{y}}_0 \tag{4.104}$$

where $\bar{W} \in \mathbb{R}^{2(\tau_{max}+1)mn \times (2\tau_{max}+1)mn}$ is given by

$$\bar{W} = \begin{bmatrix} 0_{2n\times 2n} & \cdots & 0_{2n\times 2n} & I_{2n} & 0_{2n\times 2n} & \cdots & \cdots & 0_{2n\times 2n} \\ \vdots & \cdots & \cdots & 0_{2n\times 2n} & \ddots & 0_{2n\times 2n} & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 0_{2n} & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & 0_{2n\times 2n} \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0_{2n\times 2n} & I_{2n} \\ \vdots & \cdots & \cdots & 0_{2n\times 2n} & I_{2n} & I_{2n} & 0_{2n\times 2n} & 0_{2n\times 2n} \\ \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & 0_{2n\times 2n} \\ 0_{2n\times 2n} & \cdots & \cdots & \cdots & \cdots & 0_{2n\times 2n} & I_{2n} & I_{2n} \\ -\gamma_1\bar{L}_{\tau_{max}} & \cdots & -\gamma_m\bar{L}_{\tau_{max}} & \cdots & -\gamma_1\bar{L}_0 & \cdots & -\gamma_{m-1}\bar{L}_0 & I_{2n}-\gamma_m\bar{L}_0 \end{bmatrix} \tag{4.105}$$

with

$$[\bar{L}_r]_{ij} = \begin{cases} -\bar{a}_{ij}, & i \neq j \text{ and } \bar{\tau}_{ij} = r, \\ \sum\limits_{k=1,k\neq i}^{2n} \bar{a}_{ik}, & i = j \text{ and } \bar{\tau}_{ij} = r \\ 0 & \text{otherwise} \end{cases}$$

for $r = 1, 2, \ldots, \tau_{max}$.

The following lemma gives the relation between the trajectories of the original delayed system in (4.98)-(4.99) and the trajectories of the extended delayed system in (4.102)-(4.103).

Lemma 4.5. *The states* $\mathbf{x}(k)$ *of the delayed system* (4.98)-(4.99) *can be obtained from its extended delayed system* (4.102)-(4.103) *as* $x_i^{(q)}(k) = \bar{x}_i^{(q)}(k)$ *for all* $k \geq 0$ *by placing* $\bar{x}_i^{(q)}(l) = -\bar{x}_{i+n}^{(q)}(l)$ *for* $i = 1, 2, \ldots, n$, $q = 1, 2, \ldots, m$ *and* $l = 0, 1, \ldots, \tau_{max}$.

Proof. For the extended system (4.102), the input $\bar{u}_i(k)$ can be re-expressed as

$$\bar{u}_i(k) = \sum_{q=1}^{m} \gamma_q \left[\sum_{j=1}^{n} \bar{a}_{ij} \left(\bar{x}_j^{(q)}(k-\tau_{ij}) - \bar{x}_i^{(q)}(k) \right) + \sum_{j=1}^{n} \bar{a}_{i,j+n} \left(\bar{x}_{j+n}^{(q)}(k-\tau_{ij}) - \bar{x}_i^{(q)}(k) \right) \right],$$

$$\bar{u}_{i+n}(k)=\sum_{q=1}^{m}\gamma_q\left[\sum_{j=1}^{n}\bar{a}_{i+n,j+n}\left(\bar{x}_{j+n}^{(q)}(k-\tau_{ij})-\bar{x}_{i+n}^{(q)}(k)\right)+\sum_{j=1}^{n}\bar{a}_{i+n,j}\left(\bar{x}_{j}^{(q)}(k-\tau_{ij})-\bar{x}_{i+n}^{(q)}(k)\right)\right] \tag{4.106}$$

for $i=1,2,\ldots,n$. Let $\hat{x}_i^{(q)}(k)=\bar{x}_i^{(q)}(k)+\bar{x}_{i+n}^{(q)}(k)$ for $i=1,2,\ldots,n$ and $q=1,2,\ldots,m$. Then the dynamics of $\hat{x}_i^{(q)}(k)$ can be expressed as

$$\begin{aligned}\hat{x}_i^{(q)}(k+1)&=\bar{x}_i^{(q)}(k+1)+\bar{x}_{i+n}^{(q)}(k+1)\\&=\bar{x}_i^{(q)}(k)+\bar{x}_i^{(q+1)}(k)+\bar{x}_{i+n}^{(q)}(k)+\bar{x}_{i+n}^{(q+1)}(k)\\&=\hat{x}_i^{(q)}(k)+\hat{x}_i^{(q+1)}(k)\end{aligned} \tag{4.107}$$

for $q=1,2,\ldots,m-1$ and

$$\begin{aligned}\hat{x}_i^{(m)}(k+1)&=\bar{x}_i^{(m)}(k+1)+\bar{x}_{i+n}^{(m)}(k+1)\\&=\bar{x}_i^{(m)}(k)+\bar{u}_i(k)+\bar{x}_{i+n}^{(m)}(k)+\bar{u}_{i+n}(k)\\&=\hat{x}_i^{(m)}(k)+\sum_{q=1}^{m}\gamma_q\left[\sum_{j=1}^{n}\bar{a}_{ij}\left(\bar{x}_j^{(q)}(k-\tau_{ij})\bar{x}_i^{(q)}(k)\right)\right.\\&\quad+\sum_{j=1}^{n}\bar{a}_{i,j+n}\left(\bar{x}_{j+n}^{(q)}(k-\tau_{ij})-\bar{x}_i^{(q)}(k)\right)+\sum_{j=1}^{n}\bar{a}_{i+n,j+n}\left(\bar{x}_{j+n}^{(q)}(k-\tau_{ij})-\bar{x}_{i+n}^{(q)}(k)\right)\\&\quad\left.+\sum_{j=1}^{n}\bar{a}_{i+n,j}\left(\bar{x}_j^{(q)}(k-\tau_{ij})-\bar{x}_{i+n}^{(q)}(k)\right)\right].\end{aligned}$$

Note from Definition 3.1 that $\bar{a}_{i+n,j}=\bar{a}_{i,j+n}$ and $\bar{a}_{ij}=\bar{a}_{i+n,j+n}$ for $i=1,2,\ldots,n$ which yields $\bar{a}_{ij}+\bar{a}_{i+nj}=\bar{a}_{i,j+n}+\bar{a}_{i+n,j+n}=\bar{a}_{ij}+\bar{a}_{i,j+n}=\bar{a}_{i+n,j}+\bar{a}_{i+n,j+n}=|a_{ij}|$. Then we have

$$\begin{aligned}\hat{x}_i^{(m)}(k+1)&=\hat{x}_i^{(m)}(k)+\sum_{q=1}^{m}\gamma_q\left[\sum_{j=1}^{n}|a_{ij}|\bar{x}_j^{(q)}(k-\tau_{ij})+|a_{ij}|\bar{x}_{j+n}^{(q)}(k-\tau_{ij})\right.\\&\quad\left.-|a_{ij}|\bar{x}_i^{(q)}(k)-|a_{ij}|\bar{x}_{i+n}^{(q)}(k)\right]\\&=\hat{x}_i^{(m)}(k)+\sum_{q=1}^{m}\gamma_q\sum_{j=1}^{n}\hat{a}_{ij}\left(\hat{x}_j^{(q)}(k-\tau_{ij})-\hat{x}_i^{(q)}(k)\right)\end{aligned} \tag{4.108}$$

for $i = 1, 2, \ldots, n$ where $\hat{a}_{ij} = |a_{ij}|$.

For analysis purposes, we define the augmented state vector $\hat{\mathbf{y}}(k) = \big[\hat{\mathbf{x}}(k)^\mathsf{T}, \hat{\mathbf{x}}(k+1)^\mathsf{T}, \ldots, \hat{\mathbf{x}}(k+\tau_{max})^\mathsf{T}\big]^\mathsf{T}$ where $\hat{\mathbf{x}}(k) = \big[\hat{\mathbf{x}}^{(1)}(k)^\mathsf{T}, \hat{\mathbf{x}}^{(2)}(k)^\mathsf{T}, \ldots, \hat{\mathbf{x}}^{(m)}(k)^\mathsf{T}\big]^\mathsf{T}$ with $\hat{\mathbf{x}}^{(q)}(k) = \Big[\hat{x}_1^{(q)}(k), \hat{x}_2^{(q)}(k), \ldots, \hat{x}_n^{(q)}(k)\Big]^\mathsf{T}$ for $q = 1, 2, \ldots, m$ and $\hat{\mathbf{y}}_0$ is the initial condition. Then (4.107)-(4.108) can be written in matrix form as

$$\hat{\mathbf{y}}(k+1) = \hat{W}\hat{\mathbf{y}}(k), \quad \hat{\mathbf{y}}(0) = \hat{\mathbf{y}}_0 \tag{4.109}$$

where $\hat{W} \in \mathbb{R}^{(\tau_{max}+1)mn \times (\tau_{max}+1)mn}$ is given by

$$\hat{W} = \begin{bmatrix} 0_n & \cdots & 0_n & I_n & 0_n & \cdots & \cdots & 0_n \\ \vdots & \cdots & \cdots & 0_n & \ddots & 0_n & \cdots & \vdots \\ \vdots & \cdots & \cdots & \cdots & 0_n & \ddots & \ddots & \vdots \\ \vdots & \cdots & \cdots & \cdots & \cdots & \ddots & \ddots & 0_n \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0_n & I_n \\ \vdots & \cdots & \cdots & 0_n & I_n & I_n & 0_n & 0_n \\ \vdots & \cdots & \cdots & \cdots & \ddots & \ddots & \ddots & 0_n \\ 0_n & \cdots & \cdots & \cdots & \cdots & 0_n & I_n & I_n \\ -\gamma_1\hat{L}_{\tau_{max}} & \cdots & -\gamma_m\hat{L}_{\tau_{max}} & \cdots & -\gamma_1\hat{L}_0 & \cdots & -\gamma_{m-1}\hat{L}_0 & I_n - \gamma_m\hat{L}_0 \end{bmatrix} \tag{4.110}$$

with

$$[\hat{L}_r]_{ij} = \begin{cases} -\hat{a}_{ij}, & i \neq j \text{ and } \tau_{ij} = r, \\ \sum\limits_{k=1, k\neq i}^{n} \hat{a}_{ik}, & i = j \text{ and } \tau_{ij} = r \\ 0 & \text{otherwise} \end{cases}$$

for $r = 1, 2, \ldots, \tau_{max}$. Since the solution of (4.104) is $\hat{\mathbf{y}}(k) = \hat{W}^k\hat{\mathbf{y}}_0$, the choice of $\hat{\mathbf{y}}_0 = [0, \ldots, 0]^\mathsf{T}$ implies $\hat{\mathbf{y}}(k) = \hat{\mathbf{y}}_0 = [0, \ldots, 0]^\mathsf{T}$ for all k, that is, $\bar{x}_i^{(q)}(k) = -\bar{x}_{i+n}^{(q)}(k)$ for $i = 1, 2, \ldots, n$ and $q = 1, 2, \ldots, m$ and all k. We substitute $\bar{x}_{j+n}^{(q)}(k) = -\bar{x}_j^{(q)}(k)$ into

(4.106) to obtain

$$\begin{aligned}\bar{u}_i(k) &= \sum_{q=1}^{m}\gamma_q\left[\sum_{j=1}^{n}\bar{a}_{ij}\left(\bar{x}_j^{(q)}(k-\tau_{ij})-\bar{x}_i^{(q)}(k)\right)+\sum_{j=1}^{n}\bar{a}_{i,j+n}\left(-\bar{x}_j^{(q)}(k-\tau_{ij})-\bar{x}_i^{(q)}(k)\right)\right]\\ &= \sum_{q=1}^{m}\gamma_q\left[\sum_{j=1}^{n}(\bar{a}_{ij}-\bar{a}_{i,j+n})\bar{x}_j^{(q)}(k-\tau_{ij})-(\bar{a}_{ij}+\bar{a}_{i,j+n})\bar{x}_i^{(q)}(k)\right] \qquad (4.111)\end{aligned}$$

for $i = 1, 2, \ldots, n$. From Definition 3.1, we get

$$\begin{aligned}\bar{u}_i(k) &= \sum_{q=1}^{m}\gamma_q\left[\sum_{j=1}^{n}\left(\frac{|a_{ij}|+a_{ij}}{2}-\frac{|a_{ij}|-a_{ij}}{2}\right)\bar{x}_j^{(q)}(k)-\left(\frac{|a_{ij}|+a_{ij}}{2}+\frac{|a_{ij}|-a_{ij}}{2}\right)\bar{x}_i^{(q)}(k)\right]\\ &= \sum_{q=1}^{m}\gamma_q\sum_{j=1}^{n}a_{ij}\left(x_j^{(q)}(k-\tau_{ij})-\mathrm{sgn}(a_{ij})x_i^{(q)}(k)\right) \qquad (4.112)\end{aligned}$$

for $i = 1, 2, \ldots, n$. Note that (4.112) is equal to (4.99). By the choice of $\bar{x}_i^{(q)}(l) = -\bar{x}_{i+n}^{(q)}(l)$, $i = 1, 2, \ldots, n$, $q = 1, 2, \ldots, m$ and $l = 0, 1, \ldots, \tau_{max}$, one can recover the trajectories $\mathbf{x}(k)$ which completes the proof. □

In the calculation of the clusters for a higher-order integrator discrete-time system with nonuniform delay, the following result is obtained.

Theorem 4.6. *Suppose that the extended system* (4.102)-(4.103) *contains* $\bar{l}_p$ *PLSs and* $\bar{l}_s$ *SLSs. Then the delayed system* (4.98)-(4.99) *represented by a signed directed graph forms* $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ *clusters if the multiplicity of the eigenvalue 1 of* $\bar{W}$ *is* $m\bar{l}_p$ *and the remaining eigenvalues are inside the unit circle where* l' *is the number of subgraphs that does not contain any agent of the original system.*

Proof. The characteristic polynomials of W, $\hat{W}$ and $\bar{W}$ can be respectively computed as

$$\Delta(\lambda) = \det\left(\lambda^{\tau_{max}(m-1)}\left[\lambda^{\tau_{max}}(\lambda-1)^m I_n + \sum_{l=1}^{m}\gamma_l(\lambda-1)^{l-1}\mathcal{L}\right]\right),$$
$$\hat{\Delta}(\lambda) = \det\left(\lambda^{\tau_{max}(m-1)}\left[\lambda^{\tau_{max}}(\lambda-1)^m I_n + \sum_{l=1}^{m}\gamma_l(\lambda-1)^{l-1}\hat{\mathcal{L}}\right]\right), \tag{4.113}$$
$$\bar{\Delta}(\lambda) = \det\left(\lambda^{\tau_{max}(m-1)}\left[\lambda^{\tau_{max}}(\lambda-1)^m I_{2n} + \sum_{l=1}^{m}\gamma_l(\lambda-1)^{l-1}\bar{\mathcal{L}}\right]\right)$$

where $\mathcal{L} = \sum_{r=0}^{\tau_{max}} L_r\lambda^{\tau_{max}-r}$, $\hat{\mathcal{L}} = \sum_{r=0}^{\tau_{max}} \hat{L}_r\lambda^{\tau_{max}-r}$ and $\bar{\mathcal{L}} = \sum_{r=0}^{\tau_{max}} \bar{L}_r\lambda^{\tau_{max}-r}$.

From Chapter 3, $\rho(\bar{L}_r) = \rho(L_r) \cup \rho(\hat{L}_r)$ $r = 1, 2, \ldots, \tau_{max}$. Hence, $\forall\lambda$ that is a root of $\Delta(\cdot)$ or $\hat{\Delta}(\cdot)$, we conclude that λ is a root of $\bar{\Delta}(\cdot)$. The characteristic polynomial $\bar{\Delta}(\lambda)$ in (4.113) has at least $m\bar{l}_p$ roots at $\lambda = 1$. For a choice of γ_l $(l = 1, 2, \ldots, m)$ which ensures the remaining eigenvalues of $\bar{W}$ to be less than 1 in magnitude, the extended system in (4.102)-(4.103) is stable. The stability of (4.102)-(4.103) implies the stability of (4.98)-(4.99) and consequently the agents in the extended delayed system form $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ clusters by using Definition 3.1. Since l' clusters of the extended delayed system do not consist of agents in the original delayed system, we conclude that the number of clusters for original delayed system is calculated as $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$. □

Remark 4.4. Theorem 4.6 demonstrates that bounded nonuniform delay has no impact on the number of clusters or the members within each cluster so long as the extended delayed system is stable. Consequently, the system (4.98)-(4.99) achieves $\bar{l}_p + \bar{l}_s - l'$ clusters without any changes to their composition.

4.4. Numerical Analysis

Consider the network in Figure 4.6 where negative and positive communication flows are represented by red and black arrows, respectively. Note that the network has no structural restrictive elements (structurally balanced, strongly connected, spanning tree etc.).

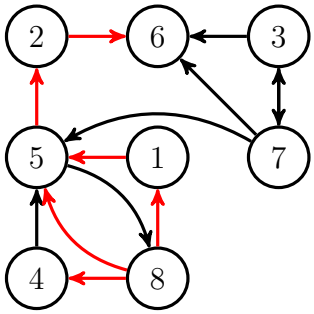


Figure 4.6. A network consisting of 8 agents and 13 edges

In order to simulate the network in second-order delayed dynamics, the following system matrix is used:

$$L = \begin{bmatrix} 4 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 2 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & -3 & 7 & 0 & -2 & 1 \\ 0 & 5 & -1 & 0 & 0 & 12 & -6 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -6 & 0 & 0 & 6 \end{bmatrix}. \tag{4.114}$$

By exploiting Definition 3.1, the corresponding extended directed graph can be obtained as shown in Figure 4.7.

At this stage, PLSs and SLSs play a key role to determine clusters and their members for the constructed extended directed graph. The concepts of PLS and SLS are exploited in the calculating the number of clusters and the determination of the members of each cluster. The procedure of detection for PLSs and SLSs can be summarized as follows:

- The determination of PLS agent sets relies on the detection algorithm detailed in [13]. As per the PLS definition, each PLS is required to have a spanning tree,

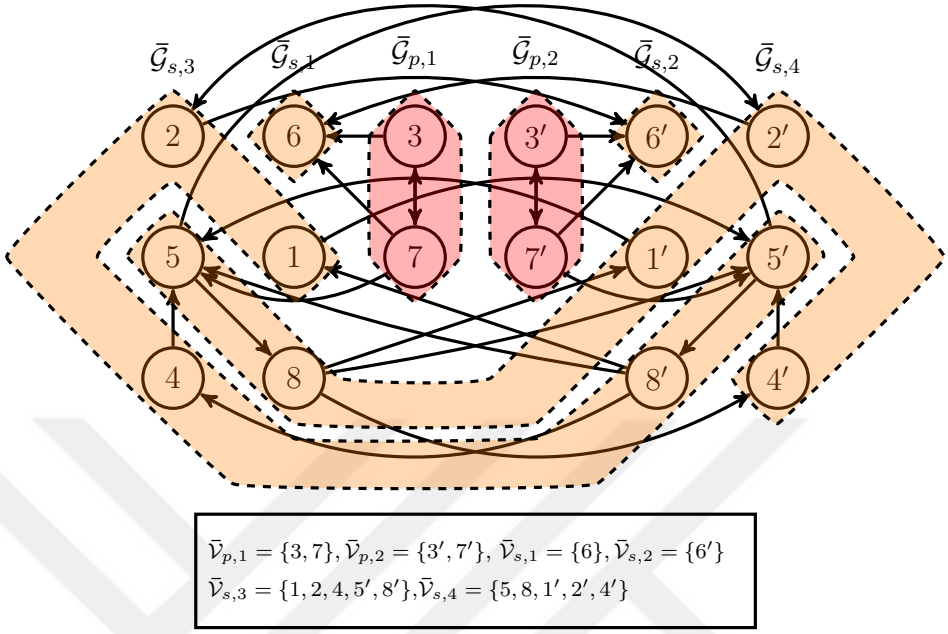


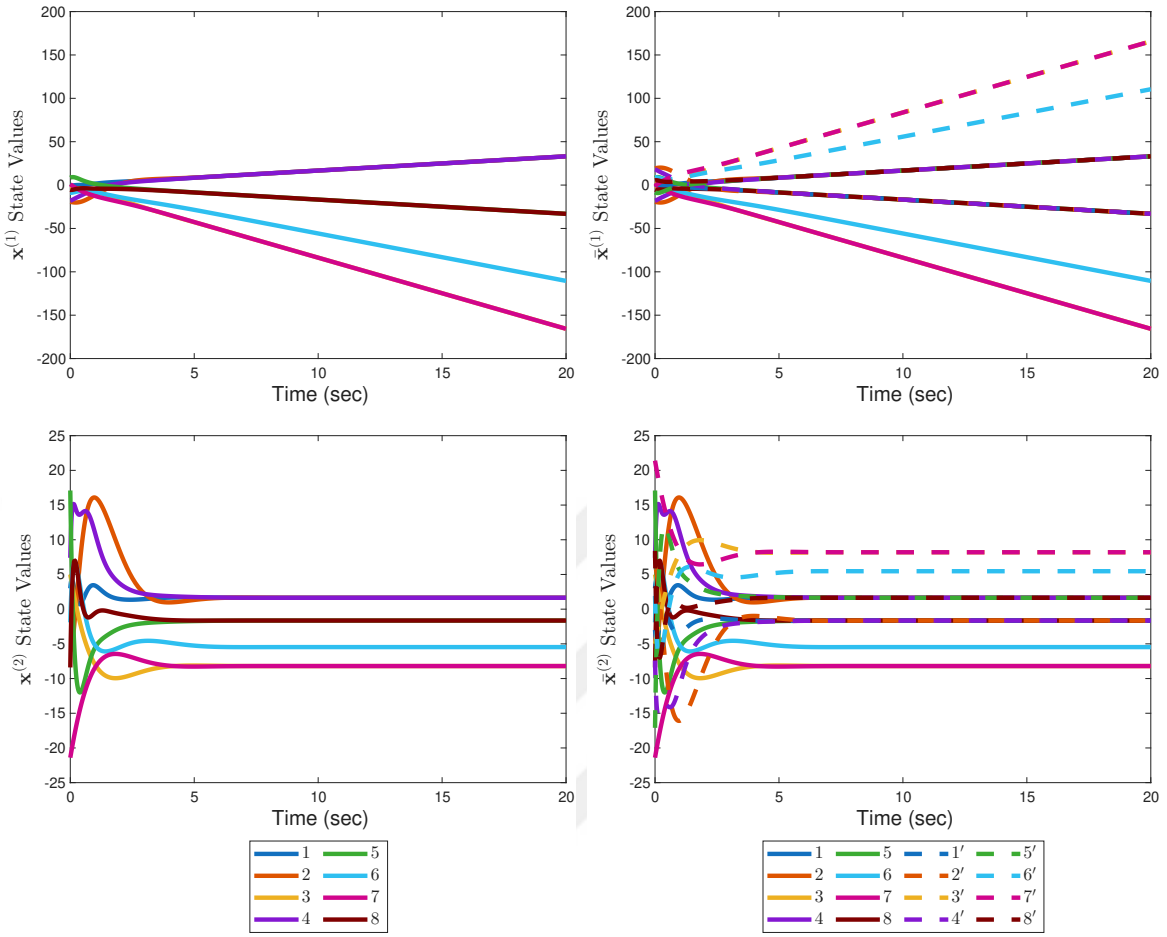
Figure 4.7. Extended directed graph for the network in Figure 4.6

ensuring that no external information reaches the cluster's members from outside.

- Following the removal of PLS agents from $\mathcal{V}$, the SLS agent sets are derived from the remaining agents using the detection algorithm described in [13]. According to the SLS definition, each SLS must have a spanning tree, and its root must receive information from at least two distinct clusters. Other agents within the subgraph do not receive external links from outside.

To this end, the clusters and their members in the extended system are found as $\bar{\mathcal{V}}_{p,1} = \{3, 7\}$, $\bar{\mathcal{V}}_{p,2} = \{3', 7'\}$, $\bar{\mathcal{V}}_{s,1} = \{6\}$, $\bar{\mathcal{V}}_{s,2} = \{6'\}$, $\bar{\mathcal{V}}_{s,3} = \{1, 2, 4, 5', 8'\}$ and $\bar{\mathcal{V}}_{s,4} = \{5, 8, 1', 2', 4'\}$ as shown in Figure 4.7.

In the absence of any delay, the choice of the control parameters as $\gamma_1 = \gamma_2 = 1$ gives a stable extended system. The simulation results are depicted in Figures 4.8(a)-4.8(b) for the signed and its corresponding extended directed graphs for $\tau = 0$. The signed network converges to 4 distinct clusters as shown in Figure 4.8(a). These clusters can be determined from its corresponding extended representation. The numbers of PLSs and SLSs of the extended system are $\bar{l}_p = 2$ and $\bar{l}_s = 4$, respectively (Figure



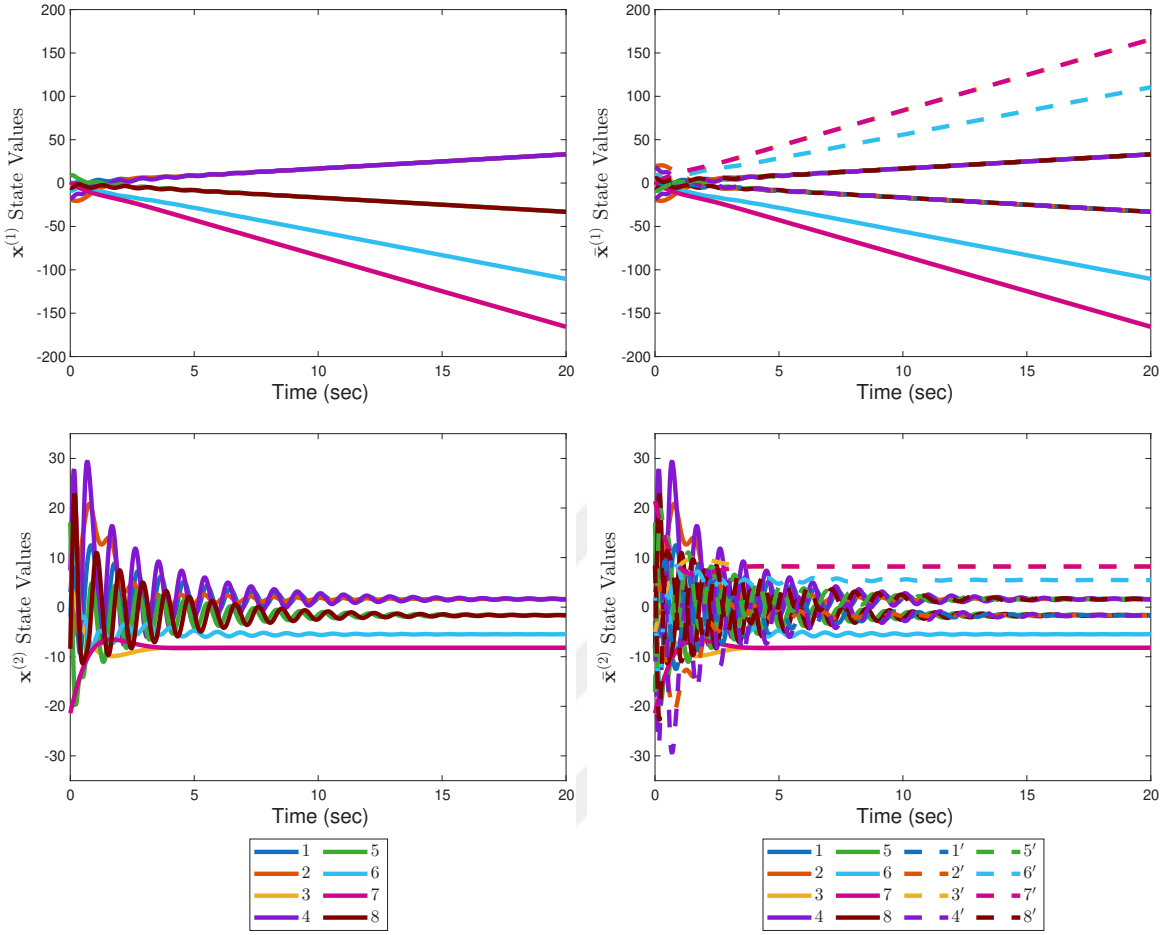
(a) Signed directed graph (b) Extended directed graph

Figure 4.8. State trajectories of agents associated with signed and extended directed graphs in Figures 4.6 and 4.7 without any delay.

4.8(b)). Furthermore, there are $l' = 2$ subgraphs ($\bar{\mathcal{V}}_{p,2} = \{3', 7'\}$, $\bar{\mathcal{V}}_{s,2} = \{6'\}$) that do not contain any agent of the original system. From the clusters in the extended system, the original clusters ($\bar{l}_p + \bar{l}_s - l' = 4$) can be obtained as $\{3, 7\}, \{1, 2, 4\}, \{5, 8\}$ and $\{6\}$.

Example 4.2. ***Continuous-Time Networks with Input Delay***

In presence of input delay for the network in Figure 4.6, by applying the Procedure 3 on Theorem 4.4, we can obtain an upper bound for τ. *The upper bound* τ^* *is calculated as* 0.0976 *for the eigenvalue* $4.6723 - j4.5784$ *of* $\bar{L}$. *By Theorem 4.2, the extended system is stable if and only if* $\tau < \tau^* = 0.0976$ *and converges to 6 distinct clusters* ($\bar{l}_p + \bar{l}_s = 2 + 4 = 6$). *In Figures 4.9(a) and 4.9(b), the trajectories of states are depicted for signed and extended delayed systems with* $\tau = 0.09$, *respectively. Since*



(a) Signed directed graph (b) Extended directed graph

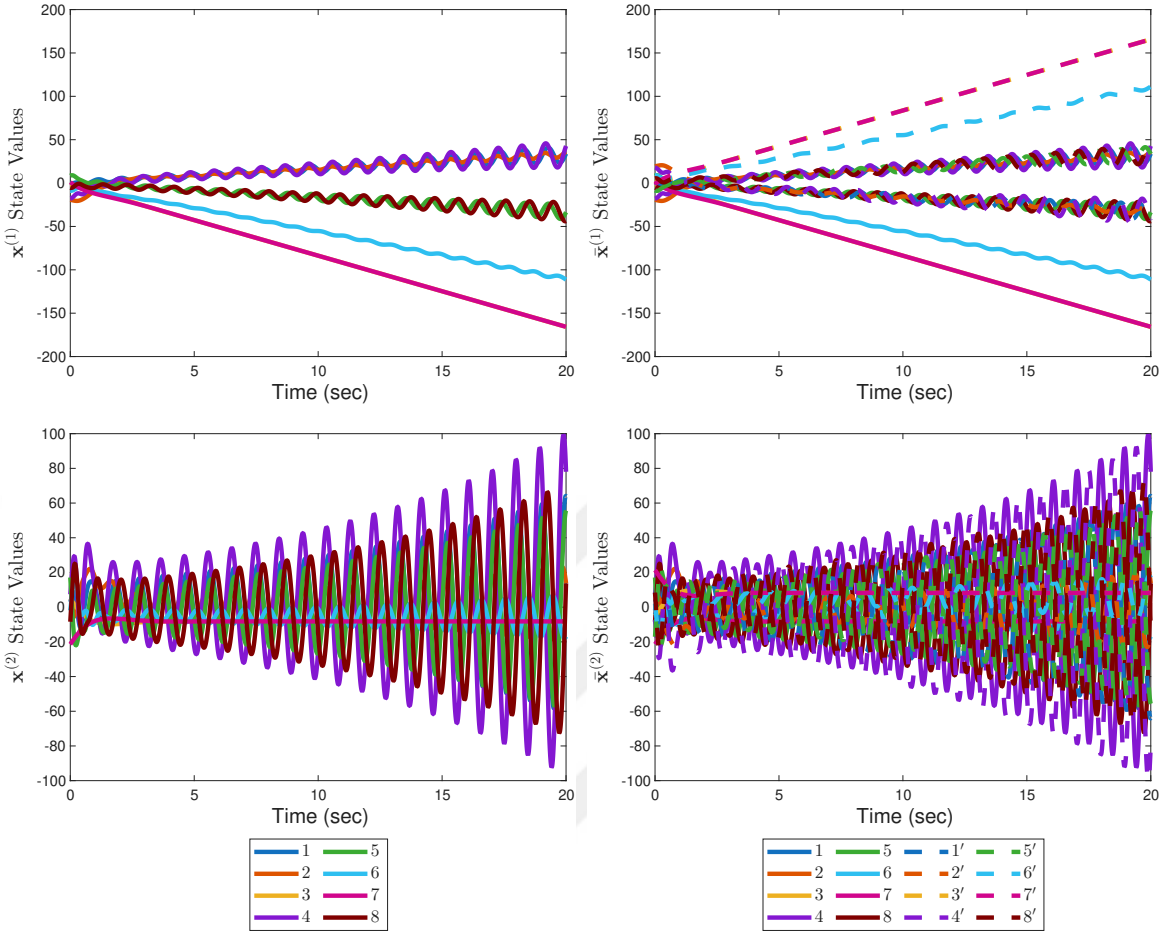
Figure 4.9. State trajectories of agents associated with signed and extended directed graphs in Figures 4.6 and 4.7 for input delay of 0.09s.

*the given delay (*0.09 *s) is smaller than the upper bound* τ^**, the network converges to 4 distinct clusters as expected. The original state trajectories in Figure 4.9(a) can be recovered from its extended system trajectories shown in Figure 4.9(b) by using Lemma 4.3 and Theorem 4.4.*

Exceeding the maximum delay bound ($\tau^* = 0.0976$ *s) destabilizes both the signed system and its extended dynamics, resulting in behavior that deviates from the expected cluster consensus as shown in Figure 4.10.*

Example 4.3. ***Continuous-Time Networks with Communication Delay***

In presence of communication delay for the network in Figure 4.6, the system

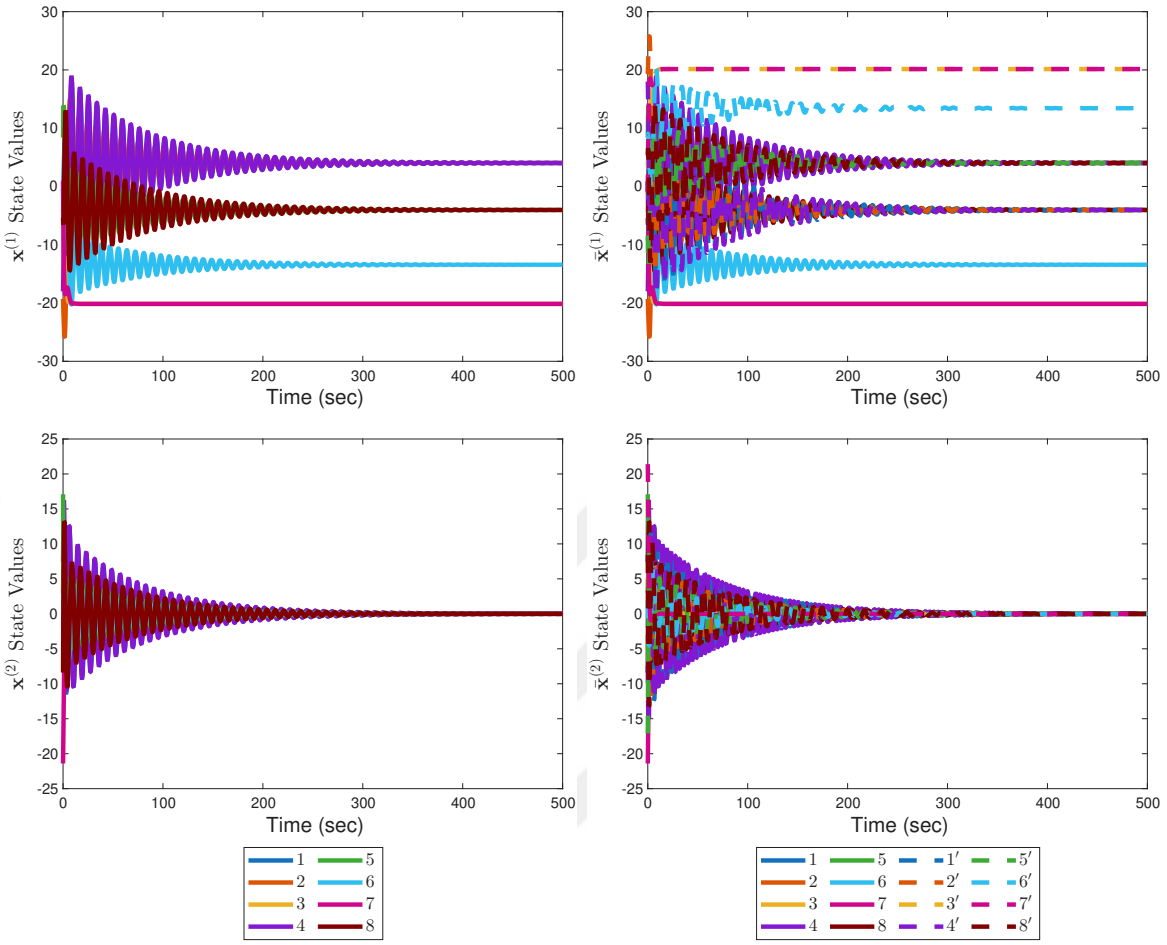


(a) Signed directed graph (b) Extended directed graph

Figure 4.10. State trajectories of agents associated with signed and extended directed graphs in Figures 4.6 and 4.7 for input delay of 0.1s.

matrix satisfying Assumption 4.2 is chosen as follows:

$$L = \begin{bmatrix} 0.8 & 0 & 0 & 0 & 0 & 0 & 0 & 0.8 \\ 0 & 0.6667 & 0 & 0 & 0.6667 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0.875 & 0 & 0 & 0 & 0.875 \\ 0.125 & 0 & 0 & -0.375 & 0.875 & 0 & -0.25 & 0.125 \\ 0 & 0.3846 & -0.0769 & 0 & 0 & 0.9231 & -0.4615 & 0 \\ 0 & 0 & -0.5 & 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & -0.8571 & 0 & 0 & 0.8571 \end{bmatrix}. \tag{4.115}$$



(a) Signed directed graph (b) Extended directed graph

Figure 4.11. State trajectories of agents associated with signed and extended directed graphs in Figures 4.6 and 4.7 for communication delay of 0.8s.

By applying the Procedure 4 on Theorem 4.5, we can determine an upper bound for τ. *The upper bound* $\tau^* = 0.8310$ *is calculated from* $\min\{\tau_1^*, \tau_2^*, \tau_3^*\}$ *where* $\tau_1^* = \infty$, $\tau_2^* = 3.5725$ *and* $\tau_3^* = 0.8310$. *By Theorem 4.3, the extended system is stable if* $\tau < \tau^* = 0.8310$ *and reaches 6 clusters* ($\bar{l}_p + \bar{l}_s = 2 + 4 = 6$). *In Figure 4.11, the trajectories of states are depicted for signed and extended delayed systems with* $\tau = 0.8$, *respectively. Since the given delay (0.8 s) is smaller than the upper bound* τ^*, *the network converges to 4 distinct clusters as expected. Due to Assumption 4.2, the states of* $\mathbf{x}^{(2)}(t)$ *converges to zero value but the states of* $\mathbf{x}^{(1)}(t)$ *converges to 4 distinct clusters. The clusters and their members are the same as in non-delayed case. The original state trajectories in Figure 4.11(a) can be recovered from its extended system trajectories shown in Figure 4.11(b) by using Lemma 4.4 and Theorem 4.5.*

4.5. Chapter Summary & Concluding Remarks

In this chapter, we have discussed the cluster consensus problem for input or communication delayed higher-order multi-agent systems evolving over any given unsigned or signed directed graph. For unsigned networks, we have stated conditions on the control parameters so that stable cluster consensus is achieved. Additionally, given a stable system, the upper limit of input or communication delay is determined explicitly. Then, we have delved into the linear cluster consensus problem concerning delayed higher-order integrator multi-agent networks described by any given signed directed graph. We have investigated the relation between signed network with delay (input or communication) and their extended dynamics, providing the convergence properties in both continuous and discrete-time domains. For a given signed system with input delay, the maximum delay bound that preserves stable dynamics has been calculated and clusters of the system have been identified. For signed systems with communication delay, the stability criteria have been derived and the cluster compositions of the systems have been determined by utilizing extended directed graph representation. For discrete-time integrator systems with nonuniform delay, it has been demonstrated that delay does not affect cluster consensus so long as the extended delayed system is stable. For all cases, graphs representing communication topologies of networks are free from any structural limitations such as the existence of a spanning tree. While the cluster agreement problem has been investigated for linearly communicated systems in this and the previous chapters, effects of nonlinearity on cluster consensus property are discussed in Chapter 5.

5. CLUSTER CONSENSUS FOR NONLINEAR DYNAMICS

In this chapter, we examine the cluster consensus challenge in networks where first and second-order systems interact through both cooperation and competition and exhibit inherent nonlinear dynamics. For a first order multi-agent network evolving over an unsigned directed graph, not necessarily containing a spanning tree, a sufficient condition based on Lyapunov theory is derived on the single controller parameter so that the system achieves cluster consensus. These stability results are extended to second-order nonlinear systems that utilize two parameters in the distributed control law. We expand the stability analysis to include first and second-order nonlinear systems represented by signed directed graphs by employing the extended graph representation.

5.1. Nonlinear Networks with Unsigned Directed Graphs

5.1.1. First-Order Nonlinear Dynamics

In this section, the goal is to derive conditions leading first-order multi-agent systems to reach cluster consensus under inherent nonlinear dynamics. Each agent, i, has the following first-order nonlinear dynamics

$$\dot{x}_i(t) = f(x_i, t) + u_i(t), \quad i = 1, 2, \ldots, n \tag{5.1}$$

where $x_i(t) \in \mathbb{R}$, $f(x_i, t)$ and $u_i(t)$ are the state, the inherent nonlinear dynamics and the control input of agent i, respectively. The control input $u_i(t)$ is described as

$$u_i(t) = \gamma_1 \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(t) - x_i(t)). \tag{5.2}$$

where $\gamma_1 > 0$ is control parameter. The control input can be rewritten by using the unsigned Laplacian matrix in the following form:

$$u_i(t) = -\gamma_1 \sum_{j \in \mathcal{N}_i} l_{ij} x_j(t). \tag{5.3}$$

When (5.1) and (5.3) are combined, the dynamics of agents are represented as

$$\dot{x}_i(t) = f(x_i, t) - \gamma_1 \sum_{j \in \mathcal{N}_i} l_{ij} x_j(t), \quad i = 1, 2, \ldots, n, \tag{5.4}$$

or equivalently in the matrix form as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(x, t) - \gamma_1 L\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{5.5}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^\mathsf{T}$, $\mathbf{f}(\mathbf{x}, t) = [f(x_1, t), f(x_2, t), \ldots, f(x_n, t)]^\mathsf{T}$ and $\mathbf{x}_0$ is the initial condition.

Remark 5.1. Note that since each agent utilizes local and neighboring agents' information in its update law, the algorithm is fully distributed. In this section, we will determine the stability conditions for which the network utilizing distributed algorithm (5.4) achieves cluster consensus.

We can rewrite (5.1) in terms of the PLS and SLS concepts as

$$\begin{bmatrix} \dot{\mathbf{x}}_p(t) \\ \dot{\mathbf{x}}_s(t) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\mathbf{x}_p, t) \\ \mathbf{f}(\mathbf{x}_s, t) \end{bmatrix} - \gamma_1 \begin{bmatrix} L_p & 0_{n_p \times n_s} \\ L_{sp} & L_s \end{bmatrix} \begin{bmatrix} \mathbf{x}_p(t) \\ \mathbf{x}_s(t) \end{bmatrix} \tag{5.6}$$

where $\mathbf{x}_p(t) \in \mathbb{R}^{n_p}$ and $\mathbf{x}_s(t) \in \mathbb{R}^{n_s}$ are the state vectors for PLSs and SLSs, respectively. The following assumption is utilized for analysis of cluster consensus.

Assumption 5.1. ([49]) *Given the function $f(x,t)$ with nonnegative constants $\eta_i \geq 0$ satisfying $\sum_{i=1}^{n_p} \eta_i = 1$, there exist a constant c such that the inequality*

$$\left| f(y,t) - \sum_{i=1}^{n_p} \eta_i f(x_i,t) \right| \leq c \left| y - \sum_{i=1}^{n_p} \eta_i x_i \right|, \quad \forall y, x_i \in \mathbb{R} \tag{5.7}$$

holds.

Remark 5.2. For $n_p = 1$, the system using (5.1) has only one PLS and the condition (5.7) reduces to the Lipschitz condition as follows:

$$|f(y,t) - f(x,t)| \leq c\,|y - x|\,, \quad \forall y, x \in \mathbb{R}.$$

The condition in Assumption 5.1 ensures that followers stay in a region spanned by leaders. [49, 51–56] All linear functions and some nonlinear functions ($x\sin(t)$, $-x\cos(t^2)$ etc.) are suitable to use the condition (5.7). Note that, if the condition (5.7) holds, then the Lipschitz condition is satisfied, but not vice-versa. Assumption 5.1 ensures that the SLSs always converge to the convex hull spanned by PLSs. Therefore, the convergence points of SLSs do not depend on the initial values of SLSs, but on the initial values of PLSs. By using this assumption, the containment control property for system with inherently nonlinear dynamics can be achieved. In Example 4, this phenomenon is simulated in multi-robot systems to illustrate the rationale behind using Assumption 5.1.

Under Assumption 5.1, consider the following transformation:

$$\begin{bmatrix} \mathbf{y}_p(t) \\ \mathbf{y}_s(t) \end{bmatrix} = \begin{bmatrix} S_p & 0_{n_p \times n_s} \\ 0_{n_s \times n_p} & I_{n_s} \end{bmatrix} \begin{bmatrix} \mathbf{x}_p(t) \\ \mathbf{x}_s(t) \end{bmatrix} \tag{5.8}$$

where $S_p = \text{blkdiag}\,\{S_1, S_2, \ldots, S_{l_p}\}$ with $S_i \in \mathbb{R}^{n_{p,i} \times n_{p,i}}$ given by

$$S_i = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & -1 \end{bmatrix}$$

for $i = 1, 2, \ldots, l_p$. Note that $S_i = S_i^{-1}$ which also implies $S_p = S_p^{-1}$. Let $\mathbf{y}_{p,i} = [y_{p,i,1}, \mathbf{e}_{p,i}^{\mathsf{T}}]^{\mathsf{T}}$ be the transformed PLS dynamics where

$$\begin{aligned} y_{p,i,1} &= x_{p,i,1}, \\ \mathbf{e}_{p,i} &= [y_{p,i,2}, y_{p,i,3}, \ldots, y_{p,i,n_{p,i}}]^{\mathsf{T}} \\ &= [x_{p,i,1} - x_{p,i,2}, x_{p,i,1} - x_{p,i,3}, \ldots, x_{p,i,1} - x_{p,i,n_{p,i}}]^{\mathsf{T}} \end{aligned}$$

for $i = 1, 2, \ldots, l_p$. Rewriting (5.6) with respect to $\mathbf{y}_p$ and $\mathbf{y}_s$, we have

$$\begin{bmatrix} \dot{\mathbf{y}}_p(t) \\ \dot{\mathbf{y}}_s(t) \end{bmatrix} = \begin{bmatrix} S_p \mathbf{f}(\mathbf{y}_p, t) \\ \mathbf{f}(\mathbf{y}_s, t) \end{bmatrix} - \gamma_1 \begin{bmatrix} S_p L_p S_p^{-1} & 0_{n_p \times n_s} \\ L_{sp} S_p^{-1} & L_s \end{bmatrix} \begin{bmatrix} \mathbf{y}_p(t) \\ \mathbf{y}_s(t) \end{bmatrix}. \tag{5.9}$$

From the above equation, the PLS and SLS dynamics can be rewritten in the following form:

$$\dot{\mathbf{y}}_p(t) = S_p \mathbf{f}(\mathbf{y}_p, t) - \gamma_1 S_p L_p S_p^{-1} \mathbf{y}_p(t), \tag{5.10a}$$

$$\dot{\mathbf{y}}_s(t) = \mathbf{f}(\mathbf{y}_s, t) - \gamma_1 L_s \mathbf{y}_s(t) - \gamma_1 L_{sp} S_p^{-1} \mathbf{y}_p(t). \tag{5.10b}$$

Since each PLS dynamics is independent of the other PLS dynamics, the overall system dynamics can be written as

$$\begin{aligned}
\dot{\mathbf{y}}_{p,1}(t) &= S_1\mathbf{f}(\mathbf{y}_{p,1},t) - \gamma_1 S_1 L_{1,1} S_1^{-1}\mathbf{y}_{p,1}(t),\\
&\vdots\\
\dot{\mathbf{y}}_{p,l_p}(t) &= S_{l_p}\mathbf{f}(\mathbf{y}_{p,l_p},t) - \gamma_1 S_{l_p} L_{l_p,l_p} S_{l_p}^{-1}\mathbf{y}_{p,l_p}(t),\\
\dot{\mathbf{y}}_s(t) &= \mathbf{f}(\mathbf{y}_s,t) - \gamma_1 L_s\mathbf{y}_s(t) - \gamma_1 L_{sp} S_p^{-1}\mathbf{y}_p(t).
\end{aligned} \tag{5.11}$$

Define $E_i = \begin{bmatrix}\mathbf{1}_{n_{p,i}-1} & -I_{n_{p,i}-1}\end{bmatrix}$ and $F_i = \begin{bmatrix}\mathbf{0}_{n_{p,i}-1} & -I_{n_{p,i}-1}\end{bmatrix}^\mathsf{T}$ for $i = 1, 2, \ldots, l_p$. In this context, each PLS dynamics can be written as the following two subsystems

$$\dot{y}_{p,i,1}(t) = f(y_{p,i,1},t) - \gamma_1 l_{i,1} F_i \mathbf{e}_{p,i}(t), \tag{5.12a}$$

$$\mathbf{e}_{p,i}(t) = \mathbf{f}_e(y_{p,i},t) - \gamma_1 E_i L_{i,i} F_i \mathbf{e}_{p,i}(t) \tag{5.12b}$$

where $l_{i,1}$ is the first row of $L_{i,i}$ and $\mathbf{f}_e(y_{p,i},t) = [f(y_{p,i,1}) - f(y_{p,i,1} - y_{p,i,2}), \ldots, f(y_{p,i,1}) - f(y_{p,i,1} - y_{p,i,n_{p,i}})]^\mathsf{T}$.

The error dynamics of all PLSs can be written in the following form:

$$\mathbf{e}_p(t) = \bar{\mathbf{f}}_e(\mathbf{y}_p,t) - \gamma_1 E L_p F \mathbf{e}_p(t)$$

where $\mathbf{e}_p(t) = \left[\mathbf{e}_{p,1}(t)^\mathsf{T}, \mathbf{e}_{p,2}(t)^\mathsf{T}, \ldots, \mathbf{e}_{p,l_p}(t)^\mathsf{T}\right]^\mathsf{T}$, $\bar{\mathbf{f}}_e(\mathbf{y}_p,t) = \left[\mathbf{f}_e(\mathbf{y}_{p,1},t)^\mathsf{T}, \ldots, \mathbf{f}_e(\mathbf{y}_{p,l_p},t)^\mathsf{T}\right]^\mathsf{T}$, $E = \text{blkdiag}\{E_1, E_2, \ldots, E_{l_p}\}$ and $F = \text{blkdiag}\{F_1, F_2, \ldots, F_{l_p}\}$. Before presenting the main result, we need the following lemmas:

Lemma 5.1. ([107]) *For any two real vectors a and b with the same dimension, the inequality*

$$2a^\mathsf{T} b \leq a^\mathsf{T}\Phi a + b^\mathsf{T}\Phi^{-1} b$$

holds where Φ is any positive definite matrix with an appropriate dimension.

Lemma 5.2. ([108]) *For a real, symmetric matrix P, the inequality*

$$\lambda_m(P)x^\mathsf{T}x \leq x^\mathsf{T}Px \leq \lambda_M(P)x^\mathsf{T}x$$

holds where $\lambda_m(P)$ and $\lambda_M(P)$ denote the minimum and maximum eigenvalues of P, respectively.

The following theorem states the conditions on the control parameter γ_1 that guarantee cluster consensus of a first-order multi-agent system with inherent nonlinear dynamics.

Theorem 5.1. *Suppose that system* (5.1) *evolves over an unsigned directed graph with l_p PLSs and l_s SLSs and there exists a positive coefficient c that satisfies Assumption 5.1. The system in* (5.1) *converges to $\mathcal{K} = l_p + l_s$ clusters if there exists a scalar control parameter γ_1 that satisfies*

$$-\gamma_1\left[(EL_pF)^\mathsf{T}P_p + P_p(EL_pF)\right] + c^2\lambda_M(P_p)I_{n_p-l_p} + P_p \prec 0, \tag{5.13a}$$

$$-\gamma_1\left[L_s^\mathsf{T}P_s + P_sL_s\right] + c^2\lambda_M(P_s)I_{n_s} + P_s \prec 0 \tag{5.13b}$$

for some symmetric positive definite matrices $P_p \in \mathbb{R}^{(n_p-l_p)\times(n_p-l_p)}$ and $P_s \in \mathbb{R}^{n_s\times n_s}$ that yield $(EL_pF)^\mathsf{T}P_p + P_p(EL_pF)$ and $L_s^\mathsf{T}P_s + P_sL_s$ positive definite, respectively.

Proof. We investigate the PLS and SLS dynamics consecutively.

Step 1 (Stability and the Number of Clusters for the PLS Dynamics): Due to the structure of L_p, each PLS dynamics can be individually investigated as

$$\begin{aligned}\dot{y}_{p,i,1}(t) &= f(y_{p,i,1},t) - \gamma_1 l_{i,1}F_i\mathbf{e}_{p,i}(t),\\ \mathbf{e}_{p,i}(t) &= \mathbf{f}_e(\mathbf{y}_{p,i},t) - \gamma_1 E_iL_{i,i}F_i\mathbf{e}_{p,i}(t).\end{aligned}$$

To prove the stability of all PLSs at once, the following candidate Lyapunov function is considered:

$$V_1(t) = \mathbf{e}_p(t)^\mathsf{T} P_p \mathbf{e}_p(t)$$

where $P_p = \text{blkdiag}\{P_1, P_2, \ldots, P_{l_p}\}$ with each P_i being also a symmetric positive definite matrix that satisfy $(E_i L_{i,i} F_i)^\mathsf{T} P_i + P_i(E_i L_{i,i} F_i) \prec 0$. Under Assumption 5.1, differentiating $V_1(t)$ yields

$$\dot{V}_1(t) = -\gamma_1 \mathbf{e}_p(t)^\mathsf{T} \left[(EL_pF)^\mathsf{T} P_p + P_p(EL_pF)\right] \mathbf{e}_p(t) + 2\mathbf{e}_p(t)^\mathsf{T} P_p \bar{\mathbf{f}}_e(\mathbf{y}_{p,i}, t).$$

Using Lemmas 5.1 and 5.2, one can obtain the following:

$$\dot{V}_1(t) \leq -\gamma_1 \mathbf{e}_p(t)^\mathsf{T} \left[(EL_pF)^\mathsf{T} P_p + P_p(EL_pF)\right] \mathbf{e}_p(t) + \mathbf{e}_p(t)^\mathsf{T} P_p \mathbf{e}_p(t) + \bar{\mathbf{f}}_e(\mathbf{y}_p, t)^\mathsf{T} P_p \bar{\mathbf{f}}_e(\mathbf{y}_p, t),$$
$$\dot{V}_1(t) \leq -\mathbf{e}_p(t)^\mathsf{T} \left[\gamma_1 \left((EL_pF)^\mathsf{T} P_p + P_p(EL_pF)\right) - c^2 \lambda_{\max}(P_p) I_{n_p - l_p} - P_p\right] \mathbf{e}_p(t)^\mathsf{T}.$$

We have $\dot{V}_1(t) < 0$ for $\mathbf{e}_p(t) \neq 0$, which implies that each PLS dynamics is asymptotically stable. The agents in a particular PLS act together i.e., $\lim_{t\to\infty} \|x_{p,i,j_1} - x_{p,i,j_2}\| = 0$ for $i = 1, 2, \ldots, l_p$ and $j_1, j_2 = 1, 2, \ldots, n_{p,i}$. This implies that one can find a total of l_p clusters in l_p PLSs.

Step 2 (Stability and the Number of Clusters for the SLS Dynamics): In this step, we will prove that the following holds

$$\lim_{t\to\infty} \left\|\mathbf{y}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \mathbf{y}_p(t)\right\| = 0. \tag{5.14}$$

Let $\mathbf{e}_s(t) = \mathbf{y}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \mathbf{y}_p(t)$. Taking derivatives gives

$$\begin{aligned}\dot{\mathbf{e}}_s(t) &= \dot{\mathbf{y}}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \dot{\mathbf{y}}_p(t) \\ &= \mathbf{f}(\mathbf{y}_s, t) - \gamma_1 L_s \mathbf{y}_s(t) - \gamma_1 L_{sp} S_p^{-1} \mathbf{y}_p(t) + L_s^{-1} L_{sp} \mathbf{f}(\mathbf{y}_p, t) - \gamma_1 L_s^{-1} L_{sp} L_p S_p^{-1} \mathbf{y}_p(t)\end{aligned}$$

$$= \mathbf{f}(\mathbf{y}_s,t)+L_s^{-1}L_{sp}\mathbf{f}(\mathbf{y}_p,t)-\gamma_1 L_s\left(\mathbf{y}_s(t)+L_s^{-1}L_{sp}S_p^{-1}\mathbf{y}_p(t)\right)-\gamma_1 L_s^{-1}L_{sp}L_pS_p^{-1}\mathbf{y}_p(t)$$
$$= \mathbf{f}(\mathbf{y}_s,t) + L_s^{-1}L_{sp}\mathbf{f}(\mathbf{y}_p,t) - \gamma_1 L_s\mathbf{e}_s(t) - \gamma_1 L_s^{-1}L_{sp}L_pS_p^{-1}\mathbf{y}_p(t).$$

Let $\omega(t) = L_pS_p^{-1}\mathbf{y}_p(t) = L_p\mathbf{x}_p(t)$. The above dynamics become as follows:

$$\dot{\mathbf{e}}_s(t) = \mathbf{f}(\mathbf{y}_s,t) + L_s^{-1}L_{sp}\mathbf{f}(\mathbf{y}_p,t) - \gamma_1 L_s\mathbf{e}_s(t) - \gamma_1 L_s^{-1}L_{sp}\omega(t).$$

Note that $\omega(t)$ converges to zero as $t \to \infty$ from Step 1 since $L_p\mathbf{x}_p(t) \to 0$. In order to analyze the stability of the above system dynamics, it is required to investigate the stability of the unforced system i.e., $\omega(t) = 0$ for all t. If the origin of unforced system is asymptotically stable, then the states of forced system converge to zero with $\omega(t) = 0$ as $t \to \infty$ (p. 107 in [109]). For this reason, choose the Lyapunov function candidate

$$V_2(t) = \mathbf{e}_s(t)^\mathsf{T}P_s\mathbf{e}_s(t)$$

for the unforced SLS dynamics i.e., $\dot{\mathbf{e}}_s(t) = \mathbf{f}(\mathbf{y}_s,t)+L_s^{-1}L_{sp}\mathbf{f}(\mathbf{y}_p,t)-\gamma_1 L_s\mathbf{e}_s(t)$. Taking derivative of $V_2(t)$ yields the following:

$$\dot{V}_2(t) = -\gamma_1\mathbf{e}_s(t)^\mathsf{T}\left[L_s^\mathsf{T}P_s + P_sL_s\right]\mathbf{e}_s(t) + 2\mathbf{e}_s(t)^\mathsf{T}P_s\left[\mathbf{f}(\mathbf{y}_s,t) + L_s^{-1}L_{sp}\mathbf{f}(\mathbf{y}_p,t)\right].$$

Using Lemmas 5.1 and 5.2, one can obtain the following:

$$\begin{aligned}\dot{V}_2(t) &\le -\gamma_1\mathbf{e}_s(t)^\mathsf{T}\left[L_s^\mathsf{T}P_s + P_sL_s\right]\mathbf{e}_s(t) + \mathbf{e}_s(t)^\mathsf{T}P_s\mathbf{e}_s(t)\\ &\quad + \left[\mathbf{f}(\mathbf{y}_s,t) + L_s^{-1}L_{sp}\mathbf{f}(\mathbf{y}_p,t)\right]^\mathsf{T}P_s\left[\mathbf{f}(\mathbf{y}_s,t) + L_s^{-1}L_{sp}\mathbf{f}(\mathbf{y}_p,t)\right],\\ \dot{V}_2(t) &\le -\mathbf{e}_s(t)^\mathsf{T}\left[\gamma_1(L_s^\mathsf{T}P_s + P_sL_s) - c^2\lambda_M(P_s)I_{n_s} - P_s\right]\mathbf{e}_s(t).\end{aligned}$$

By (5.13b), we have $\dot{V}_2(t) < 0$ for $\mathbf{e}_s(t) \neq 0$, which implies that the origin of $\mathbf{e}_s(t)$ is asymptotically stable. This implies that the following expression holds

$$\mathbf{y}_s(\infty) = -L_s^{-1}L_{sp}S_p^{-1}\mathbf{y}_p(\infty).$$

Note that original states $\mathbf{x}_p(t)$ and $\mathbf{x}_s(t)$ can be obtained from $\mathbf{x}_p(t) = S_p^{-1}\mathbf{y}_p(t)$ and $\mathbf{x}_s(t) = \mathbf{y}_s(t)$. Therefore, as $t \to \infty$, we have

$$\mathbf{x}_s(\infty) = -L_s^{-1}L_{sp}\mathbf{x}_p(\infty).$$

Let $\mathbf{x}_p^*(t) = \lim_{t\to\infty} \mathbf{x}_p(t)$ and $\mathbf{x}_s^*(t) = \lim_{t\to\infty} \mathbf{x}_s(t)$. As $t \to \infty$, $\mathbf{x}_p(t)$ converges to trajectory $\mathbf{x}_p^*(t)$ containing l_p distinct routes from Step 1. From PLS and SLS dynamics in [98], it is known that l_p distinct points at any t gives l_s clusters in $\mathbf{x}_s^*(t)$ which is a convergent trajectory for $\mathbf{x}_s(t)$.

Step 3 (Total Number of Clusters): We have l_p distinct PLSs from Step 1 and l_s distinct SLSs from Step 2. Hence, one can conclude that the total number of cluster is computed as

$$\mathcal{K} = l_p + l_s.$$

□

Remark 5.3. By using Theorem 5.1, the selection method of control parameter γ_1 can be established as follows:

- Find a symmetric positive definite matrix P_p such that $(EL_pF)^\mathsf{T}P_p + P_p(EL_pF)$ is positive definite (e.g. solve for P_p from $(EL_pF)^\mathsf{T}P_p + P_p(EL_pF) = I_{n_p-l_p}$).
- Calculate the maximum eigenvalue of P_p.
- Find a symmetric positive definite matrix P_s such that $L_s^\mathsf{T}P_s + P_sL_s$ is positive definite (e.g. solve for P_s from $L_s^\mathsf{T}P_s + P_sL_s = I_{n_s}$).
- Calculate the maximum eigenvalue of P_s.
- Choose a control parameter γ_1 that satisfies (5.13a) and (5.13b) by using data from the previous steps. (Note (5.13a) and (5.13b) can be simultaneously satisfied by choosing a high enough γ_1).

Recall that the protocol (5.4) used as update rule is fully distributed from Remark 5.1.

However, L_p and L_s global matrices should be known only for the stability analysis and the selection of control parameter γ_1.

In conclusion, any inherently nonlinear network satisfying Assumption 5.1 for first-order dynamics can be investigated in context of the cluster consensus problem. The control parameter (γ_1) can be selected by using the condition in Theorem 5.1 and the number of clusters can be calculated by applying PLS and SLS detection algorithms. In the existing literature, the total number of clusters is not explicitly computed for a multi-agent system. In this context, this is the first study that states the stability conditions and number of clusters in a first-order network with inherent nonlinear dynamics under an arbitrary unsigned directed graph.

5.1.2. Second-Order Nonlinear Dynamics

The system is modeled by a second-order dynamics as follows:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= f(x_i, v_i, t) + u_i(t) \end{aligned} \tag{5.15}$$

where $x_i(t) \in \mathbb{R}$ and $v_i(t) \in \mathbb{R}$ are the states of agent i, $f(x_i, v_i, t)$ is a second-order inherent nonlinear function and $u_i(t)$ is the control input described as follows:

$$u_i(t) = -\gamma_1 \sum_{j \in \mathcal{N}_i} l_{ij} x_j(t) - \gamma_2 \sum_{j \in \mathcal{N}_i} l_{ij} v_j(t) \tag{5.16}$$

where γ_1 and γ_2 are control parameters. The system dynamics with control input can be rewritten in matrix form as

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{v}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n \\ \mathbf{f}(\mathbf{x}, \mathbf{v}, t) \end{bmatrix} + \begin{bmatrix} 0_{n \times n} & I_n \\ -\gamma_1 L & -\gamma_2 L \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{bmatrix}, \quad \mathbf{x}(0) = \mathbf{x}_0, \mathbf{v}(0) = \mathbf{v}_0 \tag{5.17}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^{\mathsf{T}}$, $\mathbf{v}(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^{\mathsf{T}}$, $\mathbf{f}(\mathbf{x}, \mathbf{v}, t) = [f(x_1, v_1, t), f(x_2, v_2, t), \ldots, f(x_n, v_n, t)]^{\mathsf{T}}$, $\mathbf{x}_0$ and $\mathbf{v}_0$ are the initial conditions. PLS and SLS dynamics can be written as follows:

$$\begin{bmatrix} \dot{\mathbf{x}}_p(t) \\ \dot{\mathbf{v}}_p(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n_p} \\ \mathbf{f}(\mathbf{x}_p, \mathbf{v}_p, t) \end{bmatrix} + \begin{bmatrix} 0_{n_p \times n_p} & I_{n_p} \\ -\gamma_1 L_p & -\gamma_2 L_p \end{bmatrix} \begin{bmatrix} \mathbf{x}_p(t) \\ \mathbf{v}_p(t) \end{bmatrix},$$
$$\begin{bmatrix} \dot{\mathbf{x}}_s(t) \\ \dot{\mathbf{v}}_s(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n_s} \\ \mathbf{f}(\mathbf{x}_s, \mathbf{v}_s, t) \end{bmatrix} + \begin{bmatrix} 0_{n_s \times n_s} & I_{n_s} \\ -\gamma_1 L_s & -\gamma_2 L_s \end{bmatrix} \begin{bmatrix} \mathbf{x}_s(t) \\ \mathbf{v}_s(t) \end{bmatrix} + \begin{bmatrix} 0_{n_s \times n_p} & 0_{n_s \times n_p} \\ -\gamma_1 L_{sp} & -\gamma_2 L_{sp} \end{bmatrix} \begin{bmatrix} \mathbf{x}_p(t) \\ \mathbf{v}_p(t) \end{bmatrix}. \tag{5.18}$$

Remark 5.4. Note that the second-order consensus protocol (5.17) is a fully distributed algorithm since each agent exploits its own and neighbors' information. In this section, the stability conditions are derived for the network operating distributed algorithm (5.17) to achieve cluster consensus.

The following assumption is utilized for analysis of cluster consensus in second-order systems:

Assumption 5.2. ([52]) *Given the function $f(x, v, t)$ with nonnegative constants $\eta_i \geq 0$ satisfying $\sum_{i=1}^{n_p} \eta_i = 1$, there exist constants c_1, c_2 such that the inequality*

$$\left| f(y, z, t) - \sum_{i=1}^{n_p} \eta_i f(x_i, v_i, t) \right| \leq c_1 \left| y - \sum_{i=1}^{n_p} \eta_i x_i \right| + c_2 \left| z - \sum_{i=1}^{n_p} \eta_i v_i \right|, \quad \forall y, z, x_i, v_i \in \mathbb{R} \tag{5.19}$$

holds.

Remark 5.5. For $n_p = 1$, the system using (5.15) has only one PLS and the condition (5.19) reduces to the Lipschitz condition as follows:

$$|f(y, z, t) - f(x_i, v_i, t)| \leq c_1 |y - x_i| + c_2 |z - v_i|, \quad \forall y, z, x_i, v_i \in \mathbb{R}.$$

Similar assumptions are utilized in [52]. Under Assumption 5.2, consider $y = Sx$ and $z = Sv$ transformations as follows:

$$\begin{bmatrix} \mathbf{y}_p(t) \\ \mathbf{y}_s(t) \end{bmatrix} = \begin{bmatrix} S_p & 0_{n_p \times n_s} \\ 0_{n_s \times n_p} & I_{n_s} \end{bmatrix} \begin{bmatrix} \mathbf{x}_p(t) \\ \mathbf{x}_s(t) \end{bmatrix}, \quad \begin{bmatrix} \mathbf{z}_p(t) \\ \mathbf{z}_s(t) \end{bmatrix} = \begin{bmatrix} S_p & 0_{n_p \times n_s} \\ 0_{n_s \times n_p} & I_{n_s} \end{bmatrix} \begin{bmatrix} \mathbf{v}_p(t) \\ \mathbf{v}_s(t) \end{bmatrix}.$$

Let $\mathbf{y}_{p,i} = \left[y_{p,i,1}, \mathbf{e}_{p_1,i}^T\right]^\mathsf{T}$ and $\mathbf{z}_{p,i} = \left[z_{p,i,1}, \mathbf{e}_{p_2,i}^T\right]^\mathsf{T}$ be the transformed second-order PLS dynamics where

$$\begin{aligned} y_{p,i,1} &= x_{p,i,1}, \\ \mathbf{e}_{p_1,i} &= \left[y_{p,i,2}, y_{p,i,3}, \ldots, y_{p,i,n_{p,i}}\right]^\mathsf{T} \\ &= \left[x_{p,i,1} - x_{p,i,2}, x_{p,i,1} - x_{p,i,3}, \ldots, x_{p,i,1} - x_{p,i,n_{p,i}}\right]^\mathsf{T}, \\ z_{p,i,1} &= v_{p,i,1}, \\ \mathbf{e}_{p_2,i} &= \left[z_{p,i,2}, z_{p,i,3}, \ldots, z_{p,i,n_{p,i}}\right]^\mathsf{T} \\ &= \left[v_{p,i,1} - v_{p,i,2}, v_{p,i,1} - v_{p,i,3}, \ldots, v_{p,i,1} - v_{p,i,n_{p,i}}\right]^\mathsf{T}. \end{aligned}$$

Rewriting (5.18) with respect to $\mathbf{y}_p$, $\mathbf{y}_s$, $\mathbf{z}_p$ and $\mathbf{z}_s$, we have

$$\begin{aligned} \begin{bmatrix} \dot{\mathbf{y}}_p(t) \\ \dot{\mathbf{y}}_s(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{z}_p(t) \\ \mathbf{z}_s(t) \end{bmatrix}, \\ \begin{bmatrix} \dot{\mathbf{z}}_p(t) \\ \dot{\mathbf{z}}_s(t) \end{bmatrix} &= \begin{bmatrix} S_p \mathbf{f}(\mathbf{y}_p, \mathbf{z}_p, t) - S_p L_p S_p^{-1} \mathbf{z}_p(t) \\ \mathbf{f}(\mathbf{y}_s, \mathbf{z}_s, t) - L_s \mathbf{z}_s(t) - L_{sp} S_p^{-1} \mathbf{z}_p(t) \end{bmatrix}. \end{aligned} \tag{5.20}$$

From the above equations, the PLS and SLS dynamics can be rewritten in the following form:

$$
\begin{aligned}
\begin{bmatrix}\dot{\mathbf{y}}_p(t)\\ \dot{\mathbf{z}}_p(t)\end{bmatrix} &= \begin{bmatrix}0_{n_p\times n_p} & I_{n_p}\\ -\gamma_1 S_p L_p S_p^{-1} & -\gamma_2 S_p L_p S_p^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{y}_p(t)\\ \mathbf{z}_p(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n_p}\\ S_p\mathbf{f}(\mathbf{y}_p,\mathbf{z}_p,t)\end{bmatrix},\\
\begin{bmatrix}\dot{\mathbf{y}}_s(t)\\ \dot{\mathbf{z}}_s(t)\end{bmatrix} &= \begin{bmatrix}0_{n_s\times n_s} & I_{n_s}\\ -\gamma_1 L_s & -\gamma_2 L_s\end{bmatrix}\begin{bmatrix}\mathbf{y}_s(t)\\ \mathbf{z}_s(t)\end{bmatrix} + \begin{bmatrix}0_{n_s\times n_p} & 0_{n_s\times n_p}\\ -\gamma_1 L_{sp}S_p^{-1} & -\gamma_2 L_{sp}S_p^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{y}_p(t)\\ \mathbf{z}_p(t)\end{bmatrix}\\
&\quad + \begin{bmatrix}\mathbf{0}_{n_s}\\ \mathbf{f}(\mathbf{y}_s,\mathbf{z}_s,t)\end{bmatrix}.
\end{aligned}
\tag{5.21}
$$

Since each PLS dynamics is independent of the other PLS dynamics, the overall system dynamics can be written as

$$
\begin{aligned}
\begin{bmatrix}\dot{\mathbf{y}}_{p,1}(t)\\ \dot{\mathbf{z}}_{p,1}(t)\end{bmatrix} &= \begin{bmatrix}0_{n_{p,1}\times n_{p,1}} & I_{n_{p,1}}\\ -\gamma_1 S_1 L_{1,1} S_1^{-1} & -\gamma_2 S_1 L_{1,1} S_1^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{y}_{p,1}(t)\\ \mathbf{z}_{p,1}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n_{p,1}}\\ S_1\mathbf{f}(\mathbf{y}_{p,1},\mathbf{z}_{p,1},t)\end{bmatrix},\\
&\vdots\\
\begin{bmatrix}\dot{\mathbf{y}}_{p,l_p}(t)\\ \dot{\mathbf{z}}_{p,l_p}(t)\end{bmatrix} &= \begin{bmatrix}0_{n_{p,l_p}\times n_{p,l_p}} & I_{n_{p,l_p}}\\ -\gamma_1 S_{l_p} L_{l_p,l_p} S_{l_p}^{-1} & -\gamma_2 S_{l_p} L_{l_p,l_p} S_{l_p}^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{y}_{p,l_p}(t)\\ \mathbf{z}_{p,l_p}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n_{p,l_p}}\\ S_{l_p} f(y_{p,l_p},z_{p,l_p},t)\end{bmatrix},\\
\begin{bmatrix}\dot{\mathbf{y}}_s(t)\\ \dot{\mathbf{z}}_s(t)\end{bmatrix} &= \begin{bmatrix}0_{n_s\times n_s} & I_{n_s}\\ -\gamma_1 L_s & -\gamma_2 L_s\end{bmatrix}\begin{bmatrix}\mathbf{y}_s(t)\\ \mathbf{z}_s(t)\end{bmatrix} + \begin{bmatrix}0_{n_s\times n_p} & 0_{n_s\times n_p}\\ -\gamma_1 L_{sp}S_p^{-1} & -\gamma_2 L_{sp}S_p^{-1}\end{bmatrix}\begin{bmatrix}\mathbf{y}_p(t)\\ \mathbf{z}_p(t)\end{bmatrix}\\
&\quad + \begin{bmatrix}\mathbf{0}_{n_s}\\ \mathbf{f}(\mathbf{y}_s,\mathbf{z}_s,t)\end{bmatrix}.
\end{aligned}
$$

Define $E_i = \begin{bmatrix}\mathbf{1}_{n_{p,i}-1} & -I_{n_{p,i}-1}\end{bmatrix}$ and $F_i = \begin{bmatrix}\mathbf{0}_{n_{p,i}-1} & -I_{n_{p,i}-1}\end{bmatrix}^\mathsf{T}$ for $i = 1, 2, \ldots, l_p$. In this context, each PLS dynamics can be written as the following two subsystems

$$
\begin{aligned}
\begin{bmatrix}\dot{y}_{p,i,1}(t)\\ \dot{z}_{p,i,1}(t)\end{bmatrix} &= \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_{p,i,1}(t)\\ z_{p,i,1}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n_{p,i}-1}^\mathsf{T} & \mathbf{0}_{n_{p,i}-1}^\mathsf{T}\\ -\gamma_1 l_{i,1} F_i & -\gamma_1 l_{i,1} F_i\end{bmatrix}\begin{bmatrix}\mathbf{e}_{p_1,i}(t)\\ \mathbf{e}_{p_2,i}(t)\end{bmatrix} + \begin{bmatrix}0\\ f(y_{p,i,1},z_{p,i,1},t)\end{bmatrix},\\
\begin{bmatrix}\dot{\mathbf{e}}_{p_1,i}(t)\\ \dot{\mathbf{e}}_{p_2,i}(t)\end{bmatrix} &= \begin{bmatrix}0_{n_{p,i}-1\times n_{p,i}-1} & I_{n_{p,i}-1}\\ -\gamma_1 E_i L_{i,i} F_i & -\gamma_2 E_i L_{i,i} F_i\end{bmatrix}\begin{bmatrix}\mathbf{e}_{p_1,i}(t)\\ \mathbf{e}_{p_2,i}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n_{p,i}-1}\\ \mathbf{F}_e(\mathbf{y}_{p,i},\mathbf{z}_{p,i},t)\end{bmatrix}.
\end{aligned}
$$

where $\mathbf{F}_e(\mathbf{y}_{p,i},\mathbf{z}_{p,i},t) = [f(y_{p,i,1},z_{p,i,1},t) - f(y_{p,i,1}-y_{p,i,2},z_{p,i,1}-z_{p,i,2},t),\ldots,$

$f(y_{p,i,1}, z_{p,i,1}, t) - f(y_{p,i,1} - y_{p,i,n_{p,i}}, z_{p,i,1} - z_{p,i,n_{p,i}}, t)]^{\mathsf{T}}$.

The error dynamics of all PLSs can be written in the following form:

$$\begin{bmatrix} \dot{\mathbf{e}}_{p_1}(t) \\ \dot{\mathbf{e}}_{p_2}(t) \end{bmatrix} = \begin{bmatrix} 0_{n_p - l_p \times n_p - l_p} & I_{n_p - l_p} \\ -\gamma_1 E L_p F & -\gamma_2 E L_p F \end{bmatrix} \begin{bmatrix} \mathbf{e}_{p_1}(t) \\ \mathbf{e}_{p_2}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{0}_{n_p - l_p} \\ \bar{\mathbf{F}}_e(\mathbf{y}_p, \mathbf{z}_p, t) \end{bmatrix}$$

where $\bar{\mathbf{F}}_e(\mathbf{y}_p, \mathbf{z}_p, t) = \left[\mathbf{F}_e(\mathbf{y}_{p,1}, \mathbf{z}_{p,1}, t)^{\mathsf{T}}, \mathbf{F}_e(\mathbf{y}_{p,2}, \mathbf{z}_{p,2}, t)^{\mathsf{T}}, \ldots, \mathbf{F}_e(\mathbf{y}_{p,l_p}, \mathbf{z}_{p,l_p}, t)^{\mathsf{T}}\right]^{\mathsf{T}}$. Before presenting the main result on second-order networks, we need the following lemma:

Lemma 5.3. ([108]) *Let M be a symmetric matrix of the form*

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^{\mathsf{T}} & M_3 \end{bmatrix}$$

where M_1 is invertible. Then $M \succ 0$ if and only if $M_1 \succ 0$ and $M_3 - M_2^{\mathsf{T}} M_1^{-1} M_2 \succ 0$.

The following theorem states the conditions on control parameters that guarantee cluster consensus of a second-order multi-agent system with inherent nonlinear dynamics.

Theorem 5.2. *Suppose that system* (5.15) *evolves over an unsigned directed graph with l_p PLSs and l_s SLSs and there exists positive coefficients c_1 and c_2 that satisfy Assumption 5.2. The system in* (5.15) *converges to $\mathcal{K} = l_p + l_s$ clusters if there exists a pair of control parameters (γ_1, γ_2) that satisfies $\gamma_1 \geq c_1$ $\gamma_2 \geq c_2$ and*

$$\frac{\gamma_1}{\gamma_2^2} < \min\left\{\frac{1}{2\lambda_M(P_p)}, \frac{1}{2\lambda_M(P_s)}\right\} - 2 \tag{5.22}$$

for the symmetric positive definite matrices $P_p \in \mathbb{R}^{(n_p - l_p)\times(n_p - l_p)}$, $P_s \in \mathbb{R}^{n_s \times n_s}$ that are obtained from

$$(EL_pF)^{\mathsf{T}} P_p + P_p(EL_pF) = I_{n_p - l_p}, \tag{5.23a}$$

$$L_s^{\mathsf{T}} P_s + P_s L_s = I_{n_s}. \tag{5.23b}$$

Proof. Step 1 (Stability and the Number of Clusters for the PLS Dynamics): Each PLS dynamics can be individually investigated as

$$\begin{bmatrix}\dot{y}_{p,i,1}(t)\\ \dot{z}_{p,i,1}(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_{p,i,1}(t)\\ z_{p,i,1}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n_{p,i}-1}^{\mathsf{T}} & \mathbf{0}_{n_{p,i}-1}^{\mathsf{T}}\\ -\gamma_1 l_{i,1}F_i & -\gamma_1 l_{i,1}F_i\end{bmatrix}\begin{bmatrix}\mathbf{e}_{p_1,i}(t)\\ \mathbf{e}_{p_2,i}(t)\end{bmatrix} + \begin{bmatrix}0\\ f(y_{p,i,1}, z_{p,i,1}, t)\end{bmatrix},$$
$$\begin{bmatrix}\dot{\mathbf{e}}_{p_1,i}(t)\\ \dot{\mathbf{e}}_{p_2,i}(t)\end{bmatrix} = \begin{bmatrix}0_{n_{p,i}-1\times n_{p,i}-1} & I_{n_{p,i}-1}\\ -\gamma_1 E_i L_{i,i}F_i & -\gamma_2 E_i L_{i,i}F_i\end{bmatrix}\begin{bmatrix}\mathbf{e}_{p_1,i}(t)\\ \mathbf{e}_{p_2,i}(t)\end{bmatrix} + \begin{bmatrix}\mathbf{0}_{n_{p,i}-1}\\ \mathbf{F}_e(\mathbf{y}_{p,i}, \mathbf{z}_{p,i}, t)\end{bmatrix}.$$

To prove the stability of all PLSs at once, we consider the Lyapunov function candidate

$$V_1(t) = \mathbf{e}_p(t)^{\mathsf{T}} H_p \mathbf{e}_p(t)$$

where $\mathbf{e}_p(t) = \left[\mathbf{e}_{p_1}(t)^{\mathsf{T}}\ \mathbf{e}_{p_2}(t)^{\mathsf{T}}\right]^{\mathsf{T}}$ with $\mathbf{e}_{p_1}(t) = \left[\mathbf{e}_{p_1,1}(t)^{\mathsf{T}}, \ldots, \mathbf{e}_{p_1,l_p}(t)^{\mathsf{T}}\right]^{\mathsf{T}}$, $\mathbf{e}_{p_2}(t) = \left[\mathbf{e}_{p_2,1}(t)^{\mathsf{T}}, \ldots, \mathbf{e}_{p_2,l_p}(t)^{\mathsf{T}}\right]^{\mathsf{T}}$ and

$$H_p = \begin{bmatrix}\gamma_1\gamma_2 I_{n_{p-l_p}} & \gamma_1 P_p\\ \gamma_1 P_p & \gamma_2 P_p\end{bmatrix}$$

with $P_p = \text{blkdiag}\{P_1, P_2, \ldots, P_{l_p}\}$. From Lemma 5.3, $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{\lambda_M(P_p)}$ ensures that $H_p > 0$. Therefore, when $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_p)} - 2$ holds, $V(t) > 0$ if $\mathbf{e}_p(t) \neq 0$ and $V(t) = 0$ if and only if $\mathbf{e}_p(t) = 0$. $\dot{V}_1(t)$ can be computed as

$$\begin{aligned}\dot{V}_1(t) = {} & 2\gamma_1\gamma_2 \mathbf{e}_{p_1}(t)^{\mathsf{T}}\mathbf{e}_{p_2}(t) + 2\gamma_1 \mathbf{e}_{p_2}(t)^{\mathsf{T}} P_p \mathbf{e}_{p_2}(t)\\ & - \gamma_1^2 \mathbf{e}_{p_1}(t)^{\mathsf{T}}\left[(EL_pF)^{\mathsf{T}}P_p + P_p(EL_pF)\right]\mathbf{e}_{p_1}(t)\\ & - 2\gamma_1\gamma_2 \mathbf{e}_{p_1}(t)^{\mathsf{T}}\left[(EL_pF)^{\mathsf{T}}P_p + P_p(EL_pF)\right]\mathbf{e}_{p_2}(t)\\ & - \gamma_2^2 \mathbf{e}_{p_2}(t)^{\mathsf{T}}\left[(EL_pF)^{\mathsf{T}}P_p + P_p(EL_pF)\right]\mathbf{e}_{p_2}(t) + 2\left(\gamma_1 \mathbf{e}_{p_1}(t)^{\mathsf{T}} + \gamma_2 \mathbf{e}_{p_2}(t)^{\mathsf{T}}\right) P_p \bar{\mathbf{F}}_e(t).\end{aligned} \tag{5.24}$$

Note that the inequality

$$\left\|\bar{\mathbf{F}}_e(t)\right\| \leq c_1 \left\|\mathbf{e}_{p_1}(t)\right\| + c_2 \left\|\mathbf{e}_{p_2}(t)\right\|$$

holds and by choosing $\gamma_1 \geq c_1$ and $\gamma_2 \geq c_2$, we have

$$\left(\gamma_1 \mathbf{e}_{p_1}(t)^\mathsf{T} + \gamma_2 \mathbf{e}_{p_2}(t)^\mathsf{T}\right) P_p \bar{\mathbf{F}}_e(t) \leq \lambda_M(P_p) \left(\|\gamma_1 \mathbf{e}_{p_1}(t)\| + \|\gamma_2 \mathbf{e}_{p_2}(t)\|\right). \tag{5.25}$$

Together with (5.24) and (5.25), we can obtain the following inequality:

$$\dot{V}_1(t) \leq -\left[\lambda_m(Q_p) - 4\lambda_M(P_p)\right] \left(\|\gamma_1 \mathbf{e}_{p_1}(t)\|^2 + \|\gamma_2 \mathbf{e}_{p_2}(t)\|^2\right)$$

where $Q_p = \begin{bmatrix} I_{n_p - l_p} & 0_{(n_p-l_p)\times(n_p-l_p)} \\ 0_{(n_p-l_p)\times(n_p-l_p)} & I_{n_p-l_p} - \frac{2\gamma_1}{\gamma_2^2} P_p \end{bmatrix}$. Note that $\lambda_m(Q_p) - 4\lambda_M(P_p) > 0$ if and only if $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_p)} - 2$ holds. By $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_p)} - 2$, we have $\dot{V}(t) < 0$ for $\mathbf{e}_p(t) \neq 0$, which implies that each PLS dynamics is asymptotically stable. The agents in a particular PLS act together i.e., $\lim_{t\to\infty} \|x_{p,i,j_1} - x_{p,i,j_2}\| = 0$ and $\lim_{t\to\infty} \|v_{p,i,j_1} - v_{p,i,j_2}\| = 0$ for $i = 1, 2, \ldots, l_p$ and $j_1, j_2 = 1, 2, \ldots, n_{p,i}$. This implies that one can find a total of l_p clusters in l_p PLSs.

Step 2 (Stability and the Number of Clusters for the SLS Dynamics): In this step, we will prove that the followings hold

$$\begin{aligned} \lim_{t\to\infty} \left\|\mathbf{y}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \mathbf{y}_p(t)\right\| = 0, \\ \lim_{t\to\infty} \left\|\mathbf{z}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \mathbf{z}_p(t)\right\| = 0. \end{aligned} \tag{5.26}$$

Let $\mathbf{e}_{s_1}(t) = \mathbf{y}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \mathbf{y}_p(t)$ and $\mathbf{e}_{s_2}(t) = \mathbf{z}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \mathbf{z}_p(t)$. We have

$$\mathbf{e}(t) = \begin{bmatrix} \mathbf{e}_{s_1}(t) \\ \mathbf{e}_{s_2}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{y}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \mathbf{y}_p(t) \\ \mathbf{z}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \mathbf{z}_p(t) \end{bmatrix}. \tag{5.27}$$

Taking derivatives yields

$$\begin{aligned} \dot{\mathbf{e}}_{s_1}(t) &= \dot{\mathbf{y}}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \dot{\mathbf{y}}_p(t), \\ \dot{\mathbf{e}}_{s_2}(t) &= \dot{\mathbf{z}}_s(t) + L_s^{-1} L_{sp} S_p^{-1} \dot{\mathbf{z}}_p(t), \end{aligned}$$

$$\begin{aligned}\dot{\mathbf{e}}_{s_1}(t) &= \mathbf{e}_{s_2}(t),\\ \dot{\mathbf{e}}_{s_2}(t) &= \gamma_1 L_s \mathbf{e}_{s_1}(t) - \gamma_2 L_s \mathbf{e}_{s_2}(t) + \mathbf{f}(\mathbf{y}_s, \mathbf{z}_s, t) + L_s^{-1} L_{sp} \mathbf{f}(\mathbf{y}_p, \mathbf{z}_p, t)\\ &= -\gamma_1 L_s^{-1} L_{sp} L_p S_p^{-1} \mathbf{y}_p(t) - \gamma_2 L_s^{-1} L_{sp} L_p S_p^{-1} \mathbf{z}_p(t).\end{aligned}$$

Let $\omega(t) = -\gamma_1 L_s^{-1} L_{sp} L_p S_p^{-1} \mathbf{y}_p(t) - \gamma_2 L_s^{-1} L_{sp} L_p S_p^{-1} \mathbf{z}_p(t) = -\gamma_1 L_s^{-1} L_{sp} L_p \mathbf{x}_p(t)$ $- \gamma_2 L_s^{-1} L_{sp} L_p \mathbf{v}_p(t)$. The above dynamics can be expressed as follows:

$$\begin{aligned}\dot{\mathbf{e}}_{s_1}(t) &= \mathbf{e}_{s_2}(t),\\ \dot{\mathbf{e}}_{s_2}(t) &= \gamma_1 L_s \mathbf{e}_{s_1}(t) - \gamma_2 L_s \mathbf{e}_{s_2}(t) + \mathbf{f}(\mathbf{y}_s, \mathbf{z}_s, t) + L_s^{-1} L_{sp} \mathbf{f}(\mathbf{y}_p, \mathbf{z}_p, t) + \omega(t).\end{aligned}$$

Note that $\omega(t)$ converges to zero as $t \to \infty$ from Step 1. In order to analyze the stability of the above system dynamics, it is required to investigate the stability of the unforced system i.e., $\omega(t) = 0$ for all t (p. 107 in [109]). One can obtain the derivative of the unforced system's error dynamics as

$$\begin{bmatrix}\dot{\mathbf{e}}_{s_1}(t)\\ \dot{\mathbf{e}}_{s_2}(t)\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{n_s}\\ \mathbf{f}(\mathbf{y}_s, \mathbf{z}_s, t) + L_s^{-1} L_{sp} \mathbf{f}(\mathbf{y}_p, \mathbf{z}_p, t)\end{bmatrix} + \begin{bmatrix}0_{n_s \times n_s} & I_{n_s}\\ -\gamma_1 L_s & -\gamma_2 L_s\end{bmatrix} \begin{bmatrix}\mathbf{e}_{s_1}(t)\\ \mathbf{e}_{s_2}(t)\end{bmatrix}.$$

In order to prove asymptotic stability of (5.27) choose the Lyapunov function candidate

$$V_2(t) = \mathbf{e}_s(t)^\mathsf{T} H_s \mathbf{e}_s(t)$$

for the unforced SLS dynamics where $\mathbf{e}_p(t) = \left[\mathbf{e}_{s_1}(t)^\mathsf{T}\ \mathbf{e}_{s_2}(t)^\mathsf{T}\right]^\mathsf{T}$ and

$$H_s = \begin{bmatrix}\gamma_1\gamma_2 I_{n_s} & \gamma_1 P_s\\ \gamma_1 P_s & \gamma_2 P_s\end{bmatrix}.$$

From Lemma 5.3, $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{\lambda_M(P_s)}$ ensures that $H_s > 0$. Therefore, when $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_s)} - 2$ holds, $V_2(t) > 0$ if $e_s(t) \neq 0$ and $V_2(t) = 0$ if and only if $e_s(t) = 0$. $\dot{V}_2(t)$ can be

computed as

$$\begin{aligned}\dot{V}_2(t) = {} & 2\gamma_1\gamma_2\mathbf{e}_{s_1}(t)^{\mathsf{T}}\mathbf{e}_{s_2}(t) + 2\gamma_1\mathbf{e}_{s_2}(t)^{\mathsf{T}}P_s\mathbf{e}_{s_2}(t) - \gamma_1^2\mathbf{e}_{s_1}(t)^{\mathsf{T}}\left[L_s^{\mathsf{T}}P_s + P_sL_s\right]\mathbf{e}_{s_1}(t) \\ & - 2\gamma_1\gamma_2\mathbf{e}_{s_1}(t)^{\mathsf{T}}\left[L_s^{\mathsf{T}}P_s + P_sL_s\right]\mathbf{e}_{s_2}(t) - \gamma_2^2\mathbf{e}_{s_2}(t)^{\mathsf{T}}\left[L_s^{\mathsf{T}}P_s + P_sL_s\right]\mathbf{e}_{s_2}(t) \\ & + 2\left(\gamma_1\mathbf{e}_{s_1}(t)^{\mathsf{T}} + \gamma_2\mathbf{e}_{s_2}(t)^{\mathsf{T}}\right)P_s\left[\mathbf{f}(\mathbf{y}_s,\mathbf{z}_s,t) + L_s^{-1}L_{sp}\mathbf{f}(\mathbf{y}_p,\mathbf{z}_p,t)\right].\end{aligned}$$

Following a similar procedure as in Step 1, the following inequality can be obtained:

$$\dot{V}_2(t) \leq -\left[\lambda_m(Q_s) - 4\lambda_M(P_s)\right]\left(\|\gamma_1\mathbf{e}_{s_1}\|^2 + \|\gamma_2\mathbf{e}_{s_2}\|^2\right)$$

where $Q_s = \begin{bmatrix} I_{n_s} & 0_{n_s\times n_s} \\ 0_{n_s\times n_s} & I_{n_s} - \frac{2\gamma_1}{\gamma_2^2}P_s \end{bmatrix}$. Note that $\lambda_m(Q_s) - 4\lambda_M(P_s) > 0$ if and only if $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_s)} - 2$ holds, $V_2(t) > 0$ holds. By $\frac{\gamma_1}{\gamma_2^2} < \frac{1}{2\lambda_M(P_s)} - 2$ holds, $V_2(t) > 0$, we have $\dot{V}_2(t) < 0$ for $\mathbf{e}_s(t) \neq 0$, which implies that the origin of (5.27) is asymptotically stable. This implies that the following expression holds

$$\begin{aligned}\mathbf{y}_s(\infty) &= -L_s^{-1}L_{sp}S_p^{-1}\mathbf{y}_p(\infty), \\ \mathbf{z}_s(\infty) &= -L_s^{-1}L_{sp}S_p^{-1}\mathbf{z}_p(\infty).\end{aligned}$$

Note that original states $\mathbf{x}_p(t) = S_p^{-1}\mathbf{y}_p(t)$, $\mathbf{x}_s(t) = \mathbf{y}_s(t)$, $\mathbf{v}_p(t) = S_p^{-1}\mathbf{z}_p(t)$ and $\mathbf{v}_s(t) = \mathbf{z}_s(t)$. One can obtain the following

$$\begin{aligned}\mathbf{x}_s(\infty) &= -L_s^{-1}L_{sp}\mathbf{x}_p(\infty), \\ \mathbf{v}_s(\infty) &= -L_s^{-1}L_{sp}\mathbf{v}_p(\infty).\end{aligned}$$

Let $\mathbf{x}_p^*(t) = \lim_{t\to\infty}\mathbf{x}_p(t)$, $\mathbf{x}_s^*(t) = \lim_{t\to\infty}\mathbf{x}_s(t)$, $\mathbf{v}_p^*(t) = \lim_{t\to\infty}\mathbf{v}_p(t)$ and $\mathbf{v}_s^*(t) = \lim_{t\to\infty}\mathbf{v}_s(t)$. As $t\to\infty$, $\mathbf{x}_p(t)$ converges to trajectory $\mathbf{x}_p^*(t)$ and $\mathbf{v}_p(t)$ converges to trajectory $\mathbf{v}_p^*(t)$ containing l_p distinct routes from Step 1. From PLS and SLS dynamics in Reference 98, it is known that l_p points at any t gives l_s clusters in $\mathbf{x}_s^*(t)$ and $\mathbf{v}_s^*(t)$ which are convergent trajectories for $\mathbf{x}_s(t)$ and $\mathbf{v}_s(t)$, respectively.

Step 3 (Total Number of Clusters): We have l_p distinct PLSs from Step 1 and l_s distinct SLSs from Step 2. Hence, one can conclude that the total number of cluster is computed as $\mathcal{K} = l_p + l_s$ since (5.22) is satisfied. □

Remark 5.6. By applying Theorem 5.2, the selection method of control parameters γ_1 and γ_2 can be established as follows:

- Calculate the maximum eigenvalue of P_p which is a symmetric positive definite matrix that satisfies (5.23a).
- Calculate the maximum eigenvalue of P_s which is a symmetric positive definite matrix that satisfies (5.23b).
- Obtain a pair of control parameters γ_1 and γ_2 that satisfies (5.22) by using the maximum eigenvalues of P_p and P_s.

Note that the protocol (5.17), which is used as update rule, is fully distributed from Remark 5.2. However, L_p and L_s global matrices should be known only for the stability analysis and the selection of control parameters γ_1 and γ_2.

Remark 5.7. Given an unsigned directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, \mathcal{A}\}$, Theorem 5.1 and Theorem 5.2 state the cluster consensus results on the total number of clusters for first and second-order systems with inherently nonlinear dynamics, respectively. According to these theorems, the total number of clusters for both systems can be calculated as $l_p + l_s$ with the same cluster members. Although the clusters are the same, the stability conditions and the methods of the proofs are different from each other. First of all, the Lyapunov functions that we utilized in the proofs of Theorem 5.1 and Theorem 5.2 are not the same. Furthermore, Lemma 5.1 is used to prove the result of Theorem 5.1 while Lemma 5.3 is used in the proof of Theorem 5.2. Therefore, the second-order cluster consensus problem is not exactly an extension of its first-order counterpart.

5.2. Nonlinear Networks with Signed Directed Graphs

5.2.1. First-Order Nonlinear Dynamics

The objective of this section is to determine the conditions for inherently nonlinear first-order multi-agent systems represented signed directed graphs to attain cluster consensus. The nonlinear dynamics of each agent are defined as

$$\dot{x}_i(t) = f(x_i, t) + u_i(t), \quad i = 1, 2, \ldots, n \tag{5.28}$$

where $x_i(t) \in \mathbb{R}$, $f(x_i, t)$ and $u_i(t)$ are the state, the inherent nonlinear dynamics and the control input of agent i, respectively. The control input $u_i(t)$ is described as

$$u_i(t) = \gamma_1 \sum_{j \in \mathcal{N}_i} a_{ij} \left(x_j(t) - \text{sgn}(a_{ij}) x_i(t) \right) \tag{5.29}$$

where γ_1 is the control parameter and $\text{sgn}(\cdot)$ is the signum function. By utilizing the signed Laplacian matrix, the control input is expressed as

$$u_i(t) = -\gamma_1 \sum_{j \in \mathcal{N}_i} l_{ij} x_j(t). \tag{5.30}$$

Through the combination of (5.28) and (5.30), the overall dynamics can be written in the matrix form as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, t) - \gamma_1 L \mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{5.31}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^\mathsf{T}$, $\mathbf{f}(\mathbf{x}, t) = [f(x_1, t), f(x_2, t), \ldots, f(x_n, t)]^\mathsf{T}$ and $\mathbf{x}_0$ is the initial condition.

Given the signed system (5.28) with the input (5.29), its extended system dynamics associated with $\bar{\mathcal{G}}$ are defined as

$$\dot{\bar{x}}_i(t) = f(\bar{x}_i, t) + \gamma_1 \sum_{j \in \bar{\mathcal{N}}_i} \bar{a}_{ij}(\bar{x}_j(t) - \bar{x}_i(t)), \quad i = 1, 2, \ldots, 2n \tag{5.32}$$

where $\bar{x}_i(t)$ is the state of agent i. The dynamics of the above system can be expressed with the elements of the unsigned Laplacian matrix $\bar{L} = [\bar{l}_{ij}] \in \mathbb{R}^{2n \times 2n}$ as

$$\dot{\bar{x}}_i(t) = f(\bar{x}_i, t) - \gamma_1 \sum_{j \in \bar{\mathcal{N}}_i} \bar{l}_{ij}\bar{x}_j(t), \quad i = 1, 2, \ldots, 2n. \tag{5.33}$$

The subsequent matrix expression represents the extended system:

$$\dot{\bar{\mathbf{x}}}(t) = \mathbf{f}(\bar{\mathbf{x}}, t) - \gamma_1 \bar{L}\bar{\mathbf{x}}(t), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \tag{5.34}$$

where $\bar{\mathbf{x}}(t) = [\bar{x}_1(t), \bar{x}_2(t), \ldots, \bar{x}_{2n}(t)]^\mathsf{T}$, $\mathbf{f}(\bar{\mathbf{x}}, t) = [f(\bar{x}_1(t), t), f(\bar{x}_2(t), t), \ldots, f(\bar{x}_{2n}, t)]^\mathsf{T}$ and $\bar{\mathbf{x}}_0$ is the initial condition.

Remark 5.8. Extended graph representation is a tool for analyzing the dynamics of a given signed directed graph by expanding it into an unsigned directed graph.

A given *unsigned* directed graph representing a multi-agent system can be expressed by applying PLS and SLS concepts. According to the definitions of PLS and SLS in [13], an unsigned directed graph $\bar{\mathcal{G}}$ can be uniquely decomposed into $\bar{l}_p$ PLSs and $\bar{l}_s$ SLSs. In this framework, we denote the number of agents in the $i-$th PLS and $j-$th SLS as $\bar{n}_{p,i}$ $(i = 1, 2, \ldots, \bar{l}_p)$ and $\bar{n}_{s,j}$ $(j = 1, 2, \ldots, \bar{l}_s)$, respectively. The total number of agents in PLSs and SLSs are denoted by $\bar{n}_p = \sum_{i=1}^{\bar{l}_p} \bar{n}_{p,i}$ and $\bar{n}_s = \sum_{j=1}^{\bar{l}_s} \bar{n}_{s,j}$, respectively. The extended system (5.34) can be expressed as

$$\begin{bmatrix} \dot{\bar{\mathbf{x}}}_p(t) \\ \dot{\bar{\mathbf{x}}}_s(t) \end{bmatrix} = \begin{bmatrix} \mathbf{f}(\bar{\mathbf{x}}_p, t) \\ \mathbf{f}(\bar{\mathbf{x}}_s, t) \end{bmatrix} - \gamma_1 \begin{bmatrix} \bar{L}_p & 0_{\bar{n}_p \times \bar{n}_s} \\ \bar{L}_{sp} & \bar{L}_s \end{bmatrix} \begin{bmatrix} \bar{\mathbf{x}}_p(t) \\ \bar{\mathbf{x}}_s(t) \end{bmatrix} \tag{5.35}$$

where $\bar{\mathbf{x}}_p(t) \in \mathbb{R}^{\bar{n}_p}$ and $\bar{\mathbf{x}}_s(t) \in \mathbb{R}^{\bar{n}_s}$ are state vectors for PLSs and SLSs, respectively. To enable the analysis of cluster consensus, we adopt the following assumption based

on the preceding observation.

Assumption 5.3. *Given the odd function $f(x,t)$ with respect to x, there exists a constant c that satisfies the following inequality:*

$$\left| f(y,t) - \sum_{i=1}^{\bar{n}_p} \eta_i f(x_i,t) \right| \leq c \left| y - \sum_{i=1}^{\bar{n}_p} \eta_i x_i \right|, \quad \forall y, x_i \in \mathbb{R} \tag{5.36}$$

where $\eta_i \geq 0$ and $\sum_{i=1}^{\bar{n}_p} \eta_i = 1$.

Under Assumption 5.3, the following result establishes a connection between the trajectories of the system (5.28)-(5.29) represented by a signed directed graph and those of its extended representation (5.32) represented by an unsigned directed graph.

Lemma 5.4. *Under Assumption 5.3, the trajectories $\mathbf{x}(t)$ of the system (5.28)-(5.29) can be recovered from that of the system (5.32) as $x_i(t) = \bar{x}_i(t)$, $i = 1, 2, \ldots, n$ for all $t \geq 0$ by setting $\bar{x}_{i0} = x_{i0}$ and $\bar{x}_{i+n,0} = -x_{i0}$ for $i = 1, 2, \ldots, n$.*

Proof. We can express the first-order extended system in the following form:

$$\dot{\bar{x}}_i(t) = f(\bar{x}_i,t) + \gamma_1 \left(\sum_{j=1}^{n} \bar{a}_{ij}(\bar{x}_j(t) - \bar{x}_i(t)) + \sum_{j=1}^{n} \bar{a}_{i,j+n}(\bar{x}_{j+n}(t) - \bar{x}_i(t)) \right), \tag{5.37a}$$

$$\dot{\bar{x}}_{i+n}(t) = f(\bar{x}_{i+n},t) + \gamma_1 \left(\sum_{j=1}^{n} \bar{a}_{i+n,j}(\bar{x}_j(t) - \bar{x}_{i+n}(t)) + \sum_{j=1}^{n} \bar{a}_{i+n,j+n}(\bar{x}_{j+n}(t) - \bar{x}_{i+n}(t)) \right) \tag{5.37b}$$

for $i = 1, 2, \ldots, n$. Let $\hat{x}_i(t) = \bar{x}_i(t) + \bar{x}_{i+n}(t)$ for $i = 1, 2, \ldots, n$. Then, we have

$$\begin{aligned}
\dot{\hat{x}}_i(t) &= \dot{\bar{x}}_i(t) + \dot{\bar{x}}_{i+n}(t) \\
&= f(\bar{x}_i,t) + f(\bar{x}_{i+n},t) + \gamma_1 \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{a}_{i+n,j})\bar{x}_j(t) + (\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n})\bar{x}_{j+n}(t) \\
&\quad - \gamma_1 \sum_{j=1}^{n} (\bar{a}_{ij} + \bar{a}_{i,j+n})\bar{x}_i(t) + (\bar{a}_{i+n,j} + \bar{a}_{i+n,j+n})\bar{x}_{i+n}(t).
\end{aligned}$$

From Definition 3.1, we obtain

$$\bar{a}_{i+n,j} = \bar{a}_{i,j+n}, \;\; \bar{a}_{ij} = \bar{a}_{i+n,j+n} \tag{5.38}$$

for $i = 1, 2, \ldots, n$ and $|a_{ij}| = \bar{a}_{ij} + \bar{a}_{i+n,j}$. From (5.38), $\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n} = \bar{a}_{ij} + \bar{a}_{i,j+n} = \bar{a}_{i+n,j} + \bar{a}_{i+n,j+n} = |a_{ij}|$ also holds. Then, we obtain

$$\begin{aligned}\dot{\hat{x}}_i(t) &= f(\bar{x}_i, t) + f(\bar{x}_{i+n}, t) + \gamma_1 \sum_{j=1}^{n} |a_{ij}| \left(\bar{x}_j(t) + \bar{x}_{j+n}(t)\right) - \gamma_1 \sum_{j=1}^{n} |a_{ij}| \left(\bar{x}_i(t) + \bar{x}_{i+n}(t)\right) \\ &= f(\bar{x}_i, t) + f(\bar{x}_{i+n}, t) + \gamma_1 \sum_{j=1}^{n} |a_{ij}| \left(\hat{x}_j(t) - \hat{x}_i(t)\right)\end{aligned} \tag{5.39}$$

for $i = 1, 2, \ldots, n$. The dynamics of the above system is equivalent to

$$\dot{\hat{\mathbf{x}}}(t) = \hat{\mathbf{f}}(\hat{\mathbf{x}}, t) + \gamma \hat{L} \hat{\mathbf{x}}(t), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0$$

where $\hat{\mathbf{x}}(t) = [\hat{x}_1(t), \hat{x}_2(t), \ldots, \hat{x}_n(t)]^{\mathsf{T}}$, $\hat{\mathbf{f}}(\hat{\mathbf{x}}, t) = [f(\bar{x}_1, t) + f(\bar{x}_{1+n}, t), f(\bar{x}_2, t) + f(\bar{x}_{2+n}, t), \ldots, f(\bar{x}_n, t) + f(\bar{x}_{2n}, t)]^{\mathsf{T}}$, $\hat{\mathbf{x}}_0$ is the initial condition and $\hat{L}$ is the unsigned Laplacian matrix, with its entries given by

$$\hat{l}_{ij} = \begin{cases} \sum\limits_{k=1, k\neq i}^{n} |a_{ik}|, & \text{if } i = j \\ -|a_{ij}|, & \text{otherwise} \end{cases} \tag{5.40}$$

Note that $\hat{\mathbf{x}}_0 = [0, \ldots, 0]^{\mathsf{T}}$ implies that $\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_0 = [0, \ldots, 0]^{\mathsf{T}}$ for all t, i.e. $\bar{x}_i(t) = -\bar{x}_{i+n}$ for $i = 1, 2, \ldots, n$. Substituting $\bar{x}_{j+n}(t) = -\bar{x}_j(t)$ into (5.37a), we obtain

$$\dot{\bar{x}}_i = f(\bar{x}_i, t) + \gamma_1 \left(\sum_{j=1}^{n} (\bar{a}_{ij} - \bar{a}_{i,j+n}) \bar{x}_j(t) + \sum_{j=1}^{n} (\bar{a}_{ij} - \bar{a}_{i,j+n}) \bar{x}_i(t) \right).$$

From Definition 3.1, we have

$$\dot{\bar{x}}_i = f(\bar{x}_i, t) + \gamma_1 \left(\sum_{j=1}^{n} a_{ij} (\bar{x}_j(t) - \operatorname{sgn}(a_{ij}) \bar{x}_i(t) \right) \tag{5.41}$$

for $i = 1, 2, \ldots, n$. Note that (5.41) is equal to (5.28)-(5.29). By setting $\bar{x}_{i0} = x_{i0}$ and $\bar{x}_{i+n,0} = -x_{i0}$ for $i = 1, 2, \ldots, n$, the trajectories $x_i(t)$ can be recovered. This completes the proof. □

Combining PLS/SLS concepts that are applicable for *unsigned* directed graphs and Theorem 5.1, we have the subsequent result:

Theorem 5.3. *Suppose that the extended system* (5.34) *evolves over an unsigned directed graph* $\bar{\mathcal{G}}$ *with* $\bar{l}_p$ *PLSs and* $\bar{l}_s$ *SLSs and there exists a positive coefficient* c *that satisfies Assumption 5.3. Let* l' *be the number of subgraphs of the extended system that does not contain any agents of the original system* (5.31)*. The system* (5.31) *converges to* $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ *clusters if there exist a scalar control parameter* γ_1 *that satisfies*

$$\begin{aligned} -\gamma_1 \left[(\bar{E}\bar{L}_p\bar{F})^\mathsf{T}\bar{P}_p + \bar{P}_p(\bar{E}\bar{L}_p\bar{F}) \right] + c^2\lambda_M(\bar{P}_p)I_{\bar{n}_p-\bar{l}_p} + \bar{P}_p \prec 0, \\ -\gamma_1 \left[\bar{L}_s^\mathsf{T}\bar{P}_s + \bar{P}_s\bar{L}_s \right] + c^2\lambda_M(\bar{P}_s)I_{\bar{n}_s} + \bar{P}_s \prec 0 \end{aligned} \tag{5.42}$$

where $\bar{E} = \text{blkdiag}\{\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_{\bar{l}_p}\}$ *with* $\bar{E}_i = \left[\mathbf{1}_{\bar{n}_{p,i}-1} \ -I_{\bar{n}_{p,i}-1}\right]$, $\bar{F} = \text{blkdiag}\{\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_{\bar{l}_p}\}$ *with* $\bar{F}_i = \left[\mathbf{0}_{\bar{n}_{p,i}-1} \ -I_{\bar{n}_{p,i}-1}\right]$ *and* $\bar{P}_p \in \mathbb{R}^{(\bar{n}_p-\bar{l}_p)\times(\bar{n}_p-\bar{l}_p)}$ *and* $\bar{P}_s \in \mathbb{R}^{\bar{n}_s\times\bar{n}_s}$ *some symmetric positive definite matrices that yield* $(\bar{E}\bar{L}_p\bar{F})^\mathsf{T}\bar{P}_p + \bar{P}_p(\bar{E}\bar{L}_p\bar{F})$ *and* $\bar{L}_s^\mathsf{T}\bar{P}_s + \bar{P}_s\bar{L}_s$ *positive definite, respectively.*

Proof. The extended system (5.32) is stable under the condition (5.42) is satisfied and converges to $\bar{l}_p + \bar{l}_s$ by using Theorem 5.1. From Lemma 5.4, recall that the trajectories of (5.28)-(5.29) are obtained from those of (5.32). Therefore, we need to remove l' clusters whose are not in the original set. Hence, the original system (5.28)-(5.29) consists of $\bar{l}_p + \bar{l}_s - l'$ clusters. This completes the proof. □

5.2.2. Second-Order Nonlinear Dynamics

The system is modeled by second-order dynamics as follows:

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ \dot{v}_i(t) &= f(x_i, v_i, t) + u_i(t), \quad i = 1, 2, \ldots, n \end{aligned} \tag{5.43}$$

where $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$ are the states of agent i, $f(x_i, v_i, t)$ is a second-order inherent nonlinear function and $u_i(t)$ is the control input described as follows:

$$u_i(t) = \gamma_1 \sum_{j \in \mathcal{N}_i} a_{ij} \left(x_j(t) - \text{sgn}(a_{ij}) x_i(t) \right) + \gamma_2 \sum_{j \in \mathcal{N}_i} a_{ij} \left(v_j(t) - \text{sgn}(a_{ij}) v_i(t) \right) \tag{5.44}$$

where γ_1 and γ_2 are the control parameters. The system dynamics with control input can be rewritten in matrix form as

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{v}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n \\ \mathbf{f}(\mathbf{x}, \mathbf{v}, t) \end{bmatrix} + \begin{bmatrix} 0_{n \times n} & I_n \\ -\gamma_1 L & -\gamma_2 L \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \mathbf{v}(t) \end{bmatrix}, \quad \mathbf{x}(0) = \mathbf{x}_0, \mathbf{v}(0) = \mathbf{v}_0 \tag{5.45}$$

where $\mathbf{x}(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^\mathsf{T}$, $\mathbf{v}(t) = [v_1(t), v_2(t), \ldots, v_n(t)]^\mathsf{T}$, $\mathbf{f}(\mathbf{x}, \mathbf{v}, t) = [f(x_1, v_1, t), f(x_2, v_2, t), \ldots, f(x_n, v_n, t)]^\mathsf{T}$, $\mathbf{x}_0$ and $\mathbf{v}_0$ are the initial conditions.

Given the signed system (5.43)-(5.44), its extended system dynamics associated with $\bar{\mathcal{G}}$ are defined as

$$\dot{\bar{x}}_i(t) = f(\bar{x}_i, t) + \gamma_1 \sum_{j \in \bar{\mathcal{N}}_i} \bar{a}_{ij} \left(\bar{x}_j(t) - \bar{x}_i(t) \right) + \gamma_2 \sum_{j \in \bar{\mathcal{N}}_i} \bar{a}_{ij} \left(\bar{v}_j(t) - \bar{v}_i(t) \right), \quad i = 1, 2, \ldots, 2n \tag{5.46}$$

where $\bar{x}_i(t) \in \mathbb{R}$ and $\bar{v}_i(t) \in \mathbb{R}$ are the states of agent i. The dynamics of the above

system can be expressed as

$$\begin{bmatrix}\dot{\bar{\mathbf{x}}}(t)\\ \dot{\bar{\mathbf{v}}}(t)\end{bmatrix}=\begin{bmatrix}\mathbf{0}_{2n}\\ \mathbf{f}(\bar{\mathbf{x}},\bar{\mathbf{v}},t)\end{bmatrix}+\begin{bmatrix}0_{2n\times 2n} & I_{2n}\\ -\gamma_1\bar{L} & -\gamma_2\bar{L}\end{bmatrix}\begin{bmatrix}\bar{\mathbf{x}}(t)\\ \bar{\mathbf{v}}(t)\end{bmatrix}\quad \bar{\mathbf{x}}(0)=\bar{\mathbf{x}}_0,\bar{\mathbf{v}}(0)=\bar{\mathbf{v}}_0 \tag{5.47}$$

where $\bar{\mathbf{x}}(t)=[\bar{x}_1(t),\bar{x}_2(t),\ldots,\bar{x}_{2n}(t)]^{\mathsf{T}}$, $\bar{\mathbf{v}}(t)=[\bar{v}_1(t),\bar{v}_2(t),\ldots,\bar{v}_{2n}(t)]^{\mathsf{T}}$, $\bar{\mathbf{x}}_0$ and $\bar{\mathbf{v}}_0$ are the initial conditions. PLS and SLS dynamics of the second-order extended system can be written as follows:

$$\begin{aligned}\begin{bmatrix}\dot{\bar{\mathbf{x}}}_p(t)\\ \dot{\bar{\mathbf{v}}}_p(t)\end{bmatrix}&=\begin{bmatrix}\mathbf{0}_{\bar{n}_p}\\ \mathbf{f}(\bar{\mathbf{x}}_p,\bar{\mathbf{v}}_p,t)\end{bmatrix}+\begin{bmatrix}0_{\bar{n}_p\times\bar{n}_p} & I_{\bar{n}_p}\\ -\gamma_1\bar{L}_p & -\gamma_2\bar{L}_p\end{bmatrix}\begin{bmatrix}\bar{\mathbf{x}}_p(t)\\ \bar{\mathbf{v}}_p(t)\end{bmatrix},\\ \begin{bmatrix}\dot{\bar{\mathbf{x}}}_s(t)\\ \dot{\bar{\mathbf{v}}}_s(t)\end{bmatrix}&=\begin{bmatrix}\mathbf{0}_{\bar{n}_s}\\ \mathbf{f}(\bar{\mathbf{x}}_s,\bar{\mathbf{v}}_s,t)\end{bmatrix}+\begin{bmatrix}0_{\bar{n}_s\times\bar{n}_s} & I_{\bar{n}_s}\\ -\gamma_1\bar{L}_s & -\gamma_2\bar{L}_s\end{bmatrix}\begin{bmatrix}\bar{\mathbf{x}}_s(t)\\ \bar{\mathbf{v}}_s(t)\end{bmatrix}+\begin{bmatrix}0_{\bar{n}_s\times\bar{n}_p} & 0_{\bar{n}_s\times\bar{n}_p}\\ -\gamma_1\bar{L}_{sp} & -\gamma_2\bar{L}_{sp}\end{bmatrix}\begin{bmatrix}\bar{\mathbf{x}}_p(t)\\ \bar{\mathbf{v}}_p(t)\end{bmatrix}.\end{aligned} \tag{5.48}$$

For the purpose of analyzing cluster consensus, we introduce the subsequent assumption for the second-order systems:

Assumption 5.4. *Given the odd function $f(x,v,t)$ with respect to x and v, there exist constants c_1,c_2 that satisfy the following inequality:*

$$\left|f(y,z,t)-\sum_{i=1}^{\bar{n}_p}\eta_i f(x_i,v_i,t)\right|\le c_1\left|y-\sum_{i=1}^{\bar{n}_p}\eta_i x_i\right|+c_2\left|z-\sum_{i=1}^{\bar{n}_p}\eta_i v_i\right|,\forall y,z,x_i,v_i\in\mathbb{R} \tag{5.49}$$

where $\eta_i\ge 0$ and $\sum_{i=1}^{\bar{n}_p}\eta_i=1$

As in first-order systems, the following result on the relationship between the second-order system (5.43)-(5.44) and its extended counterpart (5.46) is obtained under Assumption 5.4:

Lemma 5.5. *Under Assumption 5.4, the trajectories $\mathbf{x}(t)$ and $\mathbf{v}(t)$ of the system* (5.43)-(5.44) *can be recovered from that of the system* (5.46) *as $x_i(t)=\bar{x}_i(t)$ and*

$v_i(t) = \overline{v}_i(t)$, $i = 1, 2, \ldots, n$ *for all* $t \geq 0$ *by setting* $\overline{x}_{i0} = x_{i0}$, $\overline{x}_{i+n,0} = -x_{i0}$, $\overline{v}_{i0} = v_{i0}$ *and* $\overline{v}_{i+n,0} = -v_{i0}$ *for* $i = 1, 2, \ldots, n$.

Proof. We can express the second-order extended system in the following form:

$$\begin{aligned}
\dot{\overline{x}}_i(t) &= \overline{v}_i(t), \\
\dot{\overline{v}}_i(t) &= f(\overline{x}_i, \overline{v}_i, t) + \gamma_1 \left(\sum_{j=1}^{n} \overline{a}_{ij}(\overline{x}_j(t) - \overline{x}_i(t)) + \sum_{j=1}^{n} \overline{a}_{i,j+n}(\overline{x}_{j+n}(t) - \overline{x}_i(t)) \right) \\
&\quad + \gamma_2 \left(\sum_{j=1}^{n} \overline{a}_{ij}(\overline{v}_j(t) - \overline{v}_i(t)) + \sum_{j=1}^{n} \overline{a}_{i,j+n}(\overline{v}_{j+n}(t) - \overline{v}_i(t)) \right), \qquad (5.50a) \\
\dot{\overline{x}}_{i+n}(t) &= \overline{v}_{i+n}(t), \\
\dot{\overline{v}}_{i+n}(t) &= f(\overline{x}_{i+n}, \overline{v}_{i+n}, t) + \gamma_1 \left(\sum_{j=1}^{n} \overline{a}_{i+n,j}(\overline{x}_j(t) - \overline{x}_{i+n}(t)) + \sum_{j=1}^{n} \overline{a}_{i+n,j+n}(x_{j+n}(t) - x_{i+n}(t)) \right) \\
&\quad + \gamma_2 \left(\sum_{j=1}^{n} \overline{a}_{i+n,j}(\overline{v}_j(t) - \overline{v}_{i+n}(t)) + \sum_{j=1}^{n} \overline{a}_{i+n,j+n}(\overline{v}_{j+n}(t) - \overline{v}_{i+n}(t)) \right) \qquad (5.50b)
\end{aligned}$$

for $i = 1, 2, \ldots, n$. Let $\hat{x}_i(t) = \overline{x}_i(t) + \overline{x}_{i+n}(t)$ and $\hat{v}_i(t) = \overline{v}_i(t) + \overline{v}_{i+n}(t)$ for $i = 1, 2, \ldots, n$. Then, we have

$$\begin{aligned}
\dot{\hat{x}}_i(t) &= \dot{\overline{x}}_i(t) + \dot{\overline{x}}_{i+n}(t) \\
&= \hat{v}_i(t), \\
\dot{\hat{v}}_i(t) &= f(\overline{x}_i, \overline{v}_i, t) + f(\overline{x}_{i+n}, \overline{v}_{i+n}, t) \\
&\quad + \gamma_1 \sum_{j=1}^{n} (\overline{a}_{ij} + \overline{a}_{i+n,j})\overline{x}_j(t) + (\overline{a}_{i,j+n} + \overline{a}_{i+n,j+n})\overline{x}_{j+n}(t) \\
&\quad - \gamma_1 \sum_{j=1}^{n} (\overline{a}_{ij} + \overline{a}_{i,j+n})\overline{x}_i(t) + (\overline{a}_{i+n,j} + \overline{a}_{i+n,j+n})\overline{x}_{i+n}(t) \\
&\quad + \gamma_2 \sum_{j=1}^{n} (\overline{a}_{ij} + \overline{a}_{i+n,j})\overline{v}_j(t) + (\overline{a}_{i,j+n} + \overline{a}_{i+n,j+n})\overline{v}_{j+n}(t) \\
&\quad - \gamma_2 \sum_{j=1}^{n} (\overline{a}_{ij} + \overline{a}_{i,j+n})\overline{v}_i(t) + (\overline{a}_{i+n,j} + \overline{a}_{i+n,j+n})\overline{v}_{i+n}(t).
\end{aligned}$$

From Definition 3.1, we obtain

$$\bar{a}_{i+n,j} = \bar{a}_{i,j+n}, \;\; \bar{a}_{ij} = \bar{a}_{i+n,j+n} \tag{5.51}$$

for $i = 1, 2, \ldots, n$ and $|a_{ij}| = \bar{a}_{ij} + \bar{a}_{i+n,j}$. From (5.51), $\bar{a}_{i,j+n} + \bar{a}_{i+n,j+n} = \bar{a}_{ij} + \bar{a}_{i,j+n} = \bar{a}_{i+n,j} + \bar{a}_{i+n,j+n} = |a_{ij}|$ also holds. Then, we obtain

$$\begin{aligned}
&\dot{\hat{x}}_i(t) = \hat{v}_i(t), \\
&\dot{\hat{v}}_i(t) = f(\bar{x}_i, \bar{v}_i, t) + f(\bar{x}_{i+n}, \bar{v}_{i+n}, t) \\
&\quad + \gamma_1 \sum_{j=1}^{n} |a_{ij}| \bar{x}_j(t) + |a_{ij}| \bar{x}_{j+n}(t) - \gamma_1 \sum_{j=1}^{n} |a_{ij}| \bar{x}_i(t) + |a_{ij}| \bar{x}_{i+n}(t) \\
&\quad + \gamma_2 \sum_{j=1}^{n} |a_{ij}| \bar{v}_j(t) + |a_{ij}| \bar{v}_{j+n}(t) - \gamma_2 \sum_{j=1}^{n} |a_{ij}| \bar{v}_i(t) + |a_{ij}| \bar{v}_{i+n}(t) \\
&= f(\bar{x}_i, \bar{v}_i, t) + f(\bar{x}_{i+n}, \bar{v}_{i+n}, t) + \gamma_1 \sum_{j=1}^{n} |a_{ij}| \left(\hat{x}_j(t) - \hat{x}_i(t)\right) + \gamma_2 \sum_{j=1}^{n} |a_{ij}| \left(\hat{v}_j(t) - \hat{v}_i(t)\right)
\end{aligned}$$

for $i = 1, 2, \ldots, n$. The dynamics of the above system is equivalent to

$$\begin{bmatrix} \dot{\hat{\mathbf{x}}}(t) \\ \dot{\hat{\mathbf{v}}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_n \\ \hat{\mathbf{f}}(\hat{\mathbf{x}}, \hat{\mathbf{v}}, t) \end{bmatrix} + \begin{bmatrix} 0_{n\times n} & I_n \\ -\gamma_1 \hat{L} & -\gamma_2 \hat{L} \end{bmatrix} \begin{bmatrix} \hat{\mathbf{x}}(t) \\ \hat{\mathbf{v}}(t) \end{bmatrix}, \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0, \hat{\mathbf{v}}(0) = \hat{\mathbf{v}}_0$$

where $\hat{\mathbf{x}}(t) = [\hat{x}_1(t), \ldots, \hat{x}_n(t)]^\mathsf{T}$, $\hat{\mathbf{v}}(t) = [\hat{v}_1(t), \ldots, \hat{v}_n(t)]^\mathsf{T}$, $\hat{\mathbf{f}}(\hat{\mathbf{x}}, \hat{\mathbf{v}}, t) = [f(\bar{x}_1, \bar{v}_1, t) + f(\bar{x}_{1+n}, \bar{v}_{1+n}, t), \ldots, f(\bar{x}_n, \bar{v}_n, t) + f(\bar{x}_{2n}, \bar{v}_{2n}, t)]^\mathsf{T}$, $\hat{\mathbf{x}}_0$ and $\hat{\mathbf{v}}_0$ are the initial conditions. Note that $\hat{\mathbf{x}}_0 = \hat{\mathbf{v}}_0 = [0, \ldots, 0]^\mathsf{T}$ implies that $\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_0 = [0, \ldots, 0]^\mathsf{T}, \hat{\mathbf{v}}(t) = \hat{\mathbf{v}}_0 = [0, \ldots, 0]^\mathsf{T}$ for all t, i.e. $\bar{x}_i(t) = -\bar{x}_{i+n}$, $\bar{v}_i(t) = -\bar{v}_{i+n}$ for $i = 1, 2, \ldots, n$. Substituting $\bar{x}_{j+n}(t) = -\bar{x}_j(t)$ and $\bar{v}_{j+n}(t) = -\bar{v}_j(t)$ into (5.50a), we obtain

$$\begin{aligned}
\dot{\bar{v}}_i = f(\bar{x}_i, \bar{v}_i, t) &+ \gamma_1 \left(\sum_{j=1}^{n} (\bar{a}_{ij} - \bar{a}_{i,j+n}) \bar{x}_j(t) + \sum_{j=1}^{n} (\bar{a}_{ij} - \bar{a}_{i,j+n}) \bar{x}_i(t)) \right) \\
&+ \gamma_2 \left(\sum_{j=1}^{n} (\bar{a}_{ij} - \bar{a}_{i,j+n}) \bar{v}_j(t) + \sum_{j=1}^{n} (\bar{a}_{ij} - \bar{a}_{i,j+n}) \bar{v}_i(t)) \right)
\end{aligned}$$

for $i = 1, 2, \ldots, n$. From Definition 3.1, we have

$$\begin{aligned}
&\dot{\bar{x}}_i(t) = \bar{v}_i(t), \\
&\dot{\bar{v}}_i(t) = f(\bar{x}_i, \bar{v}_i, t) + \gamma_1 \left(\sum_{j=1}^{n} a_{ij}(\bar{x}_j(t) - \mathrm{sgn}(a_{ij})\bar{x}_i(t) \right) + \gamma_2 \left(\sum_{j=1}^{n} a_{ij}(\bar{v}_j(t) - \mathrm{sgn}(a_{ij})\bar{v}_i(t) \right)
\end{aligned} \tag{5.52}$$

for $i = 1, 2, \ldots, n$. Note that (5.52) is equal to (5.43)-(5.44). By setting $\bar{x}_{i0} = x_{i0}$, $\bar{x}_{i+n,0} = -x_{i0}$, $\bar{v}_{i0} = v_{i0}$ and $\bar{v}_{i+n,0} = -v_{i0}$ for $i = 1, 2, \ldots, n$, the trajectories $x_i(t)$ and $v_i(t)$ can be recovered. This completes the proof. □

Lemma 5.5, together with PLS/SLS decompositions, lead to the following result:

Theorem 5.4. *Suppose that the extended system* (5.46) *evolves over an unsigned directed graph* $\bar{\mathcal{G}}$ *with* $\bar{l}_p$ *PLSs and* $\bar{l}_s$ *SLSs and there exists a positive coefficient* c *that satisfies Assumption 5.4. Let* l' *be the number of subgraphs of the extended system that does not contain any agents of the original system* (5.31). *The system* (5.43)-(5.44) *converges to* $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ *clusters if there exists a pair of control parameters* (γ_1, γ_2) *that satisfies* $\gamma_1 \geq c_1$, $\gamma_2 \geq c_2$ *and*

$$\frac{\gamma_1}{\gamma_2^2} < \min\left\{ \frac{1}{2\lambda_M(\bar{P}_p)}, \frac{1}{2\lambda_M(\bar{P}_s)} \right\} - 2 \tag{5.53}$$

for the symmetric positive definite matrices $\bar{P}_p \in \mathbb{R}^{(\bar{n}_p - \bar{l}_p)\times(\bar{n}_p - \bar{l}_p)}$, $\bar{P}_s \in \mathbb{R}^{\bar{n}_s \times \bar{n}_s}$ *that are obtained from*

$$(\bar{E}\bar{L}_p\bar{F})^{\mathsf{T}}\bar{P}_p + \bar{P}_p(\bar{E}\bar{L}_p\bar{F}) = I_{\bar{n}_p - \bar{l}_p}, \tag{5.54a}$$

$$\bar{L}_s^{\mathsf{T}}\bar{P}_s + \bar{P}_s\bar{L}_s = I_{\bar{n}_s} \tag{5.54b}$$

where $\bar{E} = \mathrm{blkdiag}\{\bar{E}_1, \bar{E}_2, \ldots, \bar{E}_{\bar{l}_p}\}$ *with* $\bar{E}_i = [\mathbf{1}_{\bar{n}_{p,i}-1} \;\; -I_{\bar{n}_{p,i}-1}]$ *and* $\bar{F} = \mathrm{blkdiag}\{\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_{\bar{l}_p}\}$ *with* $\bar{F}_i = [\mathbf{0}_{\bar{n}_{p,i}-1} \;\; -I_{\bar{n}_{p,i}-1}]$.

Proof. If condition (5.53) holds, the expanded system (5.46) is stable and reaches $\bar{l}_p+\bar{l}_s$ clusters, as established in Theorem 2.2. As system (5.43)-(5.44) is derived from the trajectories of (5.46) (Lemma 5.5), the required removal of l' non-original clusters reveals that the original system contains $\bar{l}_p + \bar{l}_s - l'$ clusters. □

5.3. Numerical Analysis

This section presents numerical simulations to validate the theoretical results derived in this chapter. Consider the network with 9 agents and 13 edges modeled via signed directed graph in Figure 5.1. Within the directed graph, positive communication links are depicted by black lines, while negative links are shown as red lines. The network architecture does not exhibit any specific graph-theoretic properties, such as strong connectedness, the presence of a spanning tree, or structural balance.

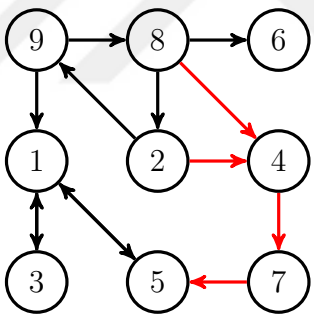


Figure 5.1. A signed directed graph with 9 agents and 13 edges.

The subsequent signed Laplacian matrix is used to simulate the network:

$$L = \begin{bmatrix} 44 & 0 & -8 & 0 & -28 & 0 & 0 & 0 & -8 \\ 0 & 16 & 0 & 0 & 0 & 0 & 0 & -16 & 0 \\ -8 & 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 16 & 0 & 24 & 0 & 0 & 0 & 8 & 0 \\ -8 & 0 & 0 & 0 & 16 & 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 & 0 & -8 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 12 & -12 \\ 0 & -20 & 0 & 0 & 0 & 0 & 0 & 0 & 20 \end{bmatrix}. \tag{5.55}$$

The extended directed graph, visualized in Figure 5.2, is derived from Definition 3.1. In the determination of cluster formation and membership within the extended directed graph, PLS and SLS concepts proposed in [13] are utilized. The clusters and members in the extended system are then: $\bar{\mathcal{V}}_{p,1} = \{2, 6, 7, 8, 9, 4'\}$, $\bar{\mathcal{V}}_{p,2} = \{4, 2', 6', 7', 8', 9'\}$, $\bar{\mathcal{V}}_{s,1} = \{1, 3\}$, $\bar{\mathcal{V}}_{s,2} = \{5\}$, $\bar{\mathcal{V}}_{s,3} = \{5'\}$, $\bar{\mathcal{V}}_{s,4} = \{1', 3'\}$. The total number of PLSs and SLSs are $\bar{l}_p = 2$ and $\bar{l}_s = 4$, respectively. Note that $\bar{\mathcal{V}}_{s,3} = \{5'\}$ and $\bar{\mathcal{V}}_{s,4} = \{1', 3'\}$ do not contain any agents from the original network ($l' = 2$).

Example 5.1. ***(First-order nonlinear dynamics)*** *We examine first-order consensus dynamics characterized by inherent nonlinear functions, as described in* (5.28).

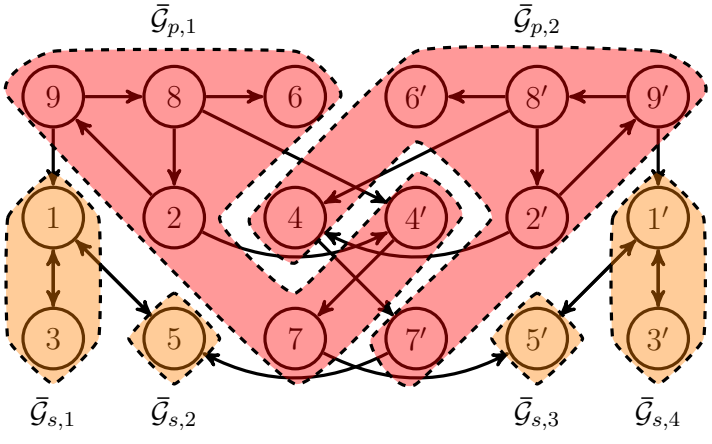


Figure 5.2. Extended directed graph for the network in Figure 5.1.

Specifically, the nonlinear function is defined as $f(x,t) = x\sin(t)$. *It is noteworthy that Assumptions 5.1 and 5.3 are satisfied with a constant* $c = 1$. *The choice of the control parameter* $\gamma_1 = 2$ *and the following matrices hold the condition* (6.23) *for the extended system:*

$$\bar{P}_p = \text{blkdiag}\{\bar{P}_1, \bar{P}_2\}$$

where

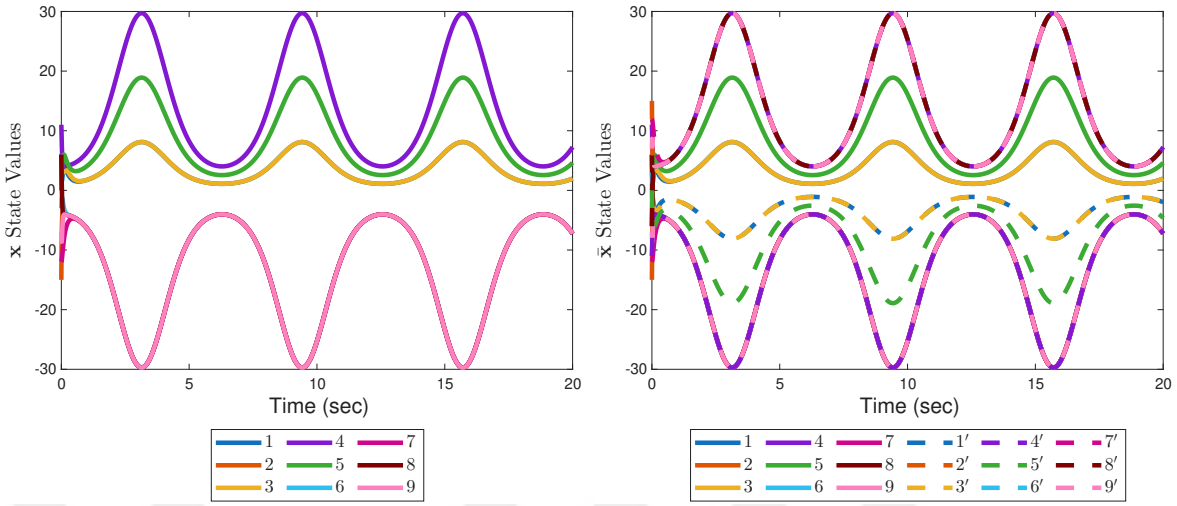
$$\bar{P}_1 = \begin{bmatrix} 0.0625 & 0 & -0.0117 & -0.0050 & 0 \\ 0 & 0.1250 & -0.0536 & -0.0268 & 0.0179 \\ -0.0117 & -0.0536 & 0.0482 & 0.0120 & -0.0116 \\ -0.0050 & -0.0268 & 0.0120 & 0.0322 & -0.0056 \\ 0 & 0.0179 & -0.0116 & -0.0056 & 0.0238 \end{bmatrix},$$

$$\bar{P}_2 = \begin{bmatrix} 0.0400 & -0.0229 & -0.0375 & 0.0140 & 0.0096 \\ -0.0229 & 0.0625 & 0 & -0.0058 & -0.0025 \\ -0.0375 & 0 & 0.1250 & -0.0464 & -0.0232 \\ 0.0140 & -0.0058 & -0.0464 & 0.0439 & 0.0132 \\ 0.0096 & -0.0025 & -0.0232 & 0.0132 & 0.0329 \end{bmatrix}$$

and

$$\bar{P}_s = \begin{bmatrix} 0.0192 & 0.0206 & 0.0226 & 0 & 0 & 0 \\ 0.0206 & 0.0831 & 0.0316 & 0 & 0 & 0 \\ 0.0226 & 0.0316 & 0.0708 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0192 & 0.0206 & 0.0226 \\ 0 & 0 & 0 & 0.0206 & 0.0831 & 0.0316 \\ 0 & 0 & 0 & 0.0226 & 0.0316 & 0.0708 \end{bmatrix}.$$

By Theorem 5.1, the first-order dynamics of the extended system reach cluster consensus for $\gamma_1 > 0.3329$. *When we simulate the original system (Figure 5.1) and its extended counterpart (Figure 5.2) for the initial states* $\mathbf{x}_0 = [6,\ -15,\ 4,\ 11,\ 0,\ -3,\ -12,\ 6, -7]^\mathsf{T}$



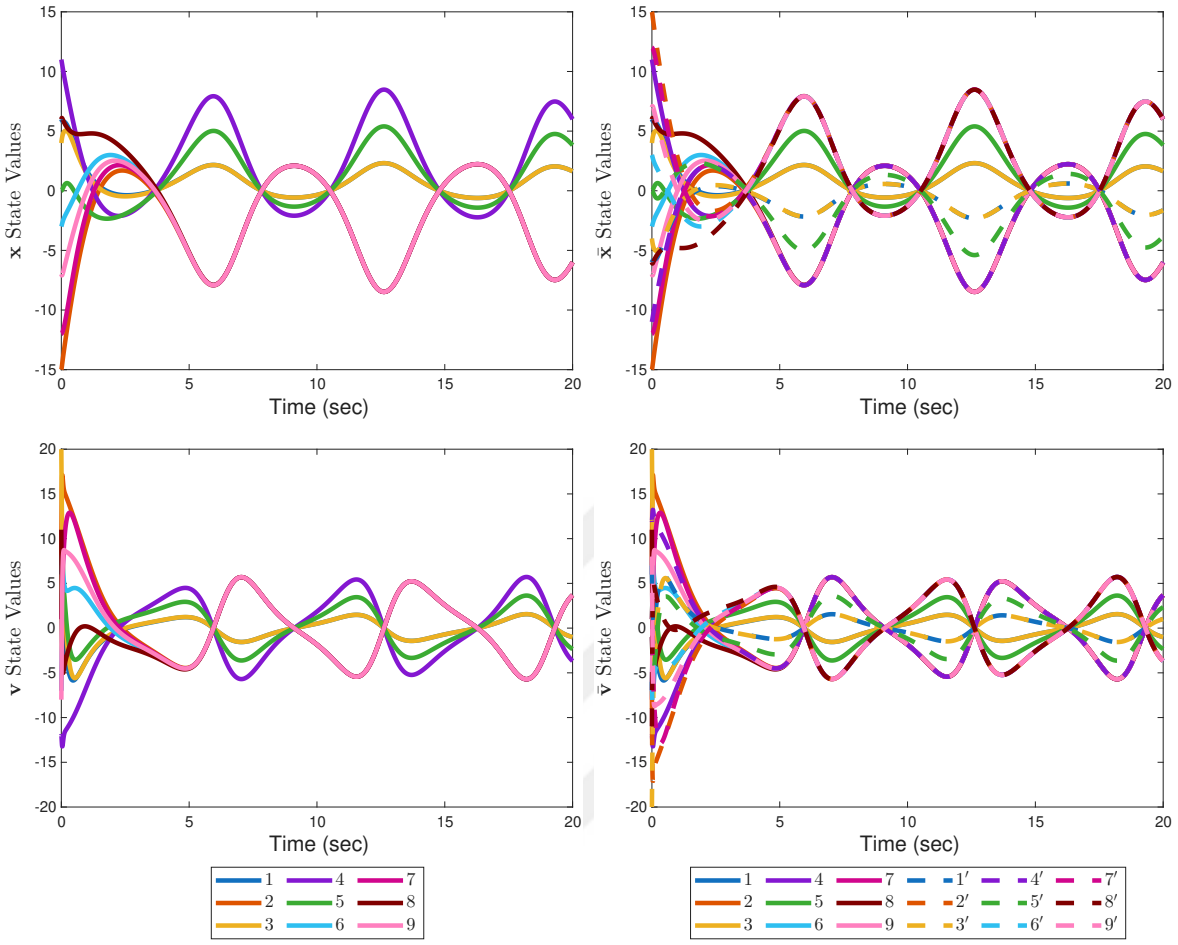
(a) Signed directed graph (b) Extended directed graph

Figure 5.3. State evolutions of first-order nonlinear dynamics for the networks in Figures 5.1 and 5.2.

and $\bar{\mathbf{x}}_0 = \left[\mathbf{x}_0^\mathsf{T},\ -\mathbf{x}_0^\mathsf{T}\right]^\mathsf{T}$*, the states trajectories can be shown as in Figure 5.3. The extended network acts as 6 distinct clusters* ($\bar{l}_p + \bar{l}_s = 2 + 4 = 6$)*. From Figure 5.2, we observe that there are* $l' = 2$ *subgraphs* ($\bar{\mathcal{V}}_{s,3} = \{5'\}$, $\bar{\mathcal{V}}_{s,4} = \{1', 3'\}$) *which do not include any agents belonging to the original system. The original state trajectories, as depicted in Figure 5.3(a), are recoverable from the extended system trajectories in Figure 5.3(b) through the application of Lemma 5.4. According to Theorem 5.3, the original system converges to 4 distinct clusters* ($\bar{l}_p + \bar{l}_s - l' = 2 + 4 - 2 = 4$)*.*

Example 5.2. ***(Second-order nonlinear dynamics)*** *We investigate a second-order consensus system with inherent nonlinear dynamics, as modeled in* (5.43)*. The nonlinear function is defined as* $f(x, v, t) = -x - v\sin(t)$*. With* $c_1 = 1$ *and* $c_2 = 1$*, Assumptions 5.2 and 5.4 are satisfied. By Theorem 5.2,* $\bar{P}_p$ *and* $\bar{P}_s$ *matrices can be chosen as in Example 5.1. The conditions on* γ_1 *and* γ_2 *for PLSs and SLSs of extended system can be obtained as*

$$\begin{aligned}\frac{\gamma_1}{\gamma_2^2} &< \min\left\{\frac{1}{2\lambda_M(\bar{P}_p)}, \frac{1}{2\lambda_M(\bar{P}_s)}\right\} - 2\\ &< \min\{3.0039, 4.2226\} - 2\\ &< 1.0039.\end{aligned}$$



(a) Signed directed graph (b) Extended directed graph

Figure 5.4. State evolutions of second-order nonlinear dynamics for the networks in Figures 5.1 and 5.2.

The simulation results presented in Figure 5.4 are obtained under the condition of $\gamma_1 = \gamma_2 = 2$ *for the initial conditions* $[\mathbf{x}_0^\mathsf{T}, \mathbf{v}_0^\mathsf{T}]^\mathsf{T} = [6,\ -15,\ 4,\ 11,\ 0,\ -3,\ -12,\ 6,\ -7,\ 7,\ -7,\ 20,\ -12,\ 1,\ 6,\ 1,\ 11,\ -8]^\mathsf{T}$ *and* $[\bar{\mathbf{x}}_0^\mathsf{T}, \bar{\mathbf{v}}_0^\mathsf{T}]^\mathsf{T} = [\mathbf{x}_0^\mathsf{T}, -\mathbf{x}_0^\mathsf{T}, \mathbf{v}_0^\mathsf{T}, -\mathbf{v}_0^\mathsf{T}]^\mathsf{T}$*, respectively. It can be seen that the extended network converges to 6 distinct clusters* ($\bar{l}_p + \bar{l}_s = 2+4 = 6$)*. Lemma 5.5 provides the methodology to reconstruct the original state trajectories from the corresponding extended system trajectories. As established by Theorem 5.4, the original system exhibits convergence to 4 distinct clusters* ($\bar{l}_p + \bar{l}_s - l' = 2+4-2 = 4$)*. Notably, the cluster configurations and members are identical to those of Example 5.1.*

5.4. Chapter Summary & Concluding Remarks

This chapter has addressed the cluster consensus problem for both first and second-order networks with inherent nonlinear dynamics described by any given unsigned or signed directed graph. First, we have analyzed convergence properties of the given nonlinear systems modeled via unsigned directed graph. Two different control parameter selection methods are provided for first and second-order systems. Second, we have investigated the connection between signed networks and their associated extended dynamic systems, and subsequently characterized their convergence behaviors. Through the implementation of this extended graph representation, we have successfully calculated the number of clusters and their members for a multi-agent network, without imposing any constraints on the directed graph topology. The obtained results in this chapter have been justified by two numerical examples. Up to this point, the cluster consensus problem has been studied for linear, delayed or inherently nonlinear networks with scalar communication weightings. In Chapter 6, the case where the communication weights are matrices rather than scalar is considered.

6. CLUSTER CONSENSUS IN MATRIX-WEIGHTED NETWORKS

This chapter investigates the matrix-weighted cluster consensus problem in directed networks without assuming restrictive topological conditions such as spanning trees or strong connectivity. Unlike previous studies, which focus primarily on scalar-weighted or undirected networks, this chapter develops a theoretical framework for matrix-weighted cluster consensus in systems where edge weights are represented by positive definite or negative definite matrices. This generalization enables the analysis of consensus dynamics in systems characterized by vector-valued states and complex interaction structures. We provide conditions to determine the number and structure of clusters that emerge in such networks by using the concepts of PLS and SLS. Additionally, the analysis is extended to discrete-time systems, where a distributed approach for step-size selection ensures stability and convergence to multiple clusters.

6.1. Matrix-Weighted Graph Representation

A multi-agent system employing a matrix weighted consensus protocol is modeled using a directed graph. Toward this goal, let $\mathbb{G} = (\mathcal{V}, \mathcal{E}, \mathbb{A})$ denote a matrix-weighted directed graph where $\mathcal{V} = \{1, 2, \ldots, n\}$ is the set of agents, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of directed edges that define communication structure, and $\mathbb{A} = [A_{ij}] \in \mathbb{R}^{nd \times nd}$ is the weight matrix associated with $\mathbb{G}$. A directed edge from agent j to agent i is denoted as $(j, i) \in \mathcal{E}$ and agent j is therefore considered a neighbor of agent i in its neighbor's set which is $\mathcal{N}_i = \{j : (j, i) \in \mathcal{E}\}$. If there exists communication link from agent j to agent i, A_{ij} is positive or negative definite matrix i.e., $A_{ij} \succ 0$ or $A_{ij} \prec 0$. Otherwise, A_{ij} is the zero matrix. It is assumed that A_{ii} is the zero matrix for $i \in \mathcal{V}$.

Considering a network of n agents with associated state vectors $\mathbf{x}_i(t) \in \mathbb{R}^d$, $i = 1, 2, \ldots, n$, the matrix-weighted consensus law is utilized as follows ([74, 76, 77, 80,

81, 83, 110]):

$$\dot{\mathbf{x}}_i(t) = \sum_{j\in\mathcal{N}_i} A_{ij}(\mathbf{x}_j(t) - \text{sgn}(A_{ij})\mathbf{x}_i(t)), \quad i = 1, 2, \ldots, n \tag{6.1}$$

where sgn$(\cdot)$ is defined for positive definite or negative definite weight matrices as follows: if $A_{ij} \succ 0$, sgn$(A_{ij}) = 1$, if $A_{ij} \prec 0$, sgn$(A_{ij}) = -1$ and $A_{ij} = 0$, sgn$(A_{ij}) = 0$ ([70, 74, 77, 80, 86]).

In matrix form, (6.1) is represented as

$$\dot{\mathbf{x}}(t) = -\mathbb{L}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{6.2}$$

where $\mathbf{x}(t) = [\mathbf{x}_1(t)^\mathsf{T}, \mathbf{x}_2(t)^\mathsf{T}, \ldots, \mathbf{x}_n(t)^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{nd}$ is the augmented state vector, $\mathbf{x}_0$ is the initial condition and $\mathbb{L} = [L_{ij}]$ is the matrix-weighted Laplacian matrix defined by

$$L_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{n} \text{sgn}(A_{ik})A_{ik}, & \text{if } i = j \\ -A_{ij}, & \text{otherwise} \end{cases} \tag{6.3}$$

Given a multi-agent system evolving over a directed graph $\mathbb{G} = (\mathcal{V}, \mathcal{E}, \mathbb{A})$ not necessarily containing a spanning tree, the objective of this chapter is to examine the convergence properties of (6.2) by investigating the topological properties of $\mathbb{G}$ and algebraic properties of $\mathbb{L}$ and to determine the number of clusters together with the cluster members. This concept is referred to as matrix-weighted cluster consensus whose definition is given below:

Definition 6.1. *(Matrix-weighted cluster consensus) For the multi-agent system utilizing the consensus law* (6.1)*, if there exist* $\mathcal{K}$ *vectors* $\mathbf{y}_p \in \mathbb{R}^d$ *and* $\mathcal{K}$ *non-empty distinct sets* $\mathcal{V}_p$ *(*$p = 1, 2, \ldots, \mathcal{K}$*) that satisfy*

$$\bigcup_{p=1}^{K} \mathcal{V}_p = \mathcal{V}, \quad \mathcal{V}_p \cap \mathcal{V}_q = \emptyset, \quad \textit{for } p \neq q, \textit{ and } p, q \in \{1, 2, \ldots, \mathcal{K}\}$$

such that $\lim_{t\to\infty}\|\mathbf{x}_i(t)-\mathbf{y}_p\|=0$ *for all* $i\in\mathcal{V}_p$*, then the multi-agent system* (6.1) *achieves cluster consensus.*

6.2. Matrix-Weighted Cluster Consensus in Continuous-Time

In this section, we first examine the matrix-weighted cluster consensus problem evolving over directed graphs without any topological assumptions where agents utilize update law (6.1) with positive-definite weighting matrices. Then we extend the results to continuous time matrix-weighted networks with positive-definite or negative-definite weighting matrices.

6.2.1. Networks with Positive Definite Weights

In this section, we analyze the matrix-weighted consensus algorithm (6.1) by utilizing the definitions of PLS and SLS in [13]. Given any multi-agent network with underlying matrix-weighted directed graph $\mathbb{G}=(\mathcal{V},\mathcal{E},\mathbb{A})$, its matrix-weighted Laplacian matrix can be reshaped by using row and column operations as follows:

$$\mathbb{L}=\begin{bmatrix}\mathbb{L}_p & 0_{n_pd\times n_sd}\\ \mathbb{L}_{sp} & \mathbb{L}_s\end{bmatrix} \tag{6.4}$$

where

$$\mathbb{L}_p=\text{blkdiag}\{\mathbb{L}_{1,1},\dots,\mathbb{L}_{l_p,l_p}\},\quad \mathbb{L}_{sp}=\begin{bmatrix}\mathbb{L}_{l_p+1,1} & \dots & \mathbb{L}_{l_p+1,l_p}\\ \vdots & \ddots & \vdots\\ \mathbb{L}_{l_p+l_s,1} & \dots & \mathbb{L}_{l_p+l_s,l_p}\end{bmatrix},$$

$$\mathbb{L}_s=\begin{bmatrix}\mathbb{L}_{l_p+1,l_p+1} & \dots & \mathbb{L}_{l_p+1,l_p+l_s}\\ \vdots & \ddots & \vdots\\ \mathbb{L}_{l_p+l_s,l_p+1} & \dots & \mathbb{L}_{l_p+l_s,l_p+l_s}\end{bmatrix}.$$

Note that in the first l_p rows of $\mathbb{L}$, only the diagonal blocks are nonzero and each $\mathbb{L}_{i,i}$, $i=1,2,\dots,l_p$ matrix is also a matrix-weighted Laplacian matrix. Let $n_{p,i}$ and

$n_{s,j}$ be the number of agents in the i-th PLS and j-th SLS for $i = 1, 2, \ldots, l_p$ and $j = 1, 2, \ldots, l_s$, respectively. We express by $n_p = \sum_{i=1} n_{p,i}$ and $n_s = \sum_{j=1} n_{s,j}$, the total number of agents in the PLSs and SLSs, respectively. Then, we can rewrite (6.2) as

$$\begin{bmatrix} \dot{\mathbf{x}}_p(t) \\ \dot{\mathbf{x}}_s(t) \end{bmatrix} = - \begin{bmatrix} \mathbb{L}_p & 0_{n_p d \times n_s d} \\ \mathbb{L}_{sp} & \mathbb{L}_s \end{bmatrix} \begin{bmatrix} \mathbf{x}_p(t) \\ \mathbf{x}_s(t) \end{bmatrix} \tag{6.5}$$

where $\mathbf{x}_p(t) \in \mathbb{R}^{n_p d}$ and $\mathbf{x}_s(t) \in \mathbb{R}^{n_s d}$ are the state vectors for the PLSs and SLSs, respectively.

In undirected graphs, the matrix-weighted Laplacian matrix $\mathbb{L}$ is symmetric and has exactly dl_p zero eigenvalues and other eigenvalues are positive [111]. Out of this case, i.e., the graph $\mathbb{G}$ is directed, the matrix-weighted Laplacian matrix $\mathbb{L}$ may not be symmetric. Therefore, the spectral properties of $\mathbb{L}$ in directed graphs may be different from those in undirected graphs. The following example clarifies this concept.

Example 6.1. *Consider the unsigned network in Figure 6.1 with $d = 2$. According to the PLS and SLS definitions, the network has $l_p = 2$ PLSs ($\mathcal{V}_{p,1} = \{2\}, \mathcal{V}_{p,2} = \{4\}$) and $l_s = 2$ SLSs ($\mathcal{V}_{s,1} = \{1\}, \mathcal{V}_{s,2} = \{3\}$).*

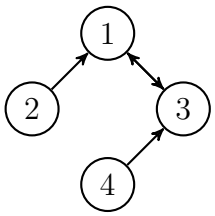


Figure 6.1. An unsigned network with 4 agents and 4 edges

The positive definite matrix-weights are chosen as

$$A_{12} = \begin{bmatrix} 4 & -4 \\ -4 & 5 \end{bmatrix}, \; A_{13} = \begin{bmatrix} 5 & -1 \\ -1 & 1 \end{bmatrix}, \; A_{31} = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}, \; A_{34} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}.$$

From the above symmetric positive definite weights, one can obtain the following non-

symmetric matrix-weighted Laplacian matrix:

$$\mathbb{L} = \begin{bmatrix} 9 & -5 & -4 & 4 & -5 & 1 & 0 & 0 \\ -5 & 6 & 4 & -5 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -5 & -4 & 0 & 0 & 6 & 6 & -1 & -2 \\ -4 & -5 & 0 & 0 & 6 & 10 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The eigenvalues of the matrix-weighted Laplacian matrix, $\mathbb{L}$*, obtained from the above matrix-weights are* $\{15.9517,\ 12.5360,\ 2.7115,\ 0,\ 0,\ 0,\ 0,\ -0.1992\}$*. Although we have* $dl_p = 4$ *zero eigenvalues, one of the other eigenvalues is negative.*

Building upon the above example, it is necessary to make the following assumption ([66, 90]):

Assumption 6.1. *The matrix-weighted Laplacian matrix* $\mathbb{L}$ *has* dl_p *zero eigenvalues, and the real parts of the other eigenvalues are positive for directed graph* $\mathbb{G}$*.*

Remark 6.1. Under Assumption 6.1, each $\mathbb{L}_{i,i}$ $(i = 1, 2, \ldots, l_p)$ contains d zero eigenvalues and the real parts of the other eigenvalues are positive so that $\mathbb{L}_p$ have dl_p zero eigenvalues. Furthermore, all eigenvalues of $\mathbb{L}_s$ has positive real parts. Note that this assumption can be satisfied for weight matrices where each agent uses a scalar multiple of the same positive definite matrix as illustrated in the following lemma.

Lemma 6.1. *Let* $P \in \mathbb{R}^{d\times d}$ *be a positive definite matrix and let the weight matrices of the network be chosen as* $A_{ij} = c_{ij}P$ *where* $c_{ij} > 0$ *if* $(j, i) \in \mathcal{E}$ *and* $c_{ij} = 0$ *otherwise. Then the matrix-weighted Laplacian matrix* $\mathbb{L}$ *of the network satisfies Assumption 6.1.*

Proof. The matrix-weighted Laplacian matrix of the network can be written as $\mathbb{L} =$

$D \otimes P$ where the elements of $D = [d_{ij}]$ are as follows

$$d_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{n} c_{ik}, & \text{if } i = j \\ -c_{ij}, & \text{otherwise} \end{cases} . \tag{6.6}$$

Since D is a Laplacian matrix of an unsigned system, l_p of its eigenvalues are zero and the remaining have positive real parts [98]. Since the eigenvalues of $D \otimes P$ are the products of eigenvalues of D and P [112], we conclude that the multiplicity of the eigenvalue 0 of $\mathbb{L}$ is equal to dl_p and the rest have positive real parts. □

Before presenting the first result of this chapter, it will be useful to introduce the following lemma:

Lemma 6.2. *Let* $\tilde{B} = \left[\begin{array}{c|c} 0_{d\times d} & 0_{d\times nd} \\ \hline * & B \end{array}\right]$ *be an* $(n+1)d \times (n+1)d$ *matrix-weighted Laplacian matrix satisfying Assumption 6.1 where* $B = [B_{ij}]_{n\times n}$ *with* $B_{ij} \in \mathbb{R}^{d\times d}$ *is an* $nd \times nd$ *matrix that satisfies* $\sum_{j=1}^{n} B_{ij} = 0_{d\times d}$ *for all* i *except for* $i = i^*$ *where* i^* *has* $\sum_{j=1}^{n} B_{i^*j} = \Phi \succ 0$. *Then*

(i) $\rho(B) \subset ORHP$

(ii) The i^**th column block of* B^{-1} *is equal to* Φ^{-1}.

Proof. The first result (i) follows from Assumption 6.1. The eigenvalues of $\tilde{B}$ can be determined as

$$\rho(\tilde{B}) = \rho(0_{d\times d}) \bigcup \rho(B).$$

According to Assumption 6.1, $\tilde{B}$ has d zero eigenvalues and the real parts of the other eigenvalues are positive. Therefore, the eigenvalues of B are in the open right half complex plane (ORHP).

For the proof of the second result (ii), consider the following equation:

$$B\begin{bmatrix} I_d \\ I_d \\ \vdots \\ I_d \end{bmatrix} = \begin{bmatrix} \Phi & 0_{d\times d} & \dots & 0_{d\times d} \\ 0_{d\times d} & I_d & \dots & 0_{d\times d} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{d\times d} & 0_{d\times d} & \dots & I_d \end{bmatrix}\begin{bmatrix} I_d \\ 0_{d\times d} \\ \vdots \\ 0_{d\times d} \end{bmatrix}.$$

Without loss of generality, we assume that $i^* = 1$. One can obtain the following equation:

$$\begin{bmatrix} I_d \\ I_d \\ \vdots \\ I_d \end{bmatrix} = B^{-1}\begin{bmatrix} \Phi & 0_{d\times d} & \dots & 0_{d\times d} \\ 0_{d\times d} & I_d & \dots & 0_{d\times d} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{d\times d} & 0_{d\times d} & \dots & I_d \end{bmatrix}\begin{bmatrix} I_d \\ 0_{d\times d} \\ \vdots \\ 0_{d\times d} \end{bmatrix},$$

$$\begin{bmatrix} I_d \\ I_d \\ \vdots \\ I_d \end{bmatrix} = B^{-1}\begin{bmatrix} \Phi \\ 0_{d\times d} \\ \vdots \\ 0_{d\times d} \end{bmatrix}.$$

Hence, the result follows. □

For a matrix-weighted network with positive definite weighting matrices, the following theorem states the relationship between the number of clusters and the properties of the network graph in terms of its PLSs and SLSs.

Theorem 6.1. *Under Assumption 6.1, the system* (6.1) *underlying unsigned directed graph* $\mathbb{G}$ *forms* $\mathcal{K} = l_p + l_s$ *clusters in the directed network where* l_p *and* l_s *are the number of PLSs and SLSs, respectively.*

Proof. From (6.5), the PLS and SLS dynamics can be rewritten in the following form:

$$\dot{\mathbf{x}}_p(t) = -\mathbb{L}_p\mathbf{x}_p(t), \tag{6.7a}$$

$$\dot{\mathbf{x}}_s(t) = -\mathbb{L}_{sp}\mathbf{x}_p(t) - \mathbb{L}_s\mathbf{x}_s(t). \tag{6.7b}$$

We examine the stability and equilibrium conditions for PLSs and SLSs individually, addressing each in consecutive steps.

Due to block diagonal structure of $\mathbb{L}_p$, each PLS dynamics can be written as

$$\mathbf{x}_{p,i}(t) = -\mathbb{L}_{i,i}\mathbf{x}_{p,i}(t), \quad i = 1, 2, \ldots, l_p$$

where $\mathbf{x}_{p,i}(t)$ is the state vector for the i-th PLS. From [91], l_p PLSs converges to l_p clusters.

We have $\rho(\mathbb{L}_s) \subset ORHP$ by Assumption 6.1. Since we also have $\mathbf{x}_p(t)$ converging to constant vectors, the system (6.7b) is stable. The equilibria of the SLS dynamics can be calculated as

$$\mathbf{x}_s(\infty) = -\mathbb{L}_s^{-1}\mathbb{L}_{sp}\mathbf{x}_p(\infty) \tag{6.8}$$

where $\mathbf{x}_p(\infty)$ and $\mathbf{x}_s(\infty)$ are the steady-states of the PLSs and SLSs, respectively. It can be shown that each row $-\mathbb{L}_s^{-1}\mathbb{L}_{sp}$ adds up to 1. Hence, we conclude that the components of $\mathbf{x}_s(\infty)$ are convex combinations of the components of $\mathbf{x}_p(\infty)$.

In this step, we show that each SLS attains different final values. The SLS dynamics can be expressed as follows:

$$\dot{\mathbf{x}}_{s,i}(t) = -\mathbb{L}_{l_p+i,l_p+i}\mathbf{x}_{s,i}(t) - \sum_{j=1,j\neq i}^{l_s} \mathbb{L}_{l_p+i,l_p+j}\mathbf{x}_{s,j}(t) - \sum_{j=1}^{l_p} \mathbb{L}_{l_p+i,j}\mathbf{x}_{p,j}(t) \tag{6.9}$$

for $i = 1, 2, \ldots, l_s$. The steady state values of the SLSs can be calculated as

$$\mathbf{x}_{s,i}(\infty) = -\sum_{j=1, j\neq i}^{l_s} (\mathbb{L}_{l_p+i,l_p+i})^{-1}\mathbb{L}_{l_p+i,l_p+j}\mathbf{x}_{s,j}(\infty) - \sum_{j=1}^{l_p}(\mathbb{L}_{l_p+i,l_p+i})^{-1}\mathbb{L}_{l_p+i,j}\mathbf{x}_{p,j}(\infty). \tag{6.10}$$

From Lemma 6.2, one can note that $(\mathbb{L}_{l_p+i,l_p+i})^{-1}\mathbb{L}_{l_p+i,l_p+j}$ and $(\mathbb{L}_{l_p+i,l_p+i})^{-1}\mathbb{L}_{l_p+i,j}$ have identical row matrices. Hence, the final values of the SLSs can be computed as

$$\mathbf{x}_{s,i}(\infty) = c_i, \quad i = 1, 2, \ldots, l_s$$

which implies that agents in a given SLS agree on a specific value, necessarily equal to the consensus value of any other layer. Consequently, the SLS dynamics lead to l_s distinct clusters for arbitrary choice of adjacency matrix. From the previous calculations, the total number of clusters is given by

$$\mathcal{K} = l_p + l_s.$$

□

Theorem 6.1 guarantees that, when Assumption 6.1 holds, any unsigned network generated from a structurally unrestricted signed network achieves cluster consensus.

6.2.2. Extension to Networks with Positive and Negative Definite Weights

In this section we extend the above results to multi-agent systems with positive definite and negative definite weighting matrices. Towards this goal, we generalize the extended graph notion in networks with scalar weights [36] to the matrix-weighted networks as follows.

Definition 6.2. *Given a matrix-weighted directed graph* $\mathbb{G} = (\mathcal{V}, \mathcal{E}, \mathbb{A})$ *where* $\mathcal{V} = \{1, 2, \ldots, n\}$ *is the set of agents,* $\mathcal{E}$ *is the set of edges and* $\mathbb{A} = [A_{ij}]$ *are the matrix-*

weights, its extended graph is defined as the graph $\bar{\mathbb{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}}, \bar{\mathbb{A}})$ *where* $\bar{\mathcal{V}} = \{1, 2, \ldots, n, 1', 2' \ldots, n'\}$*, and* $\bar{\mathcal{E}}, \bar{\mathbb{A}}$ *are defined as follows:*

1) If $(j, i) \in \mathcal{E}$ *and* $A_{ij} \succ 0$*, then* $\{(j, i), (j', i')\} \subset \bar{\mathcal{E}}$*.*
2) If $(j, i) \in \mathcal{E}$ *and* $A_{ij} \prec 0$*, then* $\{(j, i'), (j', i)\} \subset \bar{\mathcal{E}}$*.*
3) If $(j, i) \notin \mathcal{E}$*, then* $(j, i) \notin \bar{\mathcal{E}}$ *and* $(j', i') \notin \bar{\mathcal{E}}$*.*
4) The matrix-weights, $\bar{A}_{ij}$*, of the extended graph* $\bar{\mathbb{G}}$ *are calculated as*

$$\bar{A}_{ij} = \bar{A}_{i+n,j+n} = \frac{1}{2} A_{ij}(sgn(A_{ij}) + 1) \text{ and}$$
$$\bar{A}_{i+n,j} = \bar{A}_{i,j+n} = \frac{1}{2} A_{ij}(sgn(A_{ij}) - 1) \text{ for } i, j \in \{1, 2, \ldots, n\}.$$

Remark 6.2. Note from Definition 6.2 that the original graph may contain negative definite weights along with positive definite weights. On the other hand, the extended graph consists of purely positive definite weights.

Equipped with Definition 2 and given a multi-agent network utilizing matrix-weighted consensus algorithm (6.1) with positive definite and negative definite weight matrices, we introduce the associated extended multi-agent system with the following dynamics:

$$\dot{\bar{\mathbf{x}}}_i(t) = \sum_{j \in \bar{\mathcal{N}}_i} \bar{A}_{ij}(\bar{\mathbf{x}}_j(t) - \bar{\mathbf{x}}_i(t)), \quad i = 1, 2, \ldots, 2n. \tag{6.11}$$

Note that (6.11) can be expressed in matrix form as

$$\dot{\bar{\mathbf{x}}}(t) = -\bar{\mathbb{L}}\bar{x}(t), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \tag{6.12}$$

where $\bar{\mathbf{x}}(t) = [\bar{\mathbf{x}}_1(t)^\mathsf{T}, \bar{\mathbf{x}}_2(t)^\mathsf{T}, \ldots, \bar{\mathbf{x}}_{2n}(t)^\mathsf{T}]^\mathsf{T}$ is the augmented state vector of the extended graph and $\bar{\mathbf{x}}_0$ is the initial condition.

The following lemma reveals that the trajectories of (6.1) can be obtained from those of (6.11) by choosing the initial conditions properly.

Lemma 6.3. *The trajectories* $\mathbf{x}(t)$ *of the agents in system* (6.1) *can be recovered from those in its extended system* (6.11) *as* $\mathbf{x}_i(t) = \bar{\mathbf{x}}_i(t)$, $i = 1, 2, \ldots, n$, *for all* $t \geq 0$ *by setting* $\bar{\mathbf{x}}_{i0} = \mathbf{x}_{i0}$ *and* $\bar{\mathbf{x}}_{i+n,0} = -\mathbf{x}_{i0}$ *for all* $i = 1, 2, \ldots, n$.

Proof. The extended system represented by dynamics (6.11) can be re-expressed as

$$\begin{aligned}\dot{\bar{\mathbf{x}}}_i(t) &= \sum_{j\in\bar{\mathcal{N}}_i} \bar{A}_{ij}(\bar{\mathbf{x}}_j(t) - \bar{\mathbf{x}}_i(t)) \\ &= \sum_{j=1}^{n} \bar{A}_{ij}(\bar{\mathbf{x}}_j(t) - \bar{\mathbf{x}}_i(t)) + \sum_{j=1}^{n} \bar{A}_{i,j+n}(\bar{\mathbf{x}}_{j+n}(t) - \bar{\mathbf{x}}_i(t))\end{aligned} \tag{6.13}$$

for $i = 1, 2, \ldots, n$. Let $\hat{\mathbf{x}}_i(t) = \bar{\mathbf{x}}_i(t) + \bar{\mathbf{x}}_{i+n}(t)$ for $i = 1, 2, \ldots, n$ whose dynamics can be formulated as

$$\begin{aligned}\dot{\hat{\mathbf{x}}}_i(t) &= \dot{\bar{\mathbf{x}}}_i(t) + \dot{\bar{\mathbf{x}}}_{i+n}(t) \\ &= \sum_{j=1}^{n} \bar{A}_{ij}(\bar{\mathbf{x}}_j(t) - \bar{\mathbf{x}}_i(t)) + \sum_{j=1}^{n} \bar{A}_{i,j+n}(\bar{\mathbf{x}}_{j+n}(t) - \bar{\mathbf{x}}_i(t)) \\ &\quad + \sum_{j=1}^{n} \bar{A}_{i+n,j}(\bar{\mathbf{x}}_j(t) - \bar{\mathbf{x}}_{i+n}(t)) + \sum_{j=1}^{n} \bar{A}_{i+n,j+n}(\bar{\mathbf{x}}_{j+n}(t) - \bar{\mathbf{x}}_{i+n}(t)).\end{aligned}$$

By using Definition 6.2, we have $\bar{A}_{i,j+n} + \bar{A}_{i+n,j+n} = \bar{A}_{ij} + \bar{A}_{i,j+n} = \bar{A}_{i+n,j} + \bar{A}_{i+n,j+n} = \operatorname{sgn}(A_{ij})A_{ij}$ which can be utilized to obtain

$$\dot{\hat{\mathbf{x}}}_i(t) = \sum_{j=1}^{n} \operatorname{sgn}(A_{ij})A_{ij}(\hat{\mathbf{x}}_j(t) - \hat{\mathbf{x}}_i(t)) \tag{6.14}$$

or equivalently

$$\dot{\hat{\mathbf{x}}}(t) = -\hat{\mathbb{L}}\hat{\mathbf{x}}(t), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \tag{6.15}$$

where $\hat{\mathbf{x}}(t) = [\hat{\mathbf{x}}_1(t)^\mathsf{T}, \hat{\mathbf{x}}_2(t)^\mathsf{T}, \ldots, \hat{\mathbf{x}}_n(t)^\mathsf{T}]^\mathsf{T}$, $\hat{\mathbf{x}}_0$ is the initial condition and the matrix

$\hat{\mathbb{L}} = [\hat{L}_{ij}]$ is defined by

$$\hat{L}_{ij} = \begin{cases} \sum\limits_{k=1,k\neq i}^{n} \operatorname{sgn}(A_{ik})A_{ik}, & \text{if } i = j \\ -\operatorname{sgn}(A_{ij})A_{ij}, & \text{otherwise} \end{cases}.$$

Note that the solution of (6.15) is $\hat{\mathbf{x}}(t) = e^{-\hat{\mathbb{L}}t}\hat{\mathbf{x}}_0$. Since the choice of initial conditions for the states $\bar{\mathbf{x}}_i$ yields $\hat{\mathbf{x}}(0) = [0, \ldots, 0]^\mathsf{T}$, we conclude that $\hat{\mathbf{x}}(t) = [0, \ldots, 0]^\mathsf{T}$ for all t which implies $\bar{\mathbf{x}}_i(t) = -\bar{\mathbf{x}}_{i+n}(t)$ for $i = 1, 2, \ldots, n$. By substituting $\bar{\mathbf{x}}_{j+n}(t) = -\bar{\mathbf{x}}_j(t)$ into (6.13), we have

$$\dot{\bar{\mathbf{x}}}_i(t) = \sum_{j=1}^{n} \left((\bar{A}_{ij} - \bar{A}_{i,j+n})\bar{\mathbf{x}}_j(t) - (\bar{A}_{ij} + \bar{A}_{i,j+n})\bar{\mathbf{x}}_i(t) \right)$$

which can be simplified as

$$\dot{\bar{\mathbf{x}}}_i(t) = \sum_{j=1}^{n} A_{ij} \left(\bar{\mathbf{x}}_j(t) - \operatorname{sgn}(A_{ij})\bar{\mathbf{x}}_i(t) \right) \tag{6.16}$$

for $i = 1, 2, \ldots, n$. Since (6.16) and (6.1) are equivalent, we conclude that the trajectory $\mathbf{x}(t)$ can be recovered by setting $\bar{\mathbf{x}}_{i0} = \mathbf{x}_{i0}$ and $\bar{\mathbf{x}}_{i+n,0} = -\mathbf{x}_{i0}$ for all $i = 1, 2, \ldots, n$ which completes the proof. □

Lemma 6.3 establishes a relationship between the state of trajectories of the signed system (6.1) and its extended system (6.11) derived through Definition 6.2. By leveraging this connection, our subsequent analysis focuses on the stability and cluster structure of the extended system (6.11) to infer the clustering properties of the original signed system (6.1).

With the extended graph representation, a matrix-weighted network consisting of n agents can be transformed into a larger network consisting of solely positive definite weight matrices which allows us to use Theorem 6.1. Furthermore, when the dynamics of the extended network are chosen as in (6.11), Lemma 6.3 indicates that the analysis

of the original graph can be performed through the extended network.

Assumption 6.2. *The matrix-weighted Laplacian matrix $\bar{\mathbb{L}}$ has $d\bar{l}_p$ zero eigenvalues, and the real parts of the other eigenvalues are positive for directed graph $\bar{\mathbb{G}}$.*

Theorem 6.2. *Under Assumption 6.2, the system* (6.11) *underlying directed graph $\bar{\mathbb{G}}$ forms $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ clusters in the directed network where $\bar{l}_p$ and $\bar{l}_s$ are the number of PLSs and SLSs, respectively.*

Proof. Since $\bar{\mathbb{G}}$ consists of purely positive definite matrix weights, the proof directly follows from Theorem 6.1. □

Theorem 6.2 guarantees that, when Assumption 6.2 holds, any unsigned network generated from a structurally unrestricted signed network achieves cluster consensus. Utilizing Lemma 6.3 and Theorem 6.2 leads to the following subsequent theorem which allows us to determine the number of clusters to networks with positive or negative definite weight matrices.

Theorem 6.3. *Under Assumption 6.2, suppose that underlying directed graph $\mathbb{G}$ of the matrix-weighted directed network consists of positive and negative definite matrix weights. Then the agents utilizing* (6.1) *forms $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ clusters in the directed network where $\bar{l}_p$ and $\bar{l}_s$ are the number of PLSs and SLSs of the extended system, respectively; and l' is the number of subgraphs whose agent set consists of only a subset of $\bar{\mathcal{V}} \setminus \mathcal{V} = \{1', 2', \ldots, n'\}$.*

Proof. From Theorem 6.2 the extended system that is representation of the signed system converges to $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ clusters. Since l' subgraphs in $\bar{l}_p + \bar{l}_s$ subgraphs do not contain any agent of the signed system, the number of clusters for signed system is found as $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$. □

6.3. Extension to the Discrete-Time Systems

6.3.1. Networks with Positive Definite Weights

Consider a multi-agent system utilizing the following matrix-weighted consensus protocol in discrete-time:

$$\mathbf{x}_i(k+1) = \mathbf{x}_i(k) + u_i(k), \quad i = 1, 2, \ldots, n \tag{6.17}$$

where $\mathbf{x}_i(k) \in \mathbb{R}^d$ is the state of agent i at time step k. We introduce the control input of agent i as follows:

$$u_i(k) = \alpha \sum_{j \in \mathcal{N}_i} A_{ij}(\mathbf{x}_j(k) - \operatorname{sgn}(A_{ij})\mathbf{x}_i(k)) \tag{6.18}$$

where $\alpha > 0$ is the step size. We can express (6.17) together with (6.18) in matrix form as

$$\mathbf{x}(k+1) = (I_{nd} - \alpha\mathbb{L})\mathbf{x}(k), \quad \mathbf{x}(0) = \mathbf{x}_0$$

or equivalently

$$\mathbf{x}(k+1) = \mathbb{W}\mathbf{x}(k), \quad \mathbf{x}(0) = \mathbf{x}_0$$

where $\mathbf{x}(k) = [\mathbf{x}_1(k)^\mathsf{T}, \mathbf{x}_2(k)^\mathsf{T}, \ldots, \mathbf{x}_n(k)^\mathsf{T}]^\mathsf{T}$ is the augmented state vector, $\mathbb{W} = [W_{ij}]$ is the system matrix and $\mathbf{x}_0$ is the initial condition.

The matrix-weighted system matrix can be reshaped by using row and column operations as follows:

$$\mathbb{W} = \begin{bmatrix} \mathbb{W}_p & 0_{n_p d \times n_s d} \\ \mathbb{W}_{sp} & \mathbb{W}_s \end{bmatrix} \tag{6.19}$$

where

$$\mathbb{W}_p = \text{blkdiag}\{\mathbb{W}_{1,1}, \ldots, \mathbb{W}_{l_p,l_p}\}, \quad \mathbb{W}_{sp} = \begin{bmatrix} \mathbb{W}_{l_p+1,1} & \cdots & \mathbb{W}_{l_p+1,l_p} \\ \vdots & \ddots & \vdots \\ \mathbb{W}_{l_p+l_s,1} & \cdots & \mathbb{W}_{l_p+l_s,l_p} \end{bmatrix},$$

$$\mathbb{W}_s = \begin{bmatrix} \mathbb{W}_{l_p+1,l_p+1} & \cdots & \mathbb{W}_{l_p+1,l_p+l_s} \\ \vdots & \ddots & \vdots \\ \mathbb{W}_{l_p+l_s,l_p+1} & \cdots & \mathbb{W}_{l_p+l_s,l_p+l_s} \end{bmatrix}.$$

Note that in the first l_p rows of $\mathbb{W}$, only the diagonal blocks are nonzero and each $\mathbb{W}_{i,i}$, $i = 1, 2, \ldots, l_p$ matrix is also a row-stochastic matrix. We can rewrite (6.17) as

$$\begin{bmatrix} \mathbf{x}_p(k+1) \\ \mathbf{x}_s(k+1) \end{bmatrix} = \begin{bmatrix} \mathbb{W}_p & 0_{n_p d \times n_s d} \\ \mathbb{W}_{sp} & \mathbb{W}_s \end{bmatrix} \begin{bmatrix} \mathbf{x}_p(k) \\ \mathbf{x}_s(k) \end{bmatrix} \tag{6.20}$$

where $\mathbf{x}_p(k) \in \mathbb{R}^{n_p d}$ and $\mathbf{x}_s(k) \in \mathbb{R}^{n_s d}$ are the state vectors for the PLSs and SLSs, respectively.

Before presenting the results on discrete-time matrix weighted networks, it will be useful to provide the following lemma:

Lemma 6.4. *Let* $\tilde{D} = I_{(n+1)d} - \alpha\tilde{B} = \left[\begin{array}{c|c} I_d & 0_{d\times nd} \\ \hline * & D \end{array}\right]$ *be an* $(n+1)d \times (n+1)d$ *matrix with* $\tilde{B} = \left[\begin{array}{c|c} 0_{d\times d} & 0_{d\times nd} \\ \hline * & B \end{array}\right]$ *satisfying Assumption 6.1 where* $D = [D_{ij}]_{n\times n}$ *with* $D_{ij} \in \mathbb{R}^{d\times d}$ *be an* $nd \times nd$ *matrix that satisfies* $\sum_{j=1}^{n} D_{ij} = I_d$ *for all* i *except for* $i = i^*$ *where* i^* *has* $\sum_{j=1}^{n} D_{i^*j} = I_d - \alpha\xi$ *where* α *is a positive scalar and* $\xi \succ 0$. *Then*

(i) All eigenvalues of D *are strictly less than one in magnitude if and only if the following holds:*

$$\alpha < \min_i \frac{\mathbb{R}\text{e}\{\mu_i\}}{|\mu_i|^2}$$

where μ_i are nonzero eigenvalues of B.

(ii) The i^*th column block of $(I_{nd} - D)^{-1}$ is equal to $(\alpha\xi)^{-1}$.

Proof. The first result follows from Assumption 6.1 and Lemma 6.2. According to Lemma 6.2, the eigenvalues of B are in the ORHP. The eigenvalues of D can be determined as

$$\rho\left(D\right) = \rho\left(I_{nd} - \alpha B\right) = 1 - \alpha\mu_i$$

where μ_i are nonzero eigenvalues of B. For the eigenvalues of D to be strictly one in magnitude, the following condition must be met:

$$|1 - \alpha(\mathbb{R}\mathrm{e}\{\mu_i\} + j\,\mathbb{I}\mathrm{m}\{\mu_i\})| < 1.$$

The upper bound of α can be obtained as follows:

$$\alpha < \min_i \frac{2\,\mathbb{R}\mathrm{e}\{\mu_i\}}{|\mu_i|^2}. \tag{6.21}$$

For the proof of the second result (ii), consider the following equation:

$$(I_{nd} - D)\begin{bmatrix} I_d \\ I_d \\ \vdots \\ I_d \end{bmatrix} = \begin{bmatrix} \alpha\xi & 0_{d\times d} & \dots & 0_{d\times d} \\ 0_{d\times d} & I_d & \dots & 0_{d\times d} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{d\times d} & 0_{d\times d} & \dots & I_d \end{bmatrix}\begin{bmatrix} I_d \\ 0_{d\times d} \\ \vdots \\ 0_{d\times d} \end{bmatrix}.$$

Without loss of generality, we assume that $i^* = 1$. One can obtain the following equation:

$$\begin{bmatrix} I_d \\ I_d \\ \vdots \\ I_d \end{bmatrix} = (I_{nd} - D)^{-1} \begin{bmatrix} \alpha\xi & 0_{d\times d} & \dots & 0_{d\times d} \\ 0_{d\times d} & I_d & \dots & 0_{d\times d} \\ \vdots & \vdots & \ddots & \vdots \\ 0_{d\times d} & 0_{d\times d} & \dots & I_d \end{bmatrix}, \begin{bmatrix} I_d \\ 0_{d\times d} \\ \vdots \\ 0_{d\times d} \end{bmatrix}$$

$$\begin{bmatrix} I_d \\ I_d \\ \vdots \\ I_d \end{bmatrix} = (I_{nd} - D)^{-1} \begin{bmatrix} \alpha\xi \\ 0_{d\times d} \\ \vdots \\ 0_{d\times d} \end{bmatrix}.$$

Hence, the result follows. □

For a matrix-weighted network with positive definite weighting matrices, the following theorem states the relationship between the number of clusters and the properties of the network graph in terms of its PLSs and SLSs.

Theorem 6.4. *Under Assumption 6.1, the system* (6.17) *with underlying directed graph* $\mathcal{G}$ *forms* $\mathcal{K} = l_p + l_s$ *clusters in the directed network if and only if positive constant* α *satisfies*

$$\alpha < \min_i \frac{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}}{|\lambda_i|^2} \tag{6.22}$$

where l_p *and* l_s *are the number of PLSs and SLSs, respectively; and* λ_i *are nonzero eigenvalues of the matrix-weighted Laplacian matrix* $\mathbb{L}$.

Proof. The stability of the system (6.17) implies that non-unity eigenvalues of $(I_{nd} - \alpha\mathbb{L})$ are inside the unit circle. Therefore, the stability condition is equivalent to the following inequality:

$$|1 - \alpha(\mathbb{R}\mathrm{e}\{\lambda_i\} + j\,\mathbb{I}\mathrm{m}\{\lambda_i\})| < 1$$

where λ_i are the nonzero eigenvalues of $\mathbb{L}$. The upper bound of α can be calculated as follows:

$$\alpha < \min_i \frac{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}}{|\lambda_i|^2}. \tag{6.23}$$

Under (6.23) holds, the PLS and SLS dynamics can be rewritten in the following form from (6.20):

$$\mathbf{x}_p(k+1) = \mathbb{W}_p \mathbf{x}_p(k), \tag{6.24a}$$

$$\mathbf{x}_s(k+1) = \mathbb{W}_{sp} \mathbf{x}_p(k+1) + \mathbb{W}_s \mathbf{x}_s(k). \tag{6.24b}$$

We investigate separately the stability and equilibrium conditions for the PLSs and SLSs in consecutive steps.

Due to block diagonal structure of $\mathbb{W}_p$, each PLS dynamics can be written as

$$\mathbf{x}_{p,i}(k+1) = \mathbb{W}_{i,i} \mathbf{x}_{p,i}(k), \quad i = 1, 2, \ldots, l_p$$

where $\mathbf{x}_{p,i}(k)$ is the state vector for the i-th PLS. From [91], l_p PLSs converges to l_p clusters.

We have $\rho(\mathbb{W}_s)$ are inside the unit circle. Since we also have $\mathbf{x}_p(k)$ converging to constant vectors, the system (6.24b) is stable. The equilibria of the SLS dynamics can be calculated as

$$\mathbf{x}_s(\infty) = -(I_{n_s d} - \mathbb{W}_s)^{-1} \mathbb{W}_{sp} \mathbf{x}_p(\infty) \tag{6.25}$$

where $\mathbf{x}_p(\infty)$ and $\mathbf{x}_s(\infty)$ are the steady-states of the PLSs and SLSs, respectively. It can be shown that each row $(I_{n_s d} - \mathbb{W}_s)^{-1} \mathbb{W}_{sp}$ adds up to -1. Hence, we conclude that the components of $\mathbf{x}_s(\infty)$ are convex combinations of the components of $\mathbf{x}_p(\infty)$.

In this step, we show that each SLS attains different final values. The SLS dynamics can be expressed as follows:

$$\mathbf{x}_{s,i}(k+1) = \mathbb{W}_{l_p+i,l_p+i}\mathbf{x}_{s,i}(k) + \sum_{j=1,j\neq i}^{l_s} \mathbb{W}_{l_p+i,l_p+j}\mathbf{x}_{s,j}(k) + \sum_{j=1}^{l_p} \mathbb{W}_{l_p+i,j}\mathbf{x}_{p,j}(k) \tag{6.26}$$

for $i = 1, 2, \ldots, l_s$. The steady state values of the SLSs can be calculated as

$$\begin{aligned}\mathbf{x}_{s,i}(\infty) = &\sum_{j=1,j\neq i}^{l_s} (I_{n_{s,i}d} - \mathbb{W}_{l_p+i,l_p+i})^{-1}\mathbb{W}_{l_p+i,l_p+j}\mathbf{x}_{s,j}(\infty) \\ &+ \sum_{j=1}^{l_p} (I_{n_{s,i}d} - \mathbb{W}_{l_p+i,l_p+i})^{-1}\mathbb{W}_{l_p+i,j}\mathbf{x}_{p,j}(\infty)\end{aligned} \tag{6.27}$$

for $i = 1, 2, \ldots, l_s$. From Lemma 6.4, one can note that $(I_{n_{s,i}d} - \mathbb{W}_{l_p+i,l_p+i})^{-1}\mathbb{W}_{l_p+i,l_p+j}$ and $(I_{n_{s,i}d} - \mathbb{W}_{l_p+i,l_p+i})^{-1}\mathbb{W}_{l_p+i,j}$ have identical row matrices. Hence, the final values of the SLSs can be computed as

$$\mathbf{x}_{s,i}(\infty) = c_i, \quad i = 1, 2, \ldots, l_s$$

which implies that agents in a given SLS agree on a specific value, necessarily equal to the consensus value of any other layer. Consequently, the SLS dynamics lead to $\bar{l}_s$ distinct clusters for arbitrary choice of adjacency matrix. From the previous calculations, the total number of clusters is given by

$$\mathcal{K} = l_p + l_s.$$

□

Theorem 6.4 guarantees that any unsigned network generated from a structurally unrestricted signed network achieves cluster consensus.

Remark 6.3. Note that the condition (6.22) can be derived for weight matrices where each agent uses a scalar multiple of the same positive definite matrix as illustrated in

the following lemma.

Lemma 6.5. *Let $P \in \mathbb{R}^{d\times d}$ be a symmetric positive definite matrix and let the weight matrices of the network be chosen as $A_{ij} = c_{ij}P$ where $0 < c_{ij} \leq \frac{1}{n}$ if $(j,i) \in \mathcal{E}$ and $c_{ij} = 0$ otherwise. Then, the choice of α as follows guarantees a stable system* (6.17)*:*

$$\alpha < \frac{2n}{(n-1)(\cot(\pi/n)+2)\max\limits_{k}\mu_k} \tag{6.28}$$

where μ_k are the eigenvalues of P.

Proof. This proof will establish that the expression on the right-hand side of (6.22) is greater than or equal to that of (6.28). The matrix-weighted Laplacian matrix of the network can be written as $\mathbb{L} = D \otimes P$ where the elements of $D = [d_{ij}]$ are defined as in (6.6). Let ν_j $(j = 1, 2, \ldots, n)$ and μ_k $(k = 1, 2, \ldots, d)$ be the eigenvalues of D and P, respectively. By using Kronocker product's eigenvalue property, the eigenvalues of $\mathbb{L}$ are $\lambda_i = \nu_j\mu_k$ for $j = 1, 2, \ldots, n$, $k = 1, 2, \ldots, d$ and $i = 1, 2, \ldots, nd$. In this context, the following inequality can be obtained:

$$\begin{aligned}
\min_i \frac{2\,\mathbb{R}\mathrm{e}\{\lambda_i\}}{|\lambda_i|^2} &= \min_i \frac{2}{|\lambda_i|^2/\,\mathbb{R}\mathrm{e}\{\lambda_i\}} \\
&= \frac{2}{\max\limits_i \left(|\lambda_i|^2/\,\mathbb{R}\mathrm{e}\{\lambda_i\}\right)} \\
&= \frac{2}{\max\limits_i \left(\left(\mathbb{I}\mathrm{m}^2\{\lambda_i\} + \mathbb{R}\mathrm{e}^2\{\lambda_i\}\right)/\,\mathbb{R}\mathrm{e}\{\lambda_i\}\right)} \\
&= \frac{2}{\max\limits_i \left(\mathbb{I}\mathrm{m}\{\lambda_i\}\dfrac{\mathbb{I}\mathrm{m}\{\lambda_i\}}{\mathbb{R}\mathrm{e}\{\lambda_i\}} + \mathbb{R}\mathrm{e}\{\lambda_i\}\right)} \\
&\geq \frac{2}{\max\limits_i \left(\mathbb{I}\mathrm{m}\{\lambda_i\}\dfrac{\mathbb{I}\mathrm{m}\{\lambda_i\}}{\mathbb{R}\mathrm{e}\{\lambda_i\}}\right) + \max\limits_i \left(\mathbb{R}\mathrm{e}\{\lambda_i\}\right)}.
\end{aligned}$$

Since $0 \leq d_{ij} \leq \frac{1}{n}$ for $i \neq j$, D is a *standardized Laplacian matrix* of an unsigned system. Therefore, the following three conditions hold from Proposition 1 in [113]:

(i) $\mathbb{R}\mathrm{e}\{\nu_j\} \leq \dfrac{2(n-1)}{n}$

(ii) $|\mathbb{I}\mathrm{m}\{\nu_j\}| \leq \dfrac{(n-1)}{n}$

(iii) $\dfrac{\mathbb{I}\mathrm{m}\{\nu_j\}}{\mathbb{R}\mathrm{e}\{\lambda_j\}} \leq \cot(\pi/n)$

By utilizing the conditions (i), (ii) and (iii), the subsequent inequality can be derived:

$$\begin{aligned}\min_{j,k} \frac{2\,\mathbb{R}\mathrm{e}\{\nu_j\mu_k\}}{|\nu_j\mu_k|^2} &\geq \frac{2}{\max\limits_{j,k}\left(\mathbb{I}\mathrm{m}\{\nu_j\mu_k\}\dfrac{\mathbb{I}\mathrm{m}\{\nu_j\mu_k\}}{\mathbb{R}\mathrm{e}\{\nu_j\mu_k\}}\right) + \max\limits_{j,k}\left(\mathbb{R}\mathrm{e}\{\nu_j\mu_k\}\right)} \\ &\geq \frac{2}{\left(\dfrac{(n-1)}{n}\cot(\pi/n)\right)\max\limits_k \mu_k + \dfrac{2(n-1)}{n}\max\limits_k \mu_k} \\ &= \frac{2n}{(n-1)(\cot(\pi/n)+2)\max\limits_k \mu_k}.\end{aligned}$$

Consequently, selecting the parameter α according to (6.28) ensures that (6.22) in Theorem 6.4 is satisfied which concludes the proof. □

Remark 6.4. Note that parameter selection in Lemma 6.5 depends on the size of the network (n) and the maximum eigenvalue of P. Since all agents utilize the same P matrix, all agents have information about the eigenvalues of P. Therefore, this selection method is fully distributed. Furthermore, by using Taylor series expansion of $\cot(\cdot)$, the parameter selection method can be simplified. One can obtain the following inequality from Taylor series expansion of $\cot(\cdot)$:

$$\cot(\pi/n) < \frac{n}{\pi}.$$

By replacing $\cot(\pi/n)$ by its upper bound in (6.28), we obtain a simpler, albeit more strict, bound for stability as follows:

$$\alpha < \frac{2n\pi}{(n-1)(n+2)\max\limits_i \mu_i}$$

where μ_i are the eigenvalues of P.

6.3.2. Extension to Networks with Positive and Negative Definite Weights

Given a multi-agent network utilizing matrix-weighted consensus algorithm (6.17), an extended multi-agent system dynamics evolving over $\bar{\mathbb{G}}$ can be obtained by using Definition 6.2 as follows:

$$\bar{\mathbf{x}}_i(k+1) = \bar{\mathbf{x}}_i(k) + \bar{u}_i(k), \quad i = 1, 2, \ldots, 2n \tag{6.29}$$

where $\bar{\mathcal{N}}_i$ denotes the neighbor set of agent i in $\bar{\mathbb{G}}$. The control input is

$$\bar{u}_i(k) = \alpha \sum_{j \in \bar{\mathcal{N}}_i} \bar{A}_{ij}(\bar{\mathbf{x}}_j(k) - \bar{\mathbf{x}}_i(k)). \tag{6.30}$$

Note that (6.29) and (6.30) can be expressed in matrix form as

$$\begin{aligned} \bar{\mathbf{x}}(k+1) &= (I_{2nd} - \alpha\bar{\mathbb{L}})\bar{\mathbf{x}}(k) \\ &= \bar{\mathbb{W}}\bar{\mathbf{x}}(k), \quad \bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 \end{aligned} \tag{6.31}$$

where $\bar{\mathbf{x}}(k) = [\bar{\mathbf{x}}_1(k)^\mathsf{T}, \bar{\mathbf{x}}_2(k)^\mathsf{T}, \ldots, \bar{\mathbf{x}}_{2n}(k)^\mathsf{T}]^\mathsf{T}$ is the augmented state vector of the extended graph and $\bar{\mathbf{x}}_0$ is the initial condition.

The following lemma reveals that the trajectories of (6.17) can be obtained from those of (6.29).

Lemma 6.6. *The trajectories* $\mathbf{x}(k)$ *of the agents in system* (6.17) *can be recovered from those in system* (6.29) *as* $\mathbf{x}_i(k) = \bar{\mathbf{x}}_i(k)$, $i = 1, 2, \ldots, n$, *for all* $k \geq 0$ *by setting* $\bar{\mathbf{x}}_{i0} = \mathbf{x}_{i0}$ *and* $\bar{\mathbf{x}}_{i+n,0} = -\mathbf{x}_{i0}$ *for all* $i = 1, 2, \ldots, n$.

Proof. The extended system represented by dynamics (6.29) can be re-expressed as

$$\begin{aligned}\bar{u}_i(k+1) &= \alpha \sum_{j\in\bar{\mathcal{N}}_i} \bar{A}_{ij}(\bar{\mathbf{x}}_j(k) - \bar{\mathbf{x}}_i(k)) \\ &= \alpha \sum_{j=1}^{n} \bar{A}_{ij}(\bar{\mathbf{x}}_j(k) - \bar{\mathbf{x}}_i(k)) + \alpha \sum_{j=1}^{n} \bar{A}_{i,j+n}(\bar{\mathbf{x}}_{j+n}(k) - \bar{\mathbf{x}}_i(k))\end{aligned} \tag{6.32}$$

for $i = 1, 2, \ldots, n$. Let $\hat{\mathbf{x}}_i(k) = \bar{\mathbf{x}}_i(k) + \bar{\mathbf{x}}_{i+n}(k)$ for $i = 1, 2, \ldots, n$ whose dynamics can be formulated as

$$\begin{aligned}\hat{\mathbf{x}}_i(k+1) &= \bar{\mathbf{x}}_i(k+1) + \bar{\mathbf{x}}_{i+n}(k+1) \\ &= \bar{\mathbf{x}}_i(k) + \bar{u}_i(k) + \bar{\mathbf{x}}_{i+n}(k) + \bar{u}_{i+n}(k) \\ &= \hat{\mathbf{x}}_i(k) + \alpha \sum_{j=1}^{n} \bar{A}_{ij}(\bar{\mathbf{x}}_j(k) - \bar{\mathbf{x}}_i(k)) + \alpha \sum_{j=1}^{n} \bar{A}_{i,j+n}(\bar{\mathbf{x}}_{j+n}(k) - \bar{\mathbf{x}}_i(k)) \\ &\quad + \alpha \sum_{j=1}^{n} \bar{A}_{i+n,j}(\bar{\mathbf{x}}_j(k) - \bar{\mathbf{x}}_{i+n}(k)) + \alpha \sum_{j=1}^{n} \bar{A}_{i+n,j+n}(\bar{\mathbf{x}}_{j+n}(k) - \bar{\mathbf{x}}_{i+n}(k)).\end{aligned}$$

By using Definition 6.2, we have $\bar{A}_{i,j+n} + \bar{A}_{i+n,j+n} = \bar{A}_{ij} + \bar{A}_{i,j+n} = \bar{A}_{i+n,j} + \bar{A}_{i+n,j+n} = \operatorname{sgn}(A_{ij})A_{ij}$ which can be utilized to obtain

$$\hat{\mathbf{x}}_i(k+1) = \hat{\mathbf{x}}_i(k) + \alpha \sum_{j=1}^{n} \operatorname{sgn}(A_{ij})A_{ij}(\hat{\mathbf{x}}_j(k) - \hat{\mathbf{x}}_i(k)) \tag{6.33}$$

or equivalently

$$\hat{\mathbf{x}}(k+1) = \hat{\mathbb{W}}\hat{\mathbf{x}}(k), \quad \hat{\mathbf{x}}(0) = \hat{\mathbf{x}}_0 \tag{6.34}$$

where $\hat{\mathbf{x}}(k) = [\hat{\mathbf{x}}_1(k)^\mathsf{T}, \hat{\mathbf{x}}_2(k)^\mathsf{T}, \ldots, \hat{\mathbf{x}}_n(k)^\mathsf{T}]^\mathsf{T}$, the matrix $\hat{\mathbb{W}} = [\hat{W}_{ij}] = I_{nd} - \alpha\hat{\mathbb{L}}$ and $\hat{\mathbf{x}}_0$ is the initial condition. Note that the solution of (6.34) is $\hat{\mathbf{x}}(k) = \hat{\mathbb{W}}^k\hat{\mathbf{x}}_0$. Since the choice of initial conditions for the states $\bar{x}_i$ yields $\hat{\mathbf{x}}(0) = [0, \ldots, 0]^\mathsf{T}$, we conclude that $\hat{\mathbf{x}}(k) = [0, \ldots, 0]^\mathsf{T}$ for all k which implies $\bar{\mathbf{x}}_i(k) = -\bar{\mathbf{x}}_{i+n}(k)$ for $i = 1, 2, \ldots, n$. By

substituting $\bar{\mathbf{x}}_{j+n}(k) = -\bar{\mathbf{x}}_j(k)$ into (6.32), we have

$$\bar{u}_i(k+1) = \alpha \sum_{j=1}^{n} \big((\bar{A}_{ij} - \bar{A}_{i,j+n})\bar{\mathbf{x}}_j(k) - (\bar{A}_{ij} + \bar{A}_{i,j+n})\bar{\mathbf{x}}_i(k)\big)$$

which can be simplified as

$$\bar{u}_i(k+1) = \alpha \sum_{j=1}^{n} A_{ij} \left(\bar{\mathbf{x}}_j(k) - \operatorname{sgn}(A_{ij})\bar{\mathbf{x}}_i(k)\right) \tag{6.35}$$

for $i = 1, 2, \ldots, n$. Since (6.35) and (6.17) are equivalent, we conclude that the trajectory $\mathbf{x}(k)$ can be recovered by setting $\bar{\mathbf{x}}_{i0} = \mathbf{x}_{i0}$ and $\bar{\mathbf{x}}_{i+n,0} = -\mathbf{x}_{i0}$ for all $i = 1, 2, \ldots, n$, which completes the proof. □

Building upon Lemma 6.6, which reveals a connection between the state trajectories of the signed system (6.17) and its extended version (6.29) derived from Definition 6.2, we proceed to analyze the stability and cluster structure of the extended system (6.29).

Through an analysis of the stability property of the extended system (6.29), we can state the following theorem:

Theorem 6.5. *Under Assumption 6.2, the system* (6.29) *with underlying directed graph* $\bar{\mathbb{G}}$ *forms* $\bar{\mathcal{K}} = \bar{l}_p + \bar{l}_s$ *clusters in the directed network if and only if positive constant* α *satisfies*

$$\alpha < \min_i \frac{2\,\mathbb{R}\mathrm{e}(\bar{\lambda}_i)}{|\bar{\lambda}_i|^2} \tag{6.36}$$

where $\bar{l}_p$ *and* $\bar{l}_s$ *are the number of PLSs and SLSs, respectively; and* $\bar{\lambda}_i$ *are the nonzero eigenvalues of the matrix-weighted Laplacian matrix* $\bar{\mathbb{L}}$.

Proof. Since $\bar{\mathbb{G}}$ consists of purely positive definite weights, the proof directly follows from Theorem 6.4. □

Theorem 6.5 assures us that any unsigned network constructed from a signed network without structural limitations converges to cluster consensus. The total number of clusters formed is determined by the sum $\bar{l}_p + \bar{l}_s$. Utilizing Lemma 6.6 and Theorem 6.5 lead to the following subsequent theorem:

Theorem 6.6. *Under Assumption 6.2, suppose that underlying graph $\mathbb{G} = (\mathcal{V}, \mathcal{E}, \mathbb{A})$ of the matrix-weighted directed network consists of positive and negative definite matrix weights. Then the agents utilizing* (6.17) *forms $\mathcal{K} = \bar{l}_p + \bar{l}_s - l'$ clusters in the directed network if positive constant α satisfies*

$$\alpha < \min_i \frac{2\,\mathbb{R}\mathrm{e}(\bar{\lambda}_i)}{|\bar{\lambda}_i|^2} \tag{6.37}$$

where $\bar{l}_p$ and $\bar{l}_s$ are the number of PLSs and SLSs of the extended system, respectively; l' is the number of subgraphs whose agent set consists of only a subset of $\bar{\mathcal{V}} \setminus \mathcal{V} = \{1', 2', \ldots, n'\}$ and $\bar{\lambda}_i$ are the nonzero eigenvalues of the matrix-weighted Laplacian matrix $\bar{\mathbb{L}}$.

Proof. From Theorem 6.5 the extended system that is representation of the signed system converges to $\bar{l}_p + \bar{l}_s$ clusters. Since l' subgraphs in $\bar{l}_p + \bar{l}_s$ subgraphs do not contain any agent of the signed system, the number of clusters for signed system is found as

$$\mathcal{K} = \bar{l}_p + \bar{l}_s - l'.$$

□

6.4. Numerical Analysis

Example 6.2. *Consider the signed network in Figure 6.2 consisting of 7 agents with $d = 2$, i.e., $\mathbf{x}_i \in \mathbb{R}^2$, $\mathbf{x}_i = [\mathbf{x}_{i1}, \mathbf{x}_{i2}]^\mathsf{T}$. In the given network, negative and positive communication flows are represented by red and black arrows, respectively. Note that*

the network has no structural restrictive elements (structurally balanced, strongly connected, spanning tree etc.). The matrix-weights are given as

$$A_{12} = \begin{bmatrix} -3 & -4 \\ -4 & -9 \end{bmatrix}, \quad A_{14} = \begin{bmatrix} 6 & -4 \\ -4 & 5 \end{bmatrix}, \quad A_{15} = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix},$$
$$A_{17} = \begin{bmatrix} 3 & 0 \\ 0 & 9 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} -6 & -1 \\ -1 & -3 \end{bmatrix}, \quad A_{23} = \begin{bmatrix} -3 & 4 \\ 4 & -9 \end{bmatrix}, \tag{6.38}$$
$$A_{24} = \begin{bmatrix} 5 & 2 \\ 2 & 6 \end{bmatrix}, \quad A_{26} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}, \quad A_{27} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

The eigenvalues of $\bar{\mathbb{L}}$ *constructed using extended directed graph representation with the matrix-weights in* (6.38) *are* $\{0, 0, 0, 0, 1, 1, 1, 1, 1, 1, 2, 2, 4, 4,\ 5.2798, 5.2798, 9, 9,\ 11, 11, 11, 11, 11, 11, 13, 13, 15.7202,\ 15.7202\}$.

Through the utilization of Definition 6.2, the corresponding extended directed graph can be derived as illustrated in Figure 6.3. At this point, PLSs and SLSs have significant importance in identifying clusters and their members within the extended directed graph. The concepts of PLS and SLS are utilized in computing the number of groups and ascertaining the members within each cluster. To this end, the clusters and their members in the extended system are found as $\bar{\mathcal{V}}_{p,1} = \{1, 3, 5, 6, 2'\}$, $\bar{\mathcal{V}}_{p,2} = \{2, 1', 3', 5', 6'\}$, $\bar{\mathcal{V}}_{s,1} = \{7\}$, $\bar{\mathcal{V}}_{s,2} = \{7'\}$, $\bar{\mathcal{V}}_{s,3} = \{4\}$ *and* $\bar{\mathcal{V}}_{s,4} = \{4'\}$ *as shown in Figure 6.3.*

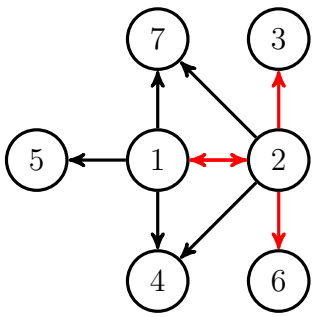


Figure 6.2. A signed network with 7 agents and 9 edges.

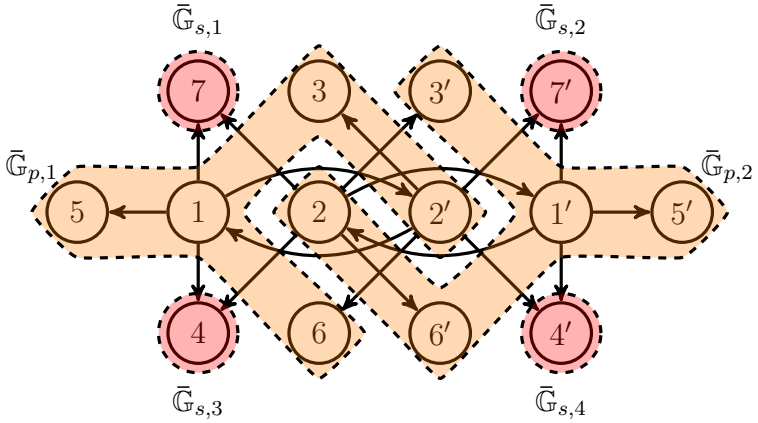


Figure 6.3. Extended directed graph for the network in Figure 6.2.

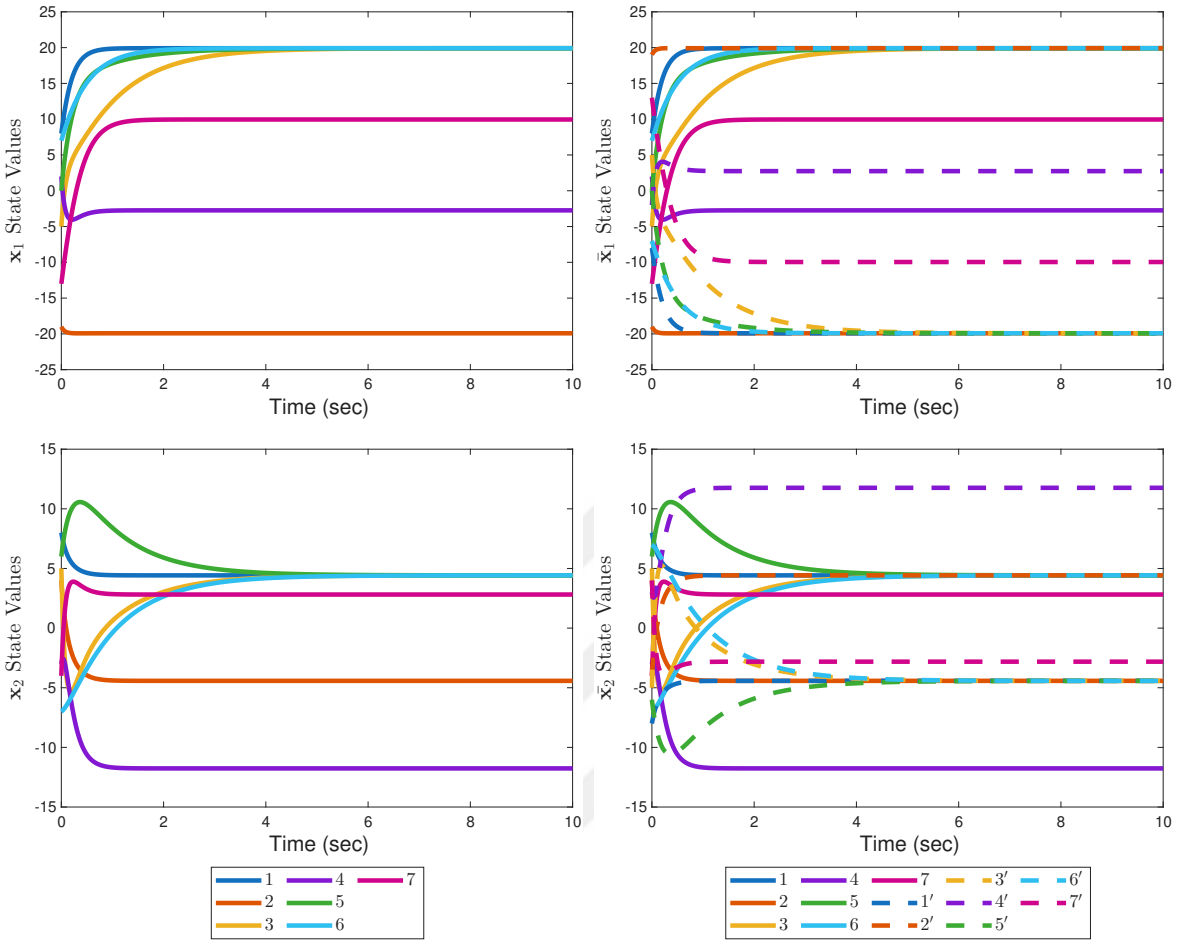
The simulation results are depicted in Figure 6.4(a) for the signed network with $\mathbf{x}(0) = \mathbf{x}_0 = [8,\ 8,\ -19,\ 4,\ -5,\ 5,\ 2,\ -3,\ 0,\ 6,\ 7,\ -7,\ -13,\ -4]^\mathsf{T}$. *The signed network converges to 4 distinct clusters as shown in Figure 6.4(a). By setting* $\bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 = \left[\mathbf{x}_0^\mathsf{T}, -\mathbf{x}_0^\mathsf{T}\right]^\mathsf{T}$ *as in Lemma 6.3, the trajectories of agents in the signed network can be recovered. The numbers of PLSs and SLSs of the extended system are* $\bar{l}_p = 2$ *and* $\bar{l}_s = 4$, *respectively (Figure 6.4(b)). As stated in Theorem 6.2, the extended network converges to 6 distinct clusters. Furthermore, there are* $l' = 2$ *subgraphs (*$\bar{\mathcal{V}}_{s,2} = \{7'\}$, $\bar{\mathcal{V}}_{s,4} = \{4'\}$*) that do not contain any agent of the original system. From the clusters in the extended system, the original clusters (*$\bar{l}_p + \bar{l}_s - l' = 4$*) can be obtained as* $\{1, 3, 5, 6\}$, $\{2\}$, $\{7\}$ *and* $\{4\}$. *The simulation results verify Theorem 6.3.*

Example 6.3. *Reconsider the network in Figure 6.2 under the framework of discrete-time dynamics. The network utilizes a system matrix of the following form:*

$$\mathbb{L} = D \otimes P$$

where

$$P = \begin{bmatrix} 6 & -4 \\ -4 & 5 \end{bmatrix}$$



(a) Signed directed graph (b) Extended directed graph

Figure 6.4. State trajectories of agents associated with signed and extended directed graphs in Figures 6.2 and 6.3.

and

$$D = \begin{bmatrix} 0.03 & 0.03 & 0 & 0 & 0 & 0 & 0 \\ 0.05 & 0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.06 & 0.06 & 0 & 0 & 0 & 0 \\ -0.03 & -0.02 & 0 & 0.05 & 0 & 0 & 0 \\ -0.04 & 0 & 0 & 0 & 0.04 & 0 & 0 \\ 0 & 0.06 & 0 & 0 & 0 & 0.06 & 0 \\ -0.05 & -0.02 & 0 & 0 & 0 & 0 & 0.07 \end{bmatrix}.$$

The system matrix of the extended network can be obtained as

$$\bar{\mathbb{L}} = \bar{D} \otimes P$$

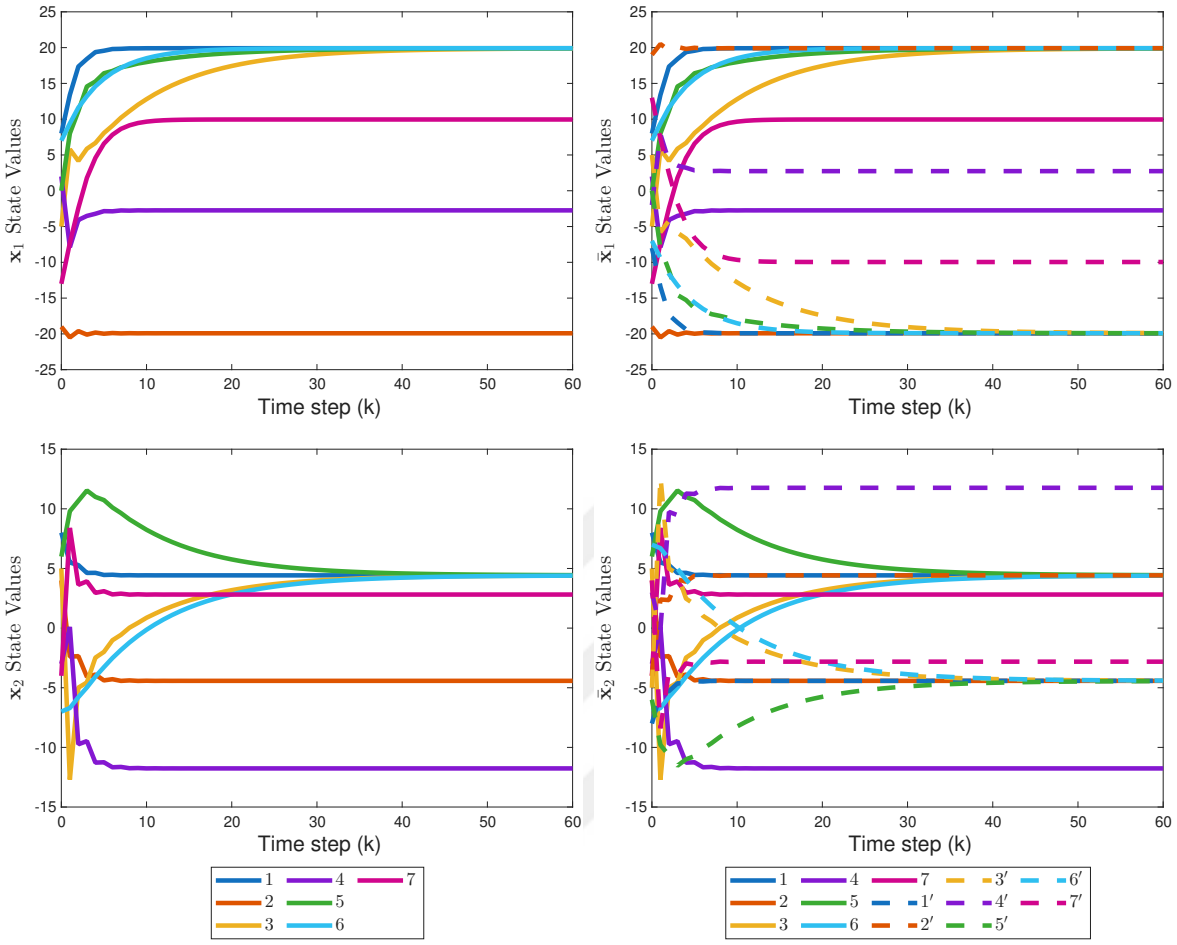
where $\bar{D}$ *is a standardized Laplacian matrix formed by Definition 6.2. From Lemma 6.5, the step size parameter for a stable extended dynamics can be calculated as*

$$\alpha < \frac{2n}{(n-1)(\cot(\pi/n)+2)\max\limits_i \mu_i} = 0.1113 \text{ for } \mu_i = 9.5311.$$

Note that α *is purely chosen in a distributed way. The influence of the parameter* α *on the system dynamics is further illustrated in Figure 6.5(a). This figure depicts the simulation results obtained for* $\alpha = 0.1$ *employing the same initial state vector* $\mathbf{x}(0) = \mathbf{x}_0 = [8,\ 8,\ -19,\ 4,\ -5,\ 5,\ 2,\ -3,\ 0,\ 6,\ 7,\ -7,\ -13,\ -4]^{\mathsf{T}}$ *as used in the aforementioned continuous-time simulation. The signed network ultimately reaches a stable state, consisting of 4 separate clusters, as depicted in Figure 6.5(a). These clusters can be determined from its corresponding extended representation as in continuous-time case by setting* $\bar{\mathbf{x}}(0) = \bar{\mathbf{x}}_0 = \left[\mathbf{x}_0^{\mathsf{T}}, -\mathbf{x}_0^{\mathsf{T}}\right]^{\mathsf{T}}$ *as in Lemma 6.6. The state evolution of the extended system is shown in Figure 6.5(b). As seen in Figure 6.5(b), the numbers of PLSs and SLSs of the extended system are* $\bar{l}_p = 2$ *and* $\bar{l}_s = 4$*, respectively. The simulation results confirm the validity of Theorem 6.5. Furthermore, there are* $l' = 2$ *subgraphs (*$\bar{\mathcal{V}}_{s,2} = \{7'\}$*,* $\bar{\mathcal{V}}_{s,4} = \{4'\}$*) that do not contain any agent of the original system. From the clusters in the extended system, the original clusters (*$\bar{l}_p + \bar{l}_s - l' = 4$*) can be obtained as* $\{1,3,5,6\}$*,* $\{2\}$*,* $\{7\}$ *and* $\{4\}$*. The simulation results verify Theorem 6.6.*

6.5. Chapter Summary & Concluding Remarks

In this chapter, we have investigated the cluster consensus problem for matrix-weighted multi-agent networks, as defined by any given signed directed graph. We have presented an extended graph representation for assessing the convergence characteristics of any provided signed directed graph. The utilization of this extended directed



(a) Signed directed graph (b) Extended directed graph

Figure 6.5. State trajectories of agents associated with signed and extended directed graphs in Figures 6.2 and 6.3.

graph composition and PLS/SLS concepts has allowed us to determine the clusters and their members for a matrix-weighted network without imposing any limitations on the structure of the directed graph. The obtained results for continuous-time systems have been extended to the discrete-time systems. The maximum allowable value of the step size α has been calculated for achieving matrix-weighted cluster consensus.

7. CONCLUSION

This thesis has significantly advanced the understanding and solvability of the cluster consensus problem in multi-agent networks by addressing several critical limitations present in the existing literature. Throughout this thesis, the main focus has been the investigation of cluster agreement for agents evolving over any given directed graph, both unsigned and signed, fundamentally moving beyond restrictive structural assumptions such as the necessity of spanning trees, strong connectivity or structural balance, which are often not applicable in real-world scenarios.

The research has successfully tackled the cluster consensus problem for a diverse range of system dynamics. For any-order linear systems on unsigned directed graphs, necessary and sufficient conditions for cluster agreement have been derived, calculating the number of clusters and determining their final values have been explicitly presented under these conditions.

For signed directed graphs across higher-order, delayed, and nonlinear systems a major contribution has been achieved by introducing and effectively utilizing an extended graph representation. This novel approach has provided a systematic framework to analyze networks with antagonistic interactions. It has enabled the explicit computation of the number of clusters and their members for any given signed directed graph. This method has been applied to continuous and discrete-time networks, higher-order systems, systems with input or communication delays, networks with inherent nonlinear dynamics and matrix-weighted systems.

For higher-order systems with input or communication delays on both unsigned and signed directed graphs, the thesis has provided conditions for stable cluster consensus. Through the extended graph representation, it has successfully established delay bounds for stable operation and determined convergence properties and cluster compositions in continuous and discrete-time domains. Notably, for discrete-time

systems with non-uniform delays, it has been shown that delay does not affect cluster formation if the extended system remains stable. Future research should explore methodologies for actively compensating for non-uniform and time-varying delays in continuous/discrete-time systems.

The analysis of first and second-order networks with inherent nonlinear dynamics on arbitrary unsigned and signed directed graphs has been enhanced by the extended graph representation. This enhancement has included the ability to characterize convergence behaviors and determine cluster numbers and members regardless of the graph's specific topology. Future work should focus on investigating how the extended graph representation can be adapted and applied to third-order and even higher-order inherently nonlinear systems. Furthermore, a wider range of nonlinear functions should be explored. The effect of delays on the convergence and clustering properties of inherently nonlinear networks should be analyzed in the future.

Furthermore, the thesis has addressed the less explored area of matrix-weighted cluster consensus on directed networks without restrictive topological conditions. By employing the extended graph representation and PLS and SLS concepts, the research has provided methods to determine the number and structure of clusters in such complex systems. The results have been extended to discrete-time matrix-weighted systems, including a distributed approach for step-size selection. Future research should investigate the effects of delay or nonlinearity on matrix-weighted networks. Additionally, networks with positive and negative semi definite weights should be explored in the future.

In summary, this thesis has provided a comprehensive and generalized framework for analyzing and achieving cluster consensus in multi-agent systems operating over arbitrary directed graphs with various dynamics and interaction types. By developing novel analytical tools, particularly the extended graph representation, the research has overcome significant limitations in determining cluster properties and stability conditions in complex and realistic network scenarios. The theoretical results from all

parts of this thesis have been demonstrated through illustrative numerical examples. These results have provided valuable insights and practical methods for the design and control of networked systems in diverse applications.

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