A MODEL FOR DEVELOPING THE MULTIPLICATION OF COMPLEX NUMBERS

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ABSTRACT

A MODEL FOR DEVELOPING THE MULTIPLICATION OF COMPLEX NUMBERS

This study investigated a prospective mathematics teacher's development of the multiplication of complex numbers in Cartesian, vector and polar forms during an instructional sequence involving quantitative reasoning. The teaching experiment methodology was used for the design of the instructional sequence and the conduct of the study. For this, upon completion of the articulation of the understanding of the multiplication of complex numbers by the researcher, a written pre-assessment, also used as the written post-assessment after the completion of the teaching sessions, was given to 28 prospective mathematics teachers. Following, one hour long pre-clinical interviews were conducted with two of the participants. Based on the analysis of the pre-interviews, one of the participants was selected. Teaching experiments included 4 teaching sessions upon completion of which a written post-assessment and also a post-clinical interview was conducted. Results, upon completion of the teaching sessions, showed that that in contrast to the data from the pre-interview, the prospective mathematics teacher was able to develop the multiplication of complex numbers in Cartesian, vector and polar forms in a quantitative structure in relation to each other. Particularly, results showed that she was able to explain the reasoning behind the polar form pointing to one of the complex numbers as operand vector acting on the other one. Similarly, she was able to justify the relationship between the multiplication of complex numbers in Cartesian, vector and polar forms. Results also pointed to some difficulties the participant had encountered while developing the reasoning behind the polar form of the multiplication of complex numbers. Results in juxtaposition to each other suggested a model for developing the multiplication of complex numbers quantitatively.

ÖZET

KARMAŞIK SAYILARIN ÇARPIMININ ANLAMLANDIRILMASI İÇİN BİR MODEL

Bu araştırmanın amacı bir matematik öğretmeni adayı tarafından nicel muhakeme üzerine kurulu bir öğretim sırasında karmaşık sayıların kartezyen, vektör ve polar formdaki çarpımı altında yatan matematiksel anlamların ve ilişkilerin oluşturulmasının incelenmesidir. Öğretim dizisinin tasarımı ve araştırmanın gerçekleştirilmesi için öğretim deneyi metodu kullanılmıştır. Bunun için, karmaşık sayıların çarpımının anlaşılmasına yönelik hipotetik açıklamanın araştırmacı tarafından tamamlanmasının ardından, 28 matematik öğretmeni adayına, öğretim dizisinin tamamlanmasından sonra yazılı son-test olarak da kullanılan yazılı ön-test uygulanmıştır. Sonrasında iki öğretmen adayıyla bir saat uzunluğunda yapılandırılmış ön-mülakat yapılmıştır. Ön-mülakatların analizine bağlı olarak bir öğretmen adayı çalışmaya katılımcı olarak seçilmiştir. Öğretim deneyleri 4 oturumdan oluşmuş olup öğretim deneylerinin bitiminden hemen sonra bir yazılı son-test ve bir de yapılandırılmış son-mülakat uygulanmıştır. Sonuçlar göstermiştir ki öğretim oturumlarının tamamlanmasının ardından matematik öğretmeni adayı karmaşık sayıların kartezyen, vektör ve polar şekildeki çarpımlarını birbiriyleriyle ilişkili niceliksel bir yapı içerisinde anlamlandırabilmiştir. Özellikle, sonuçlar göstermiştir ki matematik öğretmeni adayı karmaşık sayılardan birini diğeri üzerinde işlem yapan vektör olarak belirterek polar formdaki çarpım altında yatan mantıksal nedeni açıklayabilmiştir. Benzer şekilde, matematik öğretmeni adayı karmaşık sayıların kartezyen, vektör ve polar formlardaki çarpımları arasındaki ilişkiyi de gerekçeleyebilmiştir. Sonuçlar katılımcının karmaşık sayıların polar formda çarpımı altında yatan mantıksal nedeni anlamlandırması sırasında bazı zorluklarla karşılaştığını da göstermektedir. Sonuçlar birbirleriyle bağlantılı olarak karmaşık sayıların çarpımının nicel olarak anlamlandırılması için bir model öne sürmüştür.

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LIST OF SYMBOLS

$\overrightarrow{z_1}$	The vector z_1
α	Alpha
β	Beta
θ	Theta

LIST OF ACRONYMS/ABBREVIATIONS

CBMS	Conference Board of Mathematical Sciences
CCSS	Common Core State Standarts
cos	Cosine
cot	Cotangent
sin	Sine
tan	Tangent

1. INTRODUCTION

Complex numbers have a big significance in almost every branch of mathematics and some practices within science such as electromagnetism and quantum mechanics (Nemirovsky, Rasmussen, Sweeney and Wawro, 2012). According to CCSSM (2010), students should be able to represent the complex numbers and operations of complex numbers in Cartesian and polar forms using Complex Plane since "the concept of complex number is one of the most characteristic examples of those concepts that have algebraic and geometric representations at the same time, and both of them are necessary for an overall understanding" (Panaoura, Elia, Gagatsis and Giatilis, 2006, p. 702). In particular, students are expected to explain how different representations of complex numbers indicate the same number as well as realizing the geometric meanings under the complex number addition, multiplication and conjugation. In this regard, Karakok, Soto-Johnson and Dyben (2015) argued that:

"The document recommends that students know how to use multiple representations (e.g., algebraic and geometric) to express complex numbers using different forms (e.g., rectangular, polar) and to perform arithmetic operations. Furthermore, students should be able to make connections among these different forms" (p.318).

Taking into consideration these calls, there has been some research on (prospective) mathematics teachers' (Conner, Rasmussen, Zandieh and Smith, 2007; Karakok *et al.*, 2015; Nemirovsky, *et al.* 2012) and students' (Nordlander and Nordlander, 2012; Panaoura *et al.*, 2006; Soto-Johnson and Troup, 2014) algebraic and geometric approaches in solving complex number tasks (Soto-Johnson and Troup, 2014; Panaoura *et al.*, 2006). Among these studies, although only Conner *et al.* (2007) and Nemirovsky *et al.* (2012) focused on prospective teachers' conceptualization of the arithmetic of complex numbers, results of the other studies also reported on how participants made sense of the multiplication and division of complex numbers. In general, results pointed to three important issues: (i) prospective teachers could not represent the multiplication of complex numbers although they could represent the addition of complex numbers

on the Complex Plane (Conner *et al.*, 2007); (ii) teachers only could explain both algebraically and geometrically multiplication by "i" (Karakok, *et al.* 2015; Nemirovsky *et al.*, 2012) and (iii) while students could state the geometric meaning of the multiplication of complex number as rotation and dilation (Soto- Johnson and Troup, 2014) and teachers could state the division of complex numbers as the division of two vectors (Karakok, *et al.* 2015), neither students nor teachers could justify the reasons (Karakok, *et al.* 2015; Soto- Johnson and Troup, 2014).

Particularly, Karakok, Soto-Johnson and Dyben (2013, 2015) studied three high-school in-service mathematics teachers' conceptualization of Cartesian and exponential forms of complex numbers and representations of these forms algebraically and geometrically. Results showed that, one of the teachers, Melissa, could not make connections between algebraic and geometrical meaning of the multiplication of a complex number with the imaginary number i. She "relayed a complex number in Cartesian form as an algebraic process performed on i" (p. 4), while the other two teachers, Aaron and Troy, "were able to consider multiple representations of the Cartesian form simultaneously" (p. 5). That is, they could state that the imaginary number i was equal to the square root of - 1 and also it carried the point (x , y) to the point (-y, x) at the same time. Regarding the division of complex numbers, when Aaron and Troy explained the equation ($\frac{r_1.e^{i\theta_1}}{r_2.e^{i\theta_2}} = \frac{r_1}{r_2}.e^{i(\theta_1-\theta_2)}$), they only could explain it algebraically by referring to the law of exponents. While Aaron could state the geometrical meaning as the division of vector 1 to the vector 2, where vector 2 was operated on vector 1, Troy was able to state that division of vectors referred to repeated subtraction. Still, both were not able to provide any justification for their statements they found that:

"...eight of the ten students successfully used the plane to perform addition. Of these eight, five used what was essentially vector summation, while three more added component-wise along the axes. None of the ten students had a model for multiplication" (p. 4).

The study also showed that before the instruction while the eight prospective mathematics teachers explained the multiplication of a real number by -1 as a reflection

and iteration, after the instruction some of them could mention the rotation of 180° on the real number line. In fact, Karakok, *et al.*, (2015) argued that such perception might be

"a result of focusing on the real number line rather than the entire Complex Plane and may have contributed to their inability to illustrate how multiplication by the complex number x + yi results in a rotation and a dilation of the other factor" (p. 330).

In addition to these studies, Soto-Johnson and Troup (2014) investigated two undergraduate mathematics major students' (Kelly and Abby) "reasoning on complex-valued statements" (p. 109) such as why is $z, \overline{z} = |z|^2$? Soto-Johnson and Troup (2014) said that:

"In their explanation, Kelly and Abby once again uttered that the product of two complex numbers should produce a rotation dilation, where the dilation is determined by the magnitude of "whatever you are multiplying by" and the rotation is determined by a rotation of the "arctangent of the angle formed by the ray". Although they could verbally reason geometrically regarding the multiplication of two complex numbers, neither was able to create an associated diagram for the algebraic inscriptions" (p. 116).

Aforementioned studies showed that, both prospective/in-service teachers and the students had comparted understandings regarding the multiplication of complex numbers such that they either could calculate the multiplication without being able to show the result geometrically or even if they did it algebraically and geometrically their justifications only included multiplying with "i". It is in this regard that, in this study, working with a prospective secondary mathematics teacher, I investigate how the connections between algebraic and geometric representations of the multiplication of complex numbers could be made via transitions between Cartesian, vector and polar forms. Particularly, in this study, I report on a prospective mathematics teacher's development of the mathematical meaning and connections under the multiplication of complex numbers in algebraic, vector and polar forms once the complex number multiplication is associated with the operations on vectors. In this respect, in this

study I explore a prospective mathematics teacher's development of the multiplication of complex numbers in Cartesian, vector and polar form through quantitative reasoning (Thompson, 1994, 2011).

Quantitative reasoning addresses the significance of students' realization of conditions and measurable quantities of the conditions (Moore, Carlson and Oehrtman, 2009). Thompson (1994) stated: "A person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conception entails the quality's measurability" (p. 7). In this regard, quantitative reasoning has two roles in problem solving activities: (i) "to provide the content for the algebraic expressions so that the power of that notation can be exploited" and (ii) "to support reasoning that is flexible and general in character but does not necessarily rely on symbolic expressions" (Thompson, 2007, p. 12). In this study, I utilize quantitative reasoning in order to facilitate the conceptual understanding of the multiplication of complex numbers by recognizing the complex numbers as measurable quantities and the multiplication of the complex numbers as an operation on quantities. Therefore, a prospective mathematics teacher's development of the meaning of multiplication of complex numbers will be examined through a teaching experiment (Steffe and Thompson, 2000) which includes an instructional sequence of integrating both the algebraic and geometrical meaning of complex number multiplication. Particularly, the focus will be on the prospective teacher's constructing the relationship among the polar, vector and algebraic forms of the multiplication of two complex numbers, z = a + bi and w = x + yi.

2. LITERATURE REVIEW

In this section, first, I briefly define complex numbers. Secondly, I explain the importance of the transition between different representations of complex numbers on students' understanding. Next, I present previous research results sharing some examples from prospective teachers' and students' understanding of the multiplication of complex numbers. Finally, I describe theoretical framework of the study building on the implications of quantitative reasoning.

2.1. Complex Numbers

In this part, I refer to the definition of complex numbers from two perspectives: The formal definition from the mathematicians' perspective and the operational definition from the students' perspective.

From the mathematicians' point of view "a complex number should be conceptualized as one number, i.e., the expression a + ib is a single entity combining a real number and an imaginary number" (Nordlander and Nordlander, 2012, p. 633). Thus, the complex number a + ib, (i. e., a + bi), could be identified in the plane as a particular point (a, b), whose real component is "a" and imaginary component is "b" (Nemirovsky *et al.*, 2012). Therefore, the x-axis of the plane is the real axis and the y-axis is the imaginary axis, so the " image points of the real numbers 'a' are on the x-axis" and the "image points of the 'pure imaginary numbers' 'bi' are on the y-axis" (Panaoura *et al.*, 2006, p. 683). According to the formal definition of mathematicians, the Cartesian form of a complex number a + bi where a and b are real numbers and $i = \sqrt{-1}$, is purely arithmetic and the addition between real and imaginary parts is symbolic (Panaoura *et al.*, 2006). In this respect, as the researchers point to, the formal definition of complex numbers might hinder students' making sense of the operational and geometrical aspects of complex numbers. However, Karekok *et al.* (2015) argued that "it is important that high school students and prospective teachers have multiple and frequent opportunities to explore the geometric and active aspects of complex

numbers" (p. 349). In addition, pointing to the operational characteristic of complex numbers, Sfard (1991) argued that the 1^{st}step towards conceptualization of complex numbers is to recognize that $i = \sqrt{-1}$.

Acknowledging the importance of the formal definition and taking into consideration Sfard (1991) and Karakok *et al.* (2015)'s calls on the operational aspects of complex numbers, Sarac and Karagoz Akar (2017) and Sarac (2016) have pointed to the operational definition of complex numbers. Particularly, in a series of teaching experiments Sarac and Karagoz Akar (2017) have attempted to develop the conceptual understanding of complex numbers through quantitative reasoning on the part of a prospective secondary mathematics teacher. They found out that the quantitative operations of dilation, reflection and rotation could trigger the prospective teachers' development of the components of the Cartesian form of complex numbers (i. e., the meaning of "a" and "b") meaningfully from the roots of the quadratic equations. Particularly, their study showed that as a result of engaging in the quantitative operations of reflection, dilation and rotation, a prospective teacher was able to define complex numbers "as the elements of the set of the roots of any quadratic equation with real coefficients".

Thus, in this research study, as well as acknowledging the formal definition of complex numbers (Nordlander and Nordlander, 2012; Panaoura *et al.*, 2006), I also embrace the operational definition of complex numbers as the elements of a well-defined set (Sfard, 1991) such that complex numbers are the elements of the set of the roots of any quadratic equation with real coefficients (Sarac, 2016; Sarac and Karagoz Akar, 2017).

2.2. Representations of Complex Numbers

In this part, first, I mention the importance of representations in mathematics, especially for the understanding of complex numbers. Secondly, I show how the experts change their algebraic and geometric conceptions of complex numbers according to the complex number operations. Finally, I briefly describe high school students' transitions

between algebraic and geometric representations of complex numbers during their engagement on complex number tasks.

An important educational objective for pupils is to identify and efficiently use various forms of presentation of the same mathematical concept, and move flexibly from one system of presentation to another by using different approaches (Panaoura *et al.*, 2006, p. 702). Sfard (1991) stated:

"Unlike material objects, however, advanced mathematical constructs are totally inaccessible to our senses - they can only be seen with our mind's eyes. Indeed, even when we draw a function or write down a number, we are very careful to emphasize that the sign on the paper is but one among many possible representations of some abstract entity, which by itself can be neither seen nor touched. ... Being capable of somehow "seeing" these invisible objects appears to be an essential component of mathematical ability..." (p. 3).

Therefore, representations in mathematics have a big significance, since differently from other sciences mathematical concepts "such as function, vector, complex number, is not perceptually or instrumentally accessible" (Panaoura *et al.*, 2006, p. 686). Particularly, Glas (1998) says that vectors and complex numbers are two examples of constructs in mathematics, which are based on a formal regulations within an organized body of nonsense signs that do not have a meaning in terms of "informal or quasi-empirical" (p. 367). Moreover, Nordlander and Nordlander (2012) asserted "the abstract and theoretical nature of the complex numbers, and possibly the lack of visualization, constitute an obstacle in understanding the use of the concept" (p. 633).

Soto-Johnson, Oehrtman, Noblet, Roberson and Rozner (2012) also reported on how a group of experts consisting of five PhD mathematicians and a PhD physicist conceived complex numbers: They immediately referred to vectors for complex number addition and make vector summation, perceived them in the Cartesian form for subtraction and addition, recognized in polar form for division and multiplication and defined 'z' as an operator and 'w' as the operand in the multiplication of two complex

numbers, 'z.w'. Thus, they had an opportunity to oscillate flexibly between geometric and algebraic representations of the complex numbers.

Contrary to the results from experts, Panaoura *et al.* (2006) study showed that the majority of high school students who were responsible to solve two "complex number conversion tasks" (p. 701), chose to use geometric approach rather than algebraic approach and were more successful to come up with the correct answers than students who chose to use algebraic approach. On the other hand, they were not able to switch from one representation to the other, since they were not able to perceive that algebraic and geometric representations of a complex number indicated the same number such that those representations were mathematically connected (Panaoura *et al.*, 2006). Panaoura *et al.* (2006) claimed that teachers' instructional goals and curricular objectives have an important impact on representations chosen by students to solve mathematical tasks regarding complex numbers. Nordlander and Nordlander (2012) also argued that due to the visualization challenge of complex numbers, students have difficulties in making sense the meaning of rules. Therefore, the teacher's role is crucial in order to minimize the difficulties in students' learning of complex numbers. They stated that "to introduce the concept with an innovative and visual method could be a way to enhance the teacher's professional growth as well as a better learning situation for the student" (p. 639).

Building on the results of previous research in this study, both the transition from the algebraic to the geometric and the geometric to the algebraic representation of the multiplication of complex numbers will be given prominence.

2.3. Multiplication of Complex Numbers

In this section, I start with the meaning of the multiplication of integers in order to explain the meaning of multiplication of complex numbers. Then, I explain how the multiplication of complex numbers is acknowledged in the field by referring to the previous research on the multiplication of complex numbers. The explanation of the meaning of the multiplication of integers follows:

Hollis (1967) gives some real life examples from the physical concept of movement to explain multiplication of integers. These examples show that we could utilize vectors to make the multiplication of any two integers reasonable: "a" and "b". In the multiplication "a.b", we say that "a" is the 1^{st}factor and "b" is the 2nd factor. The integer "b" indicates the direction and the magnitude of a vector; the sign of the integer "a" indicates how the direction of the "b" vector changes and the magnitude of the integer "a" indicates how many "b" vectors that are attached end to end. When the sign of first factor "a" is positive, this means we add the vector "b" end to end for "a" times without changing its direction and so we form a new vector. Moreover, when the sign of first factor "a" is negative, this means we add the vector "b" end to end for "a" times to form a new vector and then we reverse the direction of the new vector. Similarly, Arcavi and Bruckheimer (1981) assert that "to multiply a positive number is to "enlarge" the vector retaining its direction, and multiply by a negative number is to "enlarge" it and to reverse its direction" (p. 33). In this case, "the ... multiplication of two complex numbers with zero imaginary components (i.e., laying on the horizontal axis) obtain exactly the same results as the regular addition and multiplication of their real components" (Nemirovsky *et al.*, 2012, p. 297). However, the multiplication of two complex numbers with imaginary components requires perceiving complex numbers as vectors and as operations on vectors (Nemirovsky *et al.*, 2012).

Previous research showed that neither prospective nor in-service teachers have the algebraic and geometrical meaning of the multiplication of complex numbers simultaneously. That is, they have comparted understandings regarding the multiplication of complex numbers. Their comparted understandings allow them to perform the multiplication of complex numbers in two ways: Either they make the algebraic calculations without any connection with geometry or they construct the algebraic and geometrical solutions only on tasks including the multiplication of a complex number with "i".

In particular, Karekok *et al.* (2015) reported on three in-service secondary mathematics teachers' (Melissa, Troy and Aaron) conceptions of multiplication by i. The study showed that Melissa "utilized the fact that i^2 is -1" (p. 339) and was able to explain algebraically the multiplication of i.(3i) by pointing to the fact she knew. How-

ever, she could not explain the multiplication geometrically. While Troy acknowledged that the multiplication of a complex number by i resulted in "90° rotation about the origin" (p. 341), "Aaron recognized the product of a complex number by i as a 90° counterclockwise rotation and described a procedural method for verifying this fact" (p. 343). That is, he easily made algebraic multiplications by i in the Cartesian form and conceived the imaginary number as an operator that translated the real part and imaginary part of a complex number to the imaginary axis and real axis in the counterclockwise, respectively.

In addition, Conner *et al.* (2007) explored the ten prospective mathematics teachers' conceptions of the multiplication of complex numbers based on their explanations for why the real number multiplication 7.(-1) = -7. Before the study, students justified this solution by using iteration of -1. When they were asked to think on 7 rather than -1, none of them considered rotation. Only two of them justified the equation using reflection. After the study, students could think the multiplication of 7 by -1 via the 180° rotation over real number line. On the other hand, data from students' conceptions and usage of Complex Plane showed that "eight of the ten students successfully used the plane to perform addition" but "none of the ten students had a model for multiplication" (p. 4). Conner *et al.* (2007) further stated that:

"They expressed a far lower comfort level with using the plane to demonstrate complex arithmetic than they had with simple calculation. This deficit persisted through the unit of study, but improvement was seen. For example, prior to instruction, only one student was even somewhat comfortable using the Complex Plane to demonstrate arithmetic. In comparison, ..., six out of ten students were somewhat or very comfortable using the Complex Plane to demonstrate arithmetic" (p. 4).

Therefore, the study showed us that the instructions containing the Complex Plane increased the prospective mathematics teachers' conceptions of the geometric meaning of the multiplication of two real numbers.

Nemirovsky *et al.* (2012) studied with thirteen prospective secondary school teachers, who enrolled in a course based on conceptual understanding of essential mathematical ideas, communicating mathematical thinking and making sense the acceptable mathematical justifications. From the thirteen prospective secondary school teachers, three of them used the classroom floor as Complex Plane and explored the geometrical meaning of the multiplication the complex number ($2 + \frac{1}{2}$ i) by i, the result of which is ($-\frac{1}{2} + 2$ i). Results of this study showed that prospective teachers could realize the multiplication by i "preserve the distance to the origin" (p. 302) and "multiplication by i is a 90-degree rotation" instead of reflection (p. 303). In this regard, Nemirovsky *et al.* (2012) asserted that "the complex plane has a particular arithmetic" (p. 296) based on operations on vectors. For instance, while the addition of two complex numbers could be recognized as the component wise addition of two vectors, the multiplication of two complex numbers could be "obtained by (a) multiplying their magnitudes and (b) adding their angles" (Nemirovsky *et al.*, 2012, p. 296). Therefore, "multiplying any complex number by i results in another complex number with the same magnitude (because the magnitude of i is 1) rotated 90 degrees (because the angle of i is 90)" (Nemirovsky *et al.*, 2012, p. 297). In this case, the multiplication of a complex number with i ends up with merely the transition of the real part of the complex number to the positive part of the imaginary axis and the imaginary part of the complex number to the negative part of real axis. That is, the multiplication of a complex number with i through the Cartesian form results in "...a 90-degree rotation as far as coordinates go. Just like how we had (a, b) here. That point actually became (-b, a)" (Karakok *et al.*, 2015, p. 343).

Ipek (2013) also claimed that a complex number z = a + bi could be considered as a vector, whose starting point is origin (0, 0) and the end point is (a, b) and such recognition could make geometric inferences about the multiplication of complex numbers possible. On the other hand, Karakok *et al.* (2013) said that:

"This perspective may have facilitated their ability to provide a geometric interpretation of the addition of complex numbers using vectors or by decomposing the numbers into the real and imaginary components. Unfortunately, this perception of

complex numbers does not lend itself well to the geometric interpretation of complex number multiplication" (p. 2).

Thus, Karakok *et al.* (2013) argued that accepting complex numbers as a pairs of real numbers or seeing them as vectors are useful for the geometric interpretation of the addition of complex numbers, but it is not a proper perspective in order to develop the meaning of multiplication of complex numbers.

Karakok *et al.* (2015) also claimed that "perceiving complex number addition as the addition of vectors makes mathematical sense, but it is not sound mathematics to discuss the product and quotient of vectors in R^n" (p. 347). They asserted that when two complex numbers are to be added, they are accepted as vectors and addition becomes a vector summation instinctively. On the other hand, when two complex numbers are to be multiplied, it becomes difficult "to perceive one complex number as an operator and the other as a vector" (p. 342).

To the contrary, in this study, I report on how vector form of complex numbers has an important role in the development of the meaning of multiplication of complex numbers. It is in this respect that, in this study, the focus will be the realization of the multiplication of two complex numbers as the multiplication of two vectors such that one of them will represent a vector while the other one will be an operator defining the vector operations acted on the other vector. In addition, the focus will be on constructing the relationship among the polar, vector and Cartesian forms of the multiplication of two complex numbers, z = a + bi and w = x + yi.

2.4. Theoretical Framework

In this section, I describe theory of quantitative reasoning by focusing on the constructs of quantity, quantification, quantitative and numerical operations and quantitative relationships. Although not mentioned explicitly, by referring to the previous research I also point to the possible quantitative and numerical operations embedded in the different operations on complex numbers. Finally, I explain the rationale be-

hind the use of quantitative reasoning framework in this study. In order to explain quantitative reasoning, Thompson (1994) defines quantity first: He mentions quantity as a "conceptual entity" (p. 7) such that quantity refers an object's or phenomena's measurable quality. Then, "quantification is a process by which one assigns numerical values to qualities" (Thompson, 1994, p. 9). In order to attain a numerical value to a quality, the measurement could be done directly of indirectly (Thompson, 1994). Quantification, therefore, could be seen as a mapping that matches "the result of measuring and the attribute that is measured" (Moore, 2010, p. 21).

For the quantification process to occur, Moore *et al.* (2009) defined quantitative operation and quantitative relationship: A quantitative operation means derivation of a new quantity by taking into consideration two quantities at hand and "a quantitative relationship is the image of three quantities, two of which determine the 3rd by a quantitative operation" (p. 5).

Thompson (1994) made a distinction between quantitative operations and numerical operations such that quantitative operations "create quantities and numerical operations used to evaluate quantities" (p. 11). He stated:

"Quantitative and numerical operations are certainly related developmentally, but in any particular moment they are not the same even though in very simple situations children (and teachers) can found them unproblematically" (Thompson, 2011, p. 42).

For instance, the operations on numbers such that addition, subtraction, multiplication and division are numerical (arithmetic) operations and arithmetic operations are "actually used to calculate a quantity's value in a given situation" (Thomson, 1990, p. 12). Thompson (1990) also added "given a quantitative relationship, one can determine the appropriate arithmetical operation to evaluate any quantity in the relationship" (p. 27). On the other hand, "a quantitative operation is non-numerical; it has to do with the comprehension of a situation" (Thompson, 1994, p. 13). Therefore, Thompson (1994) defining quantitative operations based on actions of mind provided some examples:

"...The quantitative operation of combining two quantities additively originates in the actions of putting together to make a whole and separating a whole to make parts; the quantitative operation of comparing two quantities additively originates in the action of matching two quantities with the goal of determining excess or deficit; the quantitative operation of comparing two quantities multiplicatively originates in matching and subdividing with the goal of sharing" (p. 10).

Based on these explanations, from Thompson's perspective, "quantitative reasoning is the analysis of a situation into a quantitative structure - a network of quantities and quantitative relationships" (Thompson, 1990, p. 13). Similarly, Moore, Carlson and Oehrtman (2009) stated that:

"Quantitative reasoning refers to the mental actions of an individual conceiving of a situation (e.g. a runner in a race), constructing quantities of his or her conceived situation (e.g., time elapsed or distance ran), and both developing and reasoning about relationships between these constructed quantities (e.g., a relationship between distance ran, time elapsed, and average speed)" (p. 3).

In sum, quantities are conceived properties and they might vary from one individual to another according to their personal experiences (Moore *et al.*, 2009; Moore, 2010). Thus, Thompson (2011) stated that even if it is the most obvious attribute of an object or situation, students' conceptions of this attribute could significantly differ from teachers' conception. That is, teachers can only make inferences about the quantities established on the part of their students.

Therefore, for the concept formation, Thompson's (1994, 2011) emphasis is more on the operational aspect such that it becomes possible, from the teacher's perspective, to explicate the nature of reasoning someone might undertake. Particularly, researchers claimed that "quantitative reasoning provides conceptual content for powerful forms of representation and manipulation in algebra" (Smith III and Thompson, 2007, p. 10). It is in this respect that, while building complex number multiplication on the part of a prospective secondary mathematics teacher, in this study, I use quantitative reasoning

in two ways: while designing the sequence of tasks in the teaching experiment series and while analyzing the prospective teacher's thought processes before, during and after the teaching sessions. I elaborate on how I use quantitative reasoning framework during the analysis under the analysis section. Taking into account the previous research, I now elaborate on how I use the framework while designing the tasks in the teaching experiment series.

First, previous research has pointed to the quantitative operations involved in the formation (Sarac and Karagoz Akar, 2017) and the arithmetic (e.g., Nemirovsky *et al.*, 2012; Soto-Johnson and Troup, 2014) of complex numbers: Particularly, addition and subtraction of complex numbers corresponds to *translations*; multiplication and division of complex numbers is equivalent to the composition of *dilation and rotation*; and, the conjugation of complex numbers is equivalent to a *reflection* about the real axis (Nemirovsky , Rasmussen, Sweeney, and Wawro, 2012; Soto-Johnson and Troup, 2014; Usiskin, Peressini, Marchisotto and Stanley, 2003). Similarly, Ipek (2013) stated that vector representation of complex numbers accompanies with "the modulo and argument concepts" (p. 23) such that multiplication of a complex number z with another complex number w = (r, θ) means the rotation of the vector 'z' through the argument θ and expansion of the vector 'z' by factor r. That is, "both the rotation and expansion are centered at the origin", such that there is "no difference whether we do the rotation followed by the expansion or the expansion followed by the rotation" and when "r $<$ 1 then the expansion is in a contraction" (Ipek, 2013, p. 23). In addition, Sarac (2016) specifically worked on one prospective teacher's developing the meaning of the multiplication of a real number with the imaginary number "i".

Though, she found out that, although the participant was able to make sense of the multiplication of the real number with the imaginary number "i" as the 90° counterclockwise rotation, she was not able to make sense of it until she calculated (b.i) algebraically first. Therefore, taking into consideration the aforementioned studies reporting the importance of such operations with the arithmetic operations of complex numbers, in this study, I conjectured that the quantitative operations such as *dilating, rotation* and also *translation* plays an important role in the reconstruction of the

multiplication of complex numbers.

Secondly, I also embrace the idea that as a robust concept, complex numbers are conceptualized in a quantitative structure. I now elaborate on that: In this study, I embrace the concept of the multiplication of complex numbers within a quantitative structure as follows: When a complex number z = x + yi is multiplied with another complex number w = a + bi, z could be thought as a vector and the real and imaginary parts of w could be separated into two. In this way, one could utilize that the vector form of z could be multiplied with the real and imaginary part of w one by one through the distributive property of multiplication on addition. Then, one would be able reason that the two new vectors, a.z and bi.z, are found, where depending on the magnitudes of a and b the magnitude of the vector z would be enlarged or shrunk and also, with the existence of "i" the result would end in 90° rotation. Then, the "+" between a.z and bi.z could be seen as the vector summation and the two vectors could be the summed in order to get the product 'z.w'. In addition, one could realize that once the vector summation was utilizied, a right triangle would be formed by the vectors z, w and z.w, where the right legs would be the vectors z and w, and the hypothenuse would be the vector z.w. Then, one could use the Pythagorean Theorem and calculate and explain that the square root of the addition of the square of the magnitude of the right legs, z and w, would be equal to the magnitude of the hypothenuse, z.w. Thus, one could find that the magnitude of z.w would be equal to the multiplication of the magnitude of z with the magnitude of w. Therefore, one could conclude that when two complex numbers, z and w, were multiplied, one of the complex numbers, w, would dilate the other complex number, z, by factor of the magnitude of itself and thus, the magnitude of the resultant complex number, z.w, would be equal to the multiplication of the magnitudes of z and w. Regarding the cases of the scalars a and b, one could explain that the vectors a.z and bi.z could change in direction and magnitude. Also, one could calculate the angle of the vector z.w based on the angles of the vectors z and w. For that, one could translate the vector bi.z to the end of the vector a.z in order to be able to calculate the angle of the vector z.w based on the angles of the vector z and w. Particularly, if one could translate the vector bi.z to the end of the vector a.z, s/he could reach some concluisons within the right triangle of the vector summation

of the vectors, a.z and bi.z: (i) the vector, a.z would be on the same direction with the vector z, for all cases of a, (ii) when $a > 0$, the vector, a.z would be on the same orientation with the vector z, thus the angle of the vector, a.z would be the same with the angle of the vector, z, (iii) when $a < 0$, the vector, a.z would be on the reverse orientation with the vector z, thus the angle of the vector, a.z is 180° more from the angle of the vector, z and (iv) the angle of the vector, z.w would be equal to addition of the angle of the vector, a.z with the angle between the vector, a.z, and the resultant vector, z.w. Therefore, one could calculate the tangent of the angle between the vector, a.z and the resultant vector, z.w, in the right triangle to evaluate the angle between the vector, a.z and the resultant vector, z.w. Then, one could realize that it would be important to define the angle between the vector, a.z, and the resultant vector, z.w in terms of the angle of w, since for all cases of a and b, the angle between the vector, a.z and the vector, z.w would be equal to the angle of the vector w. Thus, one could reach the result that the vector, z.w would rotate from the vector z by the angle of the vector w. Therefore, one could say that the angle of the vector, z.w would be equal to the addition of the angles of the vectors, z and w. Thus, one could understand the reasoning behind the polar multiplication formula and could justify that the argument of the complex numbers, z.w, would be equal to the addition of the arguments of the complex numbers, z and w.

Therefore, taking into consideration of the quantitative operations embedded both in the formation and arithmetic of complex numbers, I determined a sequence of tasks that would possibly trigger the concept of the multiplication of complex numbers as explicated above on the part of a prospective teacher. I leave the discussion on determining the task sequence to the Appendix A.

3. SIGNIFICANCE OF THE RESEARCH STUDY

Smith III and Thompson (2007) emphasized "an early emphasis on developing children's ability to conceive of, reason about, and manipulate complex ideas and relationships, as an equal complement to numerical reasoning and computation" (p. 6). In other words, the teaching activities should be oriented based on "number and quantity" rather than be based on "number only" (p. 44). Similarly, Panaoura *et al.* (2006) claimed that while thinking about complex numbers "...pupils' solution methods were very much restricted to the use of the algebraic approach, even if a geometric approach seemed easier and more efficient" (p. 693). Thus, in this study, I embrace that while working on complex numbers not only algebraic approach but also the geometric approach should be simultaneously given prominence and the tasks should focus students' attention on both "number and quantity".

In addition, some operations such as translation, reflection, rotation and dilation etc. involved in the arithmetic of complex numbers were pointed to (e.g., Ipek, 2013; Nemirovsky *et al.*, 20012; Sarac, 2016; Soto-Johnson and Troup, 2014). Also, Sarac (2016) has particularly shown that a prospective secondary mathematics teacher has developed the meaning of the multiplication of a real number with the imaginary number, i, as referring to 90° counterclockwise rotation of the real number. Though, she had difficulty in representing the result geometrically. Also, researchers claimed that when two complex numbers are multiplied, it becomes difficult "to perceive one complex number as an operator and the other as a vector" (p. 342) although addition of complex numbers are thought as adding two vectors more easily (Karakok *et al.*, 2015). By the same token, results from (Conner *et al.*, 2007) study pointed that prospective teachers could not represent the multiplication of complex numbers although they could represent the addition of complex numbers on the Complex Plane (Conner *et al.*, 2007). Similarly, teachers only could explain multiplication by "i" both algebraically and geometrically (Karakok, *et al.* 2015; Nemirovsky *et al.*, 2012). In addition, even when students could state the geometric meaning of the multiplication of complex number as rotation and dilation they could not justify the reasons (Soto- Johnson and Troup,

2014). In sum, although the aforementioned studies informed the field with important results in terms of the imparted knowledge (prospective) teachers and students acquired regarding the multiplication of complex numbers, none of these research studies have solely focused on the development of the multiplication of complex numbers (i.e., (a + ib). (x + iy)) on the part of prospective teachers or students.

Taking previous research into account, this research study scrutinized a prospective secondary mathematics teacher's, Derin, construction of the meaning to multiplication of complex numbers based on addition of the vectors and multiplication of a vector by a scalar and the complex number bi. In this regard, in this study, a prospective secondary mathematics teacher's development of the consideration that multiplication of complex numbers is the same as multiplication of two vectors is investigated.

It is in this respect that this research study is thought to contribute to research and teaching in two distinct but related ways: by providing a sequence of tasks focusing on students' re-construction of the multiplication of complex numbers though quantitative reasoning and by providing a model, from the point of view of the researcher, of possible thought processes someone might engender during such development as well as providing the difficulties someone might encounter while developing such meaning.

4. STATEMENT OF THE PROBLEM

This study aims to generate a model of how a prospective secondary mathematics teacher develops the meaning of the multiplication of complex numbers based on her quantitative reasoning on vectors. Therefore, this study particularly investigates the following research questions:

(i) How does a prospective secondary mathematics teacher develop the meaning of the multiplication of complex numbers in Cartesian, vector and polar forms?
(ii) What meanings of the multiplication of complex numbers in Cartesian, vector and polar forms does a prospective secondary mathematics teacher develop during an instructional sequence involving quantitative reasoning?
(iii) What difficulties might a prospective secondary mathematics teacher encounter while developing the meaning of the multiplication of complex numbers in Cartesian, vector and polar forms.

5. METHOD

In this section, I explain the design, participants and the procedure of the study. I also explain how I conduct the analysis.

5.1. Design of the Study: Teaching Experiment Method

In the context of this study, since I explore the development of the meaning of the multiplication of Cartesian, vector and polar forms of complex numbers during an instructional sequence involving quantitative reasoning, the design of the study is chosen as teaching experiment methodology (Steffe and Thompson, 2000).

Teaching experiment methodology focuses on comprehension of the students' progress rather than only concentrate on the students' current understanding as in clinical interviews (Steffe and Thompson, 2000). Steffe and Thompson (2000) stated that teaching experiments are based on a sequence of teaching episodes. There are some components of teaching episodes: (i) "a teaching agent", (ii) "one or more students", (iii) "a witness of the teaching episodes" and (iv) "a method of recording this transpires during the episode" (p. 273).

In particular, during the teaching experiment, the teaching agents (the teacher-researcher) should remain away from his/her conceptual knowledge and operational knowledge and not try to introduce his/her own understandings to the students (Steffe and Thompson, 2000). In this regard, during the teaching episodes, for the teacher-researcher the student(s) language and actions are main sources of information. Therefore, teacher-researcher continuously tries to make interpretations from the student(s) sayings and doings in order to test his/her hypotheses which are written before the experiment. In this study, I consider the written hypotheses as the teacher- "researcher's articulation of intended or inferred student knowledge of the logical necessity involved in a particular mathematical relationship" (Simon, 2017, p. 124) since it is expected to be constructed by the teacher-researcher prior to the teachings (see Appendix A).

Yet, at some points during the teaching episodes, the teacher-researcher has to prepare new hypotheses according to the data that comes from the students and reorganize teaching experiment in order to test them (Steffe and Thompson, 2000). While generating and testing hypotheses, teacher-researcher should also take into consideration the students' "essential mistakes", since these mistakes shows what the students could and could not do (Steffe and Thompson, 2000, p. 276). Also, the teacher-researcher should concentrate on the students' reasoning to understand students' mental operations focusing on their oral and verbal explanations and actions so that s/he could decide what the student(s) should learn and how the student(s) should learn remaining of the teaching episodes by updating the teaching goals to adapt the students' ongoing mathematical activities (Steffe and Thompson, 2000). Still, the students could regulate and organize their mathematical thinking themselves. That is, the teacher-researcher might just organize the situations/teaching environments and the form of interaction with student. Eventually the teacher-researcher constructs "models of students' mathematics" (p. 268) where students' mathematics refer the ways of students' oral and written answers during the engagements in mathematical activities (i.e., tasks). Steffe and Thomson (2000) define this model construction as "mathematics of students" (p. 268) which means at the same time making inferences from students mathematical understandings. That is, the teacher-researcher builds models on students' thinking and conceptual understanding based on the generalizations, they interpret from students' mental actions that are repeatedly put into their mathematical behaviors (i.e., oral and written statements) during the teaching activities. The construction of the models of students' mathematics (Steffe and Thompson, 2000) aligns with the construct, mathematical conception, since it is also defined as "an explanatory model used to explain observed abilities and limitations of mathematics learners in terms of their (inferred) ways of knowing" (Simon, 2017, p. 120)

Steffe and Thompson (2000) also suggest that the teacher-researcher should make a connection with student(s) before the study in order to "recognize mathematical language and current students' actions" (p. 283). Thus, the communication between the researcher and student(s) might be healthier during the teaching episodes. Moreover, Steffe and Thompson (2000) argue that during the teaching episodes a witness should

be present in order to help the teacher-researcher by making alternative inferences form the student(s) activities.

It is in this respect that since the aim of this research study was to understand and explicate how a prospective secondary mathematics teacher developed the meaning of the multiplication of complex numbers in the Cartesian, vector and polar forms, the teaching experiment methodology was used to generate a model of her construction of her meanings.

In this study, another researcher was the teacher-researcher who conducted the teaching sessions and the pre- and post-clinical interviews with the participant, Derin. Also, another researcher outside of the study was behind the camera as an observer. She also was a researcher in the field of mathematics education and had master degree in mathematics. Two cameras and an audio-recorder were used to record the teaching sessions. The teacher-researcher tested the hypotheses I have written in the articulation of the understanding of the multiplication of complex numbers. During the teaching episodes/sessions, the teacher-researcher developed new hypotheses according to the inferences of the participant's sayings, doings and difficulties. Also, I and the teacher-researcher examined the new hypotheses after each of the teaching sessions and concentrated on the particaipant's reasoning in order to understand what she should learn and how she should learn. Thus, I and the teacher-researcher regulated the teaching goals and teaching environment according to the participant's needs to realize ongoing mathematical activies. Therefore, I and the teacher-reseacher aimed to build a model of the participant's conceptual understanding of the multiplication of complex numbers. Also, I and the teacher-researcher took into consideration the other researcher's comments witnessing and observing the participant's written and oral explanations, reasoning and essential mistakes.

Hipotheses of this study were:

(i) The participant will be able to realize the similarities and differences between the orientations, directions and starting points of the vectors $c.z_1$ and $di.z_1$, where z_1

is a complex number and c and d are real numbers, regarding all cases of c and d.

(ii) The participant will be able to reason on the rotation between the vectors $c.z_1$ and $di.z_1$ such that the angle between these vectors is equal to 90° regardless of all cases of c and d.

(iii) The participant will be able to explain the multiplication of $z_1 = a + bi$ and $z_2 = c + di$ in vector form referring to the real part, $c.z_1$ and imaginary part $di.z_1$ of $z_1.z_2$.

(iv) The participant will be able to reason on the geometrical representation of the multiplication of complex numbers, $z_1.z_2$ in vector form regarding all of the cases c and d.

(v) The participant will be able to think of the real and imaginary parts of the resultant vector, ac - db + i.(ad + bc), in terms of the real and imaginary parts of z_1 and z_2.

(vi) The participant will be able to focus on one of the complex numbers as a multiplier and the other as a multiplicand; rather than focussing on the product $z_1.z_2$.

(vii) The participant will be able to mention either z_1 or z_2 as the operant vector acting on the other one.

(viii) The participant will be able to reason on the angle of the vector $z_1.z_2$ referring to the angles of z_1 and z_2 regarding all the cases of c and d.

(ix) The participant will be able to evaluate the angle between the vectors $z_1.z_2$ and z_1 regarding all the cases of c and d.

(x) The participant will be able to reason on the tangent value of the angle between the vectors $z_1.z_2$ and z_1, and the angle of the vector z_2 regarding all of the cases of c and d.

(xi) The participant will be able to explain that the argument of $z_1.z_2$ is equal to addition of the arguments of z_1 and z_2 regarding the all the cases of c and d.

(xii) The participant will be able to reason on the magnitude of the vector $z_1.z_2$ in terms of the magnitudes of the vectors $c.z_1$ and $di.z_1$ using the geometric representation of the vector summation $z_1.z_2 = c.z_1 + di.z_1$ regarding all the cases of c and d.

(xiii) The participant will be able to calculate the magnitude of $z_1.z_2$ in terms of the magnitudes of z_1 and z_2 using the Pyhtagorean Theorem.

(xiv) The participant will be able to explain that the magnitude of $z_1.z_2$ is equal to the multiplication of the magnitudes of z_1 and z_2 regarding all the cases of c and d.

(xv) The participant will be able to explain the relationship between the multiplication of complex numbers in Cartesian, vector and polar forms.

The hypotheses above would be tested during and after the teaching sessions in order to develop a second order model of the participant's development of the multiplication of complex numbers.

5.2. Participants

One prospective secondary mathematics teacher, Derin (a pseudoname) participated in the study. She was at her last year of undergraduate study. Her GPA was 3,03. She took 12 mathematics courses during her undergraduate study. The courses were such that: (i) Calculus I, (ii) Calculus II, (iii) Introduction to Mathematical Structures, (iv) Matrix Theory, (v) Linear Algebra I, (vi) Discrete Mathematics, (vii) Algebra I, (viii) Statistics, (ix) Introduction to Complex Analysis, (x) Differential Equations, (xi) Introduction to Set Theory and (xii) Probability. When the overall grades that she got from these mathematics courses were examined, she got from 4 of 12 mathematics courses BB and above and the remaining 8 of 12 mathematics courses CC and below. Also, she did not fail or repeat any of these 12 mathematics courses. She got DD from the Introduction to Complex Analysis course.

In order to select the participant of the study, first, I provided a written pre-assessment (see Appendix B for the pre-assessment questions) to 28 prospective mathematics teachers who were in their fifth year of five-year undergraduate program in secondary school mathematics education program in a public university in Turkey, in 2017 where the medium of instruction is English. Based on the analysis of their written responses, I determined that some prospective teachers already knew the meaning of the multiplication of complex numbers and some of them did not have enough knowledge about the complex numbers. I chose to work with Derin, since she depicted the kind of knowledge previous research pointed to: She had imparted knowledge regarding

the multiplication of complex numbers albeit her robust understanding regarding the nature of complex numbers.

Particularly, she had following prior knowledge: She was able to define complex numbers both from the mathematicians point of view and operationally and represent them both in Cartesian, vector and polar forms; was able to define vectors and vector summation and show them geometrically; was able to multiply both a vector and a complex number, z= a + bi, with a scalar and show geometrically; was able to able to multiply a complex number z= a + bi by the imaginary number "yi" algebraically and represent geometrically; and finally was able to multiply a complex number z= a + bi with another complex number w= x + yi, algebraically. As aforementioned earlier, Derin possessed the kind of knowledge previous research indicated to.

Although having such knowledge, it was comparted in the following way: Derin had

(i) the lack of ability to represent and explain the multiplication of two complex numbers, z = a + bi and w = x + yi, in vector form,
(ii) the lack of ability to represent and explain the multiplication of two complex numbers, z = a + bi and w = x + yi, in polar form,
(iii) the lack of ability to explain the relationship among the Cartesian, vector and polar forms of the multiplication of two complex numbers, z = a + bi and w = x + yi,
(iv) the lack of ability to explain the multiplication of two complex numbers, z = a + bi and w = x + yi geometrically using the operations: dilation and rotation,
(v) the lack of ability to explain the meaning of addition of arguments of the two complex numbers when they are multiplied.
(vi) the lack of ability to explain the meaning of the multiplication of magnitudes of the two complex numbers when multiplied.

These criteria provided an opportunity to establish the development of the multiplication of complex numbers on the part of the participant with the pre-requisite

understandings and abilities needed to engage in task sequence involved in teaching sessions. Therefore, based on the hypothetically articulated concept of the multiplication of complex numbers (Steffe and Thompson, 2000) (see Appendix A), the participant's lack of understandings about the concept of the multiplication of complex numbers was planned to be overcome during the teaching experiment sessions.

5.3. Procedure

5.3.1. Data Sources

Data sources of the study are (i) scanned copies of the pre- and post-written assessments, (ii) scanned copies of student work during interviews and the teaching experiment sessions,(iii) transcripts of the videos of the pre- and post-clinical interviews, (iv) transcripts of the videos of teaching sessions and (v) scanned observer's and teacher-researcher's notes taken during teaching sessions and after the analysis of teaching sessions.

5.3.2. Data Collection

Data was collected in three phases: (i) a pre-interview after a written pre-assessment, (ii) the teaching sessions and (iii) a post-interview after a written post-assessment.

5.3.2.1. A Pre-Interview After a Written Pre-Assessment. First, in one-hour long written problem solving session, 28 prospective mathematics teachers answered the pre-assessment questions (see Appendix B for the pre-assessment questions). Then according to their answers in conformity with the criteria that I mentioned in the participants part, I and another researcher (hereafter referring "we") selected the potential participants of the study. As already mentioned, based on the analysis of their written responses, we determined that some prospective teachers already knew the meaning of the multiplication of complex numbers, some of them did not have enough knowledge

about the Cartesian, vector and polar form of the complex numbers and some of them could not define vectors and operations on vectors geometrically to participate in the teaching sessions. In this case, three prospective mathematics teachers' written pre-assessments showed that they were appropriate participants. Then, one week later, the teacher-researcher conducted one hour long task-based clinical interviews with two of three prospective mathematics teachers as its principles defined by Goldin (2000). the third participant did not volunteer to continue the study. The two interviews were videotaped and transcribed. Then, based on the analysis of the transcripts and the written pre-assessment, the participant of the study was selected. The other second participant was not chosen, since s/he could not make transitions between the vector form and the Cartesian form of the complex numbers. Also, s/he explained consistently the multiplication of complex numbers in polar form algebraically and geometrically.

5.3.2.2. The Teaching Sessions. Prior to the teaching sessions, based on the preliminary analysis of the transcripts of the clinical interview and the written pre-assessment that I and the teacher-researcher expected as outlined earlier, I prepared a hypothetical teaching sequence (see Appendix E for the hypothetical teaching sequence) based on the articulation of the concept of the multiplication of complex numbers (see Appendix A for the articulation). In fact, after my articulation of the concept, I and the teacher-researcher reached consensus on the hypothetical learning progress on the part of the participant with the desired pre-knowledge. During the articulation of the concept, I asked myself what the participant had already known and what she should knew and how she could be able to develop that knowledge about the multiplication of complex numbers.

According to the hypothetical teaching-sequence, each teaching session lasted up to 60 - 75 minutes. There were 4 teaching sessions in total that took a place within a span of 8 days. Two digital cameras and a tape-recorder were managed during the sessions. One of the cameras focused on the participant's work and another focused on the teacher - researcher work in order to prevent missing data and facilitate gathering reliable information. Then, audio-recordings and video-recordings were transcribed.

According to the collected data in each teaching session, the next teaching-session was reorganized for the needs of the participant's current understanding and mental actions in order to draw the best model of her development.

5.3.2.3. A Post-Interview After a Written Post-Assessment. Upon completion of the teaching sequence, the participant answered the written post-assessment questions same as the pre-assessment (see Appendix B for the post-assessment questions). Then, one week after the written post-assessment, the teacher-researcher conducted a 90-minute long post-interview with the participant. As in the pre-interview, two cameras and an audio-recorder were used and the recorded data was transcribed. The main purpose of the post-interview was to describe the participant's current understandings and meanings of the concept of multiplication of complex numbers and to observe the development occurred during the teaching sessions.

5.3.3. Data Analysis

Data analysis includes the following: (i) analysis of pre-interviews and post interviews, (ii) ongoing data analysis and (iii) retrospective analysis.

5.3.3.1. Analysis of Pre- and Post-Interviews. Analysis of pre-interviews was done to assess and acknowledge the participant's initial understanding (s) about (i) the complex numbers, (ii) geometric representations of complex numbers, (iii) the algebraic and geometric meanings of the multiplication of complex numbers in Cartesian form, (iv) the algebraic and geometric meanings of the multiplication of complex numbers in vector form and (v) the algebraic and geometric meanings of the multiplication of complex numbers in polar form. As the results of the analysis of pre-interviews shaped the teaching sequence properly based on the participant's pre-understanding(s) and ways of thinking, analysis of post-interviews was important to determine current understanding(s) and meanings of the participant about the algebraic and geometric meanings of the multiplication of Cartesian, vector and polar forms of the complex numbers upon completion of the teaching sequence. The analysis was also important

to draw an appropriate model of the participant's development of the meanings of the multiplication of complex numbers (Clement, 2000).

In this regard, the focus of analysis was the participant's reasoning quantitatively on the interview questions. For the analysis, I read the transcription of the interviews line-by-line with the goal of determining how the participant reasoned. In particular, the participant's answers to 'why?' questions were examined closely to articulate her understandings, her ways of reasoning and her mental operations she possibly engaged in while answering the questions about the multiplication of the complex numbers. Also, the participant's geometric representations of the multiplication of complex numbers and the meanings which she attributed to geometric representations were the focus of attention in order to describe the development effectively. In this case, Clement (2000) said that:

"... the ability to collect and analyze data on mental processes at the level of a subject's authentic ideas and meanings, and to expose hidden structures and processes in the subject's thinking that could not be detected by less open-ended techniques. These abilities are especially important because of Piaget's discovery that people have many interesting knowledge structures and reasoning processes that are not the same as academic ones they have alternative conceptions and use nonformal reasoning and learning processes. Mapping this "hidden world" of indigenous thinking is crucial for the success of instructional design. Students cannot help but use their own prior conceptions and reasoning processes during instruction, and these have strong effects on the course of instruction".

5.3.3.2. Ongoing Data Analysis. After and in between each teaching session, I and the teacher-researcher reflected on the participant's understanding and development of the meaning of the multiplication of complex numbers connecting to the concept of vectors. I and the teacher-researcher mainly focused on the following during the ongoing analysis:

(i) The mathematical issues that appeared during the teaching sessions and the way that the participant thought about and handled these issues;
(ii) Whether there was a change in the participant's thinking about these issues or in her approach at the end of the sessions and the nature of that change;
(iii) The weaknesses/difficulties the participant encountered and possible explanations for such weaknesses/difficulties;
(iv) The factors that afforded or limited the change or the development in her conception,
(v) The possible mathematical understandings and meanings of the participant that had been further investigated in the following session(s).

5.3.3.3. Retrospective Analysis. Steffe and Thompson (2000) explained the importance of retrospective analysis in teaching experiment methodology such that "Careful analysis of the videotapes offers the researchers the opportunity to activate the records of their past experiences with the students and to bring them into conscious awareness" (p. 292). Steffe and Thompson (2000) stated that as a result of retrospective analysis, model building process starts and during building a model, the teacher-researcher catches the patterns of the participant's mental operations referred in explanations.

Thus, retrospective analysis period started at the end of the data collection. For retrospective analysis, I and the teacher-researcher read the transcription of the videos of the teaching sessions line-by-line with the goal of determining the actions and explanations that show the participant's understanding(s) and ways of reasoning. Thus, the conceptual analysis method was used to qualify the participant's reasoning; to modify the existing hypotheses and to develop new hypotheses about the mental operations of the participant (Steffe and Thompson, 2000). Also, retrospective analysis allowed searching for counter examples within data in order to improve and support the claims about the participant's understanding(s) and ways of reasoning. This in turn enabled to construct a second order model of the participant's development of the meaning of the multiplication of complex numbers.

6. RESULTS

6.1. Written Pre-Assessment and Pre-Interview

In this section, I shared data regarding Derin's existing knowledge pior to the teaching sessions. Due to the lenght of the excerpts from the pre-interview, I share data in two parts. the first part includes data showing how Derin defined and represented complex numbers. Also, it included data depicting what she knew and did not know about the Cartesian, polar and vector form of multiplication of complex numbers.Then, in the second part , data shows how Derin reasoned both algebraically and geometrically while multiplying a complex number with a scalar and with pure imaginary number.

6.1.1. Derin's Definition and Representation of Complex Numbers and Multiplication of Complex Numbers

Particularly, during both the written pre-assessment and the pre-interview (see Appendix B for written pre-assessment questions and Appendix C for pre-interview questions), Derin was asked to comment on the definition and the geometric representation of complex numbers:

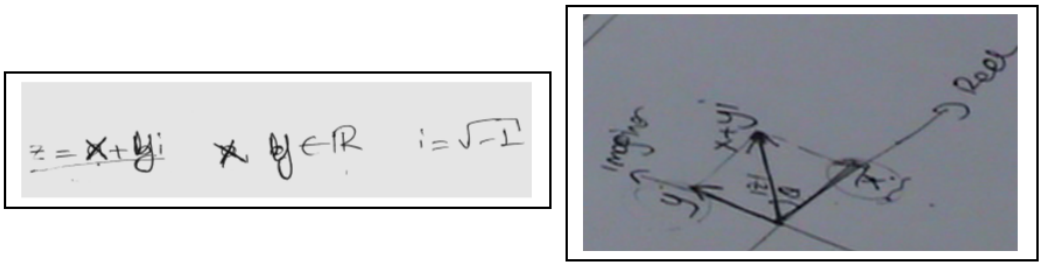
R: Could you define complex numbers?

D: Let's say z is equal to x plus yi, where x and y are real numbers. The numbers written in this way are complex numbers (see Figure 6.1a).

R: What is i here?

D: It is the square root of minus one. If I show as a point, here is x (She refers to the real part of x + yi) and there is y (She refers to the imaginary part of x + yi). If I represent x + yi in vector form, this x is one part of z (She draws a vector from the origin to the point of x). x is the part of z on the real axis. This is also a part of z on

the imaginary axis (She draws a vector from the origin to the point of y) (see Figure 6.1b).



a b

Figure 6.1. Derin defines the complex number z = x + yi and shows it in vector form and in polar form geometrically (a, b).

R: Okay. How could you define z geometrically?

D: For instance, like we say the absolute value of z for the magnitude of z. If here is theta (She refers to the angle between the vector of the complex number x + yi and the positive part of the real axis in the Figure 6.1b), we say the multiplication of the magnitude of z with cosine teta gives the magnitude of this vector (She refers to the vector on the real axis). When we multiply the magnitude of z with sine teta gives the magnitude of this vector (She refers to the vector on the imaginary axis).

As the data showed, Derin was able to define complex numbers algebraically. She also knew that the numbers written as z = x + yi where x, y ∈ R, $i^2 = \sqrt{-1}$, were complex numbers. Also, she was able to show the complex number x + yi on the Complex Plane including the real axis and the imaginary axis. She knew that x was the real part of z and y was the imaginary part of z. In addition, she was able to show the complex number z in vector form. She was able to show that the real part, x of z, was the projection vector of z on the real axis and the imaginary part, y of z, was the projection vector of z on the imaginary axis. Similarly, she was able to define complex number, z, geometrically in terms of its magnitude and argument. She was able to show that the magnitude of the vector z was defined as $|z|$ such that when the magnitude of z was multiplied with the cosine and sine of the angle between the vector,

z and the positive part of the real axis, she would get the magnitude of the real part of z on the real axis and the magnitude of the imaginary part of z on the imaginary axis.

Regarding the algebraic explanation of the multiplication of complex numbers, writing $z_1 = a + bi$ and $z_2 = c + di$, Derin stated:

D: a plus bi times c plus di. First I multiply a and c, ac, then I multiply a and di, adi. Then, bci plus bdi². bdi² is equal to -db, since i² is -1 (see Figure 6.2: She writes $z_1.z_2 = (a + bi).(c + di) = ac + adi + bci - db$). If I arrange the equation, ac minus db plus ad plus bc in the paranthesis of i (She writes ac - db + i.(ad + bc)).

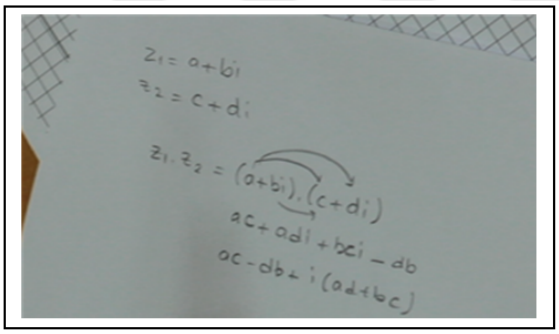


Figure 6.2. Derin multiplies the complex numbers (a + bi) and (c + di) algebraically.

As the data suggested using the distribution property of multiplication, Derin was able to multiply the components of z_1 with that of z_2. She also knew that $i^2 = -1$. She further knew that she could add the real parts and the imaginary parts of the product to write it in the original form as ac - db + i.(ad + bc). All these showed that Derin was able to calculate and define the multiplication of the Cartesian form of complex numbers algebraically.

R: Well, how would you express this geometrically?

D: Now, this one... How? Do you want me to show it on the figure?

R: Yes, sure.

D: For example (see Figure 6.3: She draws a Complex Plane). Let's say it is on the Complex Plane. For example if a plus bi is here (She marks a point on the 1st quadrant). That point shows the complex number a plus bi. For example, c plus di... If I mark the complex number c plus di in one of the quadrants of the Complex Plane. Let's say it is c plus di (She marks another point on the 1st quadrant and writes c + di). Vector representation of these, how did I do that? Now when I express it as vector, newly formed (She draws vectors starting from the origin to specified points on the 1st quadrant)? For example, y is here like (She draws a horizontal line from the point that express the number c + di to the imaginary axis). If we multiply these, a number will be formed. A number like this (She refers to the number ac - db + i.(ad + bc)). This number... I don't know where ac minus db has to be, now. Hmm... I assume that the point is somewhere on the plane, let's say it is here (She points to another point on the 1st quadrant). Result of this multiplication will be a complex number (She draws a vector from the origin to a point that she specified). Here is ac minus db (She draws a perpendicular line from the specified point to the real axis and points to the intersection point of the line and the real axis). Here is ad plus bc (She draws a perpendicular line from the specified point to the imaginary axis and points to the intersection point of the line and the imaginary axis).

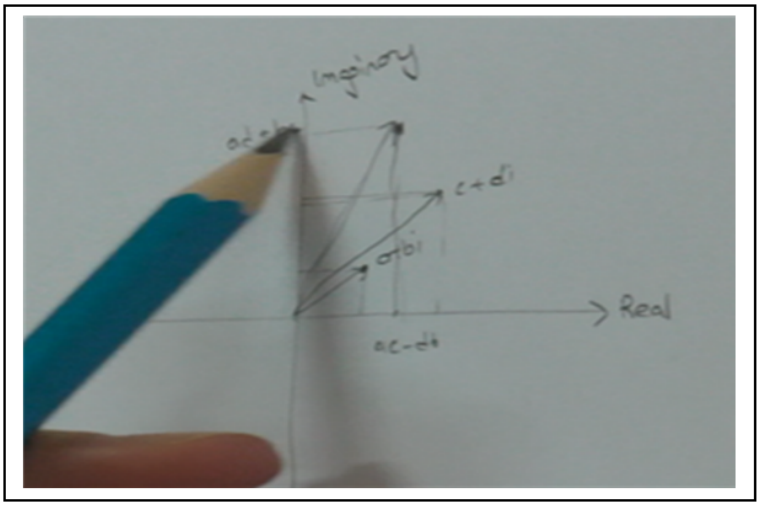


Figure 6.3. Derin multiplies the complex numbers (a + bi) and (c + di) geometrically.

Data showed that Derin was able to show the result of the multiplication of the Cartesian form of complex numbers geometrically in vector form. Though, her

representatition was not proper. That is, although she knew; and, specifically showed z_1 and z_2 as a point and further explained and showed $z_1.z_2$ as a point on the Cartesian plane, the points were not written as (a, b) or (c, d) or (ac - db, ad + bc). Secondly, her focus was on the product rather than the multiplicants. She did not mention either z_1 or z_2 as the operant vector acting on the other. Specifically, she showed and explained the result of the multiplication as a point and a vector such that she pointed to the real and imaginary parts of the product as parts of the resultant vector on the Cartesian plane. However, she did not mention the addition of the real parts and the imaginary parts referring to the vector summation such that the addition of real parts would refer to the addittion of the real parts of z_1 and z_2 as vectors and the addition of the imaginary parts would refer to the addition of the imaginary parts of z_1 and z_2 as vectors.

Although, not mentioning the resultant vector as the multiplication of two vectors seems to be a limitation on her part, it may not absolutely indicate to her lack of knowlegde. Further data is needed and will be shared to show her lack of knowledge regarding her reasoning on the multiplication of complex numbers in vector form.

Derin also was asked to explain the multiplication of complex numbers in polar form:

R: Well, how could you explain the multiplication of these two complex numbers in polar form?

D: This one? (She shows the point that she marked on the Complex Plane for $z_1.z_2$).

R: Hıhım.

D: Well. Will I write it by using ac minus db' It is r multiplied with cis teta. r multiplied with cis teta is r times cosine teta plus i sine teta (She writes r.cis θ= r (cos θ + i.sin θ)). r is here (She points to the length of the vector that she draws for

$z_1.z_2$). We will obtain that one by taking the square of these (She refers to the real and imaginary parts of the number $z_1.z_2$).

R: Write them down if you want.

D: r, the square of the real part ac minus db, plus square of ad plus bc (She writes $r = (ac - db)^2 + (ad + bc)^2$) square root of the r (She draws a square root symbol on $r = (ac - db)^2 + (ad + bc)^2$). (see Figure 6.4)

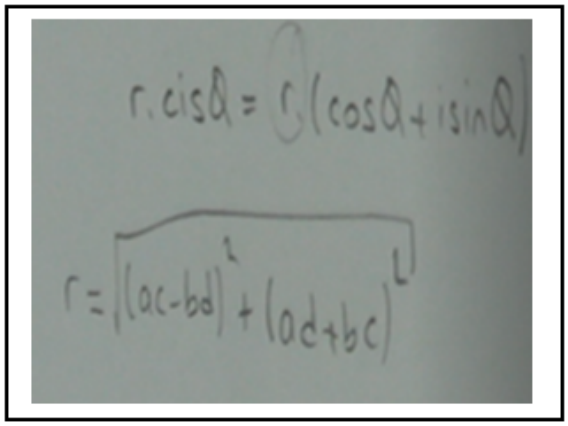


Figure 6.4. Derin shows the complex number in polar form and calculates the magnitude of $z_1.z_2$.

R: And what is ?

D: In here theta is... (She shows the angle between the vector she drew for $z_1.z_2$ and positive side of the real axis and writes θ). What did I say'I told it, but I can't remember. A moment please. Do I express it with arc or what? To find theta, tangent theta, well ad plus bc divided by ac minus db (see Figure 6.5: She writes tan tan $\theta = \frac{|ad+bc|}{|ac-db|}$).

R: Are they... (She refers to the nominator and denominator of the tan tan $\theta = \frac{|ad+bc|}{|ac-db|}$). What is ad plus bc?

D: Here (She shows the distance of the point of $z_1.z_2$ to real axis) and there (She shows the distance of the point of $z_1.z_2$ to imaginary axis). Hmm, like this (She takes the absolute values of nominator and denominator of the tan $\theta = \frac{|ad+bc|}{|ac-db|}$) (see Figure 6.5).

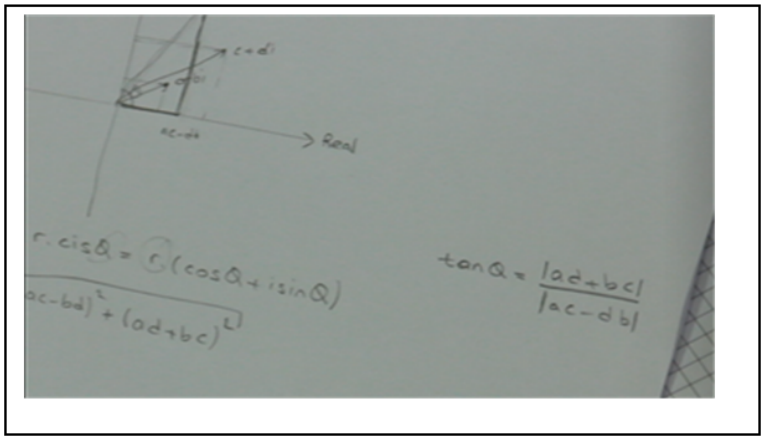


Figure 6.5. Derin calculates the tangent of the argument of the complex number $z_1.z_2$.

As the data indicated, Derin was able to state the general polar form of a complex number, r.(cos θ + i.sin θ), and also determine the polar form of $z_1.z_2$. She also knew that r referred to the magnitude of $z_1.z_2$ and the tangent of the angle referred to $\frac{|ad+bc|}{|ac-db|}$. In addition, her crossing over the real and imaginary parts and writing the expressions "ad + bc" and "ac - db" in absolute values indicated that she thought that the imaginary part, ac - db, referred to the distance to the real axis and the real part, ad + bc, refered to the distance to the imaginary axis. She pointed to this in the following excerpt. Though, while explaining the polar form, her focus was again on the product rather than the multiplicants z_1 and z_2.

Therefore the researcher further probed her to reason from the multiplicants.

R: Okay. Could you write theta, angle of the multiplication vector in terms of the angle of z_1 and z_2?

D: I assume that here is alpha (She points to the angle between the vector she drew for z_1 and the positive side of the real axis), tangent alpha (She writes tan α)

equals... Here is a (She draws a perpendicular line from the number z_1 to real axis and names the intersection point of line and the real axis as a), that point was b (She draws a perpendicular line from the number z_1 to imaginary axis and names the intersection point of line and the imaginary axis as b). Tangent alpha, b over a. I assume it as length. (She writes tan $\alpha = \frac{|b|}{|a|}$) Tangent beta (She writes tan α) is here (She points to the distance of the number z_2 to its real axis). Here is c (She draws a perpendicular line from z_2 to real axis and names the intersection point of line and the real axis as c), here is d (She draws a perpendicular line from z_2 to imaginary axis and names the intersection point of line and the imaginary axis as d). d is divided by c (She writes tan$\beta = \frac{|d|}{|c|}$) (see Figure 6.6).

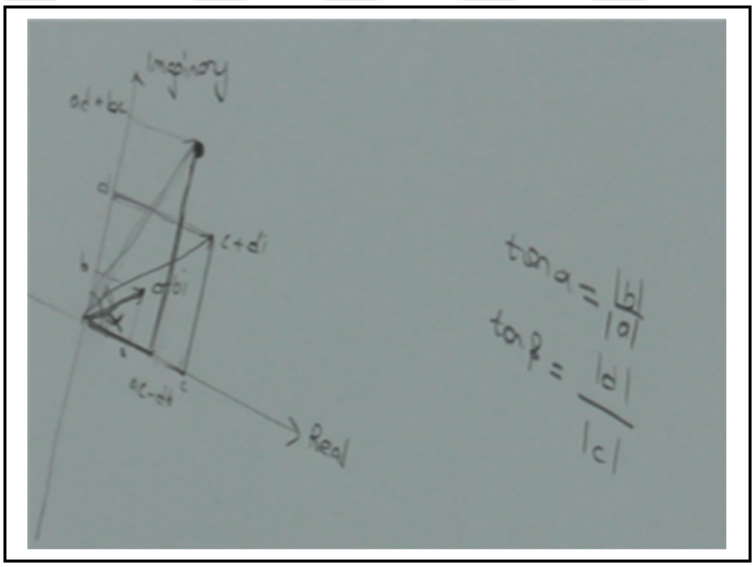


Figure 6.6. Derin calculates the tangents of the arguments of the complex numbers z_1 and z_2.

D: Tangent α plus β. Hmm, I know them but... Normally theta is α plus β, isn't it? Yes. But, its reason...

R: Yes, I will ask. Do you know its reason?

D: I don't know the reason. (She thinks about the equation that tan $\theta = \frac{|ad+bc|}{|ac-db|}$ for 30 seconds). I don't know.

As tha data showed, Derin could make the algebraic calculations in order to find the argument of the multiplication of complex numbers in polar form and say that the argument of the complex number $z_1.z_2$, theta, is equal to the addition of the arguments of the complex numbers z_1, α, and z_2, β, in polar form. Though, again, her focus was on the argument of the product.

Also, when prompted to think in terms of the multiplicants, she could not explain the reasoning behind why the argument of the product was determined as the addition of the arguments of the multiplicants. Then, the researcher probed her to further reason about multiplying the magnitudes of the multiplicants, z_1 and z_2, to determine the magnitude of the product. Discussion followed:

R: Okay. Do you know why r is the multiplication of the lenghts of z_1 and z_2? I mean this notation you see (She points to the notation $r = \sqrt{(ac-db)^2+(ad+bc)^2}$), do you know why it is equal to the multiplication of the lengths of z_1 and z_2?

D: Eer (She thinks for several seconds). Let's see, the length of z_1... (She thinks on the Complex Plane for a minute). There were questions like that (referring to her past experience). I did something like that, but I can't recall correctly right now. In the addition, it is a vector, since there is vector summation... In multiplication... We obtain this notation (She points to the notation ac - db + i.(ad + bc)) in multiplication. (She thinks for several seconds) No, I can't recall.

R: Okay. That is okay. When you think of the multiplication of complex numbers, can you explain the relation between the Cartesian and polar form of multiplication of complex numbers?

D: When we multiply the complex numbers in polar form, we multiply the magnitudes. We add up the angles.

R. Okay, would you say more?

D: Hmmm. That is all.

As the data showed, Derin was not able to explain why multiplying the magnitudes of the multiplicants, z_1 and z_2, determines the magnitude of the product although she was able to state the magnitude of r as the multiplication of the magnitudes of z_1 and z_2. Even though she thought of vector summation, she went back to the resultant vector, ac - db + i.(ad + bc), and could not explain why she would multiply the magnitudes. Though, the researcher did not ask further why she thought of vector summation. Still, I argue that Derin possibly was imagining adding z_1 and z_2 in vector form picturally. This was in fact shown in the written session (see Figure 6.7 below). However, in the case of complex multiplication, she was not able to think of the real and imaginary parts of the resultant vector, ac - db + i.(ad + bc), in terms of the real and imaginary parts of z_1 and z_2. This also suggested that she did not think of either of z_1 or z_2 as an operant vector acting on the other one. Therefore, the whole data suggested that, Derin was able to state and calculate the multiplication of z_1 and z_2 in polar form and was able to show the result as a vector, yet, she could not explain neither the reasoning behind the formula nor the relationship between the polar and the vector form of the multiplication of complex numbers.

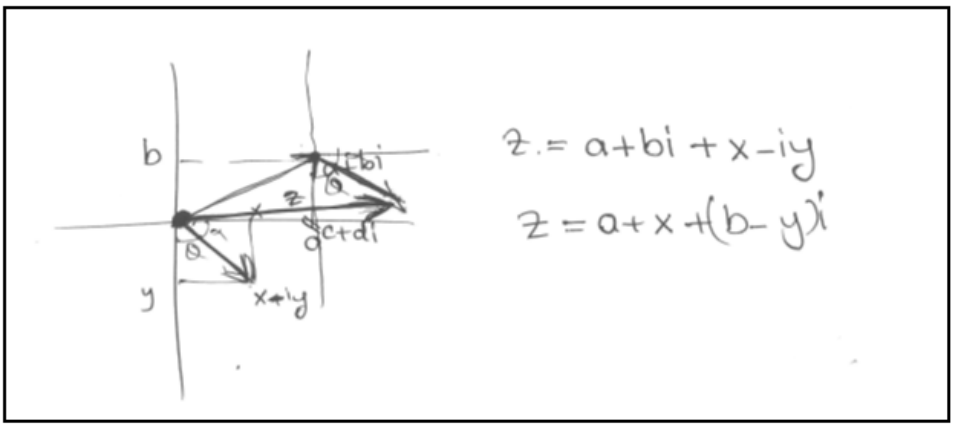


Figure 6.7. Derin's adding the vectors a + bi and x - yi.

6.1.2. Derin's Reasoning on the Multiplication of a Complex Number with a Scalar and with a Pure Imaginary Number

Derin was also asked to reason on the multiplication of complex numbers with a scalar, c and pure imaginary number, di and show it geometrically. In fact, during the written sessions, Derin was able to do these calculations both algebraically and show the result geometrically. During the interveiw, the researcher asked to get further information on how Derin reasoned. Due to the lenght of the excerpts again, I share the data in two sub-parts.

6.1.2.1. Derin's reasoning on multiplying z = a + bi with a scalar, c. *R: Well, we asked it in pre-test too. A real number, let's say c, a real number. The real number c. There is also the number z_1, a plus bi. Think about their multiplication algebraically. Can you explain this to us?*

D: Eeer, c times a plus bi (She writes c.(a + bi)). When I am multiplying these, first I multiply c with a, I obtain a number like ca plus cbi (She writes c.(a + bi) = ca + cbi).

R: Can you show this geometrically?

D: This one. What is c times a plus bi, do you want me to show it on the figure?

R: Hmm. I mean pointwise, geometrically. On the Complex Plane.

D: Here is the number a, b (She specifies a point, "a" on positive side of the real axis on the 1st quadrant and a point, "b" point on the positive side of the imaginary axis. She points to the point a + bi with the intersection of the points a and b). If I multiply it with c, for example, in here I said... a plus bi (She writes a + bi on the point that she specified). Hmm, if the real number c, is bigger than one, let's say it is two, I assume it increases the magnitude of a plus bi. Okay (She specifies another point farther from the origin than a + bi on the Complex Plane). It increases magnitude as

it is (She writes c.(a + bi) over the point that she recently specified). Hmm, if it is smaller than one, I mean between zero and one, it decreases the magnitude of a plus bi. It can be written anywhere around there (She specifies another point closer to the origin than a + bi on the Complex Plane). Hmm, it is here (She points to the point that she specified latest).

R: Can you write it down here (see Figure 6.8)?

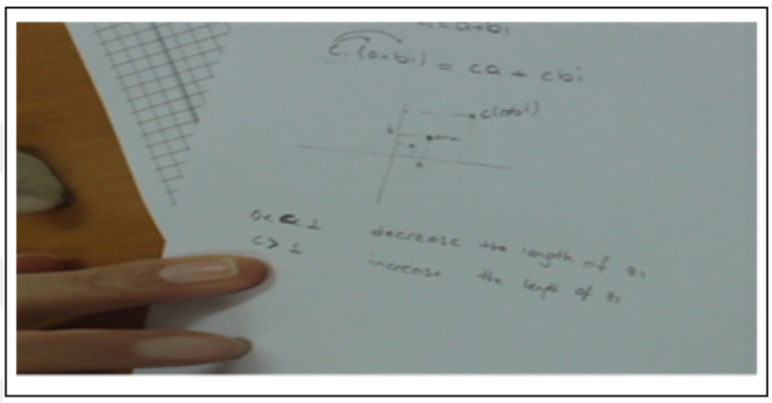


Figure 6.8. Derin writes $\theta_1 - \theta_2$ below β.

R: Okay. What did you. Could you say it again?

D: It decreased the magnitude of z_1 *(While she is telling it, she also writes next to* $0 < c < 1$*). Hmm, increase (She writes increase the length of* z_1 *next to* $c > 1$*).*

R: I will ask you something. We said c, between zero and one, and if c is bigger than one, didn't we? Well, does it affect the angle?

D: The angle doesn't change here, because it directly changes its magnitude. There was a vector in there for example (She draws the vector of a + bi) it either decreases the magnitude or increases. In addition to that I said if it is negative, it changes its direction. Should I write it too? c, if smaller than zero of course I say if smaller than one.

R: Okay. If it is smaller than zero... You also said if it is smaller than one. What did you think when you said that?

D: If it is smaller than one, it both decreases the magnitude of z_1 and changes its direction. If it is between negative one and zero and then also if it is smaller than negative one, I think I should say change (see Figure 6.9).

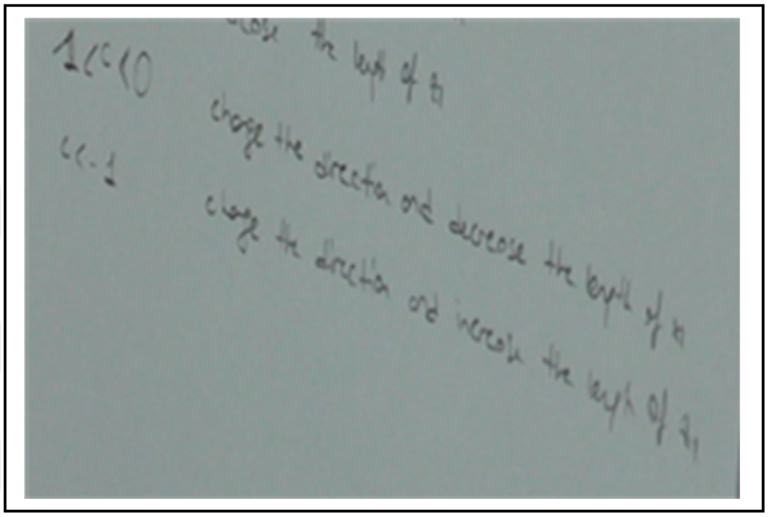


Figure 6.9. Derin writes "-1 < c < 0 change the direction and decrease the length of z_1" and she writes "c < -1 change the direction and increase the length of z_1".

R: What happens if c is zero?

D: It eliminates.

R: What do you mean it eliminates?

D: We get the number zero.

R: Okay. (The researcher focuses on the cases c is less than zero). You said it changes its direction. Do you know how many degrees does the multiplication with the scalar c change the direction of z_1?

D: A hundred and eighty degrees (She points to a vector that opposite to the vector a + bi on the Complex Plane).

R:Okay, alright.

As the data indicated, Derin could define and calculate the multiplication of a complex number, z_1 with a scalar, c. She could separate the scalar "c" into four cases according to its magnitude: $c < -1$, $-1 < c < 0$, $0 < c < 1$ and $c < 1$. She knew that c acted on the magnitude of the resultant vector such that when c was bigger than 1 and less than -1, the magnitude of z_1 increased. She also knew that when the value is between -1 and 1 not including zero the magnitude of z_1 decreased. Similarly, she knew that when c was bigger than 0, the direction of z_1 did not change, and the direction of z_1 changed 180° in the counterclockwise when "c" was smaller than zero. This suggested that she knew the impact of multiplication with c, a scalar, on the direction and magnitude the vector z_1. Unfortunately, the interviewer did not probe further why when c was smaller than -1 the vector's direction changed 180°. Though further data will indicate to that:

When further asked to show c.(a + bi) taking into consideration of z_1, Derin was able to state the following:

D: I think this and this (She points to the vectors in Figure 6.10). In here, a to the zero. In here, zero to b (see Figure 6.10).

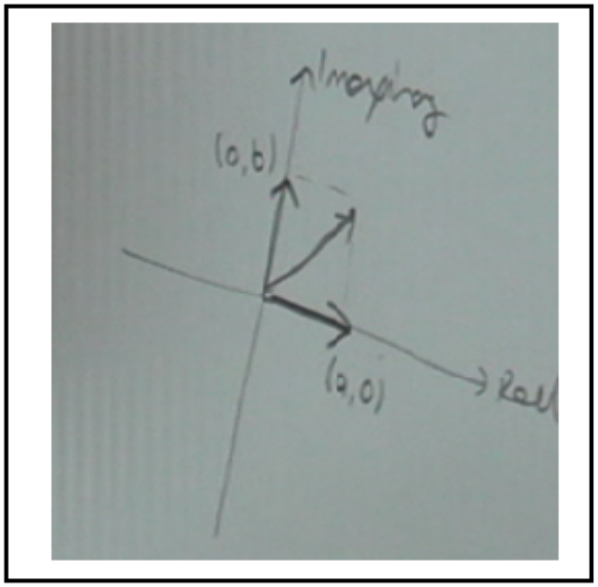


Figure 6.10. Derin draws projection vectors of the vector z_1 on the real axis and on the imaginary axis. She points to the part on the real axis and writes (a, 0). She points to the part on the imaginary axis and writes (0, b).

R: Okay, can you show us the multiplication c.(a + bi) based on these vectors?

D: Well, it increases or decreases like this (She refers to that the projection vector on the real axis will increase or decrease). (She draws a new Complex Plane and writes $0 < c < 1$ next to it). When $0 < c < 1$, if this point is (a, 0) (She draws a vector on the positive side of the real axis and wrote (a, 0) to the end of the vector), here is (She shows a point that was close to the midpoint of the vector)... That point is (ca, 0) (She draws a vector from the origin to the specified point and names the point as (ca, 0)). That one is (She draws a vector on imaginary axis) here (She specifies another point on the vector close to the midpoint of the vector and draws a vector from the origin to the point she specified). I think it was zero to b (She writes (0, b) to the end point of the vector on the imaginary axis). This is zero to bc (She writes (0, bc) to the end point of the short vector on imaginary axis).

As the data indicated, Derin knew that when $0 < c < 1$, the real and imaginary parts of the resultant vector was smaller than the real and imaginary parts of the

original vector z_1. She also knew that multiplication with a scalar between 0 and 1 did not change the direction of the projection vectors of the original vector z_1 on the real and imaginary axis. Then, she wanted to comment on the 2nd option when c > 1 and then, when -1 < c < 0:

D: If c is bigger than one (She writes c > 1 and draws a Complex Plane below it). Hmm, if it is there (She draws a vector on the real axis starting from the origin). The new one will be here (She draws a longer vector than the previous one on the real axis starting from the origin). If it is there (She draws a vector on the imaginary axis starting from the origin). The new one will be here (She draws a longer vector than the previous one on the imaginary axis starting from the origin). By the new one, I mean zero to cb (She writes (0, cb) to the end point of the 2nd vector that she drew on the imaginary axis). Zero to b (She writes (0, b) to the end point of the 1st vector that she drew on the imaginary axis). Hmm? A to the zero (She writes (a, 0) to the end point of the 1st vector that she drew on the real axis). ac to the zero (She writes (ac, 0) to the end point of the 2nd vector that she drew on the real axis).

R: When you do these multiplications, do the orientation of the real and imaginary parts of the original vector change?

D: They don't change for this one.

R: And what about the direction of the real and imaginary parts of the original vector?

D: Their direction doesn't change either. By saying direction, do you mean it being on this line (She points to the real axis)? It doesn't change. The direction doesn't change, because the angle doesn't change. I mean, the direction of the original vector doesn't change, since the directions of its real and imaginary parts don't change.

Data indicated that Derin was able to show the multiplication of complex number a + bi with a scalar c referring to its impact on the real and imaginary parts of the

vector a + bi. Particularly, she knew that when c > 1, the real and imaginary parts of z_1 increased in magnitude. Also, she was able to explain the reason behind the sameness of the direction pointing to the angle between the real and imaginary parts: She knew that the angle between them did not change, since multiplying the real and imaginary parts with a real number bigger than 1 did not change their direction. This way she knew that the angle between them did not change keeping the result vector's direction the same. All these suggeted that she could define the changes on the magnitude and the direction of real and imaginary parts in vector form when the scalar, c > 1. Also, data suggested that the starting point of real and imaginary parts of the complex number a + bi remained the same.

Derin then continued talking about when -1 < c < 0:

D: In here, if c is between negative one and zero (She writes -1 < c < 0 and draws a Complex Plane below it), if it is here (She draws a vector starting from the origin on the positive side of the real axis and writes (a, 0) to its end point). Now, it will change orientation, since it is negative, it will rotate one hundred and eighty degrees in the counterclockwise. And because it decreases the magnitude (She draws a shorter vector starting from the origin on the negative side of the real axis than the vector on the positive side). It is ac to the zero (She writes (ac, 0) to the end point of the vector). That part was zero to b (She draws a shorter vector starting from the origin on the negative side of the imaginary axis than the vector on the positive side of the imaginary axis). Zero the cb (She writes (0, cb) to the end point of the vector). It is here (She shows the resultant vector by using the vector parts on the 3rd quadrant).

R: Okay. Why?

D: Because when it rotates one hundred and eighty degrees, it is still on the same direction, because its parts also rotate one hundred and eighty degrees exactly... Hmm, it rotates one hundred and eighty degrees (She points to the Complex Plane, when -1 < c < 0 and indicates the projection vector of the vector (a, b) on the real axis). It also rotates one hundred and eighty degrees (She points to the projection vector of the

vector (a, b) on the imaginary axis).

R: What happened to the orientations and directions of the parts?

D: The orientations of the two vectors both reversed. I mean I multiplied them with negative one. They rotated one hundred and eighty in the counterclockwise and stayed on the same line. I mean this (She points to the vector c.(a + bi) on the 3rd quadrant of the Complex Plane that she drew for -1 < c < 0) is on the same direction with this one (She points to the vector a plus ib on the first quadrant of the Complex Plane that she drew for -1 < c < 0). If I must show, if here is (She points to the angle between the vector of a + bi and positive side of the real axis) theta (She writes θ to the angle). Hmm, it became like this and like that (She refers to the contrary vectors on the real axis and the imaginary axis). Since this angle is also theta (She marks the angle between the recently formed vector on the third quadrant and the negative side of the real axis, and writes θ) the direction doesn't change.
R: Hmm. You say it doesn't change. Why doesn't θ change?

D: Now this (She refers to the part of the vector a + bi on the real axis) became this (She refers to the part of the vector c.(a + bi) on the real axis). And this (She refers to the part of the vector a + bi on the imaginary axis) became this (She refers to the part of the vector c.(a + bi) on the imaginary axis). Now, multiplying it with c already changed the magnitude of a + bi. And since c is also negative, it changes the direction of a + bi.

Data showed that Derin knew that when -1 < c < 0, the magnitude of the real and imaginary parts of the resultant vector would be smaller than the real and imaginary parts of the original vector, since the absolute value of the multiplier "c" was less than 1. She knew that the real and imaginary parts of the original vector z_1 rotated 180° in the counterclockwise. Thus, she was able to say that the real and imaginary parts of the original vector stayed on the same line, but opposite orientations with the real and imaginary parts of the resultant vector because of multiplication with a negative scalar. Therefore, she was able to show that linear combination of the real and imaginary

parts of the resultant vector would be on the same line, but opposite orientation with the original vector. Also, she was able to show and explain that the angle between the original vector and the positive part of the real axis and the angle between the resultant vector and the negative part of the real axis was the same. Particularly, she reasoned that since multiplying with "c" which was less than 1 changed the magnitude and the orientation of the original vector 180° in the counterclockwise, the resultant vector had an argument that was 180° more than the argument of the original vector, since they still stayed on the same line.

Then, Derin commented on the last case asking if should continue:

D: Should I solve this one too? (She writes c < -1 and draws a Complex Plane below it. She draws two vectors starting from the origin on the positive side of the real axis and on the positive side of the imaginary axis on the Complex Plane). This one was a to the zero (She writes (a, 0) to the end point of the vector that she drew on the real axis). This one was zero to b (She writes (0, b) to the end point of the vector that she drew on the imaginary axis). It was in here (She draws the vector (a, b) on the 1st quadrant by using vector parts). Now it is big, something like negative two. It will get bigger (On the 3rd quadrant, she draws a vector starting from the origin, longer than the vector (a, b), on the same direction, but contrary orientation with (a, b)) and its parts are zero to cb (She draws a perpendicular line from the end point of the vector to the negative side of the imaginary axis and writes (0, cb) to that point), ac to the zero (She draws a perpendicular line from the end point of vector to the negative side of the real axis and writes (ac, 0) to that point).

As data showed, Derin knew multiplication with a scalar "c", when c < -1, increased the magnitude of the original vector and rotated it 180° in the counterclockwise. Thus, she was able to show that the real and imaginary parts of the original vector increased in magnitude and reversed in direction.

Then, the resercher asked for the four cases of "c" what the changing and unchanging properties were between the original vector and resultant vector:

R: Well, to sum up, could you say... What are the changing and unchanging properties of these four cases?

D: The unchanging one is its direction.

R: The direction of what?

D: Hmm, the direction of the number z_1.

R: Okay. The direction of z_1 doesn't change. In which cases?

D: In all of them. If it is bigger than zero, the orientation of z_1 doesn't change. If it is smaller than zero, the orientation of z_1 reversed as much as one hundred and eighty degrees. And the magnitude of z_1 changes unless c is one. I mean I said it there (She refers to the notes that she wrote for the situations of c).

R: Well, I will ask you another thing. We talked about its magnitude, orientation and direction. What about the starting point, when you multiply it with a real number?

D: The starting point doesn't change. We know it by them (She refers the Complex Planes). It was this here (She shows the starting point of the vector a + bi) for example when we multiplied, it became this (She points to the starting point of the vector c.(a + ib) in the case of -1 < c < 0). Starting point is still the point that zero to zero. I mean starting point doesn't change in any case. Its direction doesn't change in any case. For all cases, it remains the same. If its magnitude is different than one, it changes. It decreses if it is between zero and one. It increases if it is bigger than one. Errr... It also decreases if it is between zero and negative one. It increases if it is bigger than negative one. Errr... If it is bigger than zero, it remains on the same orientation. If it is smaller than zero, it reverses. (see Figure 6.11)

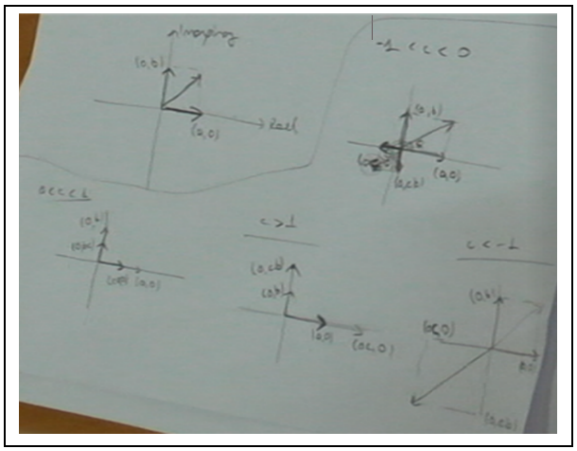


Figure 6.11. Derin shows the multiplication of z_1 with c in vector form, when -1 < c < 0, when 0 < c < 1, when c > 1 and when c < -1.

As the data indicated, considering the four cases of the scalar, c and pointing to the starting point, direction and magnitude of a complex number in vector form, Derin was able to define the changes in the original vector when it is multiplied with a scalar. She knew that the resultant vector stayed on the same line with the original vector whenever c < -1, -1 < c < 0, 0 < c < 1 and c < 1. She also knew that when c was bigger than 0, the orientation and direction of the original vector did not change, but when c was less than 0, the original vector rotated 180° in counterclockwise. Thus, the vector was on the contrary orientation and same direction. Also, she was able to state that the magnitude of the original vector increased when the absolute value of c was bigger than 1, decreased when the absolute value of c was between 0 and 1.

6.1.2.2. Derin's Reasoning on Multiplying z = a + bi with an Imaginary Number, di. Derin was also to reason on the multiplication of complex numbers with di and show it geometrically. In fact, during the written sessions, Derin was able to both algebraically and geometrically show the result of the multiplication of complex numbers with di. During the interveiw, researcher asked to get further information on how Derin reasoned:

R: I want you to multiply it with di. When you think about it can you express it algebraically?

D: Firstly, this is adi. Eee this is minus bd, since i^2 is equal to minus one (see Figure 6.12).

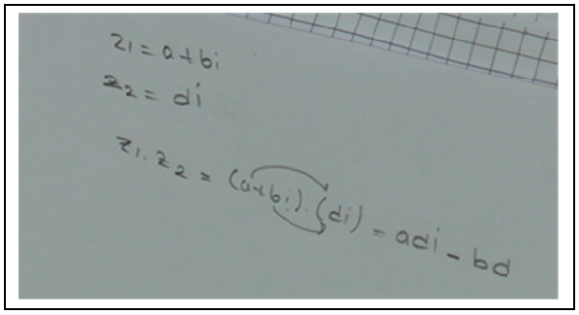


Figure 6.12. Derin multiplies a with di and writes adi and then multiplies bi with di and writes - bd and finds $z_1.z_2$ = (a + bi).di = adi - bd.

R: Can you show this geometrically?

D: Okay. Now, in here (She draws a Complex Plane and marks a point on the 1st quadrant) that is a to b (She writes (a, b) to the point that she marked). Now when we multiply them one by one, like I am multiplying this vector (She refers to the projection vector of the vector a + bi on the real axis) with this one (She refers to the vector that belongs to the number di). Us... Multiplying with the number i means rotating it 90º, because of this... If i multiplied with i, a will have been moved to this axis (She points to the imaginary axis) and because there is also multiplication with d, I will increase or decrease its magnitude (She points to the magnitude of the vector (a, 0)). We talked about it earlier. Or I will change its orientation according to case of d.

As the data showed, Derin was able to multiply the complex number a + bi with the complex number di algebraically using the distribution property of multiplication. She also knew that i^2 is equal to -1. So, she was able to write the finalized version of the resultant number as (a + bi).di = adi - bd. Data further showed that Derin knew that multipication of a + bi with di was equal to the multiplication with i and the scalar d successively. She knew that multiplication with i results with a 90° counterclockwise rotation of the real and imaginary parts of a + bi. Thus, she was able to say that the real part of a + bi was moved to the imaginary axis. Also, she knew that multiplication

with d increased or decreased the magnitude and was able to change the orientation of the real and imaginary parts of the original vector as in multiplication by a scalar, c. Then, she defined and explained the multiplication with di geometrically according to four cases of d: $d < -1$, $-1 < d < 0$, $0 < d < 1$ and $1 < d$. She started with when $0 < d < 1$:

R: Write it down please.

D: Should I write d is the same, bigger than zero and smaller than one, again? (She writes $1 > d > 0$). Normally if I multiplied it with only i, this will rotate it 90º. This a will move here (She points to the imaginary axis), I remember that.

R: If you want, write it down. Show it on the figure.

D: Okay. When we multiply a with i, this number (She marks a point on the real axis and writes (a, 0) below it) was here before, now it is here (She marks a point on the imaginary axis). Zero to a (She writes (0, a) next to the marked point. Now when we multiply it with d, because of d is a real number, it will increase or decrease its length. In here it decreased because it is between zero and one, it is in somewhere here (She draws a new Complex Plane, marks a point closer to the origin than the point (0, a) on the imaginary axis, draws the vector from the origin to that point. She writes (0, ad) to the end point of the vector). bi... it is here already (She points to the number bi on the imaginary axis on the Complex Plane that she previously drew). Now since we multiplied it with i, assume that we don't see d, I rotate it ninety degrees counterclockwise (see Figure 6.13).

R: Okay.

D: And it moved here (She points out a point on the negative side of the real axis) then if d is smaller, it is in somewhere here (She draws a vector on the negative side of the real axis on the Complex Plane that she drew latest). That is in here (She draws a longer vector than the vector she drew to the other Complex Plane on the negative side

of the real axis on the Complex Plane) zero to b (She writes (0, b) to the end point of the vector). This is bd to the zero (She writes (-bd, 0) to the end point of the other vector).

R: Hmm, okay. Where is the point in final case?

D: It is in here (She marks the point that is indicated by the linear combination of the vectors (-bd, 0) and (0, ad) and draws a vector from the origin to this point) (see Figure 6.13).

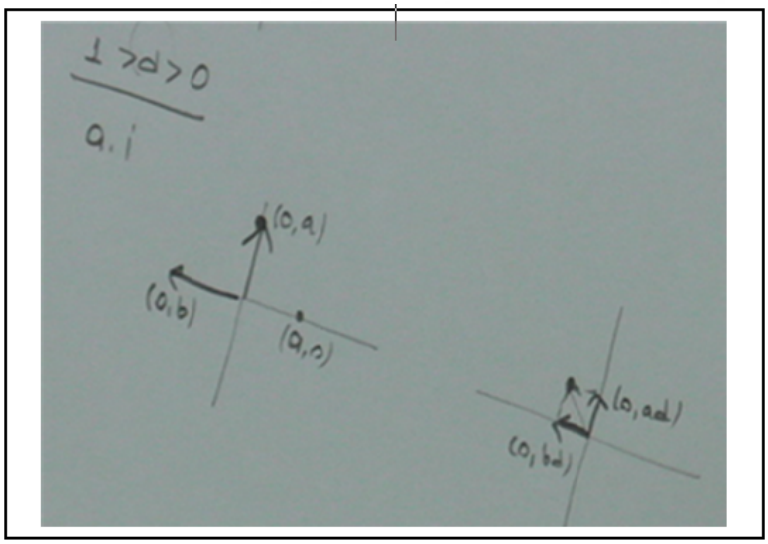


Figure 6.13. Derin shows the multiplication with di when 1 > d > 0, in vector form.

R: I have a question. When you multiplied with di when d is between zero and one and when you think about the complex number as a vector, what did change or unchange regarding to its orientation, direction and starting point?

D: Starting point didn't change. It is zero to zero as always. Its direction changed, because it rotated 90º when I multiplied it with i. Its direction changed. Eer, orientation was already changed. It was like this (She points to the vector a + bi) and it moved to here (She points to the vector -bd + adi). And also the magnitude... Its magnitude is already decreased, since it is between zero and one.

As the data indicated, Derin knew that multiplication with di when $0 < d < 1$ results with a 90° counterclockwise rotation of the real and imaginary parts of the original vector a + bi. Therefore, she was able to say the real part moved to the imaginary axis and the imaginary part moved to the real axis. She also knew that the magnitude of the real and imaginary parts of the original vector decreased since the magnitude of the scalar, d was less than 1. Thus, she was able to determine that the starting point did not change, while the direction, orientation and magnitude of the original vector z_1 changed. Thus, she knew that the direction of z_1 rotated 90° in counterclockwise and the magnitude of z_1 decreased, since $0 < d < 1$. She continued with $d > 1$ and then $-1 < d < 0$:

D: If d is bigger than one (She writes d > 1)... If d is bigger than one, its magnitude will increase. That (She points to the point (0, b)) will move to here (She marks a longer vector than the vector (0, b) on negative side of the real axis). What was it? It was bd to the zero (She writes (-bd, 0) to the end point of the vector). The other part of it comes here (She refers to a point on the imaginary axis that is farther to the origin from the point (a, 0) and draws the vector starting from the origin ending with that point). It was zero to ad (She writes (0, ad) to the end point of the vector). Because of this? Hmm, the new complex number will be here (She intersects the perpendicular lines that she marked as vectors (-bd, 0) and (0, ad) on the 2nd quadrant, marks the intersection point and draws the vector from the origin to the marked point). This is it (see Figure 6.14).

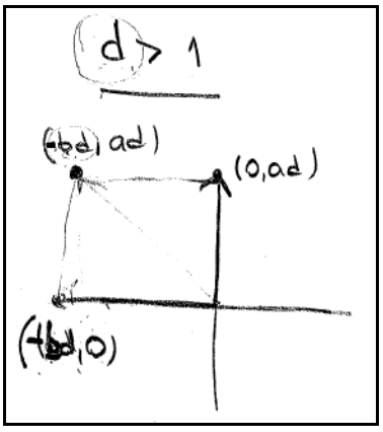


Figure 6.14. Derin shows the multiplication with di in vector form when $d > 1$.

R: Okay. What about its magnitude?

D: Its orientation changed. Its direction changed. And its magnitude increased.

As the data showed, Derin knew when d > 1, the magnitude of real and imaginary parts of the original vector increased. Also, she knew and showed that the real and imaginary parts of the original vector rotated 90° in the counterclockwise. She was able to say that the direction and orientation of the original vector changed and the magnitude of the original vector increased as a result of multiplication with di, when d > 1. Then she commented on the case, -1 < d < 0:

D: What if d is between negative one and zero (She writes -1 < d < 0 and draws a Complex Plane below it). Where was the number? It is here (She marks the complex number a + bi on the 1st quadrant of the Complex Plane and draws its projection vectors on the real axis and imaginary axis). Now what did I do it' I multiplied it with a negative number and i. That (She refers to the end point of the vector on the real axis) is a to the zero (She writes (a, 0) to the end point). This (She refers to the end point of the vector on the imaginary axis) is also zero to b (She writes (0, b) to the end point). I know that when I multiplied with i, I move this (She refers to the vector on the real axis) to this axis (She refers to the imaginary axis) because it rotates ninety degrees. I showed it there (She points to the multiplication of the number a + bi with i by using the Complex Plane of the case when 0 < d < 1). It rotated here. Then I need to rotate it one hundred and eighty degrees more, because I multiplied it with negative one. Then it will exactly move here (She extends the direction of the vector of b - ai on the 4th quadrant and specifies a point on that direction on the 4th quadrant).

R: Tell me about what you thought again?

D: Let me tell you step by step.(She points to the parts of the vector). This (She points to the projection vector on the real axis) multiplied by di. Firstly, I multiplied it with i. When I multiplied with i, it moved here. This axis. (She points to the positive side of the imaginary axis). Then? Assume that there is a point like zero to a

(She marks a point on the imaginary axis and writes (0, a)). Since I multiply it with negative one, multiplying with negative one means rotating it one hundred and eighty degrees. It came here (She points to that the point will rotate 180º around the origin and shows the arriving point on the negative side of the imaginary axis). Then it is zero to negative a (She rotates the vector 180º in the counterclockwise and marks the end point as (0, -a)). Since it moves here (She points to the end point). Since d is negative... Hmm ad. Now thinking about this part (She refers to the projection vector of a + bi on the imaginary axis), firstly I moved it here (She points to the negative side of the real axis). By multiplying with negative hmm? When I multiply it with negative, I mean d, it rotated one hundred and eighty degrees and it moved there (She indicated the positive side of the real axis). By the way magnitude is changing. Right now it is decreasing because... It is something like this (She draws a vector on the 4th quadrant). Starting point didn't change. Its direction changed. Its orientation changed. Its magnitude is also decreased (see Figure 6.15).

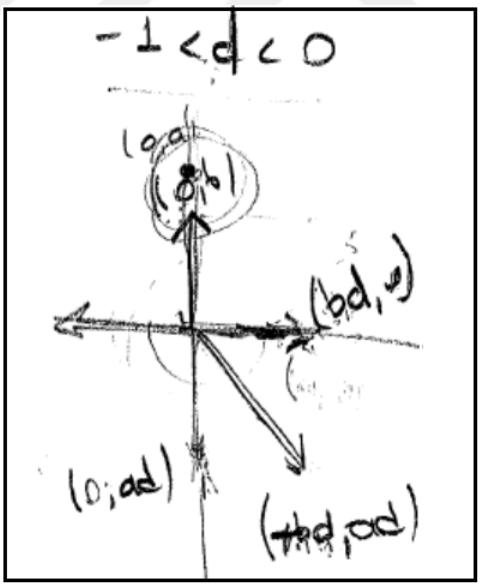


Figure 6.15. Derin shows the multiplication with di in vector form when -1 < d < 0.

As the data indicated, she knew that multiplying with di, firstly, the real and imaginary parts of the original vector z_1 rotated 90° in the counterclockwise because of the multiplication with i. Then, she knew that when -1 < d < 0, d was a negative scalar and multiplication with d resulted in 180° counterclockwise rotation of the real

and imaginary parts of i.z_1. Thus, the real part of the original vector was moved from the positive part of the real axis to the negative part of the imaginary axis and the imaginary part of the original vector was moved from the positive part of the imaginary axis to the positive part of the real axis. Thus, she was able to say that the vector z_1 was moved from the 1st quadrant to the 4th quadrant on the Complex Plane. She also was able to reason that the direction and orientation of z_1 changed and the starting point of z_1 remained the same. Also, she knew that the magnitude of z_1 decreased, since $-1 < d < 0$.

Then, she continued with the last case, when $d < -1$:

D: And if d is smaller than negative one (She writes d < -1 and draws a Complex Plane). In here, let me draw zero to a, let me start here (She draws the projection vectors of the vector a + bi on real and imaginary axes. She marks the point on the real axis as (a, 0)). Zero to b (She marks the point on the imaginary axis as (0, b)). Firstly, I will start with this part (She refers to the part on the real axis). It will rotate 90º when I multiply it with i. It moves here (She marks a point closer to the origin than the point (0, b) on the positive side of the imaginary axis). After that I multiply it with d. Since again d is negative, it rotates one hundred and eighty degrees, besides that since the absolute value of d is bigger than one, it will increase. It moves here (She draws a longer vector than the vector (a, 0) on the negative side of the imaginary axis and points to the end point of the vector). This is zero to minus ad (She writes the end point as (0, ad)). Then I think of zero to b, if I multiply it with i, it rotates ninety degrees (She points to that it will be on the negative side of the real axis). Then, I multiply it with d. It rotates one hundred and eighty degrees (She points to that it will be on the positive side of the real axis) and its magnitude increases. It will be here (She draws a vector longer than the vector (0, b) on the positive side of the real axis). It is a number like this (She marks the complex number that specified by the linear combination of the vectors and draws the vector of that complex number). Here is minus bd to the zero (She writes (bd, 0) to the final point of the vector (0, b)) (see Figure 6.16).

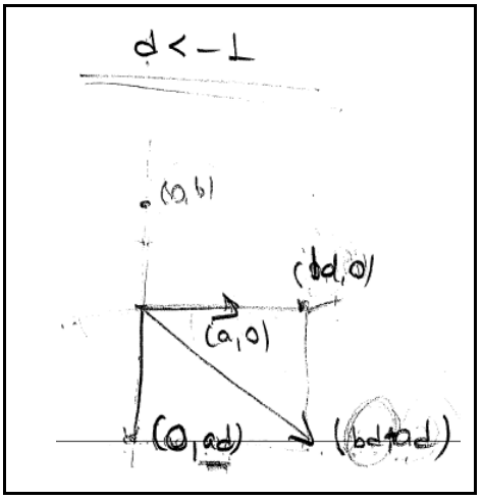


Figure 6.16. Derin shows the multiplication with di in vector form when d < -1.

R: Okay. You said when d is negative, did not you?

D: d is negative, smaller than negative one. It rotated, its direction changed. Starting point didn't change. Increased in magnitude. Its orientation changed.

As the data showed, Derin knew that when d < -1, the real and imaginary parts of the original vector rotated firstly 90° in the counterclockwise because of the multiplication with i and then rotated 180° in the counterclockwise because of the multiplication with a negative scalar d. Also, she was able to say that when d<-1, the absolute value of d was bigger than 1, the magnitude of the real and imaginary parts of the original vector increased. Also, she was able to state that when d < -1; the original vector rotated, its direction was changed, its magnitude increased, its orientation was changed, but its starting point stayed the same.

When the researcher asked again if the result of the multiplication of a + bi with a scalar, c and an imaginary number, di indicated a vector, Derin stated the following:

R: You showed like this as a point. Eer, when you multiplied with c, you got ... In here (She points to c.(a + bi) = ca + cbi) does what you obtained as a result indicate a vector?

D: Yes. It indicates.

R: Why?

D: Doesn't all the numbers specifies a vector? That's why. I mean one part of it (She points to ca) which is on the real axis, that part of it (She refers to cb) is on the imaginary axis...

R: Okay. What about when you multiply a plus ib with di?

D: Yes it does. For example this (She draws the vector (-bd, ad) on the 1st quadrant of the Complex Plane in the case of when -1 < d < 0). I mean we obtained a vector different in direction, magnitude.

As the data indicated, she knew that the multiplication of a complex number, a + bi, with a scalar, c or an imaginary number, di indicated a new vector. She knew that all complex numbers represented a vector. Thus, she was able to say that ac + bci and -bd + adi were complex numbers and also indicated vectors (ca, cb) and (-bd, ad), whose direction and magnitude were able to differ from a + bi according to the different cases of c and d.

Then, the researcher asked Derin to comment and explain again if there was a relation between multiplication with a scalar, c and an imaginary number, di.

R: When you look at it as a whole, do you see any relation between the number a plus bi multiplied with a real number and multiplied with an imaginary number? How do you relate them?

D: Here, in the multiplication with di, it rotates ninety degrees more, since we multiply also with i. For instance, c is between zero and one, and here (She examines the cases of d) what is the same with that case? This one. For instance, here (She refers to the case of 0 < c < 1), while this vector (She shows the vector of ac + bci)

is there, ad plus bdi, is farther from this vector ninety degrees, since we multiply with also i (She shows that ad + bdi rotates ninety degrees in the counterclockwise from ac + bci). If we look at the case that it is bigger than one (She refers to when c > 1). Again, this vector (She refers to the vector of ac + bci) rotates ninety degrees more in this way (She shows the rotation in the counterclockwise) in here (She refers to when d > 1), since I multiply also with i. Here it is on the thirty quadrant (She points to the vector in the case of -1 < c < 0). This (She points to the vector in the case of -1 < d < 0) is on the fourth quadrant. It means it rotates ninety degrees more.

R: Where?

D: In the cases that c is between zero and minus one and d is between zero and minus one. In here, we multiplied with c. If I assume that c and d are the same, it rotates ninty degrees more, since we multiplied with di and comes to the fourt quadrant rather than the thirty quadrant.

As the data showed, Derin knew that there was a relation between the multiplication with a scalar, c and an imaginary number, di. She reasoned that there was a relationship, because the cases $0 < c < 1$ and $0 < d < 1$; $1 < c$ and $1 < d$; and $-1 < c < 0$ and $-1 < d < 0$ were the same. In fact, she knew that for all these cases, the multiplication of a + bi with c and the multiplication of a + bi with d resulted in complex numbers on the same quadrant, since they were scalars on the same interval. Also, she knew that multiplication of a + bi with di included multiplication with d and the imaginary number, i. Thus, she was able to reason that the vector di.(a + bi) rotated 90° in counterclockwise more than the angle of the vector c.(a + bi) due to the multiplication also by i.

The researcher further probed Derin if there was a difference between the results of di.(a + bi) when a + bi was multiplied first, i, and then, d, or first by d and then, i:

R: I will ask you that firstly, we multiplied with d and then multiplied with i and so, we rotated ninety degrees. If we multiply firstly with i and then d rather than we

multiply firstly with d and then i, is there any thing different?

D: No. When I multiply firstly with i, it rotates ninety degrees first and then it rotates one hundred and eighty degrees. Thus, it again rotates two hundred and seventy degrees. On the contrary, it comes to the same point, since it rotates first one hundred and eighty degrees and then, it rotates ninety degrees, thus, it again rotates two hundred and seventy degrees.

As data showed, Derin was able to say that when the complex number a + bi is multiplied by the imaginary number di, there was not a difference in the resulted vector di.(a + bi) when a + bi was multiplied with firstly, d and then with i or firstly with i and then with d. She even explained based on the case, when d is less than 0. She knew that when she multiplied firstly with i and then with d, the original vector rotated firstly 90° in the counterclockwise and then rotated 180° in the counterclockwise. Thus the original complex numbers would be rotated totally 270° in the counterclockwise. Then, she was able to reason that when she multiplied firstly with d and then with i, the original vector rotated firstly 180° in the counterclockwise and then rotated 90° in the counterclockwise. Thus, again it would be rotated totally 270° in the counterclockwise.

In sum, data from the interview and the written session showed that Derin was able to calculate and define the multiplication of the Cartesian form of complex numbers algebraically. In addition, she was also able to multiply a complex number with a scalar, c and a complex number, di algebraically. She was also able to show the product in all these cases both as a point and as a vector. Similarly, Derin was able to state and calculate the multiplication of z_1 and z_2 in polar form algebraically and was able to show the result as a vector. However, she could not explain the reasoning behind the formula for the polar form. Particularly, she was not able to explain why multiplying the maginutes of the multiplicants, z_1 and z_2, determines the magnitude of the product although she was able to state the magnitude of r as the multiplication of the magnitudes of z_1 and z_2. Similarly, she was not able to explain why she would add the arguments in the polar form. In addition, although she was able to show and explain the point and vector representation of the product, $z_1.z_2$, both in Cartesian and

polar forms, she was not able to think of the real and imaginary parts of the resultant vector, ac - db + i.(ad + bc), in terms of the real and imaginary parts of z_1 and z_2. All these suggested that Derin was not able to think of either of z_1 or z_2 as operant vector acting on the other one. Furthermore, when even was asked to talk about the relationship between the result of the multiplication in polar form and z_1 and z_2 as vectors, she was not able to explain the relationship between the polar and vector form of the multiplication of complex numbers.

Therefore, based on the data showing both the affordances and limitations of Derin's current knowledge, I conjectured that Derin's existing knowledge would enable her to think about the multiplication of complex numbers, z_1 and z_2, by considering either of z_1 or z_2 as operant vector acting on the other one. This was because she knew the effect of multiplying a complex numbers with a scalar and a pure imaginary number, though she was not able to coordinate them to reason on the multiplication of complex numbers. Similarly, she neither was able to explain the reasoning behind the multiplication of two complex numbers in polar form nor was able to relate vector and polar forms.

6.2. Results from Teaching

6.2.1. Derin's Development of the Multiplication of Complex Numbers in Vector Form

In this section, I depict data regarding Derin's development of the multiplication of complex numbers in vector form through the first minutes of the teaching sessions. In the following paragraphs, when I say the multiplication of complex numbers in Cartesian form, I mean the resultant complex number "ac - bd + i.(ad + bc)" computed from the multiplication of complex numbers "a + bi" and "c + di" algebraically using the distribution property.

It is important to re-state what Derin knew prior to the teaching. As the data from the pre-interview showed, Derin's assimilatory scheme involved the following:

First, Derin knew that any complex number could be represented as a vector. Secondly, when asked to multiply z_1= a + bi with a scalar such as c, and with a complex number such as di, Derin was not only able to do the multiplication algebraically but also was able to show the result as vectors and pointwise.

However, she was not able to explain the multiplication of z_1 = a + bi and z_2 = c + di in vector form referring to neither the components of z_1 nor the components of z_2. Particularly, once asked to explain the Cartesian form of $z_1.z_2$ in vector form, she was only able to explain the result, $z_1.z_2$= ac - bd + i.(cb + ad) as a vector since she knew that she could represent any complex number as vectors. Though, even when prompted, her not being able to explain the resultant vector in terms of the parts of z_1 or z_2 pointed to some limitation regarding her knowledge base which included her ability to perform the multiplication of a complex number, z_1 = a + bi, with a scalar, c and a complex number, di in algebraic and vector forms; but did not include the multiplication of two complex numbers z_1 and z_2 through reasoning on either of them as operators.

In the following paragraphs, I depict data, in two parts, showing how Derin's reflection on the different cases of c and d while multiplied with z_1 = a + bi allowed her to realize z_1 as operator acting on z_1. Although, I do not claim a huge shift in terms of Derin's learning of the multiplication of two complex numbers, still, at the end of the teaching session Derin's explanations regarding how she made sense of the Cartesian form of the multiplication of z_1 and z_2, $z_1.z_2$ = ac - bd + i.(cb + ad) in vector form has changed compared to the pre-interview.

In the 1^{st}part, I mention the beginning of the teaching session. Particularly, the teaching session started with asking Derin to reflect on what she had done during the pre-interview two days ago. Particularly, the researcher asked Derin to mention one more time how she would reason if she multiplied z_1 = a + bi with a scalar, c and the pure complex number, di.

The goal was to activate what Derin knew so that she could assimilate the vector form into the Cartesian form of the multiplication of complex numbers.

As the figures below indicated Derin was able to show $z_1 = a + bi$ as a vector such that based on the projection vectors (0, a) and (b, 0) of a + bi on the real and imaginary axis, she was able to calculate and explain multipliying a + bi with c for different cases (see Figure 6.17: When $c > 1$, when $0 < c < 1$, when $1 < c < 0$ and when $0 < c < 1$).

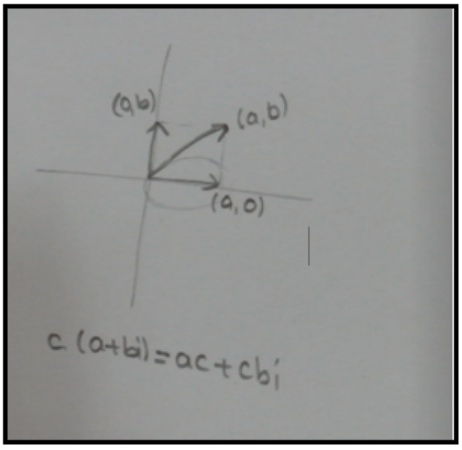


Figure 6.17. Derin's showing $z_1 = a + ib$ as a vector and multipliying c.(a + ib) algebraically.

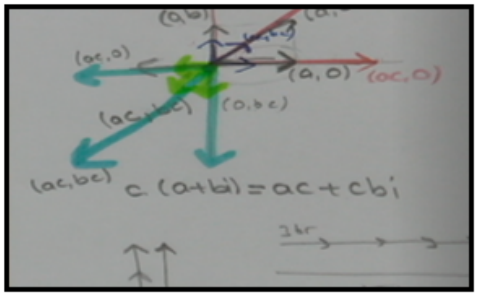


Figure 6.18. Derin's showing c.(a + ib) when $c > 1$, $0 < c < 1$, $c > 1$ and $0 < c < 1$ on the same Complex Plane.

In fact for the first case, when c was bigger than 1, Derin also stated that the orientations, directions and starting points of the projection vectors remained the same since the angles between the projection vectors (a, 0) and (0, b) of the original vector

and the projection vectors (ca, 0) and (0, cb) of the resultant vector calculated with tangent formula resulted in the ratios of $\frac{b}{a}$ and $\frac{cb}{ca}$ (see Figure 6.19). Thus, she claimed that the angles remained the same.

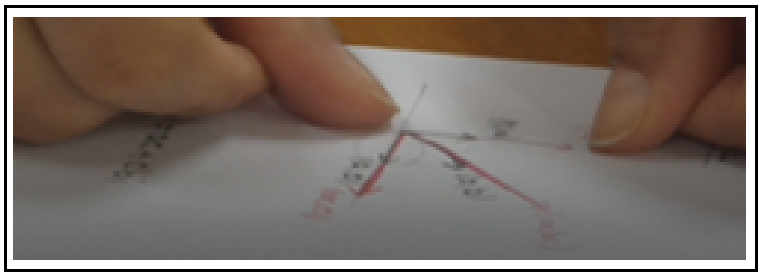
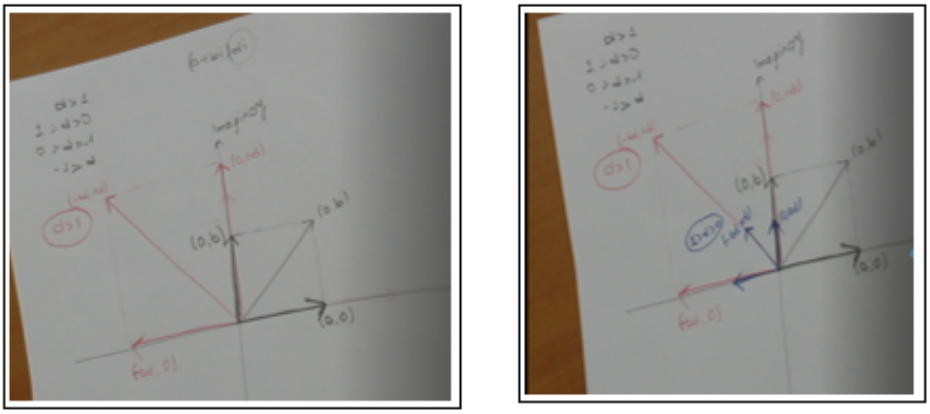


Figure 6.19. Derin's showing that the direction of (0, cb) would be the same as the direction of (0, b).

Derin also commented on the different cases of d for the multiplication of z_1 = a + ib with di. (see Figure 6.20a, b: When d > 1, when 0 < d < 1, when -1 < d < 0 and when d < 0).

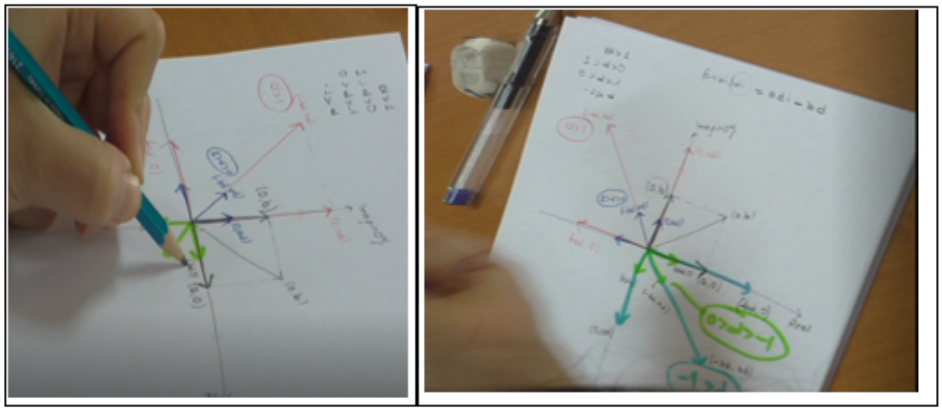


a b

Figure 6.20. Derin's showing di.(a + ib) when d > 1 and 0 < d < 1 (a, b).

For these cases, she could show the resultant vector (-bd, ad) of the multiplication di.(a + bi) by using summation of the real and imaginary parts of the original vector after their change in magnitude and 90° counterclockwise rotation. She was also able to justify that the resultant vectors when 0 < d < 1 and 1 < d stayed on the same line,

since the ratio between the magnitudes of the real and imaginary parts of the resultant vectors was the same for these two cases of d. Therefore, she was able to calculate and explain that the directions and orientations of the resultant vectors when $0 < d < 1$ and $1 < d$ were the same (see Figure 6.21a : When $1 < d$ and when $0 < d < 1$). Also for the cases of $-1 < d < 0$ and $d < -1$, she was able show the following (see Figure 6.21b: When $-1 < d < 0$ and when $d < -1$).



a b

Figure 6.21. Derin's showing di.(a + ib) when $-1 < d < 0$ and $d < -1$ (a, b).

Also, for the cases when $-1 < d < 0$ and $d < -1$, Derin was able to calculate the rotation of the original vector based on two rotations: Rotation because of multiplication by a negative scalar and rotation because of multiplication by i. She knew that the multiplication by a negative scalar resulted with a 180° counterclockwise rotation of the real and imaginary parts of the original vector. As the previous data also showed she also knew the multiplication by i resulted in a 90° counterclockwise the rotation of the two parts of the original vector. Thus, she was able to calculate and explain that the original vector was rotated 270° in total in the counterclockwise direction when $-1 < d < 0$. Also, she reasoned that the resultant vector of the case $-1 < d < 0$ stayed on the same line with the resultant vectors of the cases $0 < d < 1$ and $1 < d$ based on the opposite angles between the resultant vectors. In addition, she knew that when $-1 < d < 0$, the magnitude of the original vector decreased as in the case when $0 < d < 1$. Thus, she could show the resultant vector by using the summation of the real and imaginary parts after the change in magnitude and 270° counterclockwise rotation (see Figure 6.22b: When $-1 < d < 0$).

At that point, the researcher asked Derin to further comment on the relationship between the Cartesian form and the vector form of the multiplication of the original vector, $z_1 = a + bi$, with di. The reason was that compared to the multiplication with a scalar, c, Derin solely had been focusing on geometrical representation for the cases of di. So it was important to asses if she would be able to link the algebraic and geometrical representations. Discussion followed:

R: Can you explain the relation between the result of the multiplication in Cartesian form and this? (The researcher indicates the multiplication (a + bi).di and the vectoral demonstrations on the Complex Plane).

D: I will find algebraicaly? adi... i square, minus one... minus bd (see Figure 6.22: She expresses multiplication algebraically as adi - bd). It is also the same here (She points to the 1st input on the Complex Plane that she drew). It is also the same here, it is also the same in this one (She points to the 1st components of the vectors on the Complex Plane that she drew). The real parts are -bd. The imaginary parts are ad (She points to the imaginary part on the algebraic representation). In this, it is ad (She points to the 2nd component of the blue vector that she drew on the 4th quadrant). In this one it is also ad, and in this one, and in this one (She points to the 2nd components of all the vectors that she drew on the Complex Plane). It is all the same.

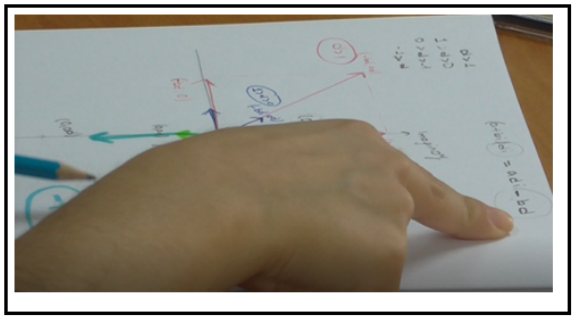


Figure 6.22. Derin's finding (a + bi).di equal to adi - bd algebraically.

As the data showed Derin was able to make connection between the Cartesian form adi - bd and vector form (-bd, ad) of the complex number di.(a + bi) = adi - bd. She was able to say that whenever d < -1, -1 < d < 0, 0 < d < 1 or 1 < d, the

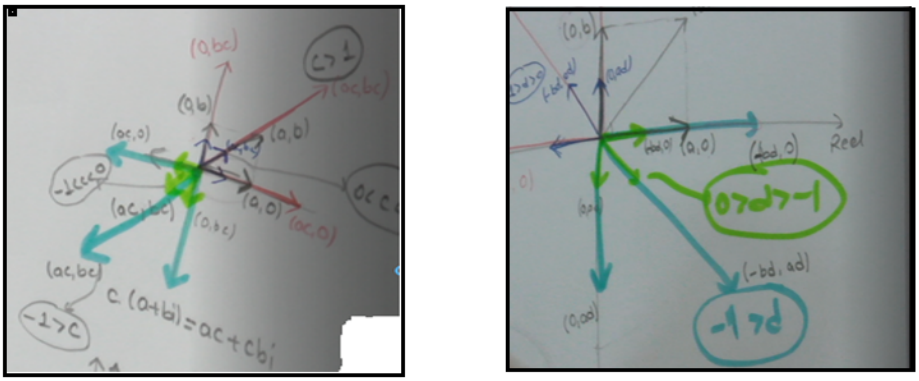
algebraic multiplication of a + bi with di gave the same result: adi - bd. She was also able to indicate that the first components of the resultant vectors were equal to -bd (the real part of the resulted complex number) and the second components of the resultant vectors were ad (the imaginary part of the resulted complex number). Therefore, she knew that the Cartesian form and the vector form of the multiplication of a complex number with di were consistent with each other.

In the second part, I depict data about Derin's reasoning on the relationship between different cases of c and d and explaining the multiplication of complex numbers in vector form. Data followed:

Thereafter, the researcher asked Derin the similarities and differences between the cases of multiplication with a scalar, c and an imaginary number, di (see Figure 6.23a, b).

R: Okay. When you compare the cases of c and di, can you tell us the similarities and differences between them? Can you explain it to us?

D: The resultant vector of the multiplication with di rotates ninety degrees more than the resultant vector of the multiplication with c. For example, when the red is on the first quadrant, it moved this to the second quadrant (She compares the vector of multiplication with c and the vector of multiplication with di). Blue is also the same. It was moved to the second quadrant from the first quadrant. In here the green was moved to the fourth quadrant from the third quadrant because we multiply it with an additional i. Blue is also the same. It was moved to the fourth quadrant from the third quadrant. Figure 6.21



a b

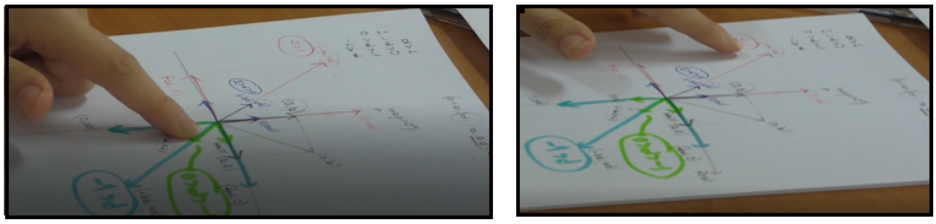
Figure 6.23. The figures Derin used mentioning the differences between the cases of multiplication with a scalar, c and an imaginary number, di (a, b).

I argue that asking Derin to reflect on the similarities and differences between the different cases of multiplication of a complex number, $z_1 = a + ib$, with a scalar, c, and a complex number in the form of di, allowed some affordances on her part: First, as the data showed that Derin was able to define the differences between the multiplication with a scalar, c and the multiplication with an imaginary number, di comparing corresponded cases: (i) $c < -1$ with $d < -1$, (ii) $-1 < c < 0$ with $-1 < d < 0$, (iii) $0 < c < 1$ with $0 < d < 1$, and (iv) $1 < c$ with $1 < d$. She knew that for all four comparisons, in the cases of d, due to the multiplication by i, the original vector rotated 90° in the counterclockwise direction more than its rotation in the cases of c. This was very important because such recognition on Derin's part could have allowed her to quantify the multiplication of two complex numbers in terms of multiplicand and multiplier. I further explain this with evidence later. Then, the researcher asked Derin the similarities between the cases of "c" and "d":

R: What are the similarities between them?

D: Hmm. The blue and the green is on the same line and the same orientation (see Figure 6.24a: She points to the blue and the green vectors which was the result of the multiplication with di). Similarly, the red and the dark blue vectors (see Figure 6.24b: She points to the red and dark blue vectors which were the results of the multiplication

with di). Besides that, their (She talks about the red and dark blue vectors) directions are here (She refers to the case of the multiplications with c) and their orientations are opposite. The blue one (She refers to dark blue) and the red one are on the same orientation, the green one and the blue one are opposite to them (She compares the direction of vectors which was the result of the multiplication with c). In here it is also the same. The blue one (She refers to dark blue one) and red one are on the same orientation. The green one and the blue one are opposite to them (She compares the directions of vectors which were obtained as a result of the multiplication with di).



a b

Figure 6.24. Derin's pointing to the vectors in blue and green and the vectors in red and dark blue (a, b).

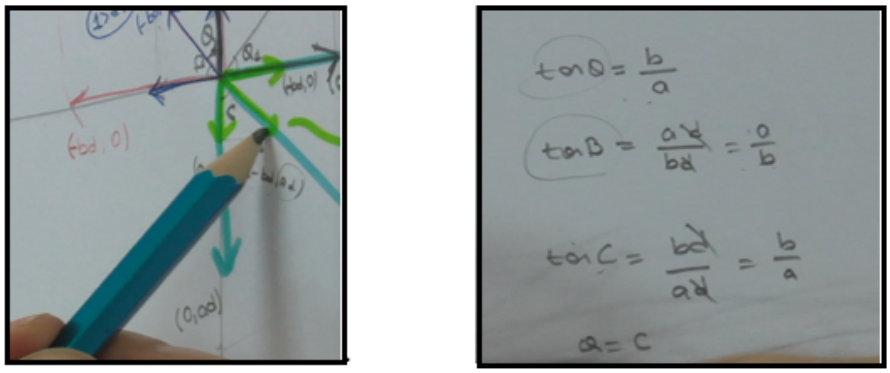
R: Well, how did you decide? What do you mean by opposite orientation?

D: I mean it is rotated one hundred and eighty degrees. It is multiplied with negative one. (She points to that when red and dark blue vectors on the 1st quadrant are rotated 180º in the counterclockwise, they will overlap with the green and the blue vectors on the 4th quadrant). Its orientation is on opposite one hundred and eighty degrees.

R: Well, how do you decide that it will be on the exact opposite one hundred and eighty degrees?

D: Here, I will say theta one (see Figure 6.25a: She writes θ_1 on the angle between the vector (a, b) and positive side of the real axis)... Now this (She refers to the angle θ_1) there (She points to the vector (a, 0)). The tangent was b over a. If I rotate them,

they will rotated ninety degrees and moved here (She points to the dark blue vector on the negative side of real axis). This also rotated ninety degrees and moved here (She points to the dark blue vector on the positive side of y-axis). When I look at here, I am looking at their parts (She points to the dark blue vectors). That angle (She points to the angle between the dark blue vector (-bd, ad) and the negative side of the real axis)... Let's call it beta (see Figure 6.25a: She names the angle as β), ad over bd, d's are cancelled, it is a over b (She calculates tangent β)... I mean tangent beta, a over b, tangent theta b over a (see Figure 6.25b: She tries to solve it using the equations she wrote). They are contrary, I mean cotangent of it is the same. Because of this, here (see Figure 6.25a: She marks the angle between the dark blue vector and positive side of the imaginary axis and names it as θ) becomes theta.



a b

Figure 6.25. Derin's writing and showing angles θ, θ_1, β, C and showing tangent θ, tangent β and tangent C (a, b).

R: What do you mean by theta?

D: That one is theta one (She corrects the angle between the dark blue vector and the positive side of the imaginary axis as θ_1). It is the same with theta one. Since it is the same with theta one... What if I do the same thing here (She points to the other vectors that she drew in blue and green)... I should do the same even there... Here is c (She marks the angle between the negative side of the imaginary axis and the vector that she drew on the 4th quadrant in green and names that angle as c) Tangent c... Here

is the opposite leg (She draws a line on the opposite of the angle c). The opposite leg. What was it? bd, bd over ad. d's are cancelled and b over a (She calculates the tangent of c using proportion of the lengths). I mean c and theta is the same (She writes θ = c). Since proportions doesn't change, I moved theta here to there (She refers to the angle c and corrects the angle c as θ). Here is also theta. Here is also theta (She points to the θ angles on the 4th and 2nd quadrants). It means they are linear (see Figure 6.26: She points to the green vector on the 4th quadrant and the linear dark blue vector on the 1st quadrant, which are linear.) They are on the same direction.

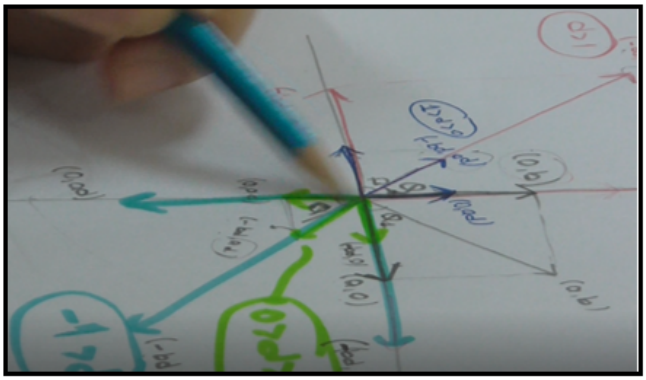


Figure 6.26. Derin's showing theta angles on the 2nd and 4th quadrants.

Secondly, as the data indicated, Derin was able to use algebra in order to show that the multiplication of a + ib with a positive scalar resulted in a vector that stayed on the same line, but in the opposite direction with the vector resulted from multiplication with a negative scalar. That is, she was able to reason that for the cases -1 < c and -1 < c < 0 and for the cases 0 < c < 1 and 1 < c, the resultant vectors were on the same direction. Also, she was able to reason that the resultant vectors for the cases 0 < c < 1 and 1 < c stayed on the same line, but were on the opposite directions with the resultant vectors when -1 < c and -1 < c < 0 because when c < 0, the vectors rotated 180° more than the resultant vectors when 0 < c. She also was able to prove her claim by calculating the angle of the vectors using tangent formula. Comparing, she further knew the same claim held for the resultant vectors when d < 0 and d > 0. Therefore, she concluded that the vectors were on the same line, but their directions were opposite to each other when the multiplier scalar was less than zero and bigger

than zero.

Based on the aforementioned data I claim the following: Asking Derin to reflect on the similarities and differences between the different cases of multiplication of a complex number, z = a + ib, with a scalar, c, and a complex number in the form of di, allowed her to reason and conclude that the resultant vectors in similar cases such as $0 < c$ and $0 < d$; and, $c < 0$ and $d < 0$ were the same with only difference of a further 90° rotation in the cases of d. That is, she reflected on the connections between multiplying a complex number, z = a + bi, with a scalar and with a complex number in the form of di. Her focusing on the connections was important to further allow her to think of the resultant vectors in terms of the original vectors. I further explain this with more evidence.

Thus, at that point during teaching, the researcher asked Derin to reason on the resultant vectors in terms of the original vector, z_1, focusing on the rotation when the multiplier scalar was less than zero and bigger than zero. Derin chose to reason on when $d < 0$ and when $d > 0$:

R: What happens when you think of rotation? What was your starting vector?

D: It is here (She points to the vector (a, b)).

R: What is the angle at its final point?

D: Hmm... Ninety plus theta (She points to the angle between the red vector on the 2nd quadrant and the positive side of the real axis). It becomes theta one.

R: What about this vector (The researcher indicates the blue vector on the 4th quadrant) then? When you think about it, how many degrees did it rotate in comparison with the 1st vector?

D: It is theta one plus... What? I guess we said two hundred and seventy. Yes, because it rotated one hundred and eighty plus ninety is two hundred and seventy. The vector was rotated theta one plus two hundred and seventy degrees (She writes θ_1 + 270 to the end of the blue vector on the 4th quadrant). The other one is theta one plus ninety (She writes θ_1 + 90 to the end of the blue vector on the 4th quadrant). Because of this, it is one hundred and eighty degrees.

R: What does that mean?

D: It means they are on the same direction but opposite orientations.

As the data showed, Derin was able to calculate and show geometrically the angle of the resultant vectors based on the rotation of the original vector. She claimed that while the original vector rotated 90° in the counterclockwise when d > 0, it rotated 270° in the counterclockwise when d < 0. Thus, she was able to calculate and explain that when d < 0, the angle of the resultant vector was $90^o + \theta_1$, where θ_1 was the angle of the original vector. Similarly, when d < 0, the angle of the resultant vector was $270^o + \theta_1$. Thus, she was able to reason that there was a 180° difference between the angels of the resultant vectors when d < 0 and when d > 0. This way she also knew that the vectors stayed on the same line, but their orientations were opposite.

It is important to comment on the following: Unfortunately the researcher did not ask Derin why she chose to start talking about the different cases of d > 0 and d < 0, when she was asked to think of the resultant vectors in terms of the original vector. However, I claim that Derin's reflection on the similarities and differences between the different cases of c and di might have allowed her to choose to focus on the cases of d, since the cases of d already have included the cases of c with only difference of a further rotation of 90°.

At that point, conjecturing that Derin's knowledge base would enable her to be able to explain, the researcher asked to reason on the multiplication $z_1.z_2$ = (a + bi).(c + di) in vector form:

R: Now think about the multiplication of these two complex numbers based on vectors. How do you explain this representation? (The researcher indicates the algebraic representation $z_1.z_2 = (a + bi).(c + di).$)

D: Based on vectors... What do you mean?

R: Here, I have a multiplication, right? Now this multiplication (The researcher points to the algebraic representation $z_1.z_2 = (a + bi).(c + di)$), think about this again. Explain us what you think.

D: It is the multiplication of two vectors.

R: Okay. You say the multiplication of two vectors. Which two vectors?

D: a plus bi and c plus di. I divide the vector into two parts. Then I multiply one by one.

R: What do you mean by one by one?

D: I mean real and imaginary parts (She points to the real and imaginary parts of the numbers z_1 and z_2). (She points to the number a in a + bi). And also the part b... When we draw (She draws a Complex Plane) I name this part as a and this part as b (see Figure 6.27: She points to two vector parts one the positive sides of the real axis and imaginary axis and names them as vector a and vector b, respectively). Then, this is c and this is d (She points to two vector parts on the negative sides of the real axis and imaginary axis and names them as vector c and vector d, respectively). Then, one by one... I multiply them with c and di, like I did before... I could not understand what you mean exactly by saying based on vectors.

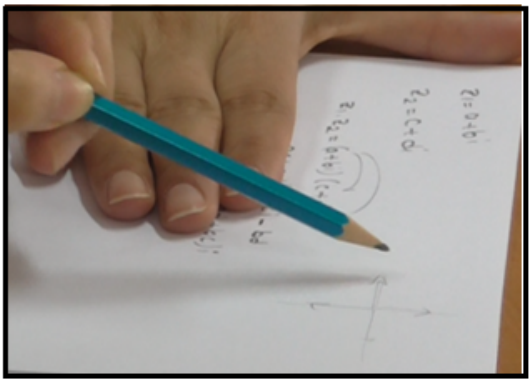


Figure 6.27. Derin's indicating the real and imaginary parts of the numbers z_1 and z_2 on the Complex Plane.

The data was important: Derin was not able to make sense of the question the 1^{st}time once asked to think of $z_1.z_2$ in terms of the multiplication of two vectors z_1 and z_2. Though, her explanation indicated that she assimilated the question to reason algebraically. Particularly, knowing to define the numbers a, bi, c and di as parts of the vectors a + bi and c + di, where each one were started from the origin and stayed on either negative side or positive side of either the real or the imaginary axis, she explained multiplying each of those parts with one another algebraically as in the Cartesian form using distribution property of multiplication. Her restating "I could not understand what you mean exactly by saying based on vectors" also indicated to her thinking algebraically rather than reasoning on the multliplication of z_1 = a + bi in terms of the parts of z_2, namely, c and di.

However, knowing that she already had reflected on the similarities and differences between different cases of c and d when multiplied with z_1, the researcher probed Derin to re-focus on multiplication with c and di algebraically and then, geometrically.

R: Alright. How do you reason on c and di? What did you say before when you think them as vectors?

D: First I multiply c with a plus bi (She writes c.(a + bi)). Then, the point a, a to the zero, how can I represent it algebraically? Then, ca to bi (She defines multiplication as ca + bi). Then, hmm... Okay (She adds c). Alright. On the 2nd one, I multiply it with the part di (She writes di.(a + bi)). It is da, aid minus bd. Then, I combine them (She points to the expressions ca + cbi and aid - bd).

R: What do you mean by combining?

D: I add it. Let me add it like this (She draws a line and sums the vectors). Okay, ca plus cbi plus adi minus bd (She writes ca + cbi + adi - bd). Hmm... a plus bi in the paranthesis times c plus di in the paranthesis (She writes the summation of c.(a + bi) and di.(a + bi) as (a + bi).(c + di)).

R: What does this (The researcher indicates (a + bi).(c + di)) equal?

D: This is the multiplication of the complex numbers (see Figure 6.28: She writes $z_1.z_2$ under (a + bi).(c + di)).

R: What does the plus sign between these mean (The researcher indicates the plus sign between c.(a + bi) and di (a + bi) in the c.(a + bi) + di.(a + bi) in Figure 6.28)?

D: It is the combination of the vector.

R: Of which vector?

D: We obtained a vector when we multiply a plus bi with c (She points to c.(a + bi) on the equation $z_1.z_2$ = c.(a + bi) + di.(a + bi)). In here (She points to di.(a + bi) on the equation $z_1.z_2$ = c.(a + bi) + di.(a + bi)). I obtained another vector by multiplying the same vector with di. It is the vector I obtained when I sum' combine those two vectors.

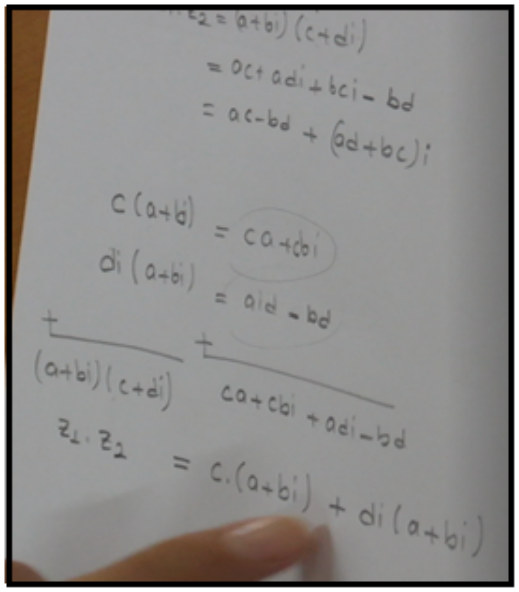


Figure 6.28. Derin's explaining the geometric meaning the plus sign between the vectors c.(a + bi) and di.(a + bi).

Derin's explanation was important: First, when asked to think of the Cartesian form of $z_1.z_2 = (a + bi).(c + di)$ in terms of multiplying with c and di with z_1, Derin was able to consider it as the addition of the vectors, $c.z_1$ and $di.z_1$. This was possible for her for two reasons: First, she already knew to multiply a + ib with c + di algebraically using the distributive property. Secondly, the algebraic manipulations of multiplying $z_1$with c and di, finding c.(a + bi) and di.(a + bi) resulted in the vectors $c.z_1$ and $di.z_1$. She also stated that the results of both c.(a + bi) and di.(a + bi) were vectors. Third, she was able to calculate and explain that z_1. z_2 = ca + cbi + adi - bd was equal to the summation of c.(a + bi) = ca + cbi and di.(a + bi) = aid - bd, algebraically. In addition, once the researher asked her the meaning of the plus sign between c.(a + bi) and di.(a + bi), she was able to reason that it referred to the combination of the vectors c.(a + bi) and di.(a + bi). That is, considering c.(a + bi) and di.(a + bi) as vectors, the plus sign between them referred to the vector summation such that she was able to assimilate z_1. z_2 as the summation of c.(a + bi) and di.(a + bi). Thefore, data showed that Derin was able to explain the multiplication of z_1 and z_2 in vector form thinking z_1 as a vector and z_2 as a multiplier acting on z_1.

Then, the researcher asked Derin to explain how she would explain $z_1.z_2$ one more time:

R: You say... Okay then. To sum it up, z_1 times z_2... Can you explain this representation (see Figure 6.29: The researcher points to the multiplication (a + bi).(c + di)) to us based on the multiplication of two complex numbers?

D: I keep the vector z_1 (She points to a + bi) and separate the vector z_2 into its real and imaginary parts as c and di. I divide it into two parts. As its real part and imaginary part. Then one of the parts, I multiply the real part with the complex number z_1. I obtain a vector. Then I multiply the complex number z_1 with the imaginary part of the complex number z_2 (She shows di). I obtain another vector. When I sum these two vectors, the vector I obtained from this summation is the complex number z_1 times z_2.

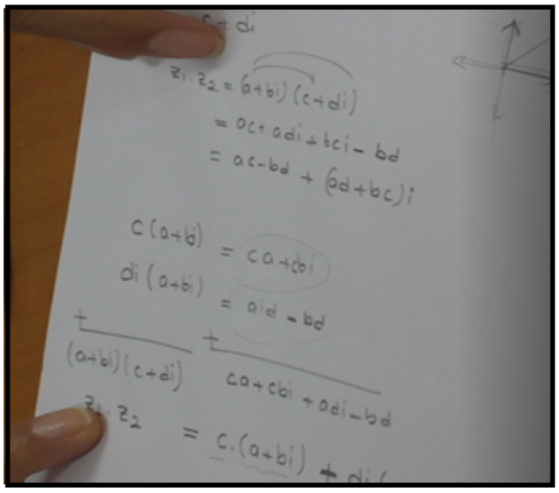


Figure 6.29. Derin's explaining the multiplication of the complex numbers in Cartesian form and in vector form.

Data was significant: Compared to the data during pre-interview prior to the teaching session, data above showed that Derin was able to explain that the multiplication $z_1.z_2$ in vector form. Particularly, she was able to assimilate z_1 as a vector and z_2 as a multiplier such that she was able to compertmentalize z_2 into two parts as "c" and "di" and multiply the vector z_1 with "c" and "di" one by one. Multipliying $c.z_1$

and di.z_1, she knew that she would get two new vectors and summation of these two vectors would give her the vector of the multiplication of the complex numbers z_1 and z_2.

In sum, from the beginning of the teaching session, Derin reasoned in the following way: When Derin was asked to multiply a + bi with a scalar c, she focused on the vector representation of a + bi. Particularly, she represented a + bi as a vector and showed its real and imaginary parts as vectors and points. Then, she multiplied the real and imaginary parts of the vector a + bi with the scalar c and showed the changes in magnitude and direction for all four cases of c, when $c > 1$, $0 < c < 1$, $1 < c < 0$ and when $0 < c < 1$. Following, she showed the resultant vector c.(a + bi) combining the resultant real and imaginary parts. Therefore, she was able to realize that in all of these four cases, when $c > 0$, the orientations, directions and starting points of the projection vectors remained the same with the original vector, but when $c < 0$, the orientations of the projection vectors were contrary, while their directions and starting points remained the same with the original vector. She was also able to show and explain the multipliation with di in a similar manner. Firstly, she thought multiplication with di as multiplication with d and then multiplication with i. In this case, multiplication with d was similar with the multiplication with c, since c and d were real scalars. So, when she multiplied a + bi with d, she accepted a + bi as a vector again and then multiplied the real and imaginary parts of the vector a + bi considering the four cases of d. Following, she rotated each part of the vector 90° in the counterclockwise because she multiplied also with the imaginary number i. Then, the researcher asked the similarities and diffferences between the cases of multiplication with c and di. In this case, she was able to realize that when c is bigger than zero and c is less than zero, the vectors had the same direction, but they differed in magnitude. Similarly, when d is bigger than zero and d is less than zero, the vectors had the same direction, but they differed in magnitude. Also, she realized that the only difference between the cases $c > 0$ and $d > 0$ or $c < 0$ and $d < 0$ was the additional 90° counterclockwise rotation because of multiplication with i. This allowed her to reason that there were 90° between the vectors c.(a + bi) and di.(a + bi) if c and d were both bigger than zero or less than zero.

I claim that Derin's reasoning on the different cases of c and d in-juxtaposition to each other both algebraically and geometrically was impotant: This was because while she was thinking different cases, the vector z_1 remained the same in the 1^{st}quadrant, while the values of c and d changed. That is, the vector, z_1, was in the 1^{st}quadrant prior to be multiplied with c and di. Then, being multiplied with c and di, its placement and magnitude changes. Change occured because of the effect of the different values of c and d acting on each part of the vector z_1. This allowed her to not only express $c.z_1$ as a single vector, but also as a linear combination of the vectors ca and cbi. Similarly reasoing on the effect of di, she was able to think of z_1 as a multiplicand, and c and di as multlipliers acting on it.

Though, after Derin's reasoning on the diferent cases of c and di, when the researcher asked her to explain the multiplication of $z_1 = (a + bi)$ and $z_2 = (c + di)$ in vector form, she could not consider the vector multiplication at first. That is, although being able to show the real and imaginary parts of a + bi and c + di as vectors, she multiplied them algebraically using the distiribution property. Then the researcher asked her to consider the multiplication of a + bi and c + di in vector form based on multiplying with c and d. At that point, she wrote c.(a + bi) = ca + bci and di.(a + bi) = adi - bd algebraically. Then, the researcher asked her to explain the multiplication (a + bi).(c + di) = ac + aci + bci - bd in terms of c.(a + bi) = ca + bci and di.(a + bi) = adi - bd. Comparing the algebraic results ac +aci + bci - bd with ca + bci and adi - bd, she realized that when she added ca + bci and adi - bd, she obtained ac + aci + bci - bd which was also equal to (a + bi).(c + di). She also knew that (a + bi).(c + di) was equal to $z_1.z_2$. Knowing previously also that c.(a + bi) and di.(a + bi) were vectors she was able to think of c.(a + bi) + di.(a + bi) as the summation of the vectors c.(a + bi) and di.(a + bi). This way, she was able connect that all these representatios; $z_1.z_2$, the algebraic expressions "ac + aci + bci - bd" and the summation of "ca + bci" and "adi - bd", and the summation of c.(a + bi) and di.(a + bi) all referred to the same thing.

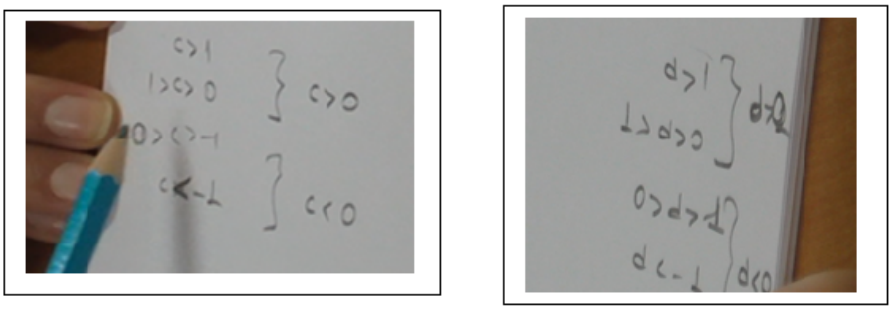
6.2.2. Derin's Development of the Multiplication of Complex Numbers in Polar Form

In this part, I depict data in five parts, regarding Derin's development of the multiplication of complex numbers in polar form, which in total took nearly 200 minutes of 4 teaching sessions.

As the data from first teaching session showed, Derin developed the vector form of the multiplication of complex numbers and its relationship to the algebraic form. Particularly, Derin was able to explain the vector form in terms of $c.z_1$ and $di.z_1$ such that she was able to explain that representatios such as $z_1.z_2$, "ac + adi + bci - bd", "ca + bci + adi - bd", and the "c.(a + bi) + di.(a + bi)" all referred to the same thing.

Therefore, the 2nd teaching session started asking Derin to reason on the angles of these vectors simultaneously. This was important to trigger the geometric representation of vector summation of $c.z_1$ and $di.z_1$ so that she could develop the reasoning behind the polar form.

At the beginning of the teaching session, in order to reason on the angles of the vectors, first, Derin wrote all the cases of c combining them writing $c > 0$; for the cases of $c > 1$ and $1 > c > 0$ and writing $c < 0$; for the cases of $c < -1$ and $0 > c > -1$ (see Figure 6.30a). Similarly, she wrote down all the cases of d combining them writing $d > 0$; for the cases of $d > 1$, $1 > d > 0$ and $d < 0$; for the cases $0 > d > -1$ and $d < -1$ (see Figure 6.30b). Then, Derin was able to categorize the cases of "c" and "d" according to the rotation of the original vector. Particularly, she knew that when $c < 0$, the original vector rotated 180° in the counter clockwise, but when $c > 0$, the original vector did not rotate. Also, she knew that when $d < 0$, the original vector rotated 270° in the counterclockwise and when $d > 0$, the original vector rotated 90° in the counterclockwise. Thus, she was able to categorize the cases of "c" and "d" according to the rotations such that for her (i) $c < -1$ and $-1 < c < 0$ meant $c < 0$, (ii) $0 < c < 1$ and $1 < c$ meant $0 < c$, (iii) $d < -1$ and $-1 < d < 0$ meant $d < 0$ and (iv) $0 < d < 1$ and $1 < d$ meant $0 < d$.



a b

Figure 6.30. Derin's combining 0 > c > -1 and c < -1 as c < 0; combining c > 1 and 1 > c > 0 as c > 0; combining 0 > d > -1 and d < -1 as d < 0 and combining d > 1 and 1 > d > 0 as d > 0, (a, b).

At that point in teaching, the reasearcher asked Derin to draw $c.z_1 + di.z_1$ reasoning on all these different cases of c and d. In addition, Derin was asked to reason on the angles of these vectors as well as the angle of the vector $z_1.z_2$. In the following subsections, I depict data regarding Derin's reasoning on the geometrical representation of $z_1.z_2$ in vector form and its angle.

6.2.2.1. Derin's reasoning for the case: $c > 0$, $d > 0$. As already mentioned, Derin was asked to show the multiplication of $z_1.z_2 = c.(a + bi) + di.(a + bi)$ geometrically based on the vectors $c.(a + bi)$ and $di.(a + bi)$ regarding the cases of c and d together and explain the angles of the vectors $c.(a + bi)$, $di.(a + bi)$ and $z_1.z_2 = c.(a + bi) + di.(a + bi)$.

Derin started talking about the case $c > 0$, $d > 0$. She first drew the vector form of $z_1.z_2$ by focusing on $c.z_1$ and $di.z_1$. Then she reasoned on the angle of $z_1.z_2$. Data followed:

R: We have two cases here. Thinking about these cases? This representation (The researcher refers to c.(a + bi) + di.(a + bi)). Can you draw these vectors considering

the different cases of c and d? You will consider the angles also.

D: Okay. When c is bigger than zero and d is bigger than zero (She writes $c > 0$, $d > 0$). Do you want me to show the cases one the figure (She draws a Complex Plane)?

R: Yes, show them as vectors on the figure. I mean explain the algebraic calculations in there in terms of vectors.

D: Okay. Here is the vector a, b (She marks a point as (a, b) on the 1st quadrant). Okay. When I multiply this (She draws the vector (a, b)) with c? Should I draw the new one on the existing one?

R: Yes, you can draw it with the red pencil. Okay.

D: (She draws a longer red vector than the vector (a, b) by crossing over the vector (a, b)). I know that they are on the same direction and same orientation. Then, if I multiply it with di when d is bigger than one, it rotates ninety degrees, since d is bigger than zero. It will be something like this (She draws a red vector on the 2nd quadrant).

R: Okay. Show me. Which part of the vector is this one (The researcher shows the red vector on the 1st quadrant) and that one (The researcher shows the red vector on the 2nd quadrant).

D: c times z (She writes c.z to the end point of the vector on the 1st quadrant) (She replaces z with z_1). This (She points to the red vector on the 2nd quadrant) is also a vector when it (She refers to z_1) multiplied with di (She writes $di.z_1$ to the end point of the red vector on the 2nd quadrant). This angle is ninety degrees. Now, here was theta (She writes θ to the angle between the vector (a, b) and positive side of the real axis). I know this isn't changed (She points to the vector $c.z_1$). That one (She points to the vector $di.z_1$), multiplying with di means rotating ninety degrees... Here was theta, it rotated ninety degrees. I mean z_1 was here (She points to the vector (a, b)), I rotated

it ninety degrees and it moved here (She points to the vector di.z_1). Because of that the angle between them is ninety degrees.

R: Okay, alright. What did you say you will do with them (The researcher refers to the vectors c.z_1 and di.z_1)?

D: I said I sum them.

R: Hmm. How will you do the summation? Can you show us?

D: I do vector summation. How can I sum this vector (She points to the vector di.z_1) without changing direction? If I move the starting point of it to the end point of this (She points to c.z_1)... I think it is something like this. When I draw like this (She moves the vector di.z_1 to the end point of the vector c.z_1 without changing its direction). Then I move from starting point of this one (She refers to the vector z_1) to that one (She points to the end point of the vector that she moved). It is like this (She draws the summation vector)(see Figure 6.31).

R: What does it show us?

D: It shows us z_1 times z_2 (She writes $z_1.z_2$ next to the summation vector).

As the data showed Derin was able to separate z_2 into two parts as "c" and "di" and multiply the vector z_1 with each of them. Though, it was interesting that Derin did not show z_2. However, she also was able to show the two multiplications on the same Complex Plane such that she stated that there was an angle of 90° between the vectors c.z_1 and di.z_1. Her drawing c.z_1 and di.z_1 (in red color) as longer than the original vector z_1 also suggested that she was thinking of d and c as bigger than 1. Further, she was able to translate the vector di.z_1 without changing its direction to the end of the vector c.z_1 and find the vector $z_1.z_2$ by making vector summation.

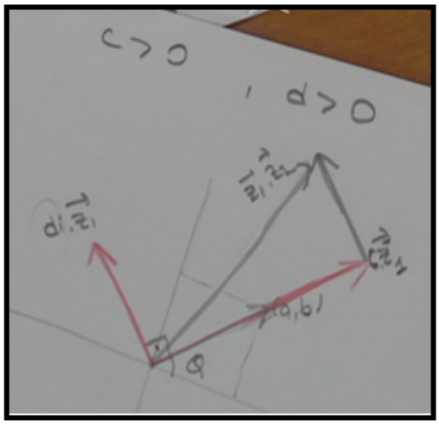


Figure 6.31. Derin's showing the summation of the vectors $c.z_1$ and $di.z_1$ resulted in the vector $z_1.z_2$.

This was because Derin had already known that once she multiplied c with z_1 and di with z_1 she would obtain two vectors to add. All these suggested that Derin was able to think of $z_1.z_2$ in vector form considering z_1 as a vector and the real and imaginary parts of z_2 as operators acting on it.

Then, the researcher asked Derin the angle of the vector of $z_1.z_2$. I depict the data in two parts: (i) Derin's difficulty in reasonning about the angle of $z_1.z_2$ and (ii) Derin's explanation of the angle of $z_1.z_2$ in terms of the angles of z_1 and z_2.

Derin's difficulty in reasonning about the angle of $z_1.z_2$ as follows:

R: Okay, what is the angle of this one (The researcher points to the vector $z_1.z_2$)?

D: Here is ninety (She points to the angle between the vector $c.z_1$ and the vector $di.z_1$, which she moved earlier, as 90º). Here is theta one. There is ninety minus theta one (see Figure 6.32: She names the angle between the vector $di.z_1$ on the 2nd quadrant and the negative side of the real axis as 90 - θ_1). I draw like this... (see Figure 6.32: She draws a line, which is paralel to the real axis, from the intersection point of the vector $c.z_1$ and the vector $di.z_1$ which she moved earlier).

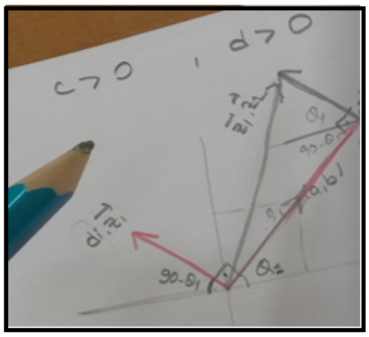


Figure 6.32. Derin's writing 90 - θ_1 on the angle between the vector di.z_1 and the negative side of the real axis.

R: Why did you draw it?

D: I drew it in parallel to here (She points to the real axis). (see Figure 6.33: She places the angles θ_1 and 90 - θ_1 in the triangle she formed.)

R: What are you thinking?

D: I am trying something to figure out the angle, but I haven't find yet. What If I name here (She names the angle between the vector $z_1.z_2$ and the vector z_1 as θ_2) as theta two (see Figure 6.33: She names the angle between the vectors $z_1.z_2$ and $di.z_1$ as 90 - θ_2).

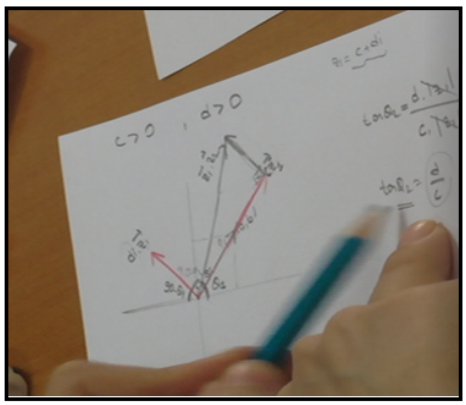


Figure 6.33. Derin's writing 90 - θ_2 on the angle between the vector $z_1.z_2$ and the vector di.z_1.

R: You call it theta two.the angle you named as ninety minus theta two.You said it is here. Okay. I will ask you this. In here (The researcher indicates the angle θ_1) with theta one. Whom does the angle theta one belong?

D: z_1

R: What about theta two?

D: z_1 times z_2. No. Theta two it is the angle of z_2, isn't it? Now, when we multiplied, the angle of z_1 times z_2 is theta one plus theta two (She points to the angle of the vector $z_1.z_2$). The angle of z_1 is theta one. I mean. I know it is z_2, theta two but I can't explain. We add up them.

R: You say you add up them, but you don't know why you do this.

D: Exactly.

R: Alright, you memorised it is theta two (The researcher indicates θ_2).

D: Yes. I wrote theta two there but.Yes, it is the angle of z_2.

As the data showed so far, although Derin was able to determine and explain $z_1.z_2$ in vector form, when $0 < c$ and $0 < d$ and knew from earlier that the argument of z_1. z_2 would be equal to $\theta_1 + \theta_2$, the sum of the arguments of z_1 and z_2, she was not able to explain the reason behind it. This was important: As the Figure 6.34 above showed, in spite of what Derin had already known and written, she was not able to think of $z_1.z_2$ in terms of z_1. That is, she did not realize that the argument of $z_1.z_2$ was obtained because of z_2's acting on z_1 such that the z_1 was rotated by the argument of z_2, θ_2.

Hereafter, I mention Derin's explanation of the angle of $z_1.z_2$ in terms of the angles of z_1 and z_2. At that point, taking into consideration that Derin knew that θ_2 was the

argument of z_2, and also that Derin's attention was already on the angle between $z_1.z_2$ and $c.z_1$, the researcher took her attention to the value of θ_2 in the triangle she had drawn. That is, the researcher conjectured that if Derin was able to evaluate how much the angle between $z_1.z_2$ and $c.z_1$ was than she could have determined the angle of $z_1.z_2$ in terms of the angles of z_1 and z_2.

R: Okay, I will ask you this... theta two in here (see Figure 6.34: The researcher is pointing to θ_2 in the triangle that Derin drew). Can you define the value of theta two?

D: Do you mean using by tangent? Here (She points to the vector $di.z_1$ which she moved) d over z_1, di, hmm is here (She points to the vector $c.z_1$) but what was that? It was c times z_1. It was d times z_1. Tangent theta two is already d over c. Tangent theta two. Now, this is its (She points to the vector $di.z_1$) length. The length of that vector (She points to the vector $di.z_1$ which she moved to 1st quadrant) over the length of the other one (see Figure 6.34). The length of this (She points to the vector $c.z_1$) is the length of the c times z_1. I cancelled the lengths of z_1. Theta two is d over c (She writes the equation).

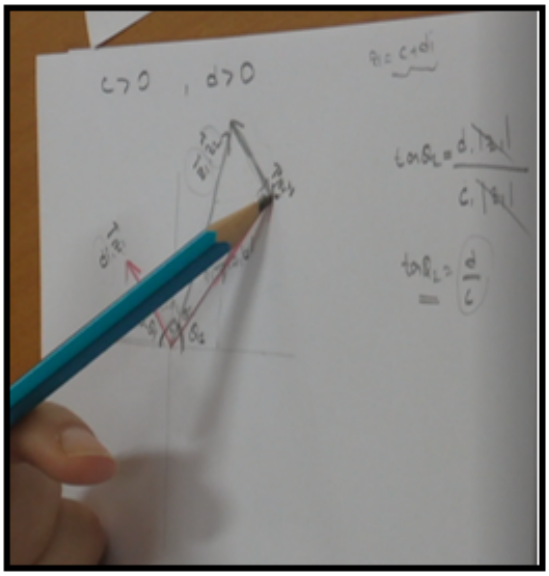


Figure 6.34. Derin's writing tangent value of θ_2.

R: Well, what does d over c mean for you?

D: Hmm... It's z_2... Because z_2 is... (She writes $z_2 = c + di$) c plus di. It gives the tangent of its angle. So it is the angle of z_2. Okay.

R: Well, can you tell me how many degrees, I mean, the angle of $z_1.z_2$, like in terms of vector z_1?

D: It rotates, yes, counterclockwise as much as theta two, the angle of z_2. It rotates as the angle of z_2. Theta one plus theta two. I mean the angle of z_1 plus the angle of z_2. Yes, exactly.

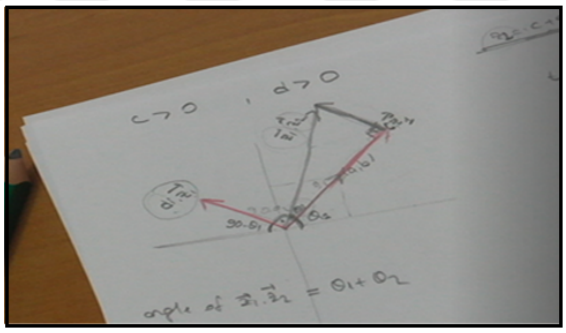


Figure 6.35. Derin's writing the angle of the vector $z_1.z_2$ is equal to the addition of the angle of z_1 and the angle of z_2, $\theta_1 + \theta_2$.

R: Okay, alright. Could you tell me why does it rotate that much?

D: What do you mean why?

R: Can you explain us why does it rotate that much?

D: Do you want me to explain what I did there (She points to the calculation of tangent θ_2)? Because, I found it out from there. I moved this vector (She points to the vector $di.z_1$) here (She points to the end point of the vector $c.z_1$). Since here (She points to the angle between the vectors $c.z_1$ and $di.z_1$) is ninety degrees, I wrote ninety there (She points to the angle between the vector $c.z_1$ and between the vector $di.z_1$ which

she moved). Here is theta two (She points to θ_2). To find how many degrees it turns, I need to figure out first what theta two is. How did I express theta two? I used the tangent formula. Tangent theta two is this one (She points to the vector $di.z_1$ which she moved) over that one (She points to the vector $c.z_1$). I mean the magnitude of this vector (She points to the vector $di.z_1$ which she moved) over the magnitude of that vector (She points to the vector $c.z_1$). I am looking at the magnitude of this vector (She points to the vector $di.z_1$ which she moved). It is equal to the magnitude of d times z_1. This over the magnitude of that vector (She points to the vector $c.z_1$). The magnitude of this vector (She points to the vector $c.z_1$) equals to the magnitude of c times z_1. I eliminate the magnitude of z_1's, since they are equal. d over c. Since d over c gives me the tangent of the angle of z_2, this is equal to the angle of z_2. I mean when I multiply z_1 with z_2, z_2 rotates z_1 as much as its angle.

R: You are saying it as much as the angle theta two, aren't you?

D: Yes, exactly.

R: Ok, I will ask you this. You said the angle of z_2 is theta two. You calculate these (The researcher points to the ratios of tangent θ_2 which belongs to the angle between the vectors $z_1.z_2$ and $c.z_1$, and the angle of the vector z_2) ratios. For example you named theta two here (The researcher points to the angle between the vectors $z_1.z_2$ and $c.z_1$). Will this one (She points to tangent $\theta_2 = \frac{|d|}{|c|}$) be the same?

D: The same.

R: How did you decide that these (The researcher refers to the angle between the vectors $z_1.z_2$ and $c.z_1$ and the angle of the vector z_2) are the same angles? Why?

D: Because their tangets are the same.

As the data indicated, when the researcher asked Derin to evaluate the value of the angle of θ_2, although hesitant, Derin thought of using the tangent formula as tan

$\theta_2 = \frac{|d|}{|c|}$. Then, once asked to reflect on what the values of d and c referred to, Derin was able to explain that it was equal to the argument of z_2. Also, after the prompt, how much the angle of $z_1.z_2$ in terms of the angle of z_1, Derin was able to reason that z_1 was rotated by the angle of z_2 when z_1 and z_2 were multiplied. She also was able to claim that it was because the tangent values of the angle of the vector z_2 and the angle between the vectors $z_1.z_2$ and $c.z_1$ were the same. Therefore, the angle of the vector $z_1.z_2$ was equal to the addition of the angles of the vector z_1 and z_2.

In sum, data from the first part showed that, Derin was also able to draw the vectors $c.z_1$ and $di.z_1$ on the Complex Plane based on her previous knowledge of the angle of the vectors $c.z_1$ when $c > 0$ and $di.z_1$ when $d > 0$. Therefore, she knew that there were 90° between the vectors $c.z_1$ and $di.z_1$ as she had reasoned before. Then, she was able to move the vector $di.z_1$ to the end of the vector $c.z_1$ without changing its direction, since they were perpendicular to each other. When she summed the vectors $c.z_1$ and $di.z_1$ using vector summation, she found the vector $z_1.z_2$. Thus, she was able to show the vector $z_1.z_2$ when $c > 0$, $d > 0$ geometrically based on the vectors $c.(a + bi)$ and $di.(a + bi)$.

Then, thinking about how to determine the angle of the vector $z_1.z_2$ when $c > 0$, $d > 0$, she named the angle of the vector z_1 as θ_1 and the angle between the vector $c.z_1$ and $z_1.z_2$ as θ_2, since she recalled that the angle of $z_1.z_2$ was equal to the addition of the angles $\theta_1 + \theta_2$, where θ_1 was the angle of $z_1$and θ_2 was the angle of z_2. However, she was not able to explain why the angle between $z_1.z_2$ and $c.z_1$ would be equal to the angle θ_2. Then, being asked to determine how much the angle between $z_1.z_2$ and the $c.z_1$, measured, she thought of evaluating it using the tangent of that angle: finding it as $\frac{d}{c}$. At that point, she was not still able to say that the vector z_1 was rotated by the angle of the vector z_2. Then, reasoning on what $\frac{d}{c}$ represented, Derin was able to state that it would be equal to the tangent value of the angle of the vector z_2. Though, only after reasoning about evaluating the angle of $z_1.z_2$ in terms of the angle of z_1, Derin was able to reason $\theta_1 + \theta_2$ would be the angle of the vector $z_1.z_2$, which was the angle of z_2 more of the angle of z_1. That is, I claim that focusing on how much the angle of $z_1.z_2$, $\theta_1 + \theta_2$, was in terms of the angle of z_1, θ_1, allowed her to think of the

vector z_1 being moved to its new position $z_1.z_2$. This then allowed her to claim that the vector, z_1, was rotated. Thus, she was able to calculate and explain the angle of the vector $z_1.z_2$ = c.(a + bi) + di.(a + bi) based on the angles of the vectors c.(a + bi) and di.(a + bi) such that the angle of the vector $z_1.z_2$, $\theta_1 + \theta_2$ was equal to the addition of the angles of the vector z_1, θ_1 and the vector z_2, θ_2. At that point, Derin wanted to continue with the case when $c > 0$ and $d < 0$.

6.2.2.2. Derin's reasoning for the case: $c > 0$, $d < 0$. Derin again first drew the vector form of $z_1.z_2$ by focusing on the vectors $c.z_1$ and $di.z_1$. Then she reasoned on the angle of $z_1.z_2$ based on the angles of the vectors $c.z_1$ and $di.z_1$. Data followed:

D: Now, I will look at when c is bigger than zero and d is smaller than zero.

R: Okay.

D: (She draws a Complex Plane). The 1st vector, z_1, is here (She points to the vector z_1 on the 1st quadrant). Since c is bigger than zero, the magnitude of this vector is increased. And in the case when d is smaller than zero... A moment please, I am distracted... When d is smaller than zero, it rotates two hundred and seventy degrees since we multiply it with both i and negative. So it (She points to the angle between the vector z_1 and positive side of the real axis as θ_1) will rotate two hundred and seventy degrees. It moves here (She points to the red vector on the 4th quadrant). Now I combine these two. I move this one (She points to the vector on the 4th quadrant) here (She points to the end point of the red vector on the 1st quadrant). I mean I do vector summation like this (She moves the vector $di.z_1$ to the end point of the vector $c.z_1$). This is ninety degrees (She points to the angle as 90° between the vector $c.z_1$ on the 1st quadrant and the vector $di.z_1$ which she moved to the end point of the $c.z_1$) (see Figure 6.34).

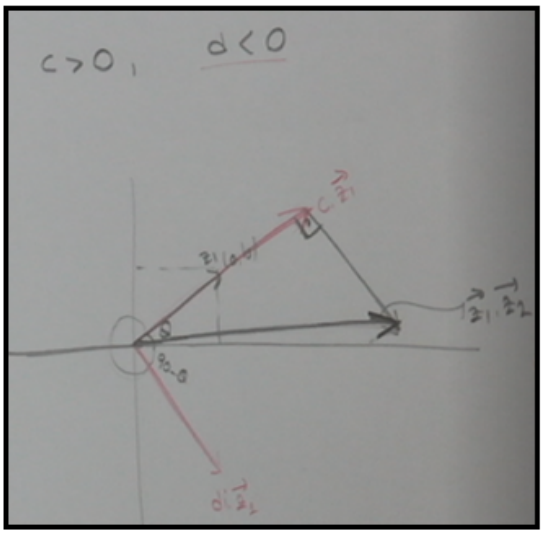


Figure 6.36. Derin's showing the vector $z_1.z_2$ as the summation of the vectors $c.z_1$ and $di.z_1$ when $c > 0$, $d < 0$.

R: How do you know?

D: It isn't. Okay, I don't know. How do I know? Because I rotated it two hundred and seventy degrees (She points to the angle from the vector z_1 to the vector $di.z_1$).

R: Alright.

D: Here was theta. That one (She points to the angle between the vector $di.z_1$ on the 4th quadrant and positive side of the real axis) is, two hundred and seventy, theta... I subtract it from three hundred and sixty... ninety minus theta, ninety degrees (She points to the angle between the vectors $c.z_1$ and $di.z_1$ as ninety degrees). I move that ninety degrees to here (She points to the angle of 90° on the 1st quadrant). Then, when I combine like that (She draws the vector $z_1.z_2$), now it's done. I add them up and I obtained this vector (see Figure 6.37).

As Derin's drawing indicated she seemed to think of the absolute values of c and d as bigger than 1. Therefore, she drew vectors (in red colors) such that the magnitude of the original vector z_1 increased by the multiplication with c and d. She further

was able to reason that while the multiplication with c, when $c > 0$, did not change the direction of the original vector z_1, the multiplication with di rotated the original vector 180° in the counter clockwise because of the multiplication with d, when $0 > d$ and then, 90° in the counter clockwise because of the multiplication with i. Thus, the original vector z_1 rotated totally 270° in the counter clockwise position because of the multiplication of di, when $d < 0$. That is how she knew that the angle between the vectors $c.z_1$ and $di.z_1$ was 90°. Then, she was able to sum the vectors $c.z_1$ and $di.z_1$ using vector summation. Again knowing that she found two vectors to add, namely $c.z_1$ and $di.z_1$, she was able to translate the vector $di.z_1$ to the end of the vector $c.z_1$ without changing its direction and obtain the vector $z_1.z_2$.

Then, Derin continued with the angle of the vector $z_1.z_2$: I depict the data in two parts: (i) Derin's difficulty in reasonning about the angle of $z_1.z_2$ and (ii) Derin's explanation of the angle of $z_1.z_2$ in terms of the angles of z_1 and z_2.

Derin's difficulty in reasonning about the angle of $z_1.z_2$ followed:

D: Which angle will I work on? That one (She points to the angle between the vector $z_1.z_2$ and the positive side of the real axis). I will name here as theta one then (First she writes θ and than replaces it with θ_1) because its sum is theta one. I called the angle of z_1 as theta one. That angle is theta one (She points to the angle between the vector z_1 and the positive side of the real axis). Let's say here is theta two (She writes θ_2 to the angle between the vector $z_1.z_2$ and the positive side of the real axis). How can I find this theta two? Now, I tried to find that one (She points to the angle θ_2) in this case (She points to when $c > 0$, $d > 0$). Why? To find out how much it (She refers to z_1) rotated. I mean to find out where it (She points to the vector z_1) moves. Because it was here (She points to the vector z_1 when $c > 0$, $d > 0$). Where did it move at the end? It, the vector, moved there (She points to the vector $z_1.z_2$ when $c > 0$, $d > 0$). To find out how much did it move... At first, the angle is as much as theta one (She returns to when $c > 0$, $d < 0$). Then it rotated like this (She signs rotating counterclockwise). I multiplied it with something and it moved here (She points to the vector $z_1.z_2$). I want to find out that (She points to the angle θ_2) to find out how much

did that angle rotated. I want to find out what theta two is.

R: Hmm.

D: So I will find theta two first. Hmm, how should I find this? This figure is so ugly. Theta two. I did something like this (see Figure 6.37: She moves the vector $c.z_1$ to the end point of the vector $di.z_1$ on the 4th quadrant and completes the figure with a rhomboid). It became something like a rhomboid. That one (She points to the angle between the vectors $c.z_1$ and $di.z_1$) is ninety. This one (She marks the angle between the vector $di.z_1$ and the vector that she moved as ninety) is also ninety. Here (She points to when $c > 0$, $d < 0$) if I look at how much it was rotated... The angle was theta one. It moved from here to the final point. Then, I will look at how much it rotated, where it moved. I mean where is my initial point? My initial point is here, theta one (She points to the angle θ_1 when $c > 0$, $d < 0$). Where is my final point? If I write here theta two (She points to the angle θ_2), the difference between... How could I express it? There is a difference as much as theta one minus theta two. I mean it moved from here (She points out the vector z_1) to here (She points to a rotation in the counterclockwise starting from z_1 to the $z_1.z_2$) (see Figure 6.35).

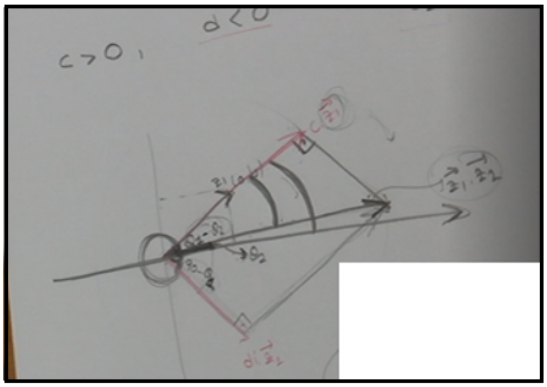


Figure 6.37. Derin's calculating the rotation of the vector z_1 when $c > 0$, $d < 0$.

R: Well, okay. How can you explain theta one minus theta two?

D: I look at how much it rotates. Theta one minus theta two? This (She points to the angle θ_1 - θ_2 on the 1st quadrant). When this vector (She points to the vector

z_1), z_1, multiplied with z_2, it moves from here (She refers to the point where the vector z_1 is) to here (She points to the end point of the vector $z_1.z_2$). Is that right? Yes. It moved clockwise. It moved from here (She points to the vector $c.z_1$) to here (She points to the vector $z_1.z_2$).

R: What did you say theta two means?

D: I mean here (She points to the angle θ_2 between the vector $z_1.z_2$ and positive side of the real axis).

R: What do you mean by here? What is this angle?

D: I mean I don't know theta two, but I can think it as the angle that moves theta one minus theta two from the first case to the second case. (Waiting for several seconds) No, I don't know. (She refers to θ_2).

As the data showed, reflecting on her reasoning for the case of $c > 0$, $d > 0$, she knew that she needed to find out how much the original vector z_1 was rotated to end up with $z_1.z_2$. Derin's statements "*How can I find this theta two? Now, I tried to find that one (the angle θ_2)... Why? To find out how much it (referring to z_1) rotated. I mean to find out where it moves. Because it was here (She points to the vector z_1 when $c > 0$, $d > 0$). Where did it move at the end? It, the vector, moved there (She points to the vector $z_1.z_2$ when $c > 0$, $d > 0$). To find out how much did it move...*" were very interesting indicating that she was reasoning on evaluating how much z_1 was rotated. Though while Derin earlier named the angle of the vector z_1 as θ_1 and the angle of the vector z_2 as θ_2, she named the angle of $z_1.z_2$ as θ_2 for this case. Then, she focused on the difference between the angle of the vector z_1 and the angle of the vector $z_1.z_2$, since she knew that the vector z_1 multiplied with the vector z_2 was rotated to the final placement of the vector $z_1.z_2$. This also suggested that she already had known that z_2 was the operator acting on z_1 when multiplied with z_1. She was also able to say that the vector z_1 was rotated in the clockwise and became the vector $z_1.z_2$ after the multiplication with the vector z_2. Therefore, writing θ_1 referrring to the angle of z_1

and θ_2 referring to the angle of $z_1.z_2$, she was able to figure out that z_1 was rotated by θ_1 - θ_2. Also once asked one more time how she would explain θ_1 - θ_2, stating that it referred to how much z_1 was rotated she further reasoned that z_1 was rotated in clockwise direction. Though, she still had difficulty in explaining how to represent θ_2.

Realizing Derin's difficulty, the researcher asked her again to explain how to evaluate θ_1 - θ_2. Discussion followed:

R: Then, when you say the angle between the vectors z_1 and z_1 times z_2... How can you find theta one minus theta two?

D: There is the tangent there (She points to the right triangle on the 1st quadrant). (She writes tangent θ_1 - θ_2.) The magnitude of here (She points to the vector di.z_1 that she moved to the 1st quadrant) is di times z_1. Its (She points to the vector c.z_1 on the 1st quadrant) magnitude is, because I am talking about tangent, c times z_1. This one and this one are cancelled (She cancels $|z_1|$'s). So, tangent theta one minus theta two becomes d over c. d over c means to me... It means the angle of z_2. So, tangent theta one minus theta two is indicating the angle of z_2. Hmm. Should I name this as b or beta? (She writes tangent β) equals to d over c, I mean this... How should I write the angle of z_2 (see Figure 6.38)?

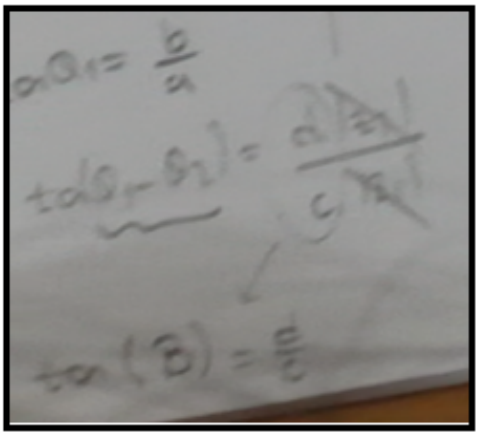


Figure 6.38. Derin's calculating the tangent θ_1, tangent (θ_1 - θ_2) and the tangent β.

R: Write it down. In final case, if you write it in terms of the angle of z_2?

D: Yes, If I write it in terms of the angle of z_2, it rotated as much as z_2. In the clockwise.

R: So, at the end, the angle of z_1 times z_2... Could you explain it in terms of the angles of z_1 and z_2?

D: Okay, I could. This (She writes θ_2) is the angle of the latest. Hmm. This is the angle at the final case. What was the angle at first case? It was theta one. Then... What do I need to find? I am distracted so much. What do you want me to explain? It was theta two. In terms of what? In terms of beta and theta one. What is theta one? The angle of what? $|z_1|$'s... It is the angle of z_1 and beta is the angle of z_2. The angle of z_1 and the angle of z_2... How can I express this at the end? The thing I called theta two... Hmm, wait a minute. I know what beta is. I know it is theta one minus theta two (see Figure 6.39: She writes θ_1 - θ_2 below β). Then, How can I express theta two? Okay. I am so confused right now. Okay. (She writes equals to β next to θ_1 - θ_2). I don't need to write theta two. beta minus theta one... minus... (She erases what she wrote). It is theta one minus beta (see Figure 6.39: She writes $\theta_2 = \theta_1$ - β).

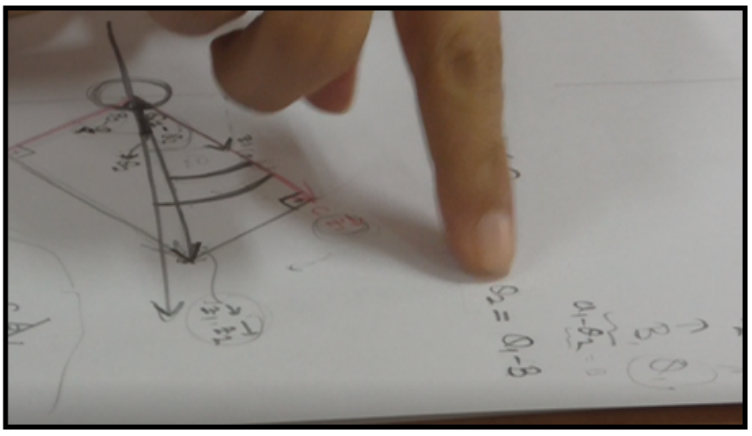


Figure 6.39. Derin writes θ_1 - θ_2 below β.

As the data showed, only after Derin was able to determine the value of tan (θ_1 - θ_2) as equal to $\frac{d}{c}$, she was able to realize that it was equal to the angle of z_2. Though,

only after she reasoned on θ_1 - θ_2 in terms of the angle of z_2, named as β, she was able to determine that θ_2 was equal to θ_1 - β. This suggested that she was able to show that the angle of $z_1.z_2$, θ_2, was equal to the subtraction of the angle of z_2 from the angle of z_1, the original vector. Data further suggested that determining and explaining what θ_2 was difficult for Derin. This was specifcially evident in her statement "*I am so confused right now*".

At that point the researcher asked her to explain again:

R: Now, tell us again what you did there.

D: Alright. Now my first angle, z_1's, was theta one. This theta one... There (She points to the angle θ_1 on the 1st quadrant). Here (She marks the angle θ_1 between the vector c.z_1 and the positive side of the real axis with a big arc). Then I named the angle of z_2 as β. And that one is here (She marks the angle θ_1- θ_2 between the vectors c.z_1 and $z_1.z_2$ with a big arc).

R: How did you decide it is beta?

D: How did I decide? Because we said... We found the angle there (She points to the angle between the vectors z_1 and $z_1.z_2$). It is tangent of here (She points to the angle θ_1 - θ_2 in tangent of θ_1 - θ_2). Since we found the tangent, it is the magnitude of here (She points to the vector di.z_1 on the 1st quadrant) divided by the magnitude of here (She points to the vector c.z_1 on the 1st quadrant). It is d over c. d over c gives me the tangent of z_2. So I decided that here (She points to the angle between the vector c.z_1 and $z_1.z_2$ on the 1st quadrant) is equal to the angle of z_2. So here is beta (She writes β to the angle between the vectors z_1 and $z_1.z_2$), I mean it was equal to the angle of z_2. At final case z_1 times z_2 moves to here. How do I find that (She points to the angle between the vector $z_1.z_2$ and positive side of the real axis)? It is theta one minus beta.

R: Okay, then could you express the angle of z_1 times z_2 in terms of the angle of z_1 and z_2? You said here (She points to $\theta_2 = \theta_1 - \beta$). Could you say it again in general?

D: Hmm... What was the angle of z_1 times z_2? Its angle was theta two. I can express it as the difference between them, I mean theta one minus β. The angle of z_1 minus the angle of z_2. I rotated it as much as the angle of z_2.

R: As much as the angle of z_2 . Why did it become negative?

D: Eer, because I rotated it in the clockwise. Hmm... (She writes "angle of $z_1.z_2$") Equals to... It was theta one minus theta two (see Figure 6.40: She writes angle of z_1. $z_2 = \theta_1 - \beta$).

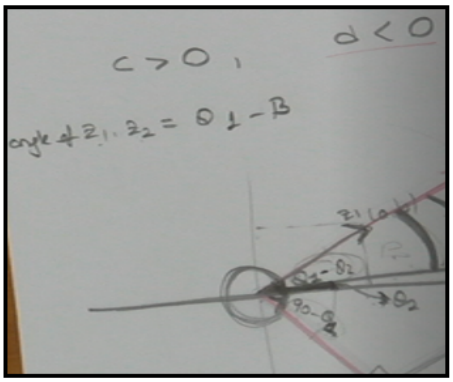


Figure 6.40. Derin's writing the angle of the vector $z_1.z_2$ in terms of the angle of the vector z_1 and the vector z_2 when $c > 0$, $d < 0$.

R: Okay. Can you tell me again what theta one and theta two was?

D: Theta one was the angle of z_1 and theta two was the angle of z_2.

R: Well. What about here?

D: Here (She refers to when $c > 0$, $d < 0$) theta one is the angle of z_1. β (She explains the β in $\theta_1 - \beta$) is the angle of z_2. It became negative because it rotated in

the clockwise. (She explains the minus in θ_1 - β). It moved to that side (She shows clockwise rotation).

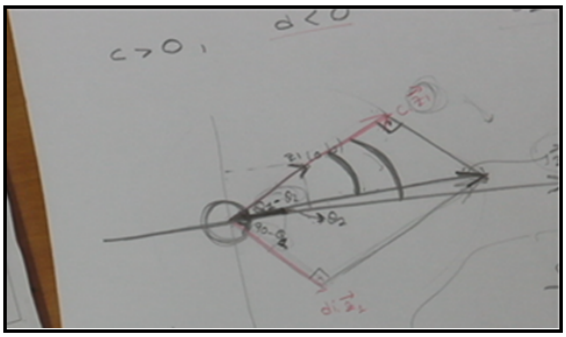


Figure 6.41. Derin's calculating and explaining the angle of the vector $z_1.z_2$, when c > 0, d < 0.

Data pointed to the importance of the use of the value of tan (θ_1 - θ_2) for Derin. Calculating and explaining the angle of rotation θ_1 - θ_2 using the tangent formula allowed Derin to realize that the angle of rotation was equal to the angle of the vector z_2. Therefore, she was able to claim that the vector z_1 was rotated in the clockwise by the angle of the vector z_2. This not only allowed her to make sense of the relationship between the angle of the original vector, θ_1, the angle of $z_1.z_2$, θ_2, and the angle of the vector z_2, β but also to explain what was pointing to the underlying reasons. She was able to explain that θ_2 was equal to the difference between θ_1 and β such that the difference was because of the rotation of z_1 in the clockwise. This was important because as the previous data has shown she was not able to explain the argument of $z_1.z_2$, θ_2, earlier.

Still, it is important to point out that Derin has not realized the following: The equation in Figure 6.42, the angle of $z_1.z_2 = \theta_1$- β, where θ_1 is the argument of z_1, and β is the argument of z_2, did not coincide with the generalized polar form in which the arguments of the two vectors z_1 and z_2 were added. In the following part, Derin explained the angle of $z_1.z_2$ in terms of the angles of z_1 and z_2. Discussion followed:

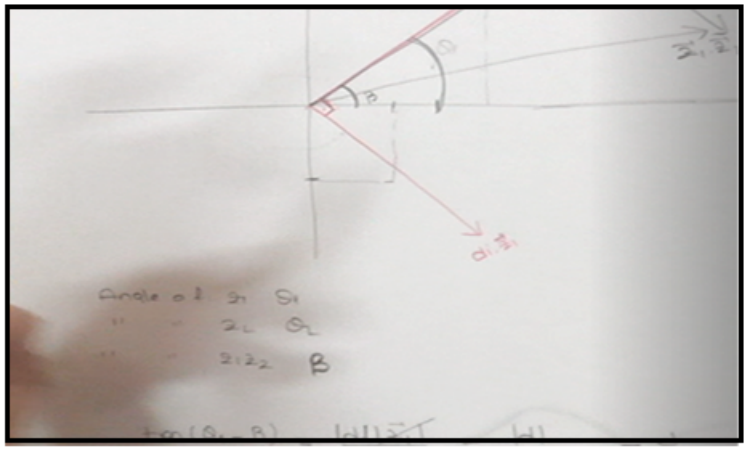


Figure 6.42. Derin's drawing a new Complex Plane, renaming the angles of z_1, z_2 and $z_1.z_2$ and pointing z_2 on the 4th quadrant.

R: When you think on the vector z_2 (She points to the vector z_2 on the 4th quadrant), where the ratio -$\frac{d}{c}$ refers to?

D: Hmm... Three hundred and sixty minus here (She points to the angle of z_2). (see Figure 6.43: Derin's writing tan (360 - θ_2) = -$\frac{d}{c}$). The tangent of this gives me -$\frac{d}{c}$. This is also minus tangent theta two. (see Figure 6.43: Derin's writing - tan (θ_2) = -$\frac{d}{c}$).

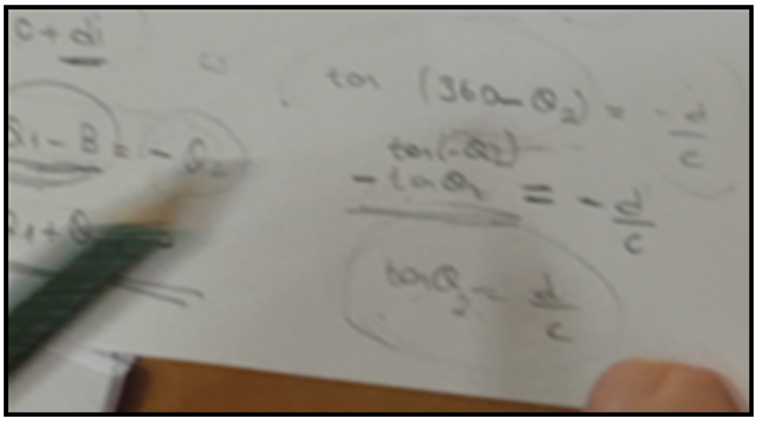


Figure 6.43. Derin's writing tan (360 - θ_2) = -$\frac{d}{c}$ and - tan (θ_2) = -$\frac{d}{c}$.

D: Equal... It is equal (She says excitedly), I mean it is minus theta two (Figure 6.44: Derin' writing θ_1 - β = - θ_2). This (She refers to the angle θ_1) minus this (She

refers to the angle β), the angle of $z_1.z_2$. I mean beta (Figure 6.44: She's calculating the angle of $z_1.z_2$ as $\theta_1 + \theta_2 = \beta$).

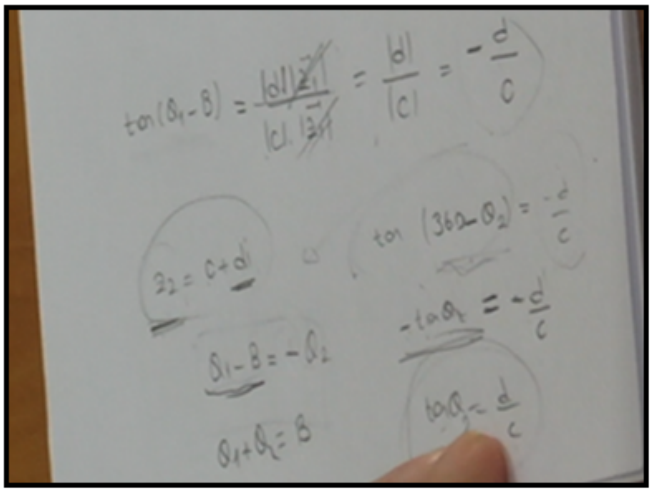


Figure 6.44. Derin's finding the angle of $z_1.z_2$ is equal to the sum of the angle of z_1 and the angle of z_2.

Data pointed to the importance of thinking of where the vector z_2 was so that Derin could find the value of tan (θ_1 - β), - $\frac{d}{c}$ based on the vector z_2. Calculating and explaining the ratio of - $\frac{d}{c}$ in terms of tangent of the acute angle between the vector z_2 and the real axis allowed Derin to realize that the tangent of θ_1 - β was equal to the tangent of the acute angle between the vector z_2 and the real axis, which was defined as 360 - θ_2. Therefore, she was able to realize that the tangent of θ_1 - β was also equal to tangent of - θ_2 using her previous knowledge of trigonometry, the tangent of 360 - θ_2 was equal to minus the tangent of θ_2 and also the tangent of - θ_2, tan (360 - θ_2) = - tan (θ_2) = tan (- θ_2). Thus, she was able to show that θ_1- β was equal to - θ_2, since they were acute angles and then, she was able to conclude that β was equal to θ_1+ θ_2. In this case, she was able to realize that when $c > 0$, $d < 0$, the multiplication of $z_1.z_2$ resulted with the rotation of z_1 in the clockwise as much as the argument of z_2.

Still, it is important to point out that Derin had difficulties while she was showing tan (360 - θ_2) = - tan (θ_2) = tan (- θ_2). This could be because of some limitations of her knowledge of trigonometry.

In sum, data suggested that, Derin was able to show the multiplication of the vectors z_1 and z_2 based on the vectors c.z_1 and di.z_1 geometrically when c > 0, d < 0. She was able to draw the vectors c.z_1 when c > 0 and di.z_1 when d < 0. Then, she was able to do vector summation between these two vectors by using the angle of 90° between them and obtain the vector $z_1.z_2$. Then, she was able to name the angles of z_1 as θ_1, z_2 as β and $z_1.z_2$ as θ_2 and show them on the Complex Plane. This way, she was able to determine that there was an angle of θ_1 - θ_2 between the vector z_1 and the vector $z_1.z_2$. She also calculated tan (θ_1 - θ_2) as $\frac{d}{c}$. Thus, she was able to conclude that the angle of θ_1 - θ_2 was as much as β, since the vector z_1 moved to the position of the vector $z_1.z_2$, she was able to say that the rotation was in the clockwise. This way, she was able to find θ_1 - θ_2 = β, and thus, θ_2= θ_1 - β. Though, this did not coincide with the general formula of the polar form as in the previous case c > 0, d > 0. In this case, the researcher asked her to think about where the vector z_2 was. This was very important for Derin to figure out that the angle of the vector z_2 was 360 - θ_2, which was also equal to - θ_2. Then, she was also able to realize that the tangent of the angle between the vector z_2 and the positive side of the real axis was equal to -$\frac{d}{c}$ since d was smaller than zero and c was bigger than zero. Then she also wa able to explain tan (360 - θ_2) = - tan (θ_2) = tan (- θ_2). This then allowed her to determine that the angle of θ_1 - β was equal to - θ_2. Thus, she was able to write the equation that θ_1 - β = - θ_2 and find $\beta = \theta_1 + \theta_2$. That is, she was able to reach the result that the angle of the vector $z_1.z_2$ was equal to the addition of the angle of z_1 and the angle of z_2.

6.2.2.3. Derin's reasoning for the case: c < 0, d > 0. Derin continued working on the case c < 0, d > 0. Again, she first drew the vector form of $z_1.z_2$ by focusing on the vectors c.z_1 and di.z_1. Then she reasoned on the angle of $z_1.z_2$ based on the angles of the vectors c.z_1 and di.z_1. Data followed:

D: When c is smaller than zero... d is bigger than zero (She writes c < 0, d > 0 and draws a Complex Plane below it).(She draws the vector z_1 on the Complex Plane) Now, When I multiply the complex number z_1 with negative c, it rotates one hundred and eighty degrees and it moves here (She points to the red vector that she drew on the

3rd quadrant), its part moves here. I multiply this (She points to the number d) with i since it is positive so it rotated just ninety degrees and becomes like this (She points to the the red vector that she drew on the 2nd quadrant). Ok. What was this (She points to the red vector that she drew on the 2nd quadrant)? This is di times z_1. That (She points to the vector on the 3rd quadrant) is c times z_1. Now I will move this d to find the vector, to sum. Like this (She moves the vector di.z_1 to the end point of the di.z_1). It becomes something like this (She draws the vector which is the summation of c.z_1 and di.z_1). The new vector. I mean the vector z_1 times z_2. (see Figure 6.45)

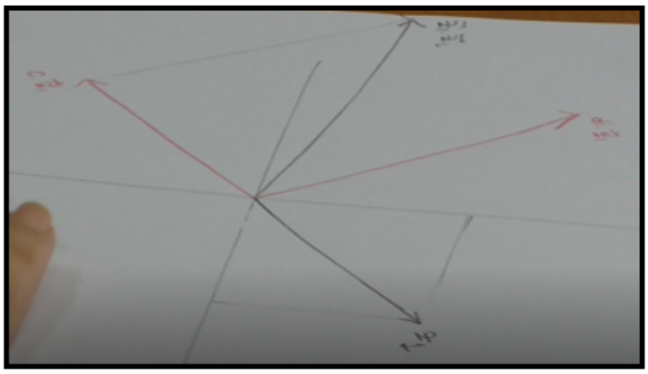


Figure 6.45. Derin's showing the multiplication of $z_1.z_2$ in vector form, when c < 0, d > 0.

As the data indicated, Derin knew that when c < 0, d > 0, the multiplication of the vector z_1 with the real part of z_2, "c" resulted with 180° counter clockwise rotation of z_1, since "c" was a negative scalar. She also knew that multiplication of the vector z_1 with the imaginary part of z_2, "di" resulted with 90° counterclockwise rotation of z_1, since "d" was a positive scalar and "i" caused to 90° counterclockwise rotation. Then, she was able to move the vector "di.z_1" to the end of the vector "c.z_1" and make vector summation in order to find the vector $z_1.z_2$.

Then, the researcher asked Derin the angle of the vector of $z_1.z_2$. I depict the data in two parts again: (i) Derin's difficulty in reasonning about the angle of $z_1.z_2$ and (ii) Derin's explanation of the angle of $z_1.z_2$ in terms of the angles of z_1 and z_2.

Derin's difficulty in reasonning about the angle of $z_1.z_2$ followed:

D: I will name that one (She points to the acute angle between the vector z_1 and the positive side of the real axis) as theta one again, the angle of z_1 (She waits for several seconds).

R: You are thinking about something...

D: No. I mean I am thinking about what I should do? Should I show z_2 first? But where is z_2? What was z_2? It was c plus di. d is positive and it is here (She points to a point on the positive side of the imaginary axis). c is negative and here (She points to a point on the negative side of the real axis). That means z is somewhere in here.z_2. It is somewhere in here (She points to the complex number where the points c and d intersect on the 2nd quadrant). What will I find here? The angle. The angle of z_1 times z_2. That angle is here (She marks the angle starting from the positive side of the real axis to the vector $z_1.z_2$) Alright. What should I do to find here? I should find there (see Figure 6.46: She points to the angle between the vectors $z_1.z_2$ and $c.z_1$).

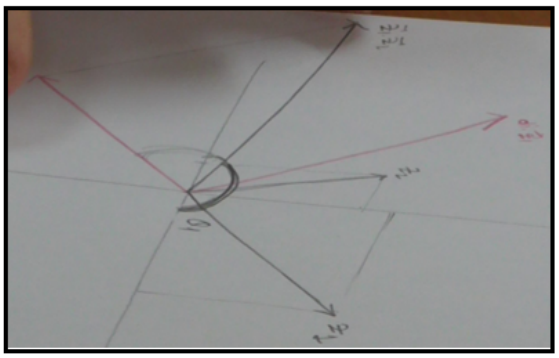


Figure 6.46. Derin's showing the angles of the vectors, z_1, $z_1.z_2$ and the angle between the vectors $c.z_1$ and $z_1.z_2$ on the Complex Plane.

R: Okay. You know. Which point should you find now?

D: I need to find this (She points to the angle of $z_1.z_2$). So I need to find that one (She points to the angle between the vector $z_1.z_2$ and negative side of the real axis) but

how can I? How can I find this one? So I need to use all of this one (She points to the angle between the vectors $z_1.z_2$ and $c.z_1$). I will start from tangent. If I find. Then... I will find. What if I find here (She points to the angle between the vectors $z_1.z_2$ and $c.z_1$) and name it as "k" (see Figure 6.47).What was this (She points to the vector $di.z_1$ that she moved to the 3rd quadrant)? The absolute value of d times the absolute value of z_1 is divided by the absolute value of c times the absolute value of z_1. Since c is less than zero, the result will be minus d over c (She writes tan $k = \frac{|d|.|z_1|}{|c|.|z_1|} = -\frac{d}{c}$). Here. Now if I consider z_2, what does minus d over c give me? Now here is the angle of z_2. It is ninety plus something (She points to the angle between the vector z_2 and positive side of the imaginary axis). I mean it is here (She points to the angle between the vector z_1 and positive side of the imaginary axis). What is that one (She points to the projection of number z_2 to imaginary axis). It is negative d, isn't it? Negative d. No, c. Here is d. There is c (She points to the projection of the number z_2 to real axis). Its tangent is c over d.

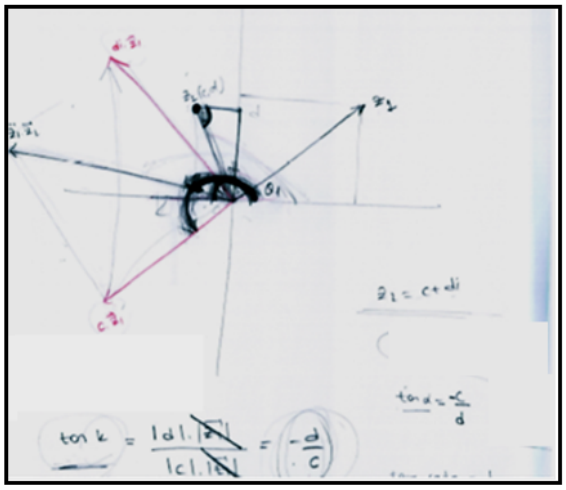


Figure 6.47. Derin names the angle between the vectors $z_1.z_2$ and $c.z_1$ as k and the angle between the vector z_2 and the positive side of the imaginary axis as α.

R: Okay. Its's tangent you say. Where did you look at, again?

D: I looked at there (She points to the angle between the vector z_1 and the positive side of the imaginary axis). If I look at the tangent of ninety plus alpha... It should be equal to there (She refers to minus d over c) (see Figure 6.48 above).

R: You named here (The researcher indicates the angle between the vector z_2 and the positive side of the imaginary axis) as alpha, didn't you?

D: Yes, I named here (She points to the angle between the vector z_1 and the positive side of imaginary axis) as alpha. I find tangent alpha. It is negative c over d. The tangent ninety plus α is minus tangent alpha. It is minus c over d. Wrong, I did something wrong I guess.

As the data indicated, Derin was able to define the angles of the vectors z_1, z_2 and $z_1.z_2$ as θ_1, θ_2 and β, respectively. Derin knew that she needed to find out the angle of the vector $z_1.z_2$ based on the angles of the vectors z_1 and z_2 from the previous cases. In addition, she focused on the angle between $z_1.z_2$ and $c.z_1$, naming it as k. Though what was important was that Derin knew from the previous case that she needed to find where z_2 was. Locating it reasoning that d was bigger than zero and c was smaller than zero, she named the angle between the vector z_2 and the positive side of the imaginary axis as α; thus, she could say that the angle of the vector z_2 was equal to $90 + \alpha$. She was able to calculate that the tangent of k was equal to - $\frac{d}{c}$ (see Figure 6.48 above). Then, she was able to calculate the tangent of α as - $\frac{c}{d}$ and thinking the tangent of $90 + \alpha$ was equal to the tangent of α, she was able to say that the tangent of $90 + \alpha$ was equal to -$\frac{c}{d}$. Here, determining the tangent of the angle of the vector z_2, $90 + \alpha$, as - $\frac{c}{d}$ confused her, since she expected to find out the tangent value of the angle of the vector z_2 to be equal to -$\frac{d}{c}$.

Rationale follows: If Derin could have focused on the vector z_2 naming its angle as θ_2 in order to define the value of tangent k, - $\frac{d}{c}$, she would have found that the tangent of the angle 180° - θ_2 would equal to - $\frac{d}{c}$, which was also equal the value of the tangent k. Since, k and 180° - θ_2 were acute angles, they should be the same, since their tangent values were the same. Thus, Derin could have found that 180° - θ_2 = k. Then, she needed to focus on the angle of the vector $z_1.z_2$. If she thought of the angle of the vector $z_1.z_2$, say β, she would have determined β in terms of the angles of θ_1 and θ_2 such that β + k was equal to $\theta_1 + 180^o$ (see Figure 6.48 above), β + k = θ_1 + 180°. Thus, she woud have calculated that $\beta = \theta_1 + 180^o$ - k. If she replaced 180° -

θ_2 with k, she could have found that $\beta = \theta_1 + 180^o$- $(180^o$- $\theta_2) = \theta_1 + \theta_2$. This would have allowed her conclude that z_1 would be rotated by the angle of z_2 when multliplied with it. Therefore, this data further suggested that although Derin knew what she needed to find out: The angle of $z_1.z_2$ based on the angle of the vector z_2 and z_1, still she had difficulty in reasoning about how to make sense of it in terms of how much z_1 was rotated when multiplied with z_2. Though it is important to state that, although the researcher did not ask Derin why she found out the tangent value of k, the angle between $z_1.z_2$ and $c.z_1$, further data will indicate to how she has reasoned.

Derin's explaination of the angle of $z_1.z_2$ in terms of the angles of z_1 and z_2 followed:

Realizing how much Derin had difficulty in reasoning about the argument of $z_1.z_2$, the researcher asked Derin to refocus on what she needed to find out. Derin started with the angle of the vector z_2 again. Discussion followed:

R: What was the angle of z_2? Can you say again?

D: Ninty plus alpha.

R: Alright. Which angle are you trying to express?

D: That angle (She points to the angle between the vector z_2 and the negative side of real axis) one hundred and eighty minus theta two. I look at tangent of this (She writes tan 180° - θ_2). Negative d over c (see Figure 6.48 below). One hundred and eighty minus theta two equals to k in terms of the angle of z_2.

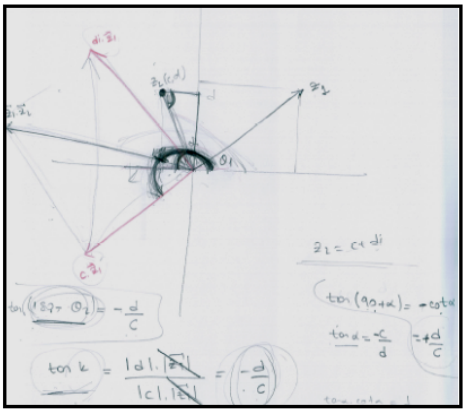


Figure 6.48. Derin's writing tan (180° - θ_2) = -$\frac{d}{c}$.

R: What do you want to do? I mean, explain again.

D: There (She points to the angle starting from the positive side of real axis to the vector $z_1.z_2$ (She thinks). There (She points to the angle between the vector z_2 and the negative side of real axis). (She becomes silent for several seconds) Here is k (She points to the angle k on the Complex Plane). Here is the angle of $z_1.z_2$ (She points to the angle of the vector $z_1.z_2$). Now this is equal to one hundred and eighty plus theta one (She points to the angle starting from the positive side of real axis to the vector $c.z_1$) I mean... Now here (She refers to the angle of the vector $z_1.z_2$) I called theta two was the angle of z_1. Theta two plus k. What do I obtain? (She points to the angle starting from the positive side of the real axis to the vector $c.z_1$). One hundred and eighty plus theta one, there (She points to 180° + θ_1 - k) one hundred and eighty plus theta one minus one hundred and eighty minus theta two. One hundred and eighty plus theta one minus one hundred and eighty plus theta two. It is theta one plus theta two (see Figure 6.47 below).

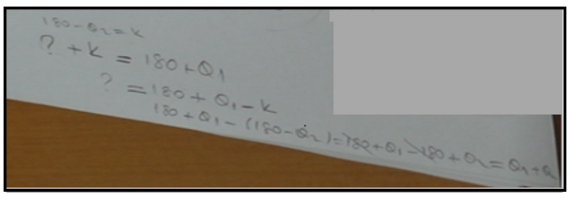


Figure 6.49. Derin's calculating the angle of the vector $z_1.z_2$ in terms of the angles of the vectors z_1 and z_2.

Data pointed to the following: First, realizing that thinking of the argument of z_2 as $90^o + \alpha$ in terms of the imaginary axis did not help her, after the 2nd prompt asking her to re-focus on what the angle of z_2 was, Derin reasoned on the angle of z_2 in terms of the negative side of the real axis. That is, she was able to think of the angle of z_2 as θ_2 and wanted to calculate the tangent value of it using 180° - θ_2. Therefore, Derin was able to find out this value as equal to -$\frac{d}{c}$ since d was bigger than zero and c was smaller than zero. Derin also knew that this value was equal to the tangent value of k. This allowed her to conclude that 180° - θ_2 = k, since she possibly used the tangent equality of the acute angles. Though, the researcher knew that Derin's stating the angle, k, in terms of the argument of z_2 would not be enough to reason on the angle of $z_1.z_2$ in terms of the angles of z_1 and z_2; how much z_1 was rotated when multiplied with z_2. This allowed Derin to think that k + ? = $180^o + \theta_1$. How she reasoned follows: She knew that the sum of the angle of $z_1.z_2$ and k would be equal to the angle of $c.z_1$. The angle of $c.z_1$ would also be equal to the total of the angle of z_1 plus 180° since multiplying with c would cause z_1 to be rotated 180°. Therefore, calculating the angle of $z_1.z_2$, she was able to determine that it would be equal to $180^o + \theta_1$ - k. Knowing from earlier that k was also equal to 180° - θ_2, in the equation, she was able to substitue 180° - θ_2 in the place of k. This way, she algebraically determined that the argument of $z_1.z_2$ was equal to $\theta_1 + \theta_2$.

Knowing how hard it was for Derin to come up with the equation, the resaercher asked Derin to re-explain how she reasoned:

R: If you want, sum it up from the beginning.

D: (She points to the angle of the vector $z_1.z_2$). That angle (She points to the angle starting from the positive side of the real axis to the vector $z_1.z_2$). To find that one, I used this angle. I used k.

R: Okay. Why did you use that one? Can you tell us?

D: Because that... To find here (She points to the angle of the vector $z_1.z_2$ on the Complex Plane). So I didn't move this one (She points to the vector $c.z_1$). I moved this one (She points to the vector $d.z_1$ which she moved to the 3rd quadrant). When, I moved this (She points to the vector $c.z_1$) to there (She draws a line in order to show she adds the vector $c.z_1$ to the end of the vector $di.z_1$), here is (She points to the angle between the vector $z_1.z_2$ and the vector $di.z_1$)... It could be equal to ninety (She points to the angle between the vectors $di.z_1$ and z_1) plus theta one plus here (She points to the angle between $z_1.z_2$ and $di.z_1$ on the 2nd quadrant). Since, we used there, the angle k. It was that one hundred and eighty theta two equals to k. Why? This one (She points to tan k = $-\frac{d}{c}$) minus d over c. This is z_2'... Since... what we want to find? I look at what the proportion of minus d over c is equal to in terms of the angle of z_2. Then, we should look at z_2. I looked at z_2. Where does the proportion of minus d over c come from? From which angle does the proportion of minus d over c come? It should be tangent, so in order to be equal here (She points to the opposite leg of the right triangle of z_2) over here (She points to the adjacent leg of the right triangle of z_2), we take this angle (She points to the acute angle on the top on the right triange on the 2nd quadrant). Since, d is positive and c is negative, the tangent of this (She points to the acute angle on the top on the right triange on the 2nd quadrant) gives me minus d over c, doesn't it? Yes, indeed. Then, I should find this angle (She points to the acute angle on the top on the right triange on the 2nd quadrant). This angle (She points to the acute angle on the top on the right triange on the 2nd quadrant) is here (She points to the angle between the vector z_2 and the real axis on the 2nd quadrant based on the opposite angles rule). How could I define here (She points to the angle between the vector z_2 and the real axis on the 2nd quadrant based)? One hundred and eighty... Ninety minus α is equal to one hundred and eighty minus theta two. Yes indeed. Then, one hundred and eighty minus theta two equals to k. Then, here, the proportions of minus d over c are the same (She shows the tangent of one hundred and eighty minus theta two and the tangent of k). Then, I foud out k in terms of theta one from there. Then, what are we looking for? Hmmm, here, namely the question mark. The question mark is equal to the angle of the vector $z_1.z_2$, isn't it? This part (She shows the question mark) plus k is equal to one hundred and eighty plus theta one. Why? Since here is theta one and here is one hundred and eighty degrees (She shows the angle between the vector z_1 and the vector

c.z_1). Then, we find the question mark from here by replacing k with one hundrd and eighty minus theta two, it becomes theta one plus theta two (see Figure 6.50).

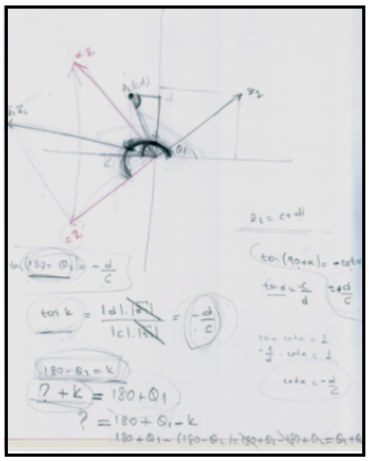


Figure 6.50. Derin's calculating the angle of $z_1.z_2$ based on the angles of z_1 and z_2 in the case of c < 0 and d > 0.

Data showed that Derin was able to show that the argument of the multiplication of complex numbers "$z_1.z_2$" in polar form resulted in the addition of the arguments of the complex numbers "z_1" and "z_2" ,when c < 0, d > 0. Particularly, her reasoning was as follows: First, she knew why she found k and why she needed to find out the tangent value of it. She knew that she needed k in order to determine the argument of $z_1.z_2$. She calculated the tangent of k based on the triangle of $z_1.z_2$, which she got moving the vector di.z_1 to the end of the vector c.z_1. She was able to say that tan $k = \frac{|d|.|z_1|}{|c|.|z_1|} = -\frac{|d|}{|c|}$, since c < 0, d > 0. She also was able to get the ratio of $-\frac{d}{c}$ by means of the point representation (c, d) and the vector representation of z_2. She was able to realize that if the angle between the vector z_2 and the positive part of the imaginary axis was equal to α, the tangent of the explementary angle of α, 90° - α was equal to - $\frac{d}{c}$. In this case, she knew that alternate interior angles were equal and the explementary angle of α, and the angle between the vector z_2 and negative part of the real axis, which was 180° - θ_2, were alternate interior angles. Thus, the tangent of 180° - θ_2 was equal to - $\frac{d}{c}$. Then, she was able to calculate and explain that k = 180° - θ_2, since she again possibly tought they were acute angles and their tangents were equal, as before. Secondly, reasoning that multiplied with c, z_1 was rotated 180° and ending

up with the argument of c.z_1 as $180^o + \theta_1$, which was also equal to the total of the angle of the vector $z_1.z_2$ and k. Therefore, she concluded that "? + k = $180^o + \theta_1$". In this case, she was able to calculate that "? = $180^o + \theta_1$ - k" and replace "k" with "180° - θ_2". Thus, she was able to get "? = $\theta_1 + \theta_2$" and so, she was able to reach again the general formula of polar form for the arguments: the angle of the vector $z_1.z_2$ was equal to the summation of the angels of the vectors z_1 and z_2.

In sum, Derin was able to show the multiplication of z_1. z_2 = c.(a + bi) + di.(a + bi) geometrically based on the vectors c.(a + bi) and di.(a + bi) regarding the cases of c < 0 and d < 0. She first drew the vector form of $z_1.z_2$ by focusing on the vectors c.z_1 and di.z_1. She showed that the vector $z_1.z_2$ was equal to the summation of the vectors c.z_1 and di.z_1 geometrically. Then she reasoned on the angle of $z_1.z_2$ based on the angles of the vectors z_1 and z_2. She named the angles of the vectors z_1, z_2 and $z_1.z_2$ as θ_1, θ_2 and ?, respectively. Then, she thought that how she could define the angle of $z_1.z_2$,"?" based on θ_1 and θ_2. She looked at the the Complex Plane and realized that the angle of the vector c.z_1 was equal to $180^o + \theta_1$, since multiplication with c when c < 0 resulted with a 180° counterclockwise rotation of the vector z_1. Then, she realized that the angle of c.z_1 was also equal to addition of the angle of the vector $z_1.z_2$ with the angle between the vector c.z_1. Then, she decided to define the angle between the vector c.z_1 and $z_1.z_2$. She named the angle between the vector c.z_1 as k and calculated tangent of k, as - $\frac{d}{c}$. Then, she focused on the angle of the vector z_2 in order to define the angle k in terms of it. Then, using the opposite angles rule, she said that 180° - θ_2 was equal to 90° - α and then, she found that the tangent of 180° - θ_2 was equal to the tangent of k. Since they were acute angles and their tangents were the same, the angle k was equal to 180° - θ_2. Then she re-focused on the vector c.z_1. She said that the addition of the angle of the vector $z_1.z_2$ and k gave the angle of the vector c.z_1, which was the addition of the angle of z_1 and the angle between the vectors c.z_1 and z_1, that was equal to 180°. Thus, she obtained the equality that "? + k = $180^o + \theta_1$". Therefore, she was able to find "?" such that "? = $180^o + \theta_1$ - k". Then, she subsituted "180° - θ_2" for "k and obtain" "? = $180^o + \theta_1$ - k = $18^o + \theta_1$ - (180° - θ_2) = $180^o + \theta_1$ - 180° - $\theta_2 = \theta_1 + \theta_2$" and so, she was able to find that the angle of the vector $z_1.z_2$ was equal to the summation of the angles of the vectors z_1 and z_2.

6.2.2.4. Derin's reasoning for the case: c < 0, d < 0. Data followed:

R: Shall we continue with the fourth case? Is it okay?

D: (see Figure 6.51 below: She writes c < 0, d < 0 and draws a Complex Plane. She draws the vector z_1 on the 1st quadrant) Since c is negative... I move (She draws a red vector on the same line but opposite direction with the vector z_1 on the 3rd quadrant). d is also negative. One hundred and eighty... It will rotate two hundred and seventy degrees so it will be somewhere here (She draws a red vector on the 4th quadrant). di times z_1... c times z_1... Now this one (She points to the vectors $c.z_1$ and $di.z_1$). I move this one (She points to the vectors $c.z_1$ and $di.z_1$) to sum up (She moves the vector $di.z_1$ to the end point of the vector $c.z_1$). It becomes like this (She draws the summation vector). It is z_1 times z_2 (She writes $z_1.z_2$ to the end point of the summation vector).

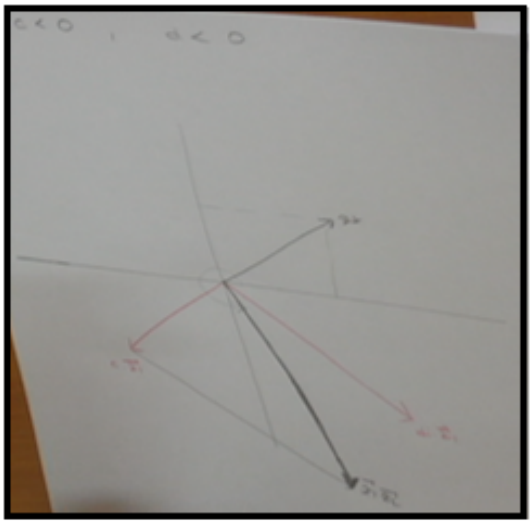


Figure 6.51. Derin's drawing the vectors z1, c.z_1 and di.z1 on the Complex Plane and showing the vector $z_1.z_2$ as the summation of the vectors c.z_1 and di.z_1 when c < 0, d < 0.

As the data showed, compared to her reasoning earlier, for Derin it was much easier to reason when c < 0, d < 0. Particularly, she reasoned that the vector c.z_1 was the rotated form of z_1 by 180° in the counterclockwise, since c < 0 and the vector di.z_1 was the rotated form of z_1 by 270° in the counterclockwise, since d < 0. Also, she

knew that she was able to obtain the vector $z_1.z_2$ utilizing vector summation when she moved the vector $di.z_1$ to the end of the vector $c.z_1$. Then, Derin continued:

D: Now I wonder about the angle of this one (She points to the angle of the vector $z_1.z_2$)(see Figure 6.52: Derin names the angle of the vector z_1 as θ_1). How can I find the angle of this (She points to the angle of the vector $z_1.z_2$)? What if I say two hundred and seventy plus that (She points to the angle between the vector $z_1.z_2$ and the negative side of the imaginary axis)? From here (She points to the positive side of real axis) minus here (She points to the angle between the vector $z_1.z_2$ and positive side of the real axis on the 4th quadrant). It doesn't matter if I find either here (She points to the angle between the vector $z_1.z_2$ and positive side of the real axis on the 4th quadrant) or here (She points to the angle between the vector $z_1.z_2$ and the negative side of y-axis on the 4th quadrant).

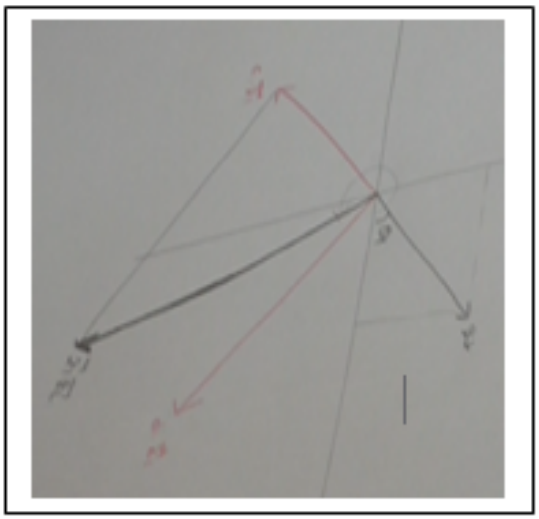


Figure 6.52. Derin's reasoning on the angle of the vector $z_1.z_2$ when $c < 0$, $d < 0$.

R: Now what do you need to find the angle there (She points to the acute angle between the vector $z_1.z_2$ and the positive side of the x-axis on the 4th quadrant)?

D: I need the tangent of this angle (She points to the acute angle between the vector $z_1.z_2$ and the positive side of the x-axis on the 4th quadrant). Hmm. If I look at that the angle (She points to the angle between the vectors $c.z_1$ and $z_1.z_2$), the tangent

of this angle. I know here (She points to the angle of the vector of z_1) theta one, I add one hundred and eighty degrees to here (She finds the angle of the vector $c.z_1$). If I also find here (She points to the angle between the vectors $c.z_1$ and $z_1.z_2$) that means I obtain the angle of z_1 times z_2. So I will name here (She points to the angle between the vectors $c.z_1$ and $z_1.z_2$) as k. Tangent k is the length of here (She points to the vector $di.z_1$ which she moved to the 3rd quadrant) divided by the length of here (She points to the vector $c.z_1$)... d over z_1 divided by c over z_2. Sorry I mean z_1. I cancelled z_1's. Since it is both negative, it is d over c. Hmm tangent k (see Figure 6.53: She writes $\tan k = \frac{d}{c}$). Hmm, what was the angle of here (She points to the vector $z_1.z_2$)? Theta one plus one hundred and eighty plus k. This is the angle of z_1 times z_2.

R: Alright.

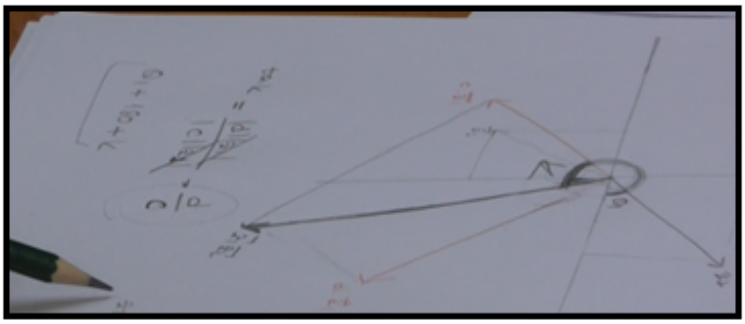


Figure 6.53. Derin's calculating the tangent of k and the angle of the vector $z_1.z_2$.

D: Now, hmm... How can I reach d over c, I will look at z_2. It is c plus di. What is the tangent of z_2? Let me find z_2. Since they are both negative (She points to the numbers c and d), they're somewhere around here (She marks a complex number, z_2 on the 3rd quadrant and draws the vector z_2). It is here (She points to the 3rd quadrant). The tangent of it... Hmm, if I do that, here (She points to the vector z_2)... What is that length (She points to the imaginary part of the vector z_2)? It is d. What is that length (She points to the real part of the vector z_2)? It is c. Which angle should I look at to obtain d over c? I should look at that angle (see Figure 6.54a: She points to the angle between the vector z_2 and the negative side of the real axis). So the angle of z_2 is one hundred eighty plus that angle (She points to the angle between the vector z_2 and

the negative side of the real axis) which I don't know. Let me call it m. The angle of z_2 *is one hundred and eighty plus m. That was tangent m (She writes tan* $m = \frac{d}{c}$*) (She does calculations). Theta two minus one hundred and eighty (She writes tan* $(\theta_2 - 180^o) = \frac{d}{c}$*). So, what equals to what? d over c... Then, which one is equal to which one? k equals to... k equals to theta two minus one hundred and eighty (see Figure 6.54b).*

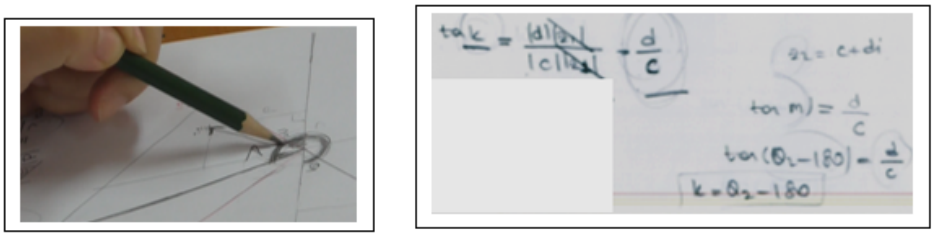


Figure 6.54. Derin's calculating k based on the angle of the vector z_2 (a, b).

Since it equals... Hmm... Here (She points to $\theta_1 + 180^o + k$). I mean this was my angle. To find it. I mean this angle (She writes the angle of $z_1.z_2$). This equals to theta one plus one hundred and eighty plus theta two minus one hundred and eighty. This is the same again (see Figure 6.55). I mean theta two is the angle of z_2. Here (She points to the angle θ_2). I rotated this (She points to the vector z_1). It moves here (She points to the vector $z_1.z_2$).

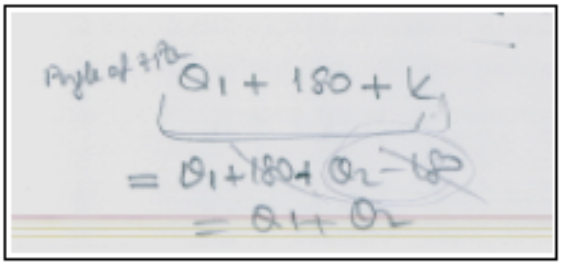


Figure 6.55. Derin's calculating the angle of the vector $z_1.z_2$ in terms of the angles of the vectors z_1 and z_2.

As data showed, Derin was able to focus on the angle of $z_1.z_2$. Specifically, she was able to name the angle of the vector z_1 as θ_1, the angle between the vectors $c.z_1$ and z_1 as 180°, because of the rotation of the vector z_1 multiplied with a negative scalar,

c, and the angle between the vector $z_1.z_2$ and the vector $c.z_1$ as k. Therefore, she was able to define the angle of the vector $z_1.z_2$ as $\theta_1 + 180^o$ + k based on all these angles. Then, she calculated the tangent value of of k as equal to $\frac{|d|.|z1|}{|c|.|z1|} = \frac{d}{c}$, since c < 0, d < 0. Then, in order to define k in terms of θ_2, she was able to use the angle between the vector z_2 and negative side of the imaginary axis. She named the angle between the vector z_2 and negative side of the imaginary axis as m. Then, she was able to calculate the tangent value of m as equal to $\frac{d}{c}$. She was able to define also that θ_2 was equal to 180° + m, based on the rotation of the vector z_2 from the positive part of the real axis and thus, she was able to write m = θ_2 - 180°. This way she was able to claim that the tangent value of m and the tangent value of k were equal, since m and k were acute angles. Therefore, she concluded that the angles m and k should also be equal. Replacing k with θ_2 - 180° she concluded that the angle of the vector $z_1.z_2$ was equal to "$\theta_1 + 180^o$ + k = $\theta_1 + 180^o$+ θ_2 - 180°= $\theta_1 + \theta_2$". She even stated what $\theta_1 + \theta_2$ represented such that the vector z_1 was rotated by the angle of the vector z_2 when multiplied with z_2 to find the angle of the vector $z_1.z_2$. This again suggested that Derin was able to think of z_1 as the multliplicand and z_2 as the operant vector acting on z_1.

In sum, Derin showed the multiplication of $z_1.z_2$ = c.(a + bi) + di.(a + bi) geometrically based on the vectors c.(a + bi) and di.(a + bi) regarding the case of c < 0 and d < 0. Derin first drew the vectors $c.z_1$ and $di.z_1$ by calculating the angles of the vector $c.z_1$ as 180° and the angle of the vector $di.z_1$ as 270° in the counterclockwise. Then, she drew the vector $z_1.z_2$ by focusing on the summation of the vectors $c.z_1$ and $di.z_1$. Then, she started to reason on the angle of the vector $z_1.z_2$ based on the angles of the vectors z_1 and z_2. She said that she was able to reach the angle of the vector $z_1.z_2$ by adding the angle of the vector z_1, the angle between the vector $c.z_1$ and the vector z_1 and the angle between the vector $c.z_1$ and the vector $z_1.z_2$. She named these angles such that the angle of the vector z_1 as θ_1, the angle between the vector $c.z_1$ and the vector z_1 as 180° and the angle between the vector $c.z_1$ and the vector $z_1.z_2$ as k. Thus, she was able to say that the angle of the vector $z_1.z_2$ was equal to $\theta_1 + 180^o$ + k. Then, she focused on the angle k. She used the tangent formula and found that the tangent of k was equal to $\frac{d}{c}$. Then, she evaluated the angle k in terms of the angle of the vector z_2, since she wanted to explain the angle of the vector $z_1.z_2$ based on the

angles of the vectors the vectors z_1 and z_2. Therefore, she named the angle between the vector z_2 and the negative side of the real axis as m and found that the tangent of m was equal to $\frac{d}{c}$. Thus, the tangents of k and m were equal. Again, by the same reasoning in the previous three cases, she said that the acute angles, k and m, were equal, since their tangents were equal. Then, she re-defined the angle m based on the angle of the vector z_2 such that m = θ_2 - 180°. Thus, she was able to say that k is equal to θ_2 - 180°. Therefore, she was able to subsitute θ_2 - 180° for k in the equation of the angle of the vector $z_1.z_2$, θ_1 + 180° + k. Thus, she was able to find that the angle of the vector $z_1.z_2$ was equal to θ_1 + θ_2, the additon of the angle of the vector z_1 and the vector z_2. Thus, she was able to conclude that the vector z_1 was rotated as much as the angle of the vector z_2 in order to find the placement of the vector $z_1.z_2$ as in remaining three cases.

6.2.3. Derin's Reasoning on the Relationship Between the Cartesian, Vector and Polar Forms

After Derin's reasoning on the different cases regarding the angle of the vector $z_1.z_2$, the researcher asked Derin to explain the relationship between the multiplication of complex numbers z_1 and z_2 in vector form and in polar form. Firstly, Derin explained the relation between the multiplication of complex numbers z_1 and z_2 in Cartesian form and in vector form. Then, she explained the relation between the multiplication of complex numbers z_1 and z_2 in Cartesian form, vector form and polar form.

Derin's explanation of the relation between the Cartesian form and vector form followed:

R: We said z_1 equals to a plus bi.z_2 equals to c plus di. Can you multiply them in Cartesian form again?

D: (She writes z_1. z_2 = (a + bi).(c + di) and calculates the equation z_1. z_2 = (a + bi).(c + di) = ac + adi + bci - bd = ac - bd + i.(ad + bc)).

R: Hmm... Can you express this multiplication to us in vector form?

D: When I multiply the vector z_1 with the vector z_2... I separate the vector z_2 into its parts. When I sum the two vectors, which I obtain when I multiply this (She points to the vector z_1) with the real part of the vector z_2, and when I multiply this (She points to the vector z_1) with the imaginary part of the vector z_2, I obtain a vector. This vector gives me the vector z_1 times z_2. (She writes $(a + bi).c + (a + bi).di = z_1.z_2$).

R: Alright. Well. Why?

D: I multiplied with the real part of z_2. Then, I multiplied with the imaginary part of z_2. And the vector I obtained by sum these up gaves me the vector z_1 times z_2.

Data showed that compared to the pre-interview and the beginning of the teaching sessions, Derin's explanation regarding vector form of the algebraic form of the multiplication of complex numbers was different: She reasoned that she would consider z_2 as a vector made up of real and imaginary components. Considering z_1 as a vector, she would then use the distributive property of multiplication and multiply the real part of z_2 with z_1 and the imaginary part of z_2 with z_1. This way she knew that both the algebraic expressions $c.z_1$ and $di.z_1$ would be vectors. Then, she would add them as vectors to come up with the $z_1.z_2$ vector.

Derin's explanation of the relation between Cartesian form, the polar form, vector form followed:

R: How could you express this multiplication (She points to $(a + bi).c + (a + bi).di = z_1.z_2$) in polar form?

D: Its expression in terms of r and theta, right? What was z_1? It was r times cis theta. I write it down like this (She writes $z_1 = r.cis\ \theta_1$). And z_2 is, r_2 times cis theta two (She writes $z_2 = r_2.cis\ \theta_2$). I mean c and d. I did it by separating it (She refers to z_2) into its real and imaginary parts... Now, z_1 times z_2... What did I do here? I

found it there. Multiplication is enough to find its length. The angles here are changing according to the cases, but adding the angle is enough. Like cis theta one plus theta two. (She writes $z_1.z_2 = |r_1|.|r_2|.cis\ (\theta_1 + \theta_2)$*).*

R: Okay okay. (The researcher refers to $|r_1|$ *and* $|r_2|$*) What are these?*

D: Their lengths.

R: And what is r_1*?*

D: r_1 *is directly... r*$_1$ *times* r_2*. Directly magnitudes. Lengths. (see Figure 6.56: Derin replaces* $r_1.r_2$ *with* $|r_1|.|r_2|$ *in the equation that* $z_1.z_2 = |r_1|.|r_2|.cis\ (\theta_1 + \theta_2)$*).*

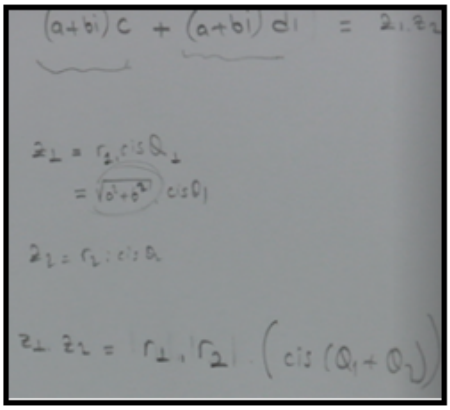


Figure 6.56. Derin's writing $z_1.z_2 = r_1.r_2.\text{cis}\ (\theta_1 + \theta_2)$.

R: Alright. I will ask you again. Can you explain the relation between polar multiplication and vector multiplication of two complex numbers? I mean, what is the relation between this one (see Figure 6.57: The researcher points to $(a + bi).c + (a + bi).di = z_1.z_2$*) and that one (*$z_1.z_2 = r_1.r_2.cis\ (\theta_1 + \theta_2)$*).*

D: By saying relation... What kind of a relation I need to explain?

R: Explain one in terms of the other. I mean explain algebra in terms of polar or look at polar form and explain algebra. How can I say... Look at this (The researcher

indicates $z_1.z_2 = r_1.r_2.cis\ (\theta_1 + \theta_2)$) and explain us the algebra or look at this (The researcher indicates $(a + bi).c + (a + bi).di = z_1.z_2$) and explain in polar form. Is it clear or? D: Alright.

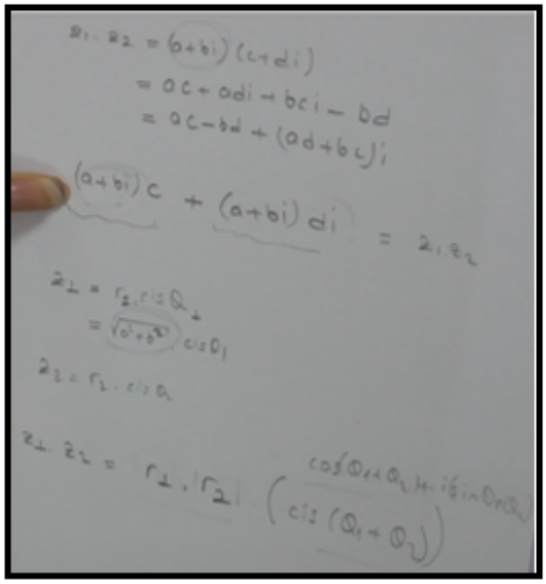


Figure 6.57. The researcher points to the multiplication of z_1 and z_2 in vector form and in polar form.

D: r1 is the length of this, a plus bi. r_2 is the length of c plus di. It is the length of z_2. I mean multiplication of their lengths... Multiplication of the lengths of the complex numbers z_1 and z_2. Here is r_1 times r_2.

R: Well, why? You should explain it with reasons, alright? I mean why?

D: We found them a little while ago. Where? Okay. There it is (see Figure 6.58: She points to the paper where she did calculations of the case $c > 0$, $d > 0$ on).

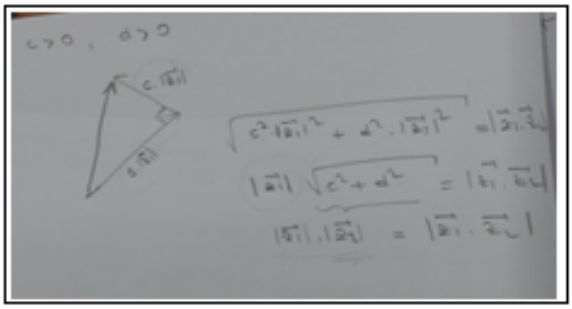


Figure 6.58. Derin's explaining the multiplication of the magnitudes of the vectors z_1 and z_2 in order to find the magnitude of the vector $z_1.z_2$.

When we separate this into its parts (She points the vector $z_1.z_2$ on the paper where she did calculations of the case $c > 0$, $d > 0$ on), that part (see Figure 6.59: She points to both the $c.z_1$ part of the vector $z_1.z_2$ and the part $(a + bi).c$ of the vector $(a + bi).c + (a + bi).di = z_1.z_2$) and this part (see Figure 6.59: She points to both the $di.z_1$ part of the vector $z_1.z_2$ and the $(a + bi).di$ part of the $(a + bi).c + (a + bi).di = z_1.z_2$ at the same time) are always perpendicular.

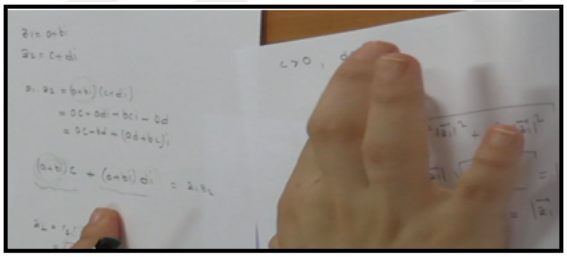


Figure 6.59. Derin's matching the parts $c.z_1$ and $di.z_1$ part of the vector $z_1.z_2$ and the parts $(a + bi).c$ and $(a + bi).di$ of the vector $(a + bi).c + (a + bi).di = z_1.z_2$.

R: Why are they always perpendicular?

D: Because the angle between those parts are always ninety degrees. When we rotated, this angle (She points to the angle between the vectors $c.z_1$ and $di.z_1$) was always ninety degrees. When I add their magnitudes up (see Figure 6.60 above: She points to the vectors $c.z_1$ and $di.z_1$ on the paper that she calculates the magnitudes in the case $c > 0$, $d > 0$). I mean this... To find that magnitude (She points to the vector $z_1.z_2$ on the vector summation), when I found these (She points to the magnitude of the

c.z_1 part and di.z_1 part of the vector $z_1.z_2$), it gave me the the magnitude of z_1 times the magnitude of z_2, I mean these (She points to (a + bi).c and (a + bi).di). I added up the vectors and since there were ninty degrees between them, I calculated using the Pythagorean Theorem and found here (She points to the magnitude of $z_1.z_2$). I found here (She points to $r_1.r_2$ in the polar formula $z_1.z_2 = r_1.r_2.cis\ (\theta_1 + \theta_2)$). Eer... The angle of $z_1.z_2$ is as I found before. When I add the angle of z_2 to the angle of z_1, I mean I rotated it as much as the angle of z_2, I found the angle of the complex number $z_1.z_2$.

As the data showed that Derin was able to calculate and explain the multiplication the complex numbers z_1 and z_2 in vector form and in polar form. She knew that in vector form, the real part "c.z_1" and the imaginary part "di.z_1" of the vector $z_1.z_2$ was perpendicular to each other for all four cases of c and d. As data have shown earlier, this was due to her abstraction of reasoning on the similarities and differences between the different cases of c and d. Using Pythagorean Theorem, she was then able to calculate and explain the magnitude of $z_1.z_2$ by using the magnitudes of the vectors c.z_1 and di.z_1 through vector summation on the right triangle these vectors formed. This suggested an important deduction on her part: Derin's reasoning on the Cartesian form of the multiplication of complex numbers z_1 and z_2 in terms of the components of z_1 and z_2 and explaining the resultant vector $z_1.z_2$ through the vectors c.z_1 and di.z_1 allowed her to also explain the reasoning behind the multiplication of the magnitudes of z_1 and z_2 in polar form. That is, she was able to justify the reason behind the polar formula of the multiplication of complex numbers z_1 and z_2. Particularly, the magnitudes of the complex numbers z_1 and z_2 were multiplied to find the magnitude of $z_1.z_2$ because of the magnitudes of c.z_1 and di.z_1 in vector form. Secondly, she was able to explain the reasoning behind the addition of the arguments of z_1 and z_2 in polar form when z_1 and z_2 were multiplied. Referring to the earlier calculations and reasoning she had done to find the angle of the vector $z_1.z_2$ for all four cases of c and d, she was able to reason that the addition of the arguments was because of the rotation of the vector z_1 by the angle of the vector z_2. Therefore, data showed that Derin thought of the algebraic form of the multiplication of complex numbers both in vector form. Also, she was able to explain the polar form in terms of the vector form such that the vector z_1 was the multiplicand and the vector z_2 was the multiplier as operator acting on the vector z_1.

It is important to say that: When, the researcher asked Derin the relation between the multiplication of complex numbers z_1 and z_2 in the Cartesian form and polar form, she was pointed to the equation that (a + bi).c + (a + bi).di = $z_1.z_2$. This probably directed Derin to think about the relation between the multiplication of complex numbers z_1 and z_2 in vector form and in polar form. Therefore, she did not mention then the relation between the multiplication of complex numbers in Cartesian form and polar form. Had the researcher asked her the relation between the Cartesian form and polar form, she could have explained. Therefore, neither I could claim that Derin could explain the relation between the multiplication of complex numbers in Cartesian form and polar form nor that she could not.

6.3. Post-Interview

In this section, I depict data after the completion of the teaching sessions, regarding Derin's knowledge of the multiplication of complex numbers in Cartesian form, in vector form and in polar form and the relation between these forms. The researcher asked Derin to explain the Cartesian form of the multliplication of two complex numbers, z_1= a + bi and z_2 = x + yi, algebraically and geometrically. Derin explained the multiplication of two complex numbers, z_1 = a + bi and z_2 = x + yi in Cartesian form algebraically and geometrically for all of the cases of x and y.

6.3.1. Derin's Knowlegde of the Multiplication of Complex Numbers in Cartesian Form

In this section, I depict data in two parts: (i) Derin's explanation of the multiplication of the Cartesian form of complex numbers algebraically and (ii) Derin's explanation of the multiplication of the Cartesian form of complex numbers geometrically.

Derin's explanation of the multiplication of the Cartesian form of complex numbers algebraically followed:

R: Let's say z_1 is a plus bi.z_2 is x plus iy. Could you explain the multiplication of these two complex numbers in Cartesian form?

D: (She writes $z_1.z_2 = (a + bi).(x + yi) = ax + ayi + bxi - by$).

R: Okay. Could you explain it to us? What did you do? How did you think?

D: Hmm... I separated z_1 into its parts. a and b... Its real and imaginary parts. Then, I multiplicated each one with z_2 and I summed them up. So I obtain z_1 times z_2 (see Figure 6.60).

R: Alright. Could you explain your, I mean that part to us (The researcher points to ax + ayi + bxi - by)? What is it?

D: Okay, okay. (see Figure 6.60: She arranges the statement : ax + ayi + bxi - by and rewrites as ax - by + i.(ay + bx)). ax minus by... This (She points to the part ax - by) is its real part. That one (She points to the part ay + bx) is the imaginary part. I keep the complex number z_2 the same (She points to x + yi) in here. I separated z_1 (She points to the number a + bi) into its real and imaginary parts. After that I multiplied its real part (She points to a) with z_2. I obtained a complex number (She points to the complex number ax + ayi). Then, I multiplied with its imaginary part (She points to bi) and I summed them up.

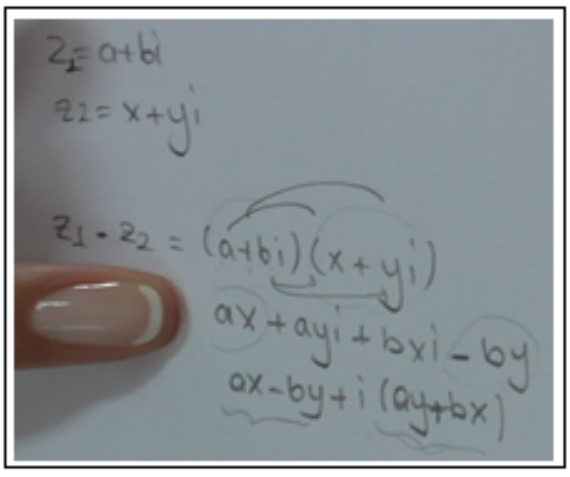


Figure 6.60. Derin's explaining the multiplication of z_1 and z_2 in Cartesian form.

R: Okay. About which point did you say you summed them up?

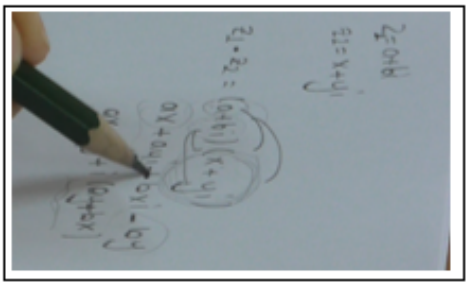


Figure 6.61. Derin's crossing the + sign between the expressions ax + ayi and bx + byi several times.

D: That one (see Figure 6.62: She points to the plus sign between the statements ax + ayi and bxi - by in the statement ax + ayi + bxi - by). I obtained a complex number here (see Figure 6.62: She points to the statement ax + ayi). I obtained also a complex number here (She points to the statement bxi - by). Summation of two complex numbers... Hmm, I expressed it like this (She points to the statement ax - by + i.(ay + bx)).

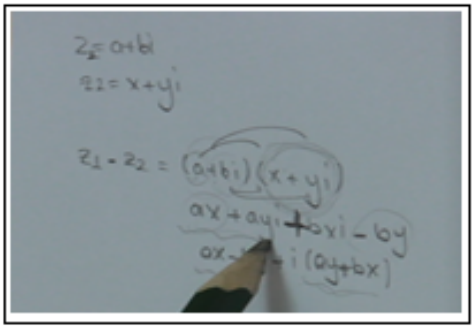


Figure 6.62. Derin's pointing to the complex number, ax + ayi.

Data indicated to several important points: First, while in the pre-interview, Derin's focus was on the product rather than the multiplicands, in the post-interview, she was able to focus on the multiplicands z_1 and z_2 rather then the poduct $z_1.z_2$. Secondly, in the pre-interview, she was not able to think about one of the complex numbers as a multiplier and the other as a multiplicand. Though, in the post interview, she was able to think z_1 as a multiplier and z_2 as a multiplicand. The reason was as follows:

Particularly, she reasoned on the complex number, z_1, as having two parts, real and imaginary. Then, she multiplied z_2 with each of them obtaining two complex numbers, a.(x + yi)and bi.(x + yi). Her reasoning on the algebraic expression that ax + ayi + bxi - by represented the addition of two vectors such that the vector addition was the "+" sign between the expressions "ax + ayi" and "bxi - by" showed that she was able to sum these two complex numbers to get the complex number $z_1.z_2$. In sum, compared to the pre-interview, in which Derin was not able to think of the real and imaginary parts of the resultant vector, ax - by + i.(ay + bx), in terms of the real and imaginary parts of z_1 and z_2, data from the post-interview showed that she was able to think of the real and imaginary parts of the resultant vector, ax - by + i.(ay + bx), in terms of the real and imaginary parts of z_1 and z_2.

Then the researcher asked Derin to talk about what she had done geometrically. Derin's explanation of the multiplication of the Cartesian form of complex numbers geometrically followed:

R: Could you explain this multiplication to us geometrically?

D: Okay. Hmm... To show by drawing figure? Or writing in terms of radius and theta... In polar form. But geometrically we...

R: What do you understand when I say geometrically? Tell me.

D: I don't... I understand drawing the figure and showing it upon that figure.

R: Okay. Please continue as you understand.

D: So, hmm... z_1 equals to a plus bi.z_2 equals to x plus yi. I show the complex number, for example, a, b in here (She draws a Complex Plane). In there (She specifies a point on the 1st quadrant) a to the b (She names the specified point as (a, b)). I showed it like this (She draws a vector starting from origin and ending in the point (a, b)). Hmm... x plus yi can be on any of the quadrants. There were cases. We analyzed

them. Hmm x... I analyzed the case x is bigger than zero and y is bigger than zero. If x is bigger than zero, y is smaller than zero. We got 4 cases like this (Figure 6.63: She writes x > 0, y > 0; x > 0, y < 0; x < 0, y < 0; x < 0, y > 0). Now what... Let me start with the 1st case (She points to the case x > 0, y > 0). If x is bigger than zero, y is bigger than zero, the complex number z_2 will be here (She draws a vector representing the complex number z_2 on the 1st quadrant). Hmm x to the y (She writes (x, y) to the end point of the vector that she drew). I will do something here. This one... I did by separating (She points to the multiplication in Cartesian form). In here I will separate z_2, multiplicate with the complex number a plus bi, then I will sum them up in vector form and I will obtain it. Or a to b in the same way. It doesn't matter. I can separate a to b and multiply with z_2. Or I can separate this one (She points to x + yi) and multiply with z_1. If I sum the results that I obtain, it is the same complex number. Since I am analyzing this case now (She refers to x > 0, y > 0), I will separate this one (She points to z_2 = x + yi). This (She refers to the vector z_2) will remain the same and I will rotate this one (She refers to the vector z_1).

In the pre-interview, Derin was not able to mention either z_1 or z_2 as the operand vector acting on the other and the resultant vector $z_1.z_2$ as the multiplication of two vectors, since her focus was on the product rather than the multiplicands. On the contrary, in the post-interview, she was able to accept either z_1 as the operand vector acting on z_2 or z_2 as the operand vector acting on z_1 and say that there was not any difference between the results of the two multiplications. Therefore, she was able to say that the operand vector would produce two new vectors by dilating and rotating the other vector such that the vector summation of these two newly formed vectors would give the resultant vector, $z_1.z_2$. That is, she was able to focus on the multiplicands rather than the product.

Derin then continued with her explanation of multiplication of complex numbers geometrically regarding all the cases of x and y. I depict data in two parts for each case of x and y: (i) her geometric explanation of multiplication of complex numbers in vector form referring to the angle of the vector $z_1.z_2$ and (ii) her geometric explanation of multiplication of complex numbers in polar form referring to the argument and the

magnitude of $z_1.z_2$. Data followed:

6.3.1.1. Derin's explaining $z_1.z_2$ when $x > 0$, $y > 0$. Derin's explanation of $z_1.z_2$ in vector form when $x > 0$, $y > 0$ followed:

D: Now Its (She points to the vector (a ,b) angle is theta one. We are trying to write it in polar form. We said it will be geometrically but... They're the same anyway. So... Since I am trying to write it in polar form, I need to find the angle of z_1 times z_2. Where is that? And its length. Then I will write it in polar form. So... I named the angle of z_1 as theta one (She names the angle between the vector (a, b) and the positive side of the real axis as θ_1. Then she writes angle of z_1, θ_1). Then, let's call the angle of z_2 as theta two (She writes angle of z_2, θ_2). I am seeking the angle which is the multiplication of these two. I don't know this one (She writes angle of $z_1.z_2$ and puts a question mark next to it). I will find this. I have the complex number z_2 there (She points to the point (x, y). I separated z_2 into its parts and I will find it part by part. If I multiplicate z_1 with x, it will be on this orientation because x is a real number and bigger than zero (She draws a longer vector than the vector z_1 by crossing on the vector z_1). Its direction will not change and I think its length increases. I guess it will be bigger than one. It moves here. Now I will multiplicate this one (She refers the vector z_1) with yi. So it will rotate just 90º, because i means rotating 90º and y is a positive real number. It moves here (She draws a vector on the 2nd quadrant).

R: What does move here?

D: Hmm, part of ay. Now this (She points to the point (a, b)) a to b is that complex number.(She writes (ax, bx) to the end point of the vector that she drew by crossing the vector z_1 on the 1st quadrant).

R: Well, why?

D: Because, If I think about this as a part (She points to the parts of the vector z_1 on the real and imaginary axes)... It (She points to the part of the vector z_1 on the the real axis) increases as much as x. And here (She points to the part of z_1 on the the imaginary axis), it makes y increase as much as x. After that I obtain this (She points to the point (ax, bx)) when I combine them. These become ay to by (She writes (ay, by) to the end point of the vector on the 2nd quadrant). Then I will sum them up in vector form. Now... No, it is... If that part, a (She points to the point x on the positive side of the real axis) rotates 90º, it will be here (She points to a point on the positive side of the imaginary axis). So it will be on this side. If the other one (She means the point b on the positive side of the imaginary axis) rotates 90º, it will move here (She points to a point on the negative side of the real axis). So it will be by to ay. Their signs... Both... A moment... Will it be negative to express positive since they are positive? (She replaces the point (by, ay) with (-by, -ay)). That (She points to by) is negative. This (She points to ay) is positive, since it is already positive (She replaces the point with (-by, ay)). It is negative by to ay. Okay. Now I will look at its angle. First, I will add them and find z_1 times z_2. It becomes like this (She moves the vector on the 2nd quadrant to the end point of the vector on the 1st quadrant). I moved this one (She points to the vector (-by, ay)). This is now the vector z_1 times z_2 (She draws a vector from the origin and to the end point of the vector (-by, ay)). I need to know this angle to reach the vector z_1 times z_2 (She draws the angle starting from the positive side of the real axis to the vector $z_1.z_2$). If I know this angle (She points to the angle between the vectors z_1 and $z_1.z_2$), sum it up with theta one, I can find the angle of z_1 times z_2 (see Figure 6.63 below).

As the data indicated, Derin was able to think of and show the resultant vector $z_1.z_2$ in terms of $z_1$and z_2. For that, she reasoned that $z_1 = a + bi$ would be the multiplicand and $z_2 = x + iy$ would be the multiplier acting on z_1. Also, she was able to explain and show the resultant vector $z_1.z_2$ as a point discussing also the effect of multiplying with yi such that she knew that multiplication with "i" would result in 90° rotation of the vector z_1. Though, first, she did have difficulty in finding

out how to represent x.(a + bi) and yi.(a + bi). Only after re-focusing in which quadrants the points would be placed, she was able to correctly determine that the resultant vectors would be (ax - by, bx + ay) pointwise. She even commented that she needed to determine the angle of $z_1.z_2$ in terms of the angle of z_1 and the angle between $z_1.z_2$ and z_1. Though, most importantly data showed that Derin stated that geometrically showing and explaning where $z_1.z_2$ was either as a point or in polar form corresponded to the same thing. Derin's explaining and flexibly transiting among all these forms simultaneously suggested that she had conceptualized the Cartesian form of the multliplication of complex numbers as a single entity such that, all those different representations, Cartesian form, vector form and polar form, showed the same number.

Derin's explanation of $z_1.z_2$ in polar form when $x > 0$, $y > 0$ followed:

R: Can you show us the point that you called the angle above?

D: This is the angle of z_1 times z_2. (She points to the angle of the vector z_1) I know. I need to find this (She points to the angle between the vectors z_1 and $z_1.z_2$). Let's call it β (She writes β). How can I find this β? I have z_2 here. From tangent β... I am looking at tangent β... Okay, since it is ninety (She points to the right angle between the vectors (ax, bx) and (-by, ay)). I moved this one (She points to the right angle) here (She points to the right angle between the vector (ax, bx) and the vector (-by, ay) that she moved to the 1st quadrant)... The angle isn't changed since it is right triangle.. Now the length of here (She points to the vector that she moved to the 1st quadrant)... Hmm... (She points to the vector (ax, bx)). If I find its length (She points to the vector (ax, bx)).. Root... The square of this (She points to ax). It was the square of ax plus square of this (She points to bx). How can I express it? Square of a plus square of b... And also square of x. I removed x. It will be absolute value. It is positive here. I mean a square plus b square is the length of z_1. The length of z_1 times the length of x... How can I express the length of this (She points to the vector that she moved to the 1st quadrant)? In the same way. Square root of a square plus b square times y. These are cancelled (see Figure 6.63: She cancels the lengths of the vector z_1). Absolute value of y over absolute value x... Since they are both positive, y

over x. Tangent is y over x. Now if I look at z_2... x plus yi. I can see the tangent of its angle is y over x. Hmm what was its angle? I named it as theta two. β equals theta two. No, wait a moment. I am thinking... A moment please. It means y over x here... Since theta two is acute angle here, theta equals to β. So here is tangent theta two y over x (She points to the number $z_2 = x + yi$). Since it is positive (She points to the statement $x > 0$, $y > 0$), theta two equals to β (see Figure 6.61 below). So I can express the angle of z_1 times z_2 here as theta one plus theta two.

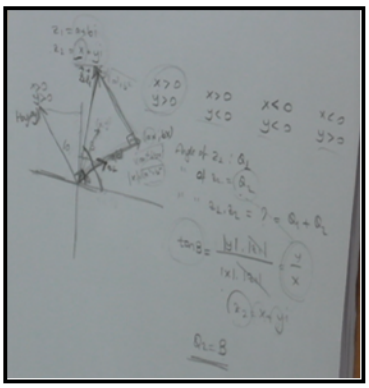


Figure 6.63. Derin calculates and explains the argument of $z_1.z_2$ in terms of the angles of the vectors z_1 and z_2 when $x > 0$, $y > 0$.

If I look at its length (She points to $z_1.z_2$ on the case of $x > 0$, $y > 0$) here... I found its length (She points to the vector $x.z_1$) and its length (She points to the vector $y.z_1$) already. I can write the length of z_1 times z_2 like this square of y times square of a plus square of b, plus square of x times square of a plus square of b (Figure 6.64: She writes $|z_1.z_2| = \sqrt{y^2.(a^2+b^2)+x^2.(a^2+b^2)}$. I found it by using Pythagorean Theorem. Since there is a right angle here (She points to the right angle on the 1st quadrant). I will find the length of that (She points to the vector $z_1.z_2$). That is square of x plus square of y, square of a plus square of b (She writes $\sqrt{x^2+y^2}.\sqrt{a^2+b^2}$).

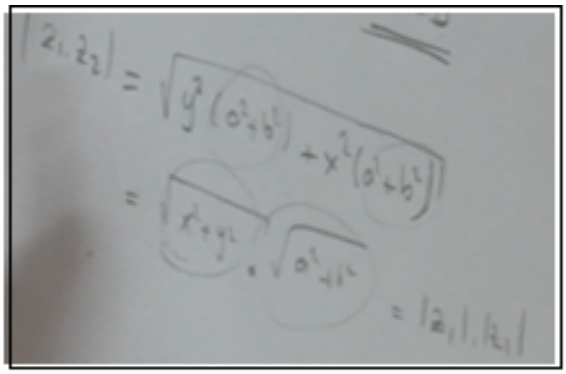


Figure 6.64. Derin calculates and explains the magnitude of $z_1.z_2$ in terms of the magnitudes of z_1 and z_2.

R: How did you come to that line? (The researcher asks how she pass from first line to second line on the statement $|z_1.z_2|$).

D: Hmm, I did... I took the common factor paranthesis. In the paranthesis of a square plus b square times y square plus x square. Then, I separated the roots, since they were multiplies. This is the length of z_2 (She points to $\sqrt{x^2+y^2}$) and this is the length of z_1(She points to$\sqrt{a^2+b^2}$)).

In the pre-interview, Derin was not able to explain the relationship between the polar and the vector form of the multiplication of complex numbers. In the post-interview, data showed that she was able to explain the relationship between the polar and the vector form of the multiplication of complex numbers. Particularly, she reasoned that multiplying z_1 with yi, z_1 was rotated 90°. Then, she reasoned that transition of the vector yi.(a + bi) to the end of the vector x.(a + bi) kept the angle 90° between these two vectors the same, creating a right triangle. This allowed her to determine the tangent value of the angle, β, between $z_1.z_2$ and z_1. Calculating it, she found it as $\frac{y}{x}$. She also knew that the tangent value of the argument of the vector z_2, θ_2, was equal to $\frac{y}{x}$ since it was in the 1^{st}quadrant. Knowing that both β and θ_2 were acute angles, she determined that since their tangent values were the same, the angles would be the same. This allowed her to conclude that the vector z_1 was rotated by the angle z_2 once multiplied with it. Data also showed that, contrary to the pre-interview

where she did not know why she multiplied the magnitudes of the vectors z_1 and z_2 in polar form, Derin was able to justfiy the reasoning behind it: She reasoned that since she added the two vectors $x.z_1$ and $yi.z_1$, this would mean that she had to determine the magnitudes of these vectors and add them using the Pythagorean Theorem. Doing the algebraic manipulations, she was able to obtain that adding these vectors yielded to the multiplication of the magnitudes of the vectors z_1 and z_2. Thus, the whole data showed that, when x and y were bigger than zero, Derin was able to not only explain the reasoning behind the polar formula for the mutliplication of complex numbers but also was able to relate the algebraic form, the vector form, the point representation and the polar form.

Once asked Derin was also able to explain why she needed to examine the argument of $z_1.z_2$:

R: Okay, okay. Alright. I will ask you a question. Hmm... I will ask you first... You said something here. You said theta one to the angle of z_1 and theta two to the angle of z_2. You said you had to look at the angle of z_1 times z_2. Why do you look at the angle? Why did you feel obliged to look at its angle?

D: Since I need to express it in polar form. Expressing in polar form means looking at the angle and looking at its length. So I found its angle. I also found its length, magnitude. I will express by using them now.

R: What if you don't express in polar form, do you feel obliged to find theta one and theta two? Why do you need to find the angle?

D: I thought that expressing geometrically means writing in polar form. Because why do we look at polar form? Geometrically... Expressing geometrically... What if I do something here. That side was z_1 times z_2 here (She draws a $z_1.z_2$ vector on the Complex Plane. Its angle was theta one plus theta two. Its length is z_1 times z_2. To find the value of the newly formed complex number. If there is z_1 here and for example there is z_2 there (Referring to Figure 6.65 above: She points to the vectors z_1 and z_2

that she drew for the cases x > 0, y > 0). When I do the multiplication, this one (She points to the complex number z_1) effects that one (She points to the complex number z_2), it will rotate and to find the new angle formed there, when this (She points to the complex number z_1) multiplicated with this (She points to the complex number z_2), one will increase the length of other one. I am looking at its magnitude to find the length of newly formed vector. It will also have a magnitude, a starting point and other things. To find the exact direction and magnitude.

Again, data pointed to the following: First, Derin knew that the reason why she examined the argument of $z_1.z_2$ was to explain the resultant vector in polar form. That is, she reasoned that the result of the multliplication of z_1 and z_2 was also a vector with magnitude and direction. Therefore, in order to explain and show the effect of z_2 on z_1 she needed to explain the argument of $z_1.z_2$ in terms of z_1 and z_2. This again suggested that, contrary to the pre-interview, in which Derin neither was able to explain the reasoning behind the polar form nor was she able to explain the relationship between the vector and the polar forms, in the post-interview she was able to reason on both of them.

Researcher also asked Derin to reason about the other cases:

6.3.1.2. Derin's explaining $z_1.z_2$ when $x > 0$, $y < 0$. Derin's explanation of $z_1.z_2$ in vector form when $x > 0$, $y < 0$ followed:

R: Let's continue with other cases.

D: (She writes the case 0 < x, y < 0). x is bigger than zero, y is smaller than zero. Okay. I assume that the complex number z_1, a plus bi, is on the 1st quadrant. It doesn't matter anyway. Hmm, what do I do when x is bigger than zero and y is smaller than zero? I keep z_1. I separate z_2 into its real and imaginary parts. After that I look at its effect on z_1 and sum it up in vector form, I obtain the complex number z_1 times z_2, or the vector. Now I know that x is a real number. And it is also bigger than

zero. I increased its length because I thought it is also bigger than one. Direction is not changed (She draws a vector longer than z_1 and on the same direction with the vector z_1 on the 1st quadrant). They were on the same direction. I show it like this because it is big (She refers to x is bigger than 1). Than it is... I can express it in this way (She writes $|x|.|z_1|$ to the end point of the vector she drew recently). Or directly x (She replaces $|x|$ with x). The length of the newly formed vector. I will express it like this (She replaces the statement at the end point of the vector with $x.\vec{z}_1$.

R: Okay. What if you express it pointwise?

D: Like before?

R: Okay, okay. Sure

D: ax and bx (She writes (ax, bx) to the end point of the vector). If y is smaller than zero... Hmm.. z_2 is x plus iy. If I multiply it with both negative and i, it rotates two hundred and seventy degrees. Hmm, one hundred and eighty (She points to the extension of the vector $x.\vec{z}_1$ on the 3rd quadrant) multiplication with negative one. It rotates also 90º. It moves here (She points to the vector on the 4th quadrant). Since it rotates total two hundred and seventy degrees, here is ninety (She points to the angle between the vector $x.\vec{z}_1$ and the vector on the 4th quadrant). Here is the vector of y times z_1 (She writes $y.\vec{z}_1$ to the end point of the vector on the 4th quadrant). If I express it pointwise, this part, because I multiply with y. So it wil be zero to ay (She writes (0, ay) to the point on the negative side of the imaginary axis). It is already negative, since y is negative. I am looking here. Here I multiply b with yi. It is by. y is negative but it is in the positive, it is negative by (She writes (-by, 0) to the point on the positive side of x-axis). So it is negative by to ay (She writes (-by, ay) to the end point of the vector that she drew on the 4th quadrant). I will find the vector $z_1.z_2$ by using these. I will sum them up in vector form. If I move that one (She moves the vector $x.\vec{z}_1$ to the end point of the vector $y.\vec{z}_1$... z_1 times z_2 becomes something like this (She draws a vector starting from the origin and ending at the end point of the vector $x.\vec{z}_1$ that she moved to the 4th quadrant).

R: You said "that one". What did you move?

D: x times z_1.

R: Okay, okay. Where did you move it?

D: I moved it here (She points to the vector that she moved to the end point of the vector $y.\vec{z}_1$ on the 4th quadrant). It is already ninety (She points to the angle between the vectors $x.\vec{z}_1$ and $y.\vec{z}_1$ on the 4th quadrant as ninety). To find the angle of here (She points to the vector $z_1.z_2$), we called the angle of z_1 as theta one (She writes "Angle of $z_1 = \theta_1$"). The angle of z_2 is theta two (She writes "Angle of $z_2 = \theta_2$"). I don't know what z_1 times z_2 is. I will look for it now. Now, here was theta one (She points to the angle of the vector z_1). How can I express this (She points to the angle of the vector $y.\vec{z}_1$)? Two hundred and seventy plus theta one. Plus... And if I know here (She points to the angle between $y.\vec{z}_1$ and $z.\vec{z}_1.z.\vec{z}_2$)... I can find the angle of z_1 times z_2. I name here as m (She writes m to the angle between $y.\vec{z}_1$ and $z.\vec{z}_1.z.\vec{z}_2$). I will get help from the equation. First I will find tangent m. The length of this (She points to the vector $x.\vec{z}_1$ that she moved to the 4th quadrant) times x times z_1. The length of that one (She points to the vector $y.\vec{z}_1$) is y times z_1. Since y is negative and x is positive, it is x over y. How did I express this? Theta one plus two hundred and seventy degrees plus m. I will express this in terms of theta one and theta two. So I need to find the m in terms of theta two in a way (see Figure 6.63 below).

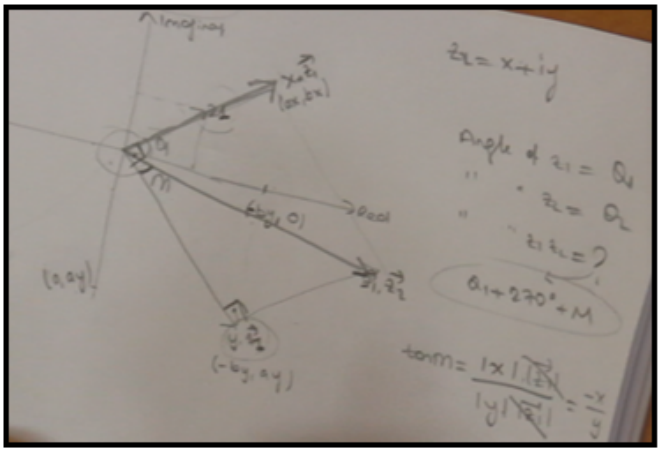


Figure 6.65. Derin's calculating the angle of the vector $z_1.z_2$.

As the data showed, Derin was able to show $z_1.z_2$ in terms of the vectors z_1 and z_2. That is, leaving z_1 in the 1^{st}quadrant and thinking of z_2 in real and imaginary parts, she explained the effect of z_2 on z_1. She particularly reasoned that if she assumed that the value of x was bigger than 1, then $x.z_1$ would be longer than z_1. Similarly, assuming that y was smaller than -1 and also taking into consideration of the effect of "i", she knew that the vector $yi.z_1$ would be 270° rotated form of z_1. Therefore, she was able to determine where $z_1.z_2$ would be placed in the 4th quadrant; specifically, the pointwise representation would be (ax - by, bx + ay). Also, she translated the vector $x.z_1$ to the end of the vector $yi.z_1$. Therefore, she also was able to determine the argument of $z_1.z_2$ as the total of θ_1 and 270° and m, where m represented the angle between $z_1.z_2$ and $yi.z_1$. She also knew that tan value of this as $-\frac{x}{y}$.

When the researcher asked why she needed to write the argument of $z_1.z_2$ in terms of the arguments of z_1 and z_2, Derin was also able to explain too:

Derin's explanation of $z_1.z_2$ in polar form when $x > 0$, $y < 0$ followed:

R: Why do you need to express it in terms of theta one and theta two?

D: Because I am looking for z_1 times z_2. I mean we are writing the vector z_1 times z_2 in terms of z_1 and z_2. Because we are trying to generalize it in that way.

R: Why are you writing in terms of z_1 and z_2?

D: Because I am trying to express in terms of complex numbers z_1 and z_2. I am trying to find a generalization. So I am using its parts.z_1 and z_2. At the end, It doesn't matter which one I use but z_2 increases the lengths of the parts of z_1 and it also changes its direction. Then I obtain z_1 times z_2 when I combine them. z_2's changing the direction and magnitude of z_1. Its effect on the z_1. Because of that we write in terms of z_1. Because it affects z_1. It changes. To look for how much it changes, what does it do...

Data showed that Derin reasoned that either of z_2 or z_1 acted on the other one in two ways: changing the magnitude and also the direction of the multiplicand. This showed that she explained the reason behind writing both the argument and the magnitude of the vector $z_1.z_2$ in terms of the arguments and the magnitudes of the vectors z_1 and z_2.

Derin then continued:

D: I name the angle of z_2 as theta two. I will look for its place. If x is bigger than zero, y is smaller than zero it is on the 4th quadrant. I may be there (She points to a point on the 4th quadrant). Like this (She draws a vector from the origin to the point that she specified for z_2 on the 4th quadrant)... Here is the place of z_2. So how can I express theta two? How can I express it as far as I know? Hmm...It starts from here (see Figure 6.66 below: She points to the positive side of the real axis) that wide angle (She points to the wide angle which extends to the vector z_2).

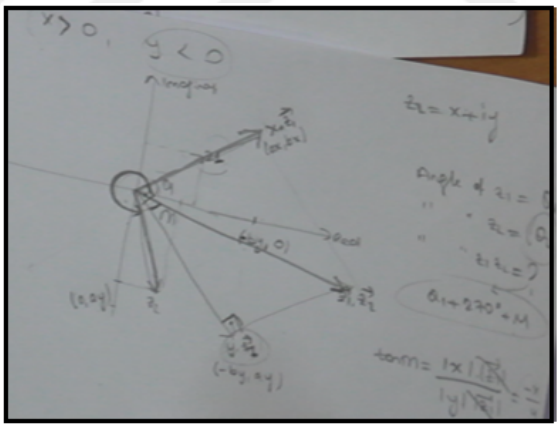


Figure 6.66. Derin's calculating the angle of the vector $z_1.z_2$ based on the angles of the vectors z_1, $yi.z_1$, $z_1.z_2$.

How can I express theta two as far as I know? Theta two is two hundred and seventy degrees plus that angle (see Figure 6.67: She points to the angle between the vector z_2 and the negative side of imaginary axis). And if I name this angle as T. I can express it as two hundred and seventy plus T. Now I can find tangent T. Hmm, tangent T... Since here is z_2, that part is x (She points to the part of the number z_2 on

real axis), here is y (She points to the part of the number z_2 on the imaginary axis). The length of here is absolute value of x (She writes $|x|$ to the length of the part of the number z_2 on the real axis), the length of here is absolute value of y (She writes $|y|$ to the length of the part of the number z_2 on the imaginary axis). The tangent here is x over y... Minus x over y. I was opposite over adjacent...

R: Why negative x over y?

D: Because y is negative. x is positive. Negative x over y. Then tangent m and tangent T will be the same. So m equals T.

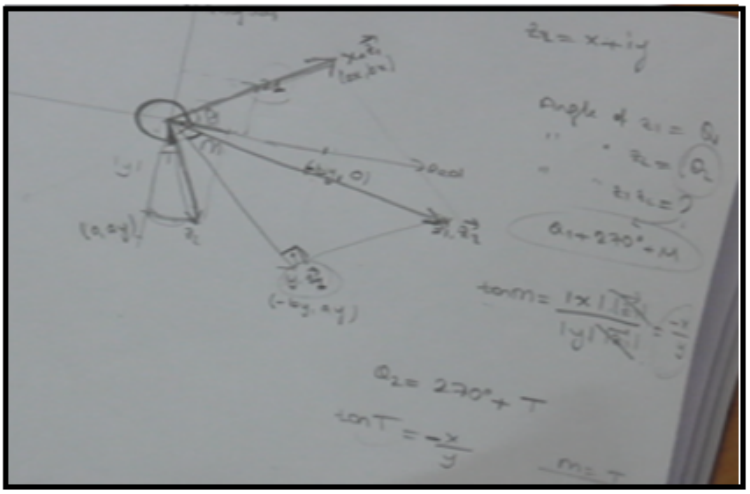


Figure 6.67. Derin's calculating the angle m in terms of the angle of the vector z_2.

If m equals T, here (She points to the statement θ_1 + 270° + m) I can write instead of m... I can replace m with T in here (She points to the statement θ_2 = 270° + T). m is theta two minus two hundred and seventy degrees (She writes θ_2 - 270° = m). Here I will write it (She points to the statement θ_2 - 270°) instead of m. It will be theta one plus two hundred and seventy plus theta two minus two hundred and seventy degrees. So the angle of z_1 times z_2 will be theta one plus theta two (see Figure 6.68).

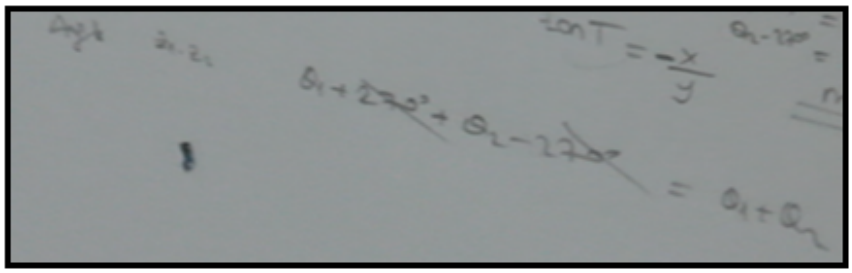


Figure 6.68. Derin's calculating the angle of the vector $z_1.z_2$ in terms of the angle of the vector z_1 and the angle of the vector z_2.

R: I will ask you something. You said negative x over y, negative x over y, when tangent m and tangent T is the same... You say you are equalizing m and t, don't you?

D: Yes. Because I assume it as acute angle here. So it has to be equal. The same.

R: You say. Alright.

D: I am sure it is an acute angle so I can say that.

R: Do you have anything to add?

D: No. If I look at the length of this, I know its 90º (She points to the angle of 90º between the vectors $y.\vec{z}_1$ and $x.\vec{z}_1$ on the 4th quadrant). Here is z_1 times z_2 (She points to the vector $z_1.z_2$). The length of here is the same (She points to the vector $x.\vec{z}_1$ x times z_1 (She writes $|x|.|z_1|$ above the vector). The length of this is y times z_1 (She writes $|y|.|z_1|$ above the vector). Now I will use Pythagorean Theorem again because it is right triangle. Like here (She points to the case $x > 0$, $y > 0$). Since the vectors are perpendicular. Since we take the absolute values of x, y and z_1, it becomes the same. So the length will be z_1 times z_2 like in the 1st case.

R: Alright. Do you have anything to say?

D: No.

Data showed the following: First, Derin was able to determine the argument of z_2 as the sum of the angles of 270° and the angle between the vector z_2 and the negative side of the imaginary axis naming it as T. Then, calculating the tangent value of it as $-\frac{x}{y}$ by pointing that the value of x was positive and the value of y was negative, she was able to determine that the tangent values of the angles of T and m were the same. Therefore, she knew that since those angles were acute angles, due to the equality of the tangent values, the angles had to be equal. This then allowed her to substitue T for m in the expression θ_1 + 270° + m where T was equal to θ_2 - 270°. Therefore, she knew that the argument of the $z_1.z_2$ had to be equal to the sum of the arguments of z_1 and z_2. She also was able to comment on the magnitude of $z_1.z_2$ in terms of $x.z_1$ and $yi.z_1$: She reasoned that since those vectors were perpendicular to each other, using Pythagorean Theorem, the magnitude of the sum of those vectors would yield to the multiplication of the magnitudes of z_1 and z_2.

Once even asked to comment on the relationship between the Cartesian form and vector form of $z_1.z_2$, she reasoned:

R: Alright. I will ask you this. At the beginning you did multiplications in Cartesian form. What can you say about them comparing the magnitude of this, the multiplication in Cartesian form and the magnitude that you find recently? (see Figure 6.69).

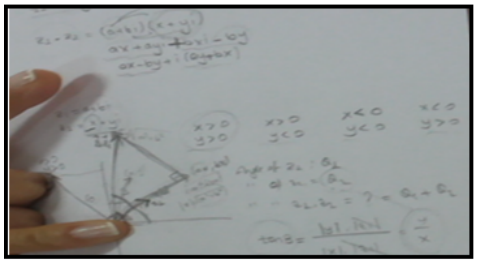


Figure 6.69. The researcher's pointing to the multiplication of z_1 and z_2 in Cartesian form and in vector form.

D: It is the same. How did we obtain the length of this (She points to the statement ax - by + i.(ay + bx)). A complex number, let's say it is here (She draws a Complex Plane and draws a vector on the 1st quadrant). How can I find it? This part (She points to the the part of the vector on the reel axis) is ax minus by to the zero (She writes (ax - by, 0)). And this part (She points to the part of the vector on the imaginary axis) is zero to ay plus by (She writes (0, ay + by)). Now this vector is this (She draws the vector (ax - by, 0)). This (She points to the vector (0, ay + by)). This (She moves the vector (0, ay + by) to the end point of the vector (ax - by, 0)). I will take the square of this (She points to the vector (ax - by, 0)). It is the same again, ninety (She marks the angle between the vectors (ax - by, 0) and (0, ay + bx) as 90º). Right triangle. I will find the square root of the square of "ax minus by" plus the square of "ay plus bx" (see Figure 6.70: She writes ($\sqrt{(ax-by)^2+(ay+bx)^2}$) (She calculates exponentials in square roots by using identities). a^2x^2 minus 2axby plus b^2y^2 plus a^2y^2 plus 2aybx plus b^2x^2 (see Figure 6.70). Now, if I take here the paranthesis of a^2, it will be x^2 plus y^2. If I take here the paranthesis of b^2, it will be x^2 plus y^2 (She writes $\sqrt{a^2(x^2+y^2)+b^2(x^2+y^2)}$). Eer.. It becomes a^2 plus b^2. It becomes x^2 plus y^2 (She writes $\sqrt{a^2+b^2}.\sqrt{x^2+y^2}$). This is z_1 (She points to $\sqrt{a^2+b^2}$) and this is z_2 (She points to $\sqrt{x^2+y^2}$) it is the same.

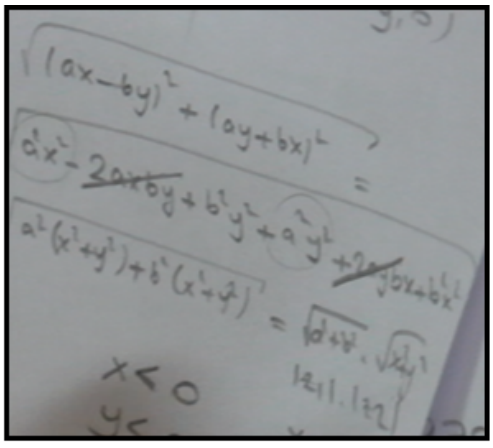


Figure 6.70. Derin's showing the magnitude of the Cartesian form of $z_1.z_2$ and the magnitude of the polar form of $z_1.z_2$ are the same.

In the pre-interview, Derin was not able to think of the real and imaginary parts of the resultant vector, ax - by + i.(ay + bx), in terms of the real and imaginary parts

of z_1 and z_2. However, data from the post-interview showed that Derin was able to think of the real and imaginary parts of the resultant vector, ax - by + i.(ay + bx), in terms of the real and imaginary parts of z_1 and z_2. Therefore, data showed that she knew that the magnitude of the Cartesian form of $z_1.z_2$ and the magnitude of the polar form of $z_1.z_2$ were the same.

The researcher even asked the following:

R: I want to ask you a question. You always look for tangents. Why do you always look for tangents?

D: Because writing in terms of tangent is the easiest. I need to express that angle in a way. Like you know, sine, cosine, tangent, cotangent... I can express in a way. It doesn't matter. But here tangent is easier.

R: Okay, okay. You said "here". Where?

D: For example the tangent of z_2... How do I express the angle of it? I call it tangent theta two. I call it y over x. I can express it as cotangent. I mean the important thing is expressing the angle. Sine, cosine, tangent, cotangent, it doesn't matter. I do it like that because it is easier.

Data was interesting such that Derin knew that she could have used any of the trigonometric values to determine and explain the reasoning behind the argument of the vector $z_1.z_2$ in terms of the arguments of z_1 and z_2. This pointed to the flexibilty in her reasoning again.

The researcher then asked Derin to continue with the other cases:

6.3.1.3. Derin's explaining $z_1 .z_2$ when $x < 0$, $y < 0$. Derin's explanation of $z_1.z_2$ in vector form when $x < 0$, $y < 0$ followed:

R: Well, shall we continue?

D: Okay, okay. Should I continue with that one (She points to the case $x < 0$, $y < 0$)? Alright. Let the complex number, vector z_1 be here (She points to the complex number z_1 as vector on the 1st quadrant). Hmm, now if I multiply this (She points to the complex number z_1) with x, since x is a negative number, it will rotate one hundred and eighty degrees counterclockwise. So it will move here (She draws the extension vector of the vector $\vec{z}_1$ on the 3rd quadrant). This is the vector x times z_1. (She writes $x.\vec{z}_1$ to the end point of the vector on the 3rd quadrant). So multiplying with y... Since it (She refers to y) is negative and I also multiply with i, it (She points to the vector z_1) will rotate two hundred and seventy degrees. It moves here (She points to the vector on the 4th quadrant). I also increased its length. It is here (She writes $y.\vec{z}_1$ to the end point of the vector). Since this (She points to $x.\vec{z}_1$) rotates one hundred and eighty and since this (She points to $y.\vec{z}_1$) rotates two hundred and seventy degrees, that one is ninety (She specifies the angle between $x.\vec{z}_1$ and $y.\vec{z}_1$ as ninety). Now I will find where the complex number z_1 times z_2 is. I will move one. It doesn't matter which one. It becomes something like this (She moves $z.\vec{z}_1$ to the end point of $y.\vec{z}_1$). It moves here (She draws the vector $z_1.z_2$ starting from the origin to the ending point of $x.\vec{z}_1$ which she moved to the 3rd quadrant).

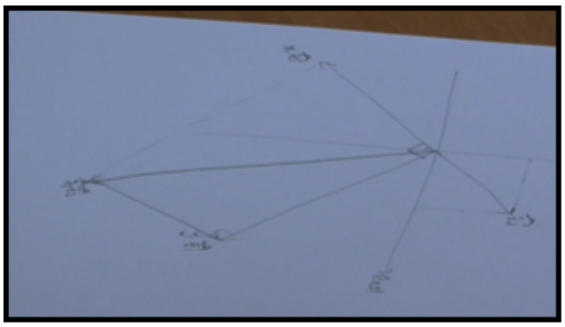


Figure 6.71. Derin's showing the multiplication of z_1 and z_2 in vector form and calculating the vector $z_1.z_2$.

As the data showed Derin was able to explain that once multiplied with x assuming that it was smaller than -1, the resultant vector $x.z_1$ was in the 3rd quadrant since z_1 was rotated 180°. Derin also claimed that its magnitude increased. Similarly,

she reasoned that since the value of y was also smaller than zero the resultant vector would be rotated 180°. Then also multiplying with "i" the final position of the vector yi.z_1 would in the 4th quadrant. Translating x.z_1 to the end of yi.z_1, Derin was able to determine the final position of $z_1.z_2$ in the 4th quadrant.

Then she continued with the argument of $z_1.z_2$: As the data will show again, for Derin the hardest case to reason about was $x < 0$, $y < 0$.

Derin's explanation of $z_1.z_2$ in polar form when $x < 0$, $y < 0$ followed:

D: I moved x times z_1 to here (She points to the ending point of the vector $y.z_1$). That one is ninety (She marks the angle between the vectors $x.z_1$ and $y.z_1$ as 90°). I obtain the complex number $z_1.z_2$. I will name the angle of this (She points to the complex number z_1) as theta one. The angle of z_2 is theta two. Now I wonder that... I need to find it. I will look for the angle of z_1 times z_2. Normally, looking from this side (She points to the negative side of the imaginary axis), how can I express the angle of here (She points to the vector $z_1.z_2$) as far as I know? Hmm, theta one plus two hundred and seventy, I name here (She points to the angle between the vectors $z_1.z_2$ and $yi.z_1$) as T. It is that one (She points to the angle of the vector $yi.z_1$) minus T. So If find T and substract it, I find the angle.

R: Well, you know. However you want.

D: So I need to find tangent T first. Tangent T, the length of here (She points to the vector $x.z_1$ that she moved to the 3rd quadrant) is x times z_1. The length of that one (She points to the vector $y.z_1$) is y times z_1 (She writes tan $T = \frac{|x|.|z_1|}{|y|.|z_1|}$). Since they (She refers to x and y) are both negative, it is positive. Tangent T is x over y (She writes tan $T = \frac{|x|.|z_1|}{|y|.|z_1|} = \frac{x}{y}$) How can I reach that x over y? Hmm... I should look where the theta two is. Since both are negative, it is on the 3rd quadrant.z_2 is somewhere there (She draws a vector on the 3rd quadrant). I will represent z_2 like one hundred and eighty plus P (see Figure 6.72: She writes θ_2=180°+P). I named that angle (She points to the angle between the vector z_2 and the negative side of the real axis) as P.

Now, tangent P is y over x but I need x over y. Then I am looking here (She points to the explementary angle of the angle P in the right angle on the 3rd quadrant). Here is R. I know that tangent R is negative x over negative y, x over y.

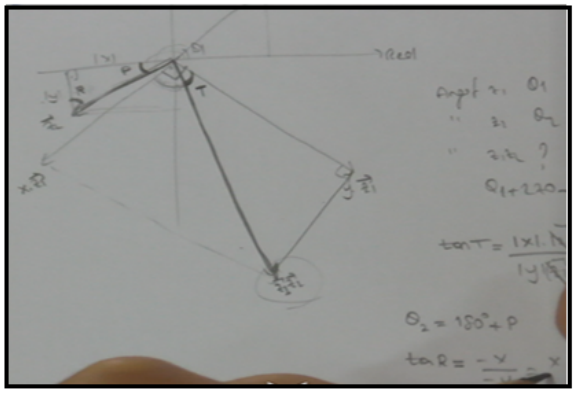


Figure 6.72. Derin's calculating the angle P based on the angle of the vector z_2.

Then R equals T. How can I express R? I can express it as ninety minus P (see Figure 6.73). I found that R and T are equals. From tangent. I understand that T and R are equals since they're acute angles. How can I express R? I can express it as ninety minus P. Okay, alright. What do I need here? I can express P as theta two minus one hundred and eighty (She writes P = θ_2 - 180º).

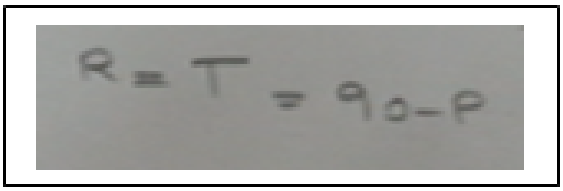


Figure 6.73. Derin's writing R = T = 90 - P.

If ninety minus theta two minus one hundred and eighty equals to T (She replaces P with θ_2 - 180º in the equation T = 90 - P and finds T = 90º - θ_2 + 180º). Then theta two is two hundred and seventy minus T (She writes θ_2 = 270º - T). So question mark equals theta one plus two hundred and seventy minues two hundred and seventy minus theta two. Then it is theta one plus two hundred and seventy minus two hundred and seventy plus theta two (She writes ? = θ_1 + 270º - (270º - θ_2) = θ_1 + 270º - 270º + θ_2 = θ_1 + θ_2) see Figure 6.74.

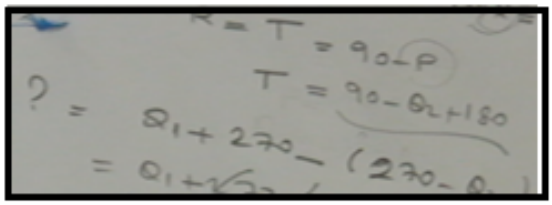


Figure 6.74. Derin's calculating the angle of the vector $z_1.z_2$ based on the angles of the vector z_1 and the vector z_2.

R: Alright. Could you show me where is the theta one plus theta two exactly?

D: Theta one plus theta two is the question mark. What was the question mark? It was the angle of the multiplication of z_1 and z_2 (She draws the angle starting from the positive side of the real axis to the vector $z_1.z_2$).

R: Alright. By expressing this (The researcher refers to the angle of the vector $z_1.z_2$) using the figure that you show us, can you show me where theta one and theta two exactly is? Where is theta one? Where is theta two?

D: Here is theta one (She draws the angle starting from the positive side of the real axis to the vector z_1). There is theta two (She draws the angle starting from the positive side of the real axis to the vector z_2).

R: Is there anything you want to say?

D: No.

R: Well. How can you describe its magnitude?

D: $z_1.z_2$ is here. I separate this again. I know that here is $y.z_1$ (She points to the vector $yi.z_1$ on the 4th quadrant). I know that there is $x.z_1$ (She points to the vector $x.z_1$ that she moved to the 4th quadrant) (She points to the right angle between the vector $yi.z_1$ and the vector $x.z_1$ which she moved to the 4th quadrant) ninety degrees. Since it

is ninety degrees between this (She points to the vector x.z_1 on the 4th quadrant) and this (She points to the vector yi.z_1 which she moved to the 4th quadrant), I can use Pythagorean Theorem and find it. It is the same. It is the same as here (She points to the case $x > 0$, $y < 0$).

Data showed the following: First, Derin determined the argument of $z_1.z_2$ thinking of the argument of z_1, the argument of yi.z_1 and the angle between $z_1.z_2$ and yi z_1. Therefore, she was able to state that the argument of $z_1.z_2$ would be equal to the sum of θ_1 and 270° from which T was subtracted. (i.e., θ_1 + 270° - T). Then she was able to calculate the tangent value of T as $\frac{x}{y}$, since x and y were both smaller than zero. Then, she knew that since x and y was smaller than zero, z_2 would be positioned in the 3rd quadrant with the argument, $\theta_2 = 180^o$ + P, where P was the angle between z_2 and the negative side of the real axis. She also was able to calculate the tangent value of P as $\frac{y}{x}$. Knowing that she needed to interpret T in terms of θ_2 so that she could represent the argument of $z_1.z_2$ in terms of the arguments of z_1 and z_2 and realizing that the tangent values of T and P were not equal, she reasoned to use the complementary angle, R, to interpret P. Therefore, she was able to write T = R = 90° - P, which also reasoning that the tangent values of T and R would be the same since they were both acute angles. She was able to substitue θ_2 - 180° for P in the equation T = 90° - P. Therefore, she concluded that $\theta_2 = 270^o$-T. Then, again substituting this for T in the equation θ_1 + 270° - T, she found out that the argument of $z_1.z_2$ was equal to the sum of the arguments of z_1 and z_2 (i.e., $\theta_1 + \theta_2$). She then also commented that when she added the vectors x.z_1 and yi.z_1 using the Pythagorean Theorem, she would have found the same magnitude for $z_1.z_2$. She even stated that the result would be the same as in the case x > 0, y < 0. All these data showed that Derin again was not only able to show where $z_1.z_2$ would be positioned, but also was able to explain the reasoning behind the polar form by pointing to the effect of z_2 on z_1.

6.3.2. Derin's Explanation of the Multiplication of the Complex Numbers in All Forms

Finally when Derin was asked to comment further on the polar form of the multliplication of the complex numbers, she commented:

R: Well. I will ask you this. How do you explain the multiplication of the complex numbers in polar form when you think about it as a whole? Use the same complex numbers again.

D: Hmm... z_1 times z_2 equals to the absolute value of z_1 times the absolute value of z_2 times cosine theta one plus theta two plus sine theta one plus theta two (She writes $z_1.z_2 = |z_1|.|z_2|.\ (\cos\ (\theta_1 + \theta_2) + i.\sin\ (\theta_1 + \theta_2))$.

R: Could you explain this part to me again? (The researcher refers to the $z_1.z_2 = |z_1|.|z_2|.\ (\cos\ (\theta_1 + \theta_2) + i.\sin\ (\theta_1 + \theta_2))$. Based on vectors. You wrote that before but...

D: Alright. My complex number is here (She draws the Complex Plane and marks a point on the 1st quadrant). This is absolute value of z_1 times absolute value of z_2 times cosine theta one plus theta two plus sine theta one plus theta two (see Figure 6.75: She writes ($|z_1|.|z_2|$). ($\cos\ (\theta_1 + \theta_2)$), $i.\sin\ (\theta_1 + \theta_2)$ next to the marked point). If I separate this into its parts (She points to the vector starting from the origin and ending in $z_1.z_2$)... (She separates the vector into its parts in real and imaginary axes. She writes ($|z_1|.|z_2|.\ (\cos\ (\theta_1 + \theta_2)$, 0) to the end point of the vector part on the real axis. She writes (0, $|z_1|.|z_2|.\ (\sin\ (\theta_1 + \theta_2)$ to the end point of the vector part on the imaginary axis. She points to the angle between the vector and the positive side of the real axis as $\theta_1 + \theta_2$. She writes $|z_1|.|z_2|$ above the vector to represent its magnitude).

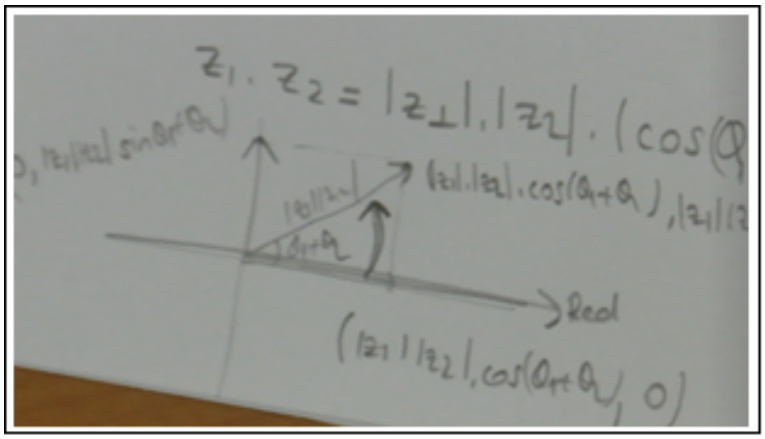


Figure 6.75. Derin's explaining $z_1.z_2 = |z_1|.|z_2|.\ \cos(\theta_1 + \theta_2) + i.\sin(\theta_1 + \theta_2)$ based on vectors.

As the data showed Derin was able to write the polar form of $z_1.z_2$ by pointing to the real and imaginary parts as a point. Also, once asked to comment on all different representations, algebraic, vector and polar forms, she stated:

R: That thing, the multiplication of the number z_1 times z_2 in Cartesian form, in vector and polar form, geometric representation... Could you talk about the relation between the multiplication of two complex numbers in Cartesian form, in vector form and in polar form... How are they relevant to each other? What does polar representation mean? What does algebraic representation mean in the multiplication of two complex numbers? What does vector representation mean? How are they relevant to each other?

D: Hmm... In Cartesian form, in vector form and... Now how do I find the number z_1 times z_2 in Cartesian form? I separate one into its real and imaginary parts, explain how it affects the other and sum them up. In the same way, I do the same in the vector form. When z_1 multiplied with z_2, what is the effect of z_2 on z_1? To find it I separate z_2 into its parts, to find how its real and imaginary parts affect z_2... I find that parts, the parts of the vector and I combine them. The complex number I find is the complex number z_1 times z_2. Expressing in polar form... It is already expressing it with its angle and magnitude. I found the angle and magnitude on the geometric

figure that I drew so I can write in polar form. I mean like this.

R: Alright. I will ask you this again. You said you had two complex numbers, didn't you (The researcher refers to the complex numbers a + bi and x + yi in the $z_1.z_2$ = (a + bi).(x + yi))? You said you separated them into its parts in vector form. Form example in the 1st one you separated a plus bi. And x plus yi here. I will ask you two things. First, you said it doesn't matter which one I separate, a + bi or x + yi.

D: Yes.

R: Why it doesn't matter?

D: Hmm, if I separate this one (She refers to a + bi) and multiply a with x plus yi, I obtain two parts like ax plus aiy. If I multiply from this side (see Figure 6.76: She multiplies from the right hand side) I obtain ax plus aiy. I obtain the same parts again.

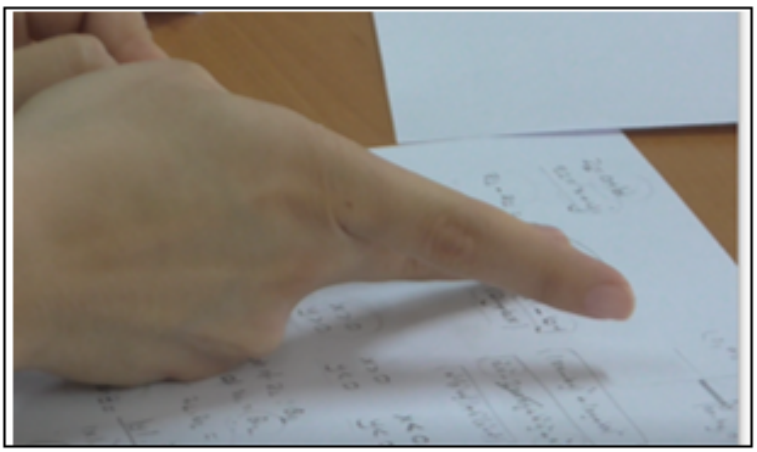


Figure 6.76. Derin's showing the results of the multiplication of z_1 and z_2 from the right hand side and multiplies from the left hand side are the same.

R: What do you mean with the same parts?

D: I mean that ax, aiy, bx, negative by. When I multiply this (She refers to a) with this (She refers to the number x + yi), I obtain ax, aiy. If I multiply from this

side (She refers to the multiplication from the left hand side). Then I continue and obtain bix, negative by. If I multiply from this side (She refers to the multiplying from right hand side) it is ax plus aiy, bxi minus by again. So I obtained four parts. Those four parts are similar to each other. They are similar because of the multiplication. So it doesn't matter which two I add in vector form. In the end I obtain the same thing when I sum all four up. When I sum the four parts, I obtain the same part again.

As the data showed, Derin was able to simultaneoulsy think about all different forms. Particularly, she reasoned that when multiplied z_1 with z_2 algebraically, she would have distributed the real and imaginary parts of either z_1 or z_2 on to the other. Further reasoning that since either of them included real and imaginary parts, the order of multiplication would not matter. Similarly, she knew that once she multiplied the real and the imaginary parts with the other vector, the result of the multiplication would again yield to vectors whom she would add to find the resultant vector. In addition, since the resultant vector had direction and also the magnitude, the polar form was the represention of the vector's direction and the magnitude. Therefore, contrary to the pre-interview, Derin was able to explain the relationship among the algebraic, vector and polar forms.

R: Okay, okay. Here you explained it in Cartesian form. Could you explain it in vector form? Thinking based on vector?

D: It doesn't matter in vector form. When you multiply this (She refers to a) with this (She referst to x) you obtain the vector ax. When you multiply this (She refers to a) with this (She refers to yi) you obtain the vector ayi... When you multiply this (She refers to bi) with this (She refers to x) you obtain bxi... When you multiply this (She refers to bi) with this (She refers to yi) you obtain negative by and I obtain the same vector when I sum them up on the Complex Plane.

R: Alright. I will tell you this. You said you obtain the vector ax by multiplying a with x. I got it. I didn't get that wrong, did I?

D: Yes.

R: Then you obtain the vector ayi by multiplying a with yi. Is each one of them a vector?

D: This is the vector ay. i just indicates that it is on the imaginary axis. They are all vectors. ax, ay, bx, negative by.

R: Well, when summing them up, which ones of them you sum up with each other and '-why?'

D: I sum up the vectors on the real axis together. I group them like that. I group the vectors on the imaginary axis together. Then I combine them (She refers to the vectors on real axis and the vectors on imaginary axis) together.

R: How do you combine them?

D: For example, one is here and one is here (see Figure 6.77: She draws two vectors both starting from the origin, one on the positive side and the other on the negative side of the real axis. The vector on the positive side is longer than the vector on the negative side). This one and this one, to specify the direction in vector form... When I sum it up, it moves somewhere here (She draws a vector starting from the origin, which is shorter than the 1st vector that she drew on the positive side). When I sum that with this (She draws two vectors starting from the origin on the positive side of imaginary axis. One of the vector is longer than the other), it becomes something like this (She draws a vector starting from the origin on the positive side of imaginary axis. Its length is as much as the other two). I sum these two (She points to the summation of the vectors on real and imaginary axes). It becomes like this (She draws the summation of the vectors on the real and the imaginary axes on the 1st quadrant).

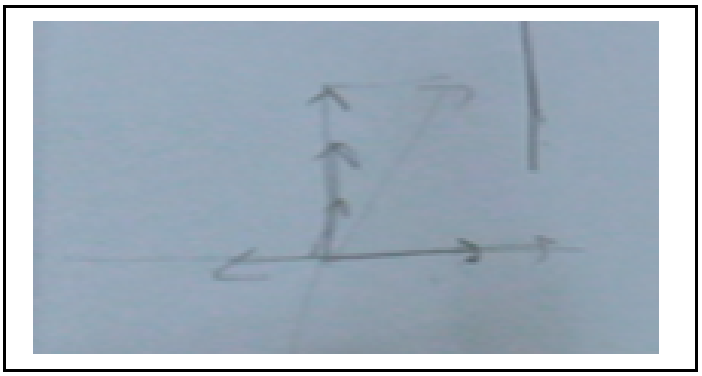


Figure 6.77. Derin shows the addition of real parts and the addition of imaginary parts of z_1 and z_2 as vectors.

Data showed the following: First, in the pre-interview, Derin was not able to mention the addition of the real parts and the imaginary parts referring to the vector addition. That is, she did not mention that the addition of real parts would refer to the addittion of the real parts of z_1 and z_2 as vectors and the addition of the imaginary parts would refer to the addition of the imaginary parts of z_1 and z_2 as vectors. However, in the post-interview, she was able to mention the addition of the real parts and the imaginary parts referring to the vector addition. That is, she knew that the addition of real parts would refer to the addittion of the real parts of z_1 and z_2 as vectors and the addition of the imaginary parts would refer to the addition of the imaginary parts of z_1 and z_2 as vectors. Secondly, in the pre-interview, Derin was not able to think of the real and imaginary parts of the resultant vector, ax - by + i.(ay + bx), in terms of the real and imaginary parts of z_1 and z_2. Contrarily, in the post-interview, she was able to think of the real and imaginary parts of the resultant vector, ax - by + i.(ay + bx), in terms of the real and imaginary parts of z_1 and z_2.

Derin even commented on the idea why she thought of the operant vector. Discussion followed:

R: Alright. Well. I will also ask you this. Why do you multiply a and bi with x plus yi part by part? You did it like this before, didn't you? You did it by separating x

plus yi to its parts for the rest. Why?

D: We could find them by separating each one when we think in vector form, but it is enough to separate just one since here (She refers to the angle between the vectors $x.z_1$ and $yi.z_1$ which she moved to the 1st quadrant, in the case of $x > 0$, $y > 0$) is 90°.

R: What do you mean by separating just one?

D: For example here (She refers to the multiplication $(a + bi).(x + yi)$). I separated $x + yi$. At first I analyzed the effect of x on z_1, then the effect of yi. Since multiplication with x and yi rotates 90°, two hundred and seventy degrees, one hundred and eighty degrees, I obtain specific angles. So it is enough to analyze it in two parts instead of separating four parts. For example what is that z_1 (She refers to the vector $x.z_1$ on the Complex Plane in the case of $x > 0$, $y > 0$)? It rotated one hundred and eighty degrees since it is multiplied with negative. It rotated two hundred and seventy degrees, if negative is multiplied with i. I obtain specific angles from them and it worked for me. It didn't be needed to separating it into four parts and analyzing and summing up each part. Because the orientation (She refers the vectors $x.z_1$ and z_1 are on the same orientation on the Complex Plane in the case of $x > 0$, $y > 0$) doesn't change when I multiplied with x. X was here, since x is a real number, its direction doesn't change when I multiplied with the part x. It rotated 90° when I multiplied with yi (She refers to the vector $yi.z_1$ on the Complex Plane in the case of $x > 0$, $y > 0$). There is a vector on the same direction and there is also a vector rotated ninety degree, I sum them. The reason is this.

Data was interesting in the following way: Derin knew that once she thought of the operand vector in two parts acting on the other vector, in each case, one of the vectors did not change the direction while the other one, changed the direction yielding a total of 90° in between the resultant vectors. Specifically, for the case $x > 0$, $y > 0$, $x.z_1$ and $yi.z_1$ were 90° apart from each other since $x.z_1$ did not change the direction and yi rotated the vector z_1 90° changing the direction. Similarly, for the case $x > 0$ and $y < 0$, $x.z_1$ would not change the direction. But yi would rotate the vector in total

of 270° changing the direction again leaving 90° between x.z_1 and yi.z_1. In addition, for the case x $<$ 0and y $>$ 0, x would rotate z_1 180° not changing the direction and yi would rotate z_1 by 90° changing the direction so yeilding to the 90° between x.z_1 and yi.z_1. Finally, when x $<$ 0 and y $<$ 0, x would rotate z_1 by 180° without changing the direction of z_1 and yi would rotate z_1 by 270° changing the direction of z_1 but leaving again 90° between x.z_1 nd yi.z_1.

7. DISCUSSION AND CONCLUSION

The purpose of this study was to investigate a prospective secondary mathematics teacher's development of the meaning of the multiplication of Cartesian, vector and polar forms of complex numbers by providing a sequence of tasks focusing on the learner's re-construction of the multiplication of complex numbers though quantitative reasoning. Results resulted in a model of a prospective secondary mathematics teacher's development of the meanings of the multiplication of complex numbers from the point of view of the researcher. Results also pointed to the possible difficulties the prospective teacher, Derin, encountered while developing the meaning of the Cartesian, vector and polar forms of the multiplication of complex numbers. I reflect on the results in three parts.

7.1. Reflection on Derin's Development of the Multiplication of Complex Numbers in Vector Form

Results showed that prior to the teaching, while reasoning on the multiplication of complex numbers, Derin was not able focus on one of the complex numbers as a multiplier and the other as a multiplicand; rather, her focus was on the product $z_1.z_2$. Also, since her focus was on the product rather than the multiplicands, she was not able to mention either z_1 or z_2 as the operand vector acting on the other one such that she was not able to consider the resultant vector $z_1.z_2$ in terms of the multiplication of two vectors, z_1 and z_2. Though, she was able to show the product, $z_1.z_2$, as a vector, since she knew that all complex numbers could also be represented as a vector. However, she was not able to think of the real and imaginary parts of the resultant vector, ac - bd + i.(ad + bc), in terms of the real and imaginary parts of $z_1 = a + bi$ and $z_2 = c + di$. Therefore, she was not able to mention the addition of the real parts and the imaginary parts referring to the vector summation. She only was able to draw four vectors for the real and imaginary parts of $z_1 = a + bi$ and $z_2 = c + di$, which started from the origin and stayed on the negative or positive side of the real or imaginary axes and she was not ble to explain ac - bd + i.(ad + bc) based on the multiplication

of the vectors of the real and imaginary parts of $z_1 = a + bi$ and $z_2 = c + di$.

Upon completion of the teaching sessions, results showed that while reasoning on the multiplication of complex numbers, Derin was able to focus on the multiplicands z_1 and z_2 rather then the product $z_1.z_2$. Thus, one of the hypotheses was verified. Particularly, she was able to think of z_1 as a multiplier and z_2 as a multiplicand as hypothesized before the teaching. Moreover, she was able to draw four vectors, which shows the vectors ax, - by, ayi and bxi where a, b, x and y were positive real numbers, on the real and imaginary axis. Then, she added first the vectors on the real axis and the imaginary axis and then, she made a vector summation between the vectors on the real axis and the imaginary axis in order to find the vector $z_1.z_2 = ax - by + i.(ay + bx)$. Data verified the hypothesis: Derin was able to think of the real and imaginary parts of the resultant vector, $ac - db + i.(ad + bc)$, in terms of the real and imaginary parts of z_1 and z_2. She was able to think of and show the resultant vector $z_1.z_2 = ax - by + i.(ay + bx)$, in terms of the addition of the real parts and the imaginary parts of $z_1 = a + bi$ and $z_2 = x + yi$ referring to the vector summation. Previous research pointed that the nature of complex numbers' being very abstract (Panaoura *et al.*, 2006; Nordlander and Nordlander; 2012) creates difficulties in the perception of the multiplication of complex numbers. Contrarily, this study showed that a prospective secondary mathematics teacher was able to multiply two complex numbers $(a + bi)$ and $(x + yi)$ by perceiving them as vectors.

Particularly results pointed that when $z_2 = c + di$, Derin's reasoning on the different cases of c and d in-juxtaposition to each other both algebraically and geometrically was important: This was because while she was thinking different cases, she kept the vector z_1 the same in the 1st quadrant, while she considered the values of c and d as changing. That is, she left the vector, z_1, in the 1^{st}quadrant prior to be multiplied with c and di. Then, multiplying with c and di, she acknowledged its original placament and magnitude as changing. She knew that the change occured because of the effect of the different values of c and d acting on each part of the vector z_1. This allowed her to not only express $c.z_1$ as a single vector but also as a linear combination of the vectors ca and cbi. Similarly reasoning on the effect of di on the original vector z_1, she was

able to explain and show the resultant vector $z_1.z_2$ as a point focusing on the effect of multiplying with di since she knew that multiplication with "i" would result in 90° rotation of the vector z_1. Thus, she was able to think of z_1 as a multiplicand and c and di as multlipliers as components of a vector, z_2, acting on it and mention z_2 as the operand vector acting on z_1 such that she was able to consider the resultant vector $z_1.z_2$ in terms of the multiplication of two vectors, z_1 and z_2. Therefore, as Soto-Johnson *et al.* (2012) said that she was able to conceive the multiplication of complex numbers as a group of experts and thus, she was able define 'z' as an operator and 'w' as the operand in the multiplication of two complex numbers, 'z.w'. In sum, focusing on the similarities and differences, reasoning on the different cases of c and d in-juxtaposition to each other allowed Derin to deduce two important conclusions: First, contrary to the results form the written pre-assessment and pre-interview, she was able to think of z_1 as a multiplicand and z_2 as a multiplier acting on it. Particularly, she assimilated $c.z_1$ and $di.z_1$ as two vectors Thus, she was able to realize the similarities and differences between the orientations, directions and starting points of the vectors $c.z_1$ and $di.z_1$, where z_1 is a complex number and c and d are real numbers, regarding all cases of c and d. This showed that, she was able to reason on the geometrical representation of the multiplication of complex numbers, $z_1.z_2$ in vector form regarding all of the cases c and d. Karakok *et al.* (2015) argued that perception one of the complex number as an operator and the other as a vector was difficult for in-service secondary mathematics teachers. Secondly, she abstracted that in all cases, the vectors $c.z_1$ and $di.z_1$ were perpendicular to each other. This allowed her to consider the addition of the two vectors $c.z_1$ and $di.z_1$ using the Pythegorean Theorem. Aforementioned studies said that teachers only could explain both algebraically and geometrically multiplication by "i" (Karakok, *et al.* 2015; Nemirovsky *et al.*, 2012), but this study showed that Derin was able to reason on the rotation between the vectors $c.z_1$ and $di.z_1$ such that the angle between them was equal to 90° regarding all cases of c and d. Also, she was able to explain the multiplication of $z_1 = a + bi$ and $z_2 = c + di$ in vector form referring to the summation of the real part, $c.z_1$ and imaginary part $di.z_1$ of $z_1.z_2$. Karakok *et al.* (2013) also argued that thinking of complex numbers as vectors might not be useful in reasoning about the multiplication of complex numbers. Particularly, they claimed: "perceiving complex number addition as the addition of vectors makes mathematical

sense, but it is not sound mathematics to discuss the product and quotient of vectors in R^n" (p. 347). This could be because of the difference between vector multiplication in R^n and in Complex Plane. In R^n, vector product/cross product of two vectors results in a new vector such that this resultant vector is perpendicular to the two vectors at the same time. That is, the resultant vector is "perpendicular to the plane containing" two vectors (Chatterjee, 2013, p. 24). However, the multiplication of two complex numbers in vector form results in a new vector which stays on the same plane with the multiplied vectors. Conner *et al.* (2007) asserted that prospective teachers could not represent the multiplication of complex numbers although they could represent the addition of complex numbers on the Complex Plane. However, results from this study showed that reasoning on one of the vectors as a multiplier comprised of two vector parts in Complex Plane provided opportunities for a prospective secondary mathematics teacher to develop the meaning of multiplication of complex numbers in vector form.

7.2. Reflection on Derin's Development of the Multiplication of Complex Numbers in Polar Form

Particular to the polar form, before the teaching sessions, Derin was not able to explain why she multiplied the magnitudes of the vectors z_1 and z_2 and added the arguments of the vectors z_1 and z_2 in polar form. However, the post-interview data verified the hypotheses of the study. Upon completion of the study, she was able to reason on the magnitude of the vector $z_1.z_2$ in terms of the magnitudes of the vectors $c.z_1$ and $di.z_1$ using the geometric representation of the vector summation $z_1.z_2 = c.z_1 + di.z_1$ regarding all cases of c and d. Particulary, she was able to calculate the magnitude of $z_1.z_2$ in terms of the magnitudes of z_1 and z_2 using the Pyhtagorean Theorem. Also, she was able to explain the magnitude of $z_1.\ z_2$ as equal to the multiplication of the magnitudes of z_1 and z_2 regarding all cases of c and d.

In addition, after the teaching sessions, she also was able to reason on the angle of the vector $z_1.z_2$ referring to the angles of z_1 and z_2 regarding all cases of c and d. Similarly, she was able evaluate the angle between the vectors $z_1.z_2$ and z_1 regarding

all cases of c and d. Thus, she was able to reason on the tangent value of the angle between the vectors $z_1.z_2$ and z_1, and the angle of the vector z_2. Though, she had difficulty in calculating the angle between the vectors $z_1.z_2$ and z_1 while reasoning on the case, $c > 0$, $d < 0$. Data suggested that this was because of her ignorence of the position of the vector z_2 on the Complex Plane, her inattention to the tangent value of the angle between the vectors $z_1.z_2$ and z_1 based on the proportion of the distances to the real and imaginary axis and her limited knowledge of trigonometric functions of the angles. Data also showed that Derin overcomed these difficulties focusing on the quantities: The vector z_2 and the tangent of the angle between the vectors $z_1.z_2$ and z_1, and the angle of z_2. Therefore, she was able to explain the argument of $z_1.z_2$ as equal to the addition of the arguments of z_1 and z_2 regarding all cases of c and d. Results of the post-interview also showed that, upon completion of the teaching sessions, Derin was able to show geometrically and explain where $z_1.z_2$ was positioned either as a point or in polar form such that she knew that they corresponded to the same thing. That is, she conceptualized the Cartesian form of the multliplication of complex numbers as a single entity such that, all representations, Cartesian form, vector form and polar form showed the same number. Therefore, she was able to explain the reasoning behind the polar form by pointing to the effect of z_2 on z_1. Specifically, she was able to justify the reasoning behind the multiplication of the magnitudes of z_1 and z_2 and the addition of their arguments in polar form. As Nordlander and Nordlander (2012) argued that due to the visualization challenge of complex numbers, students have difficulties in making sense the meaning of rules. The study showed that the vector form of the multiplication of complex numbers helped students to make sense of the meaning of the multiplication in Cartesian, vector and polar form. Aforementioned studies showed that although students could state the geometric meaning of the multiplication of complex number as rotation and dilation (Soto-Johnson and Troup, 2014) or teachers could state the division of complex numbers as the division of two vectors (Karakok, *et al.* 2015), but they could not justify the reasons (Karakok, *et al.* 2015; Soto-Johnson and Troup, 2014). In this study, the prospective mathematics teacher was able to justify her reasoning on geometric meaning of the multiplication of complex number as rotation and dilation. In particular, she was able to state the multiplication of complex numbers as the multiplication of vectors by reasoning on the magnitude and argument

qualities of the complex numbers and dilation, translation and rotation operations on the vectors during the multiplication.

7.3. Reflection on Derin's Development of the Relationship Between the Multiplication of Complex Numbers in Cartesian, Vector and Polar Forms

In the pre-interview, she was not able to explain the relationship between the polar and the vector form of the multiplication of complex numbers. In the post-interview, she was able to explain the relationship between the vector and the polar forms of the multiplication of complex numbers as hypothesized before the study. Results from (Conner *et al.*, 2007) study pointed that prospective teachers could not represent the multiplication of complex numbers although they could represent the addition of complex numbers on the Complex Plane. Similarly, Soto-Johnson and Troup (2014) asserted that although students could state the geometric meaning of the multiplication of complex number as rotation and dilation they could not justify the reasons. Contrarily, Derin was able to define the polar form by pointing to the dilation and rotation effect of the vector z_2 on the vector z_1. The results showed that the hypotheses about the development of the multiplication of complex numbers in vector form, the development of the multiplication of complex numbers in polar form and the development of the relationship between the multiplication of complex numbers in Cartesian, vector and polar form were verified at the end of the study.

8. IMPLICATIONS FOR RESEARCH AND TEACHING

Results pointed to some important implications for teaching and research. First, as the data showed Derin already knew the effect of multiplying a complex number with a scalar and a pure imaginary number. She also knew that multiplying with "i" rotated any complex number by 90°. She further knew that all the complex numbers could be represented as vectors. These were important pre-knowledge on her part to conceptualize the multiplication of complex numbers in vector form. Such prior knowledge made it possible for Derin to assimilate the similarities and differences among the different cases of c and d such that once z_1 and z_2 were multiplied where $z_2 = c + di$, she was able to deduce $c.z_1$ and $di.z_1$ as two vectors. In addition she was able to justify that these two vectors would always be perpendicular to each other. Therefore, I propose to enable high school students and pre-service teachers to acknowledge the entire pre-knowledge aforementioned prior to developing the multiplication of complex numbers. Similarly, I emphasize that reasoning on the similarities and differences between different cases of c and d both algebraically and geometrically might be propitious while developing the vector form of the multiplication of complex numbers.

By the same token, while developing the polar form of complex numbers, reasoning on the angle of $z_1.z_2$ in terms of z_1 and z_2 was very powerful for Derin. Particularly, Derin's focus on the angle between $z_1.z_2$ and z_1 and also being asked to evaluate the value of such angle enabled Derin to justify why the angle between $z_1.z_2$ and z_1 was equal to the argument of the complex number z_2. This also allowed her to reason that z_1 was rotated by the angle z_2 to be placed in the position of $z_1.z_2$. Therefore, I propose to include asking such important questions during teaching of the multiplication of complex numbers in polar form.

Although this research focused solely on a prospective secondary mathematics teachers' development of the multiplication of complex numbers, the model of the learning trajectory represented with data from the point of view of the researcher might allow to do further research with high school and pre-service teachers. In particular,

data pointed to the difficulties Derin have encountered while developing especially the reasoning behind theargument of $z_1.z_2$ in terms of the arguments of z_1 and z_2. Specifically, Derin's difficulties stemmed from her not positioning the operant vector z_2 and not evaluating the argument of it properly by taking attention to the different cases of c and d. Though, her keeping the multiplicand, z_1, in the 1^{st}quadrant and reasoning on the effect of the multiplier, z_2 by focusing on the changes in z_1 both algebraically and geometrically was propitious for her learning. Thus, learning from such empirical results, further research for the development of the polar form of the multiplication of complex numbers could be conducted to further understand the learning trajectory for the development of the multiplication of complex numbers in vector and polar forms. In addition, the results from this study showed that reasoning on the product of complex numbers in Complex Plane through vector summation was possible. It is our curiosity what possible affordances such recognition might allow a person to reason on the other representations of complex numbers (i.e., $r.e^{i\theta}$) including the complex arithmetic.

In order to experience mathematical concepts totally, teachers should have opportunities "struggling with hard problems, discovering their own solutions, reasoning mathematically, modeling with mathematics, and developing mathematical habits of mind" (CBMS, 2012). The prospective teachers should be aware of that the mathematics in high school curriculum consists of the facts and formulas and students and teachers could not justify the reasoning under them. In this case, teacher education gains importance. Prospective teachers should know more than they will teach to their students in order to prepare tasks to develop the students' reasoning skills. For instance, the prospective teachers should know the meaning of the multiplication of complex numbers in order to understand the nature of the complex numbers and the operations on the complex numbers. Thus, they could develop their reasoning skills and then, they would help their students to develop their mathematical reasoning skills in the future. Thus, the teachers will be prepared their students to realize the nature of the concept of complex numbers and operations on complex numbes thought in college.

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APPENDIX A: ARTICULATION OF THE UNDERSTANDING

A complex number z = a + bi where a and b are two real numbers, could be seen as a vector on the complex axis which accepts the x axis as the real axis, and the y axis as the imaginary axis. The vector form of the complex number z consists of two projection vectors such as $\vec{a}$ and $\vec{b}$, where $\vec{a}$ is the projection of the vector on the real axis and $\vec{b}$ is the projection of the vector on the imaginary axis.

When we multiply a complex number z = a + bi by the real number "1" (see Figure A.1), we multiply each projection vector of z by "1" and get the complex numbers z = a + bi.So, we remain the same the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ without making any change in their directions and obtain again the $\vec{a}$ and $\vec{b}$, where $\vec{a}$ is the projection of the new vector on the real axis and $\vec{b}$ is the projection of the new vector on the imaginary axis. Thus if we sum the projection vectors on the complex axis, we get the complex number z = a + bi, which is the same as at the beginning.

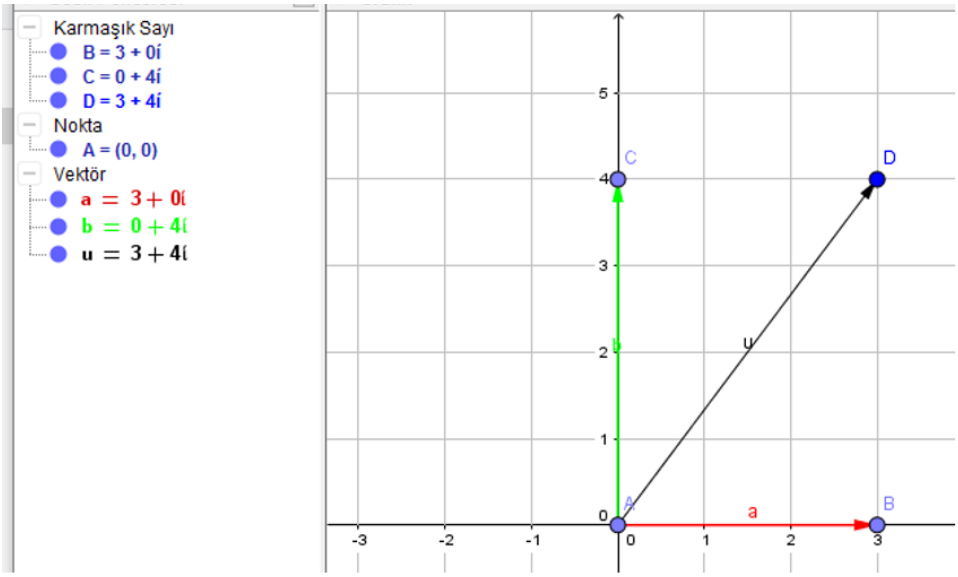


Figure A.1. Articulation of the Understanding 1.

When we multiply a complex number z = a + bi by the real number “0”, we multiply each projection vector of z by “0” and get the complex numbers z = 0 + 0i (see Figure A.2). So, we decrease the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ to zero without making any change in their directions and obtain again the $\vec{a} = \vec{0}$ and $\vec{b} = \vec{0}$, where $\vec{a}$ is the projection of the new vector on the real axis and $\vec{b}$ is the projection of the new vector on the imaginary axis. Thus if we sum the projection vectors on the complex axis, we get the complex number z = 0 + 0i, which is equal to z = 0.

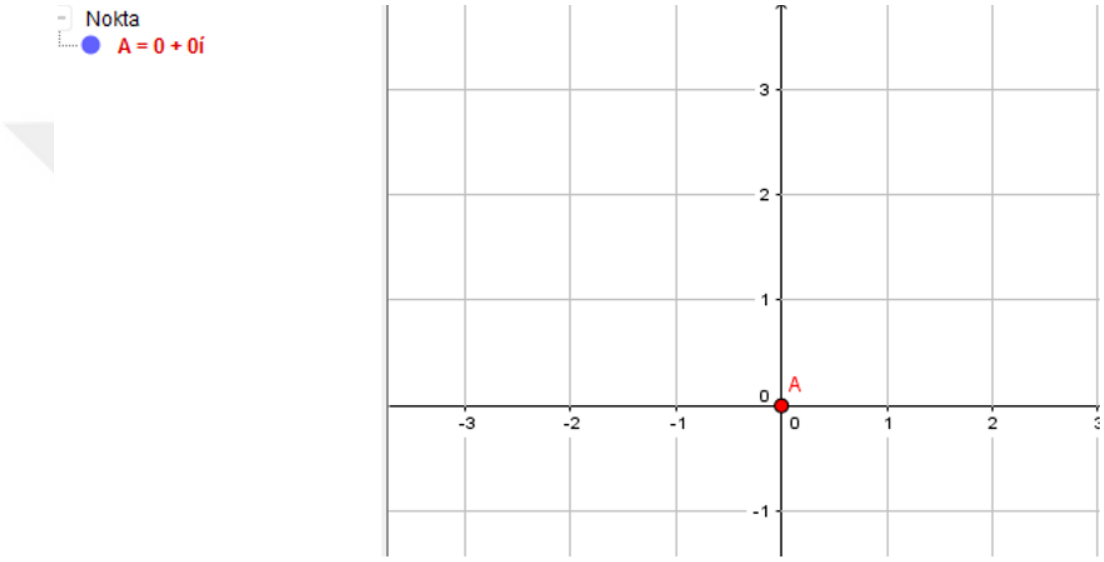


Figure A.2. Articulation of the Understanding 2.

When we multiply a complex number z = a + bi by the real number “-1” (see Figure A.3), we multiply each projection vector of z by “1” and get the complex numbers z = a + bi. So, we remain the same the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ without making any change in their directions and obtain again the $\vec{a}$ and $\vec{b}$, where $\vec{a}$ is the projection of the new vector on the real axis and $\vec{b}$ is the projection of the new vector on the imaginary axis. Then, we reverse the directions of the projection vectors and move their starting point to the origin and get $-\vec{a}$ and $-\vec{b}$. Thus if we sum the projection vectors on the complex axis, we get the complex number - z = - a - bi.

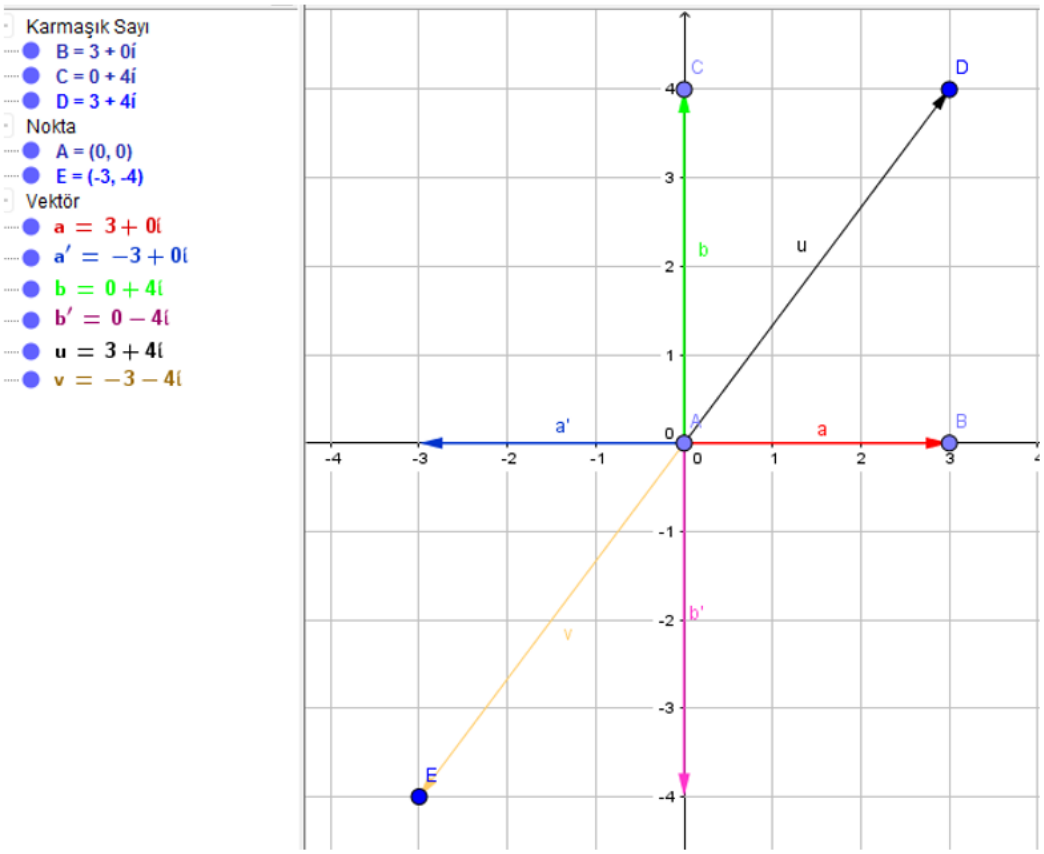


Figure A.3. Articulation of the Understanding 3.

When we multiply z = a + bi with a real number k > 1, we multiply each projection vector of z by k and get the complex number kz = ka + kbi. So, we increase the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by k, i. e., we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by k without making any change in their directions since k is positive and obtain the k$\vec{a}$ and k$\vec{a}$, where k$\vec{a}$ is the projection of the new vector on the real axis and k$\vec{b}$ is the projection of the new vector on the imaginary axis. Thus if we sum the projection vectors on the complex axis, we get the complex number kz = ka + kbi.

For example (see Figure A.4), when we multiply u = 3 + 4i with 2, we multiply each projection vector of u by 2 and get the complex number 2u. So, we increase the magnitudes of the projection vectors $\vec{a}$ = 3 + 0i and $\vec{b}$ = 0 + 4i by 2, i. e., we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by 2 without making any change in their directions and obtain the 2$\vec{a}$ = a' = 6 + 0i and 2$\vec{b}$ = b' = 0 + 8i, where 2($\vec{a}$) is the projection of the new vector on the real axis and 2$\vec{b}$ is the projection of the new vector on the imaginary

axis. Thus if we sum the projection vectors on the complex axis, we get the complex number 2u = 6 + 8i.

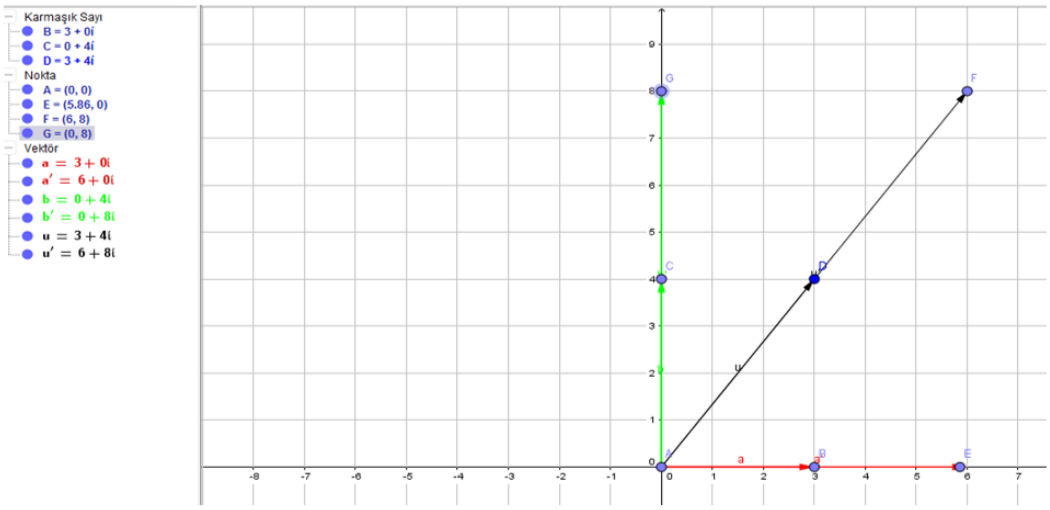


Figure A.4. Articulation of the Understanding 4.

When we multiply z = a + bi with a real number k > 0, we multiply each projection vector of z by k and get the complex number kz = ka + kbi. So, if k > 1 (see Figure A.5), we increase the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by k, i. e., when k is a positive real number, we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by k without making any change in their directions and obtain the k$\vec{a}$ and k$\vec{b}$, where k$\vec{a}$ is the projection of the new vector on the real axis and k$\vec{b}$ is the projection of the new vector on the imaginary axis. Thus if we sum the projection vectors on the complex axis, we get the complex number kz = ka + kbi. Since the directions of the projection vectors do not change, the direction of the new vector kz is the same as the direction of z, so when we look at the polar coordinates of the complex numbers z and kz, we see that the angle of kz is equal to the angle of z, but the magnitude of kz is equal to k times the magnitude of z.

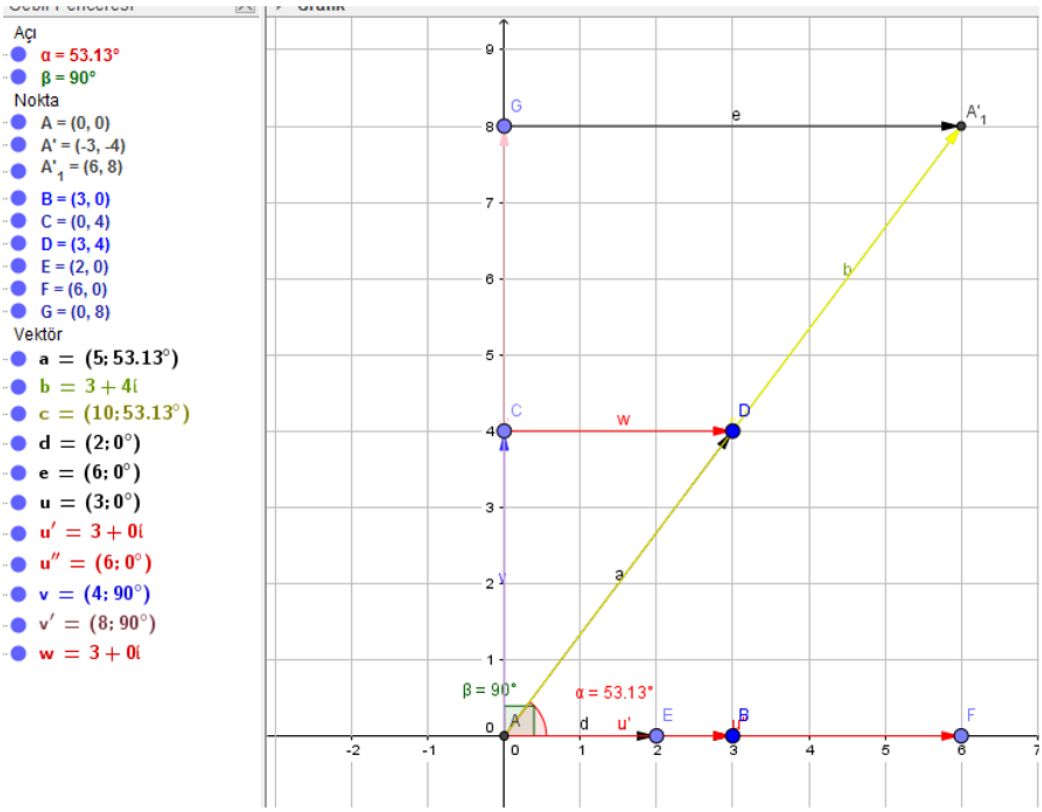


Figure A.5. Articulation of the Understanding 5.

When we multiply z = a + bi with a real number $0 < k < 1$, we multiply each projection vector of z by k and get the complex number kz = ka + kbi. So, we decrease the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by k, i. e., we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by k without making any change in their directions and obtain the k$\vec{a}$ and k$\vec{b}$, where k$\vec{a}$ is the projection of the new vector on the real axis and k$\vec{b}$ is the projection of the new vector on the imaginary axis. Thus if we sum the projection vectors on the complex axis, we get the complex number kz = ka + kbi.

For instance (see Figure A.6), when we multiply u = 3 + 4i with $1/2$, we multiply each projection vector of u by $1/2$ and get the complex number $1/2$ u. So, we decrease the magnitudes of the projection vectors $\vec{a} = 3 + 0i$ and $\vec{b} = 0 + 4i$ by half, i. e., we multiply the magnitudes of $\vec{a}$ and $\vec{b}$ by $\frac{1}{2}$ without making any change in their directions and obtain the $\frac{1}{2}\vec{a}$ = a' = 1.5 + 0i and $\frac{1}{2}\vec{b}$ = b' = 0 + 4i, where $\frac{1}{2}\vec{a}$ is the projection of the new vector on the real axis and $\frac{1}{2}\vec{b}$ is the projection of the new vector on the imaginary axis. Thus if we sum the projection vectors on the complex axis, we get the

complex number $^1/_2$ u = 1.5 + 2i.

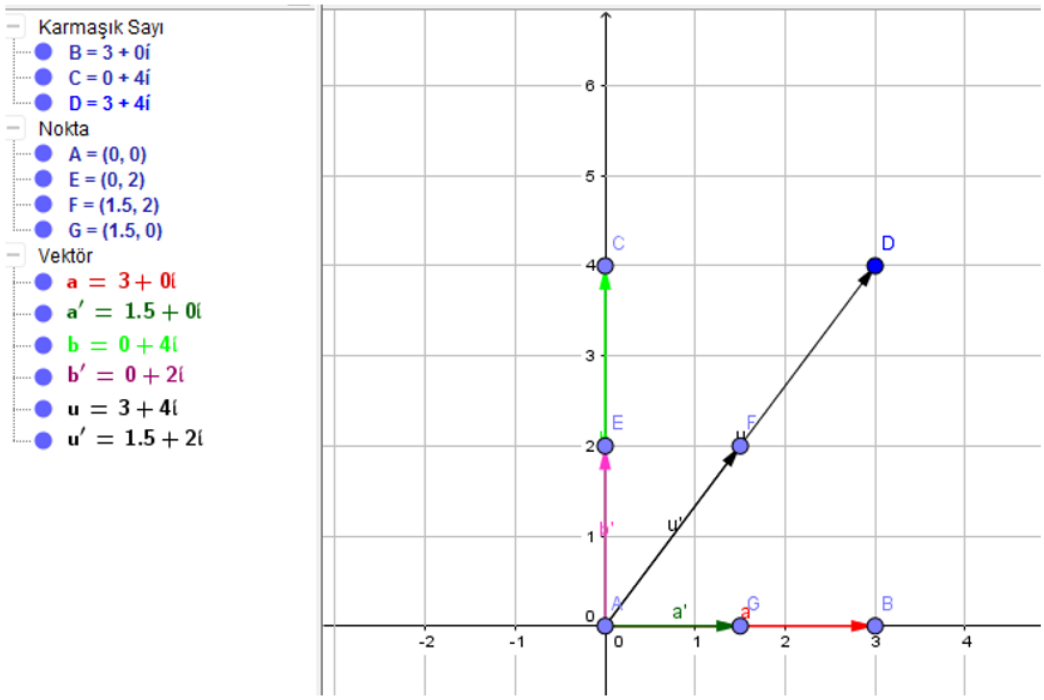


Figure A.6. Articulation of the Understanding 6.

When we multiply z = a + bi with a real number k > 0, we multiply each projection vector of z by k and get the complex number kz = ka + kbi. So, if 0 < k < 1 (see Figure A.7), thenwe decrease the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by k i. e., when k is a positive real number, we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by k without making any change in their directions and obtain the k$\vec{a}$ and k$\vec{b}$, where k$\vec{a}$ is the projection of the new vector on the real axis and k$\vec{b}$ is the projection of the new vector on the imaginary axis. Thus if we sum the projection vectors on the complex axis, we get the complex number kz = ka + kbi. Since the directions of the projection vectors do not change, the direction of the new vector kz is the same as the direction of z, so when we look at the polar coordinates of the complex numbers z and kz, we see that the angle of kz is equal to the angle of z, but the magnitude of kz is equal to k times the magnitude of z.

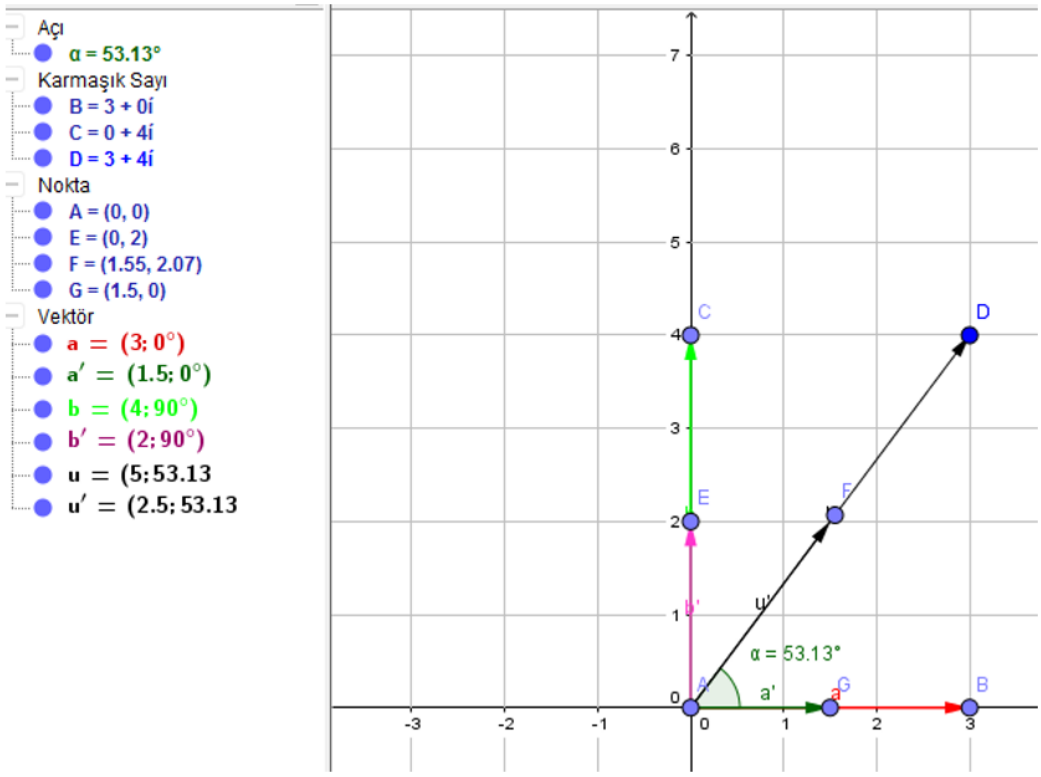


Figure A.7. Articulation of the Understanding 7.

When we multiply z = a + bi with a real number k < - 1, we multiply each projection vector of z by k and get the complex number kz = ka + kbi. So, we increase the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by k, i. e., we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by k byreversing their directions and obtain the k$\vec{a}$ and k$\vec{b}$, where k$\vec{a}$ is the projection of the new vector on the real axis and k$\vec{b}$ is the projection of the new vector on the imaginary axis. Thus if we sum the projection vectors on the complex axis, we get the complex number kz = ka + kbi.

For example (see Figure A.8), when we multiply u = 3 + 4i with -2, firstly we multiply each projection vector of u by 2 and get the complex number 2u. So, we increase the magnitudes of the projection vectors $\vec{a}$ = 3 + 0i and $\vec{b}$ = 0 + 4i by 2, i. e., we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by 2 without making any change in their directions and obtain the 2$\vec{a}$ = a' = 6 + 0i and 2$\vec{b}$ = b' = 0 + 8i, where 2$\vec{a}$ is the projection of the new vector on the real axis and 2$\vec{b}$ is the projection of the new vector on the imaginary axis. Then, we reverse the directions of the projection vectors and move their starting

point to the origin and get - $2\vec{a}$ = - 6 + 0i and - $2\vec{b}$ = 0 - 8i vectors. After that we make vector summation using - $2\vec{a}$ and - $2\vec{b}$ and obtain u" = -2u = - 6 - 8i.

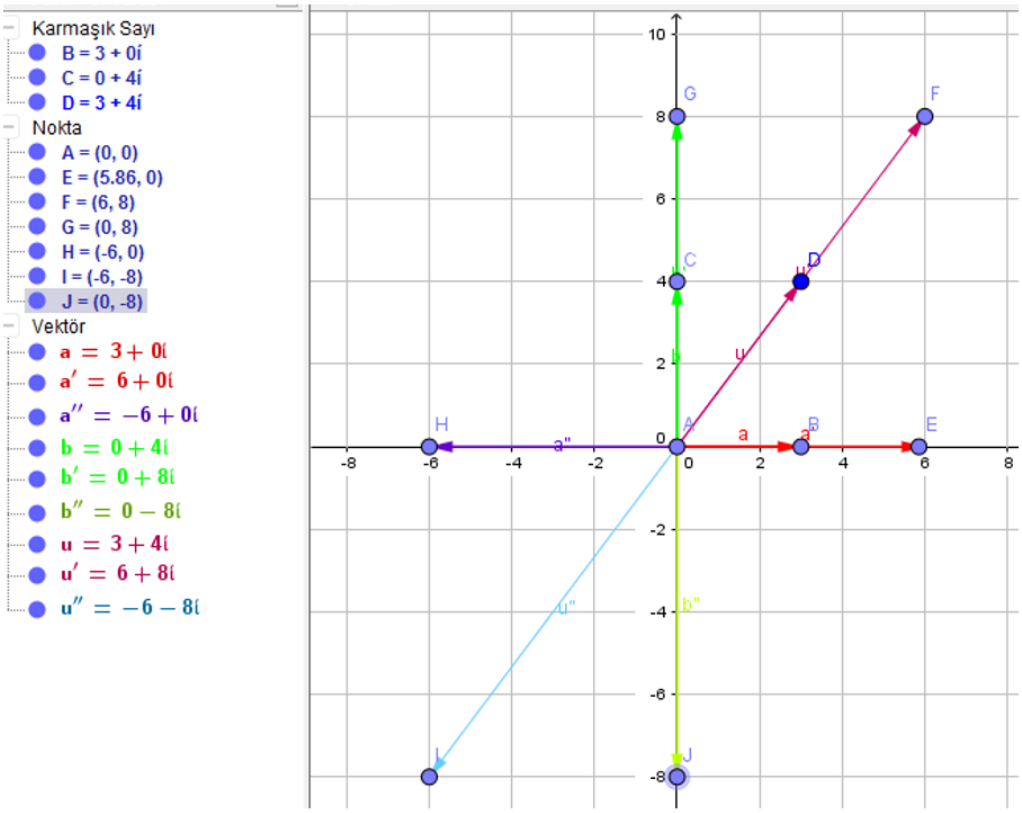


Figure A.8. Articulation of the Understanding 8.

When we multiply z = a + bi with a real number k < 0, we multiply each projection vector of z by $|k|$ and get the complex number $|k|z = |k|a + |k|bi$. So, if k < - 1 (see Figure A.9), we increase the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by $|k|$. After that we rotate each projection vector 180° counterclockwise in order to get the k$\vec{a}$ and k$\vec{b}$ vectors, i. e., when k is a negative real number, firstly we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by $|k|$ without making any change in their directions and obtain the $|k|\vec{a}$ and $|k|\vec{b}$, then we make 180° counterclockwise rotation to match $|k|\vec{a}$ and $|k|\vec{b}$ with k$\vec{a}$ and k$\vec{b}$ vectors, which are the projection of the new vector on the real axis and imaginary axis, respectively. Thus if we sum the projection vectors on the complex axis, we get the complex number kz = ka + kbi. Since the directions of the projection vectors change in 180° counterclockwise, the direction of the new vector kz is the opposite of the direction of z, so when we look at the polar coordinates of the

complex numbers z and kz, we see that the angle of kz is plus 180° from the angle of z, but the magnitude of kz is equal to $|k|$ times the magnitude of z.

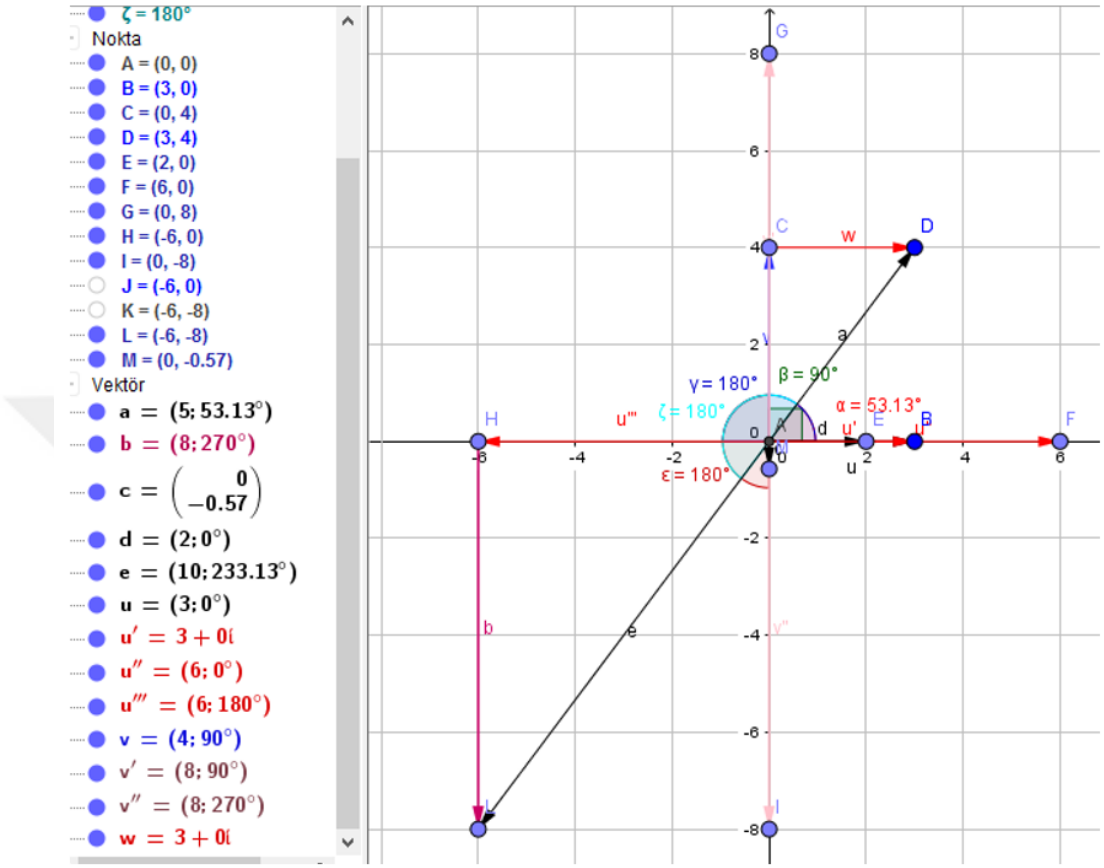


Figure A.9. Articulation of the Understanding 9.

When we multiply z = a + bi with a real number -1 < k < 0, we multiply each projection vector of z by k and get the complex number kz = ka + kbi. So, we decrease the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by $|k|$, i. e., we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by k by reversing their directions and obtain the k$\vec{a}$ and k$\vec{b}$, where k$\vec{a}$ is the projection of the new vector on the real axis and k$\vec{b}$ is the projection of the new vector on the imaginary axis. Thus if we sum the projection vectors on the complex axis, we get the complex number kz = ka + kbi.

For example (see Figure A.10), when we multiply u = 3 + 4i with -1/2, we multiply each projection vector of u by $^1/_2$ and get the complex number $^1/_2$ u. So, we decrease the magnitudes of the projection vectors $\vec{a} = 3 + 0i$ and $\vec{b} = 0 + 4i$ by half, i.

e., we multiply the magnitudes of $\vec{a}$ and $\vec{b}$ by $1/2$ without making any change in their directions and obtain the $\frac{1}{2}\ \vec{a}$ = a' = 1.5 + 0i and $\frac{1}{2}\ \vec{b}$ = b' = 0 + 2i, where $\frac{1}{2}\vec{a}$ is the projection of the new vector on the real axis and $\frac{1}{2}\vec{b}$ is the projection of the new vector on the imaginary axis. Then we reverse the directions of the vectors $\frac{1}{2}\ \vec{a}$ and $\frac{1}{2}\vec{b}$ and get - $\frac{1}{2}\vec{a}$ = a" = - 1.5 + 0i and - $\frac{1}{2}\vec{b}$ = b" = 0 - 2i. Thus if we sum the new projection vectors - $\frac{1}{2}\vec{a}$ = a" = - 1.5 + 0i and - $\frac{1}{2}\ \vec{b}$ = b" = 0 - 2i on the complex axis, we get the complex number $1/2$ u = - 1.5 - 2i.

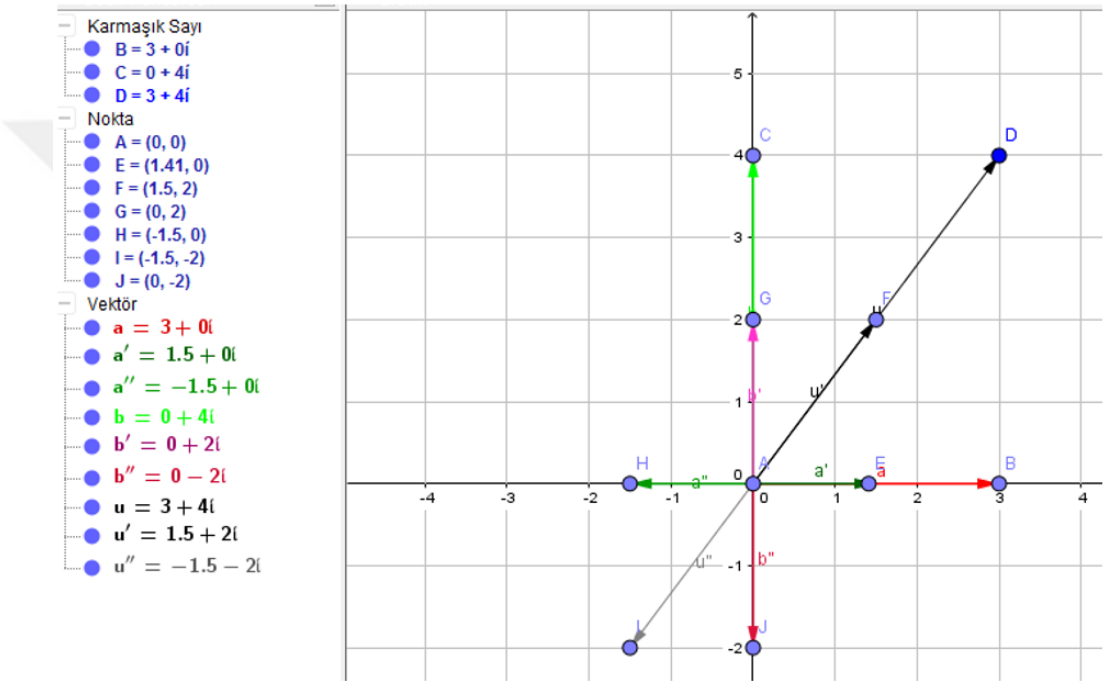


Figure A.10. Articulation of the Understanding 10.

When we multiply z = a + bi with a real number k < 0, we multiply each projection vector of z by $|k|$ and get the complex number $|k|z = |k|a + |k|bi$. So, if - 1 < k < 0 (see Figure A.11), thenwe decrease the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by $|k|$. After that we rotate each projection vector 180° counterclockwise in order to get the k$\vec{a}$ and k$\vec{b}$ vectors, i. e., when k is a negative real number, firstly we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by $|k|$ without making any change in their directions and obtain the $|k|\vec{a}$ and $|k|\vec{b}$, then we make 180° counterclockwise rotation to match $|k|\vec{a}$ and $|k|\vec{b}$ with k$\vec{a}$ and k$\vec{b}$ vectors, which are the projection of the new vector on the real axis and imaginary axis, respectively. Thus if we sum the projection vectors on the complex axis, we get the complex number kz = ka + kbi. Since the directions of

the projection vectors change in 180° counterclockwise, the direction of the new vector kz is the opposite of the direction of z, so when we look at the polar coordinates of the complex numbers z and kz, we see that the angle of kz is plus 180° from the angle of z, but the magnitude of kz is equal to $|k|$ times the magnitude of z.

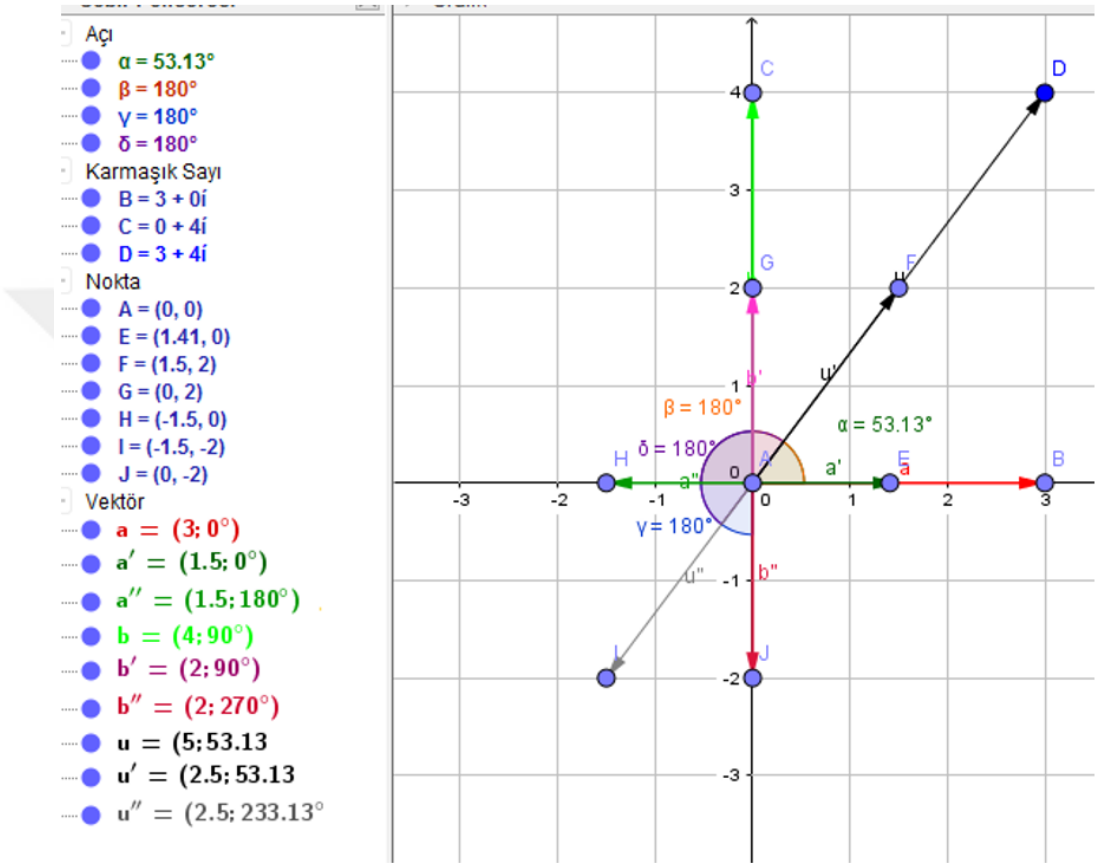


Figure A.11. Articulation of the Understanding 11.

When we multiply a complex number z = a + bi by the imaginary number "i," we rotate each projection vector by 90° in the counterclockwise. Thus, we just carry the vector on the real axis onto the imaginary axis and the vector on the imaginary axis onto the real axis without any change in their magnitude. For instance (see Figure A.12), when we take the complex number u = 3 + 4i and multiply it by i, we just move the real component 3 of u on the imaginary axis and the imaginary component 4 of u on the real axis with the help of 90° counterclockwise rotation. Then we form the new complex number u' = - 4 + 3i, which could be also gained by 90° counterclockwise rotation of the complex number u = 3 + 4i.

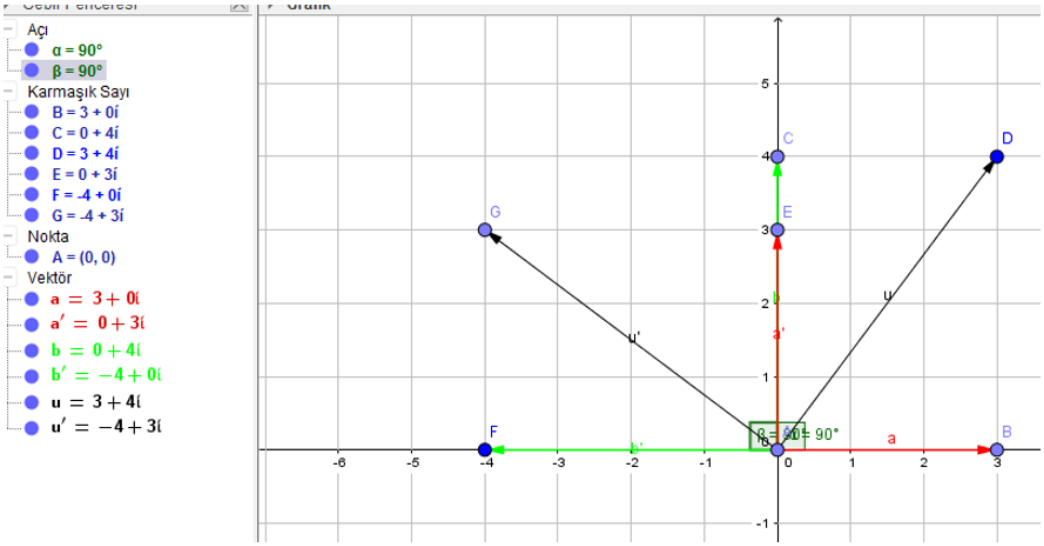


Figure A.12. Articulation of the Understanding 12.

When we multiply z = a + bi with ki where k > 1, firstly we think that we multiply z by k and get the complex number kz = ka + kbi. This means that we dilate (increase) the magnitudes of $\vec{a}$ and $\vec{b}$ by k without making any change in their directions. Then we think that we multiply kz with i so we rotate each projection vector, k$\vec{a}$ and k$\vec{b}$ by 90° in the counterclockwise. Thus, we just carry the vector on the real axis onto the imaginary axis and the vector on the imaginary axis onto the real axis without making any change in their magnitude and get the complex number that z' = - kb + kai.

For example (see Figure A.13), when we multiply u= 3 + 4i with 2i, firstly we think that we multiply u by 2 and get the complex number u' = 2u = 6 + 8i. Then we think that we multiply 2u with i and this means that we rotate each projection vector, $\vec{a}'$ = 6 + 0i and $\vec{b}'$ = 0 + 8i by 90° in the counterclockwise without any change in their magnitude. Thus, we just carry the vector a' on the real axis onto the imaginary axis and get the new vector a" = 0 + 6i. Similarly, we just carry the vector b' on the imaginary axis onto the real axis and get the vector b" = - 8 + 0i. Thus we obtain the complex number that u" = - 8 + 6i by making vector summation. We could also say that multiplication of u= 3 + 4i with 2i means that dilation of the magnitude of u by 2 and 90° counterclockwise rotation of the dilated vector to get u".

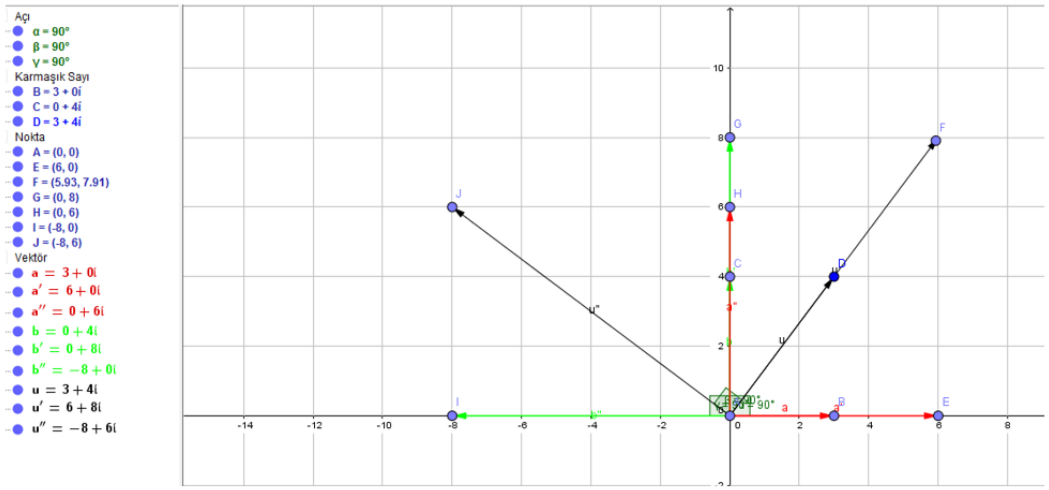


Figure A.13. Articulation of the Understanding 13.

When we multiply z = a + bi withki, where k is a positive real number, we firstly multiply each projection vector of z by k and get the complex number kz = ka + kbi. In this case, if k > 1 (see Figure A.14), we increase the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by k, i. e., when k is a positive real number, we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by k without making any change in their directions and obtain k$\vec{a}$ and k$\vec{b}$, where k$\vec{a}$ is the projection of the new vector on the real axis and k$\vec{b}$ is the projection of the new vector on the imaginary axis. Then we multiply each projection vector with i, so we rotate k$\vec{a}$ and k$\vec{b}$ vectors in 90° counterclockwise in order to get - k$\vec{b}$ on the real axisand k$\vec{a}$ on the imaginary axis, Thus if we sum the projection vectors on the complex axis, we get the complex number kz = - kb + kai. Since the directions of the projection vectors change in 90° counterclockwise, the direction of the new vector kz rotates 90° counterclockwise from the direction of z, so when we look at the polar coordinates of the complex numbers z and kz, we see that the angle of kz is equal to 90° plus the angle of z, but the magnitude of kz is equal to k times the magnitude of z.

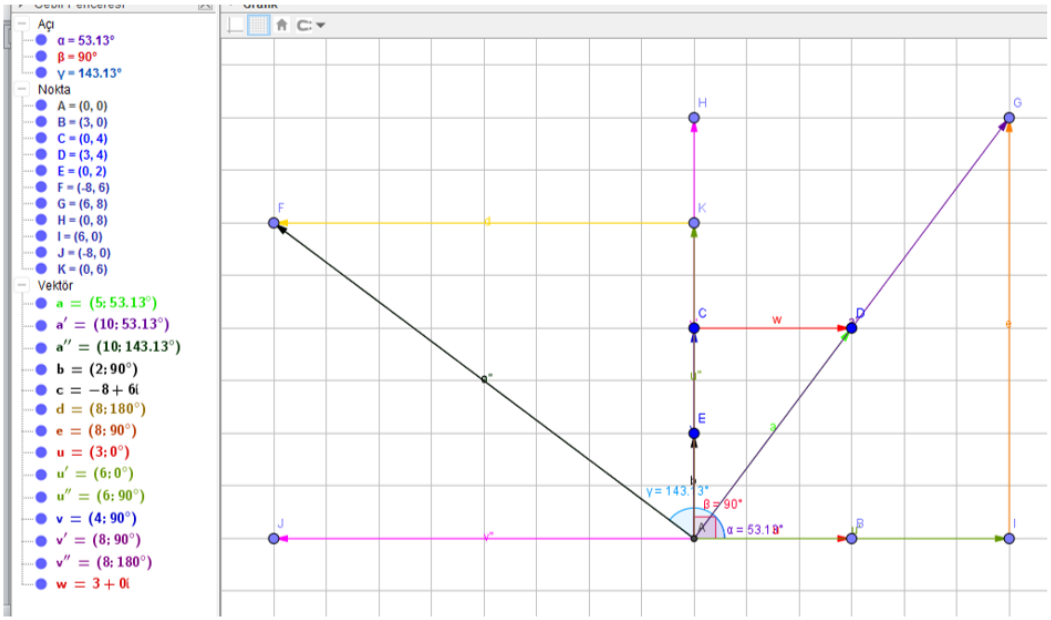


Figure A.14. Articulation of the Understanding 14.

When we multiply z = a + bi with ki where 0 < k < 1, firstly we think that we multiply z by k and get the complex number kz = ka + kbi. This means that we dilate (decrease) the magnitudes of $\vec{a}$ and $\vec{b}$ by k without making any change in their directions. Then we think that we multiply kz with i so we rotate each projection vector, k$\vec{a}$ and k$\vec{b}$ by 90° in the counterclockwise. Thus, we just carry the vector on the real axis onto the imaginary axis and the vector on the imaginary axis onto the real axis without making any change in their magnitude and get the complex number that z' = - kb + kai.

For instance (see Figure A.15), when we multiply u = 3 + 4i with $\frac{1}{2}$ i, firstly we think that we multiply u by $\frac{1}{2}$ and get the complex number u' = $1/2$ u = 1.5 + 2i. Then we think that we multiply u' = $1/2$ u with i and this means that we rotate each projection vector, $\vec{a}'$ = 1.5 + 0i and $\vec{b}'$= 0 + 2i by 90° in the counterclockwise without any change in their magnitude. Thus, we just carry the vector a' on the real axis onto the imaginary axis and get the new vector a"= 0 + 1.5i. Similarly, we just carry the vector b' on the imaginary axis onto the real axis and get the vector b" = - 2 + 0i. Thus we obtain the complex number that u" = - 2 + 1.5i by making vector summation. We could also say that multiplication of u= 3 + 4i with $1/2$ i means that

cutting in half the magnitude of u and 90° counterclockwise rotation of the half vector to get u".

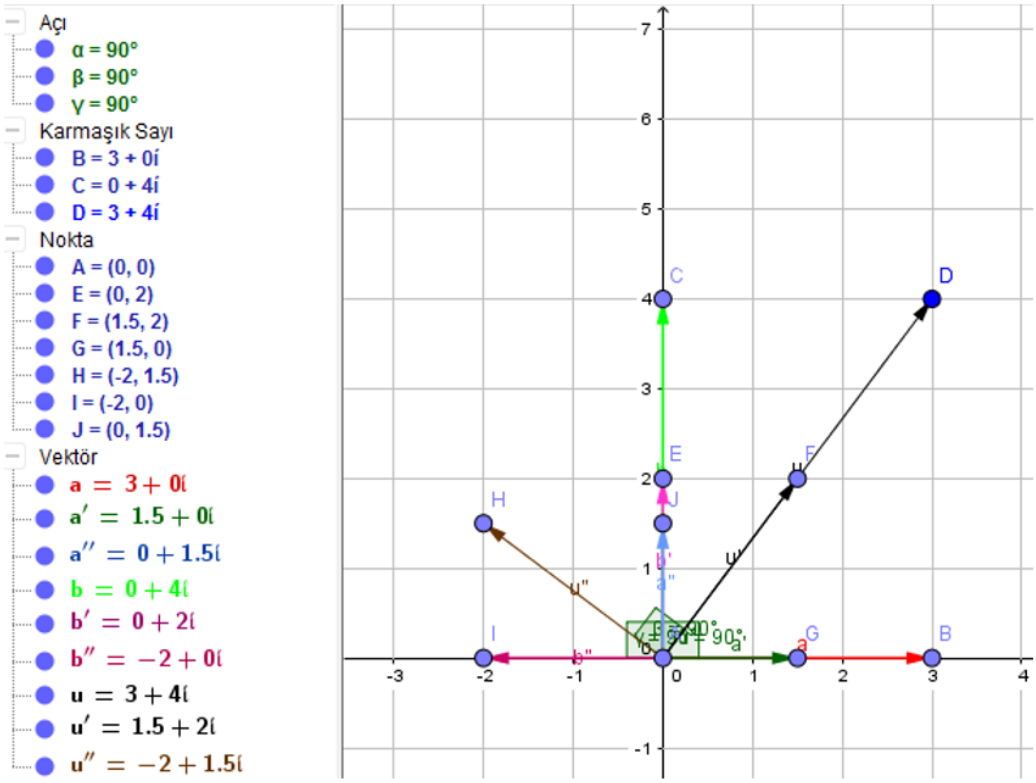


Figure A.15. Articulation of the Understanding 15.

When we multiply z = a + bi with ki, where k is a positive real number, we firstly multiply each projection vector of z by k and get the complex number kz = ka + kbi. In this case, if $0 < k < 1$ (see Figure A.16), then we decrease the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by k, i. e., when k is a positive real number, we dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by k without making any change in their directions and obtain $k\vec{a}$ and $k\vec{b}$, where $k\vec{a}$ is the projection of the new vector on the real axis and $k\vec{b}$ is the projection of the new vector on the imaginary axis. Then we multiply each projection vector with i, so we rotate $k\vec{a}$ and $k\vec{b}$ vectors in 90° counterclockwise in order to get- $k\vec{b}$ on the real axisand $k\vec{a}$ on the imaginary axis, Thus if we sum the projection vectors on the complex axis, we get the complex number kz = - kb + kai. Since the directions of the projection vectors change in 90° counterclockwise, the direction of the new vector kz rotates 90° counterclockwise from the direction of z, so when we look at the polar coordinates of the complex numbers z and kz, we see that the angle of kz is equal to

90° plus the angle of z, but the magnitude of kz is equal to k times the magnitude of z.

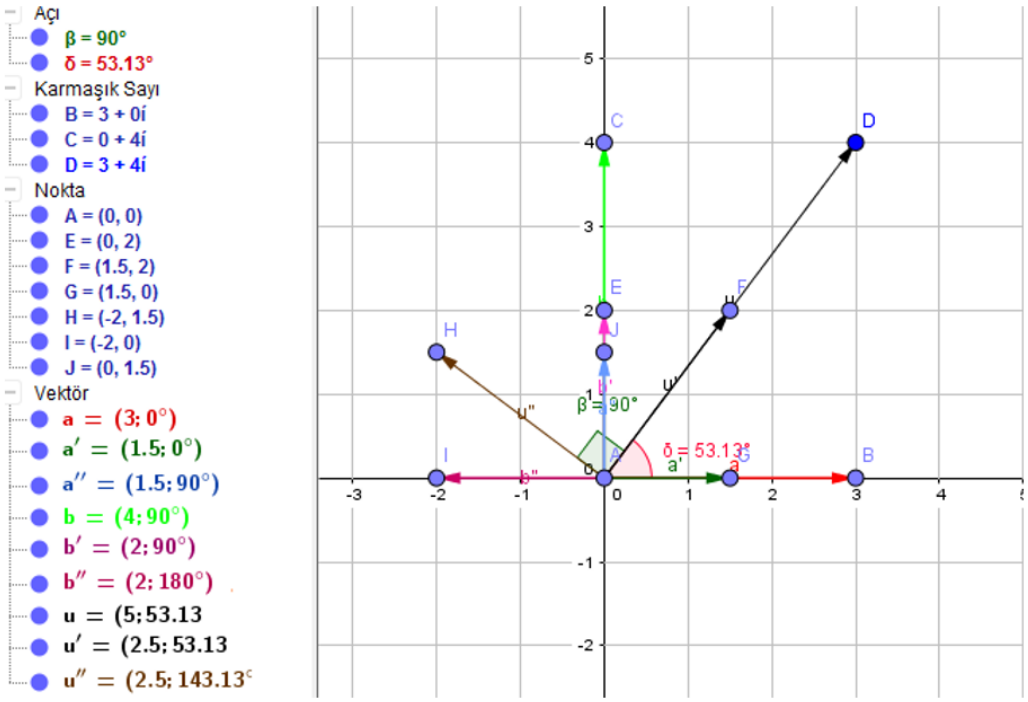


Figure A.16. Articulation of the Understanding 16.

When we multiply z = a + bi with ki where k < - 1, firstly we think that we multiply z by k and get the complex number kz = ka + kbi. This means that we dilate (increase) the magnitudes of $\vec{a}$ and $\vec{b}$ by $|k|$ by changing their directions in reverse direction. Then we think that we multiply kz with i so we rotate each projection vector, k$\vec{a}$ and k$\vec{b}$ by 90° in the counterclockwise. Thus, we just carry the vector on the real axis onto the imaginary axis and the vector on the imaginary axis onto the real axis without making any change in their magnitude and get the complex number that z' = - kb + kai.

For example (see Figure A.17), when we multiply u= 3 + 4i with - 2i, we complete three phases. Firstly we think that we multiply u by 2 and get the complex number u' = 2u = 6 + 8i. Then we reverse the directions of the projection vectors, $\vec{a}'$ = 6 + 0i and $\vec{b}'$ = 0 + 8i and move their starting points to the origin, so obtain - $\vec{a}'$ = $\vec{a}$" = - 6 + 0i and - $\vec{b}'$ = $\vec{b}$" = 0 - 8i. Secondly, we think that we multiply -2u with i and this means that we rotate each projection vector, -$\vec{a}'$ = $\vec{a}$" = - 6 + 0i and - $\vec{b}'$ = $\vec{b}$" = 0 - 8

iby 90° in the counterclockwise without any change in their magnitude. Thus, we just carry the vector a" on the real axis onto the imaginary axis and get the new vector a" = 0 - 6i. Similarly, we just carry the vector b" on the imaginary axis onto the real axis and get the vector b" = 8 + 0i. Thus, finally we obtain the complex number that u" = 8 - 6i by making vector summation. We couldalso summarize the multiplication of u = 3 + 4i with -2i such that we dilate the magnitude of u by 2, reverse its direction and then rotate 90° counterclockwise of the dilated vector -2u".

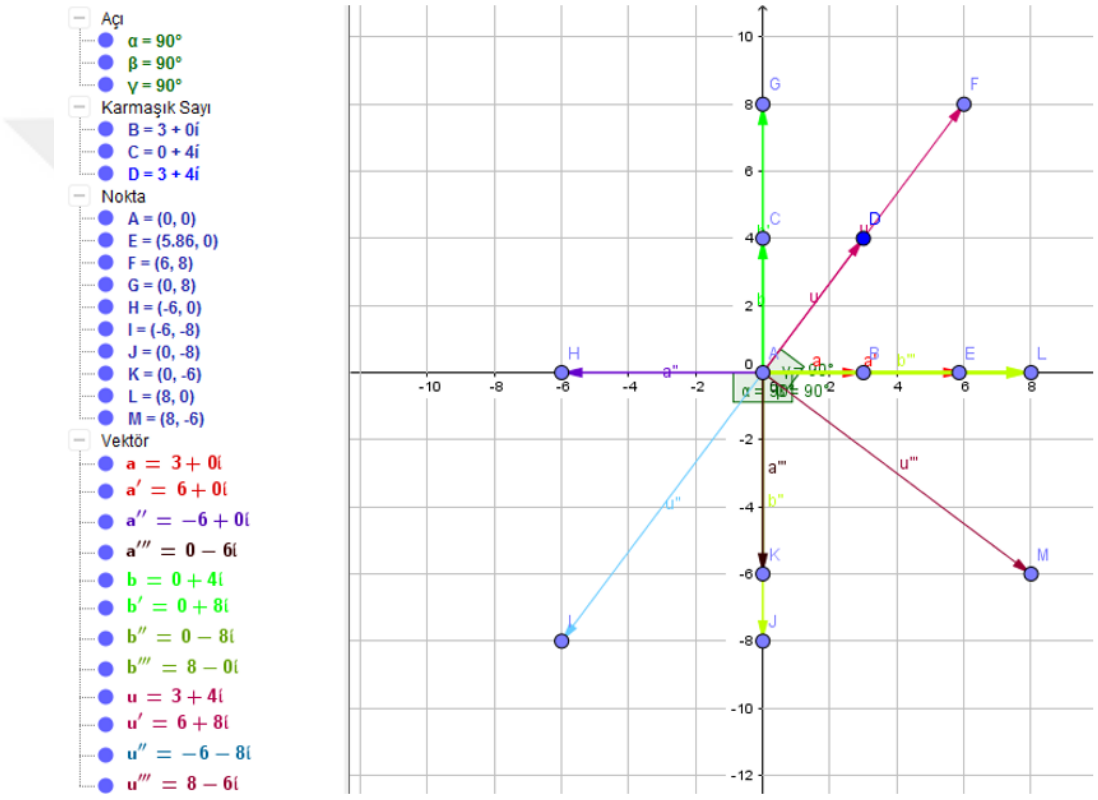


Figure A.17. Articulation of the Understanding 17.

When we multiply z = a + bi with ki, where k is a negative real number, we firstly multiply each projection vector of z by $|k|$ and get the complex number $|k|z = |k|a + |k|bi$. In this case, if k < - 1 (see Figure A.18), we increase the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by $|k|$, i. e., when k is a negative real number, we firstly dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by $|k|$ without making any change in their directions and obtain $|k|$ $\vec{a}$ and $|k|\vec{b}$, where $|k|\vec{a}$ is the projection of the new vector on the real axis and $|k|\vec{b}$ is the projection of the new vector on the imaginary axis. After that we make 180°

counterclockwise rotation to match $|k|\vec{a}$ and $|k|\vec{b}$ with k$\vec{a}$ and k$\vec{b}$ vectors, which are the projection vectors of the complex number kz.Then we multiply each projection vector with i, so we rotate k$\vec{a}$ and k$\vec{b}$ vectors in 90° counterclockwise in order to get- k$\vec{b}$ on the real axisand k$\vec{a}$ on the imaginary axis, Thus if we sum the projection vectors on the complex axis, we get the complex number kz = - kb + kai. Since the directions of the projection vectors change in firstly 180° and then 90° counterclockwise, the direction of the new vector kz rotates $180^o + 90^o = 270^o$ counterclockwise from the direction of z, so when we look at the polar coordinates of the complex numbers z and kz, we see that the angle of kz is equal to 270° plus the angle of z, but the magnitude of kz is equal to k times the magnitude of z.

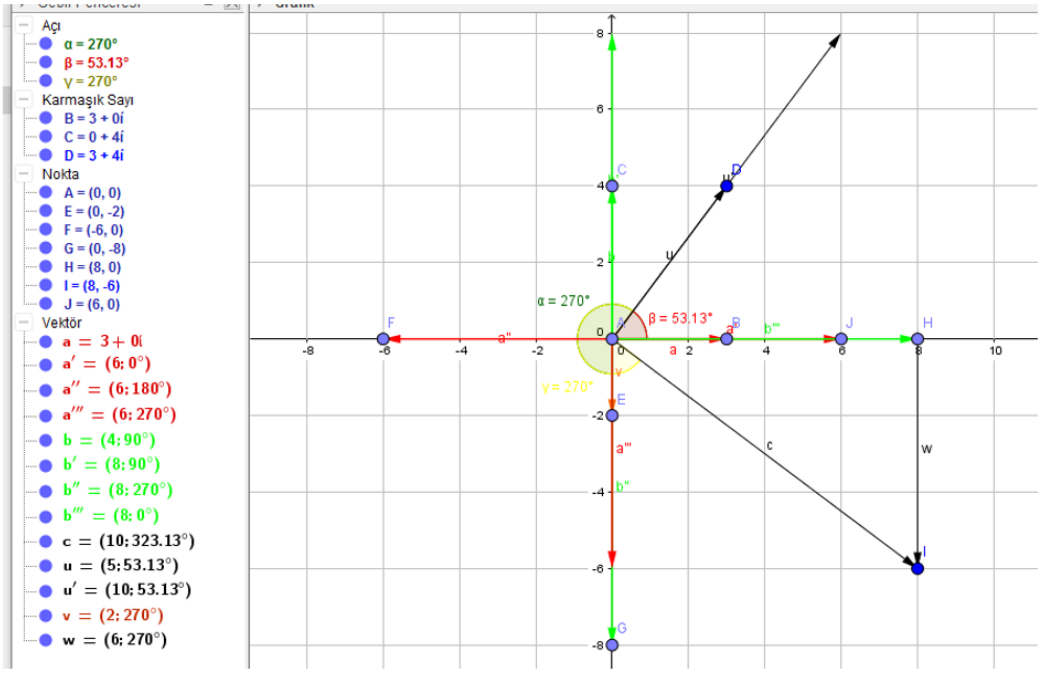


Figure A.18. Articulation of the Understanding 18.

When we multiply z = a + bi with ki where -1 < k < 0, firstly we think that we multiply z by k and get the complex number kz = ka + kbi. This means that we dilate (decrease) the magnitudes of $\vec{a}$ and $\vec{b}$ by $|k|$ by reversing their directions. Then we think that we multiply kz with i so we rotate each projection vector, k$\vec{a}$ and k$\vec{b}$ by 90° in the counterclockwise. Thus, we just carry the vector on the real axis onto the imaginary axis and the vector on the imaginary axis onto the real axis without making any change in their magnitude and get the complex number that z' = - kb + kai.

For instance (see Figure A.19), when we multiply u = 3 + 4i with - $\frac{1}{2}$ i, firstly we think that we multiply u by $\frac{1}{2}$ and get the complex number u' = $\frac{1}{2}$ u = 1.5 + 2i. After that we reverse the direction of the each projection vectors of $\vec{a}'$ = 1.5 + 0i and $\vec{b}'$ = 0 + 2i and move their starting points to the origin. Thus we obtain $\vec{a}$" = - 1.5 + 0i and $\vec{b}$"= 0 - 2i vectors. Then we think that we multiply u' = - $\frac{1}{2}$ u with i and this means that we rotate each projection vector, $\vec{a}$" = - 1.5 + 0i and $\vec{b}$"= 0 - 2i by 90° in the counterclockwise without any change in their magnitude. Thus, we just carry the vector a" on the real axis onto the imaginary axis and get the new vector a"' = 0 - 1.5i. Similarly, we just carry the vector b' on the imaginary axis onto the real axis and get the vector b"' = 2 + 0i. Thus we obtain the complex number that u"'= 2 - 1.5i by making vector summation. We could also say that multiplication of u = 3 + 4i with -$\frac{1}{2}$ i means that cutting in half the magnitude of u, reverse the direction and rotate the half vector 90° counterclockwise to get u".

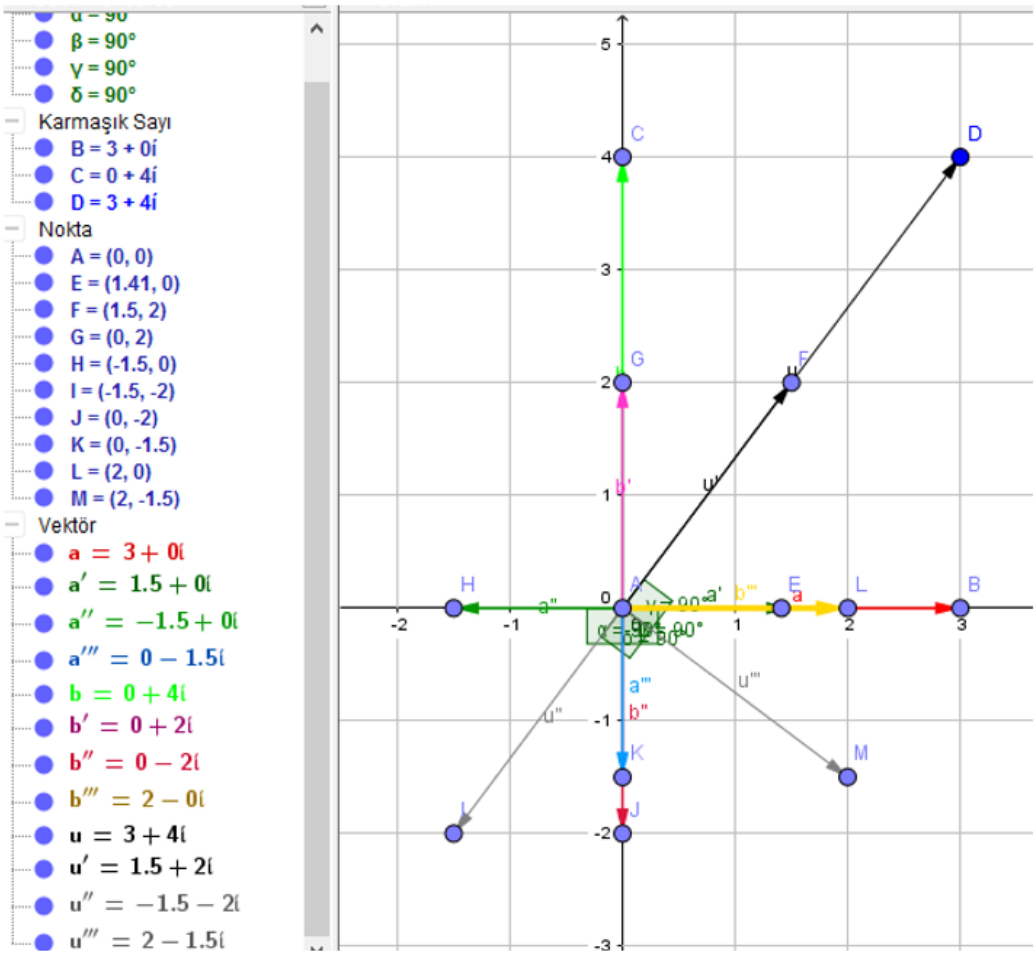


Figure A.19. Articulation of the Understanding 19.

When we multiply z = a + bi with ki, where k is a negative real number, we firstly multiply each projection vector of z by $|k|$ and get the complex number $|k|z = |k|a+|k|bi$. In this case, if -1 < k < 0 (see Figure A.20), thenwe decrease the magnitudes of the projection vectors $\vec{a}$ and $\vec{b}$ by $|k|$, i. e., when k is a negative real number, we firstly dilate the magnitudes of $\vec{a}$ and $\vec{b}$ by $|k|$ without making any change in their directions and obtain $|k|\vec{a}$ and $|k|\vec{b}$, where $|k|\vec{a}$ is the projection of the new vector on the real axis and $|k|\vec{b}$ is the projection of the new vector on the imaginary axis. After that we make 180° counterclockwise rotation to match $|k|\vec{a}$ and $|k|\vec{b}$ with k$\vec{a}$ and k$\vec{b}$ vectors, which are the projection vectors of the complex number kz.Then we multiply each projection vector with i, so we rotate k$\vec{a}$ and k$\vec{b}$ vectors in 90° counterclockwise in order to get - k$\vec{b}$ on the real axisand k$\vec{a}$ on the imaginary axis, Thus if we sum the projection vectors on the complex axis, we get the complex number kz = - kb + kai. Since the directions of the projection vectors change in firstly 180° and then 90° counterclockwise, the direction of the new vector kz rotates $180^o + 90^o = 270^o$ counterclockwise from the direction of z, so when we look at the polar coordinates of the complex numbers z and kz, we see that the angle of kz is equal to 270° plus the angle of z, but the magnitude of kz is equal to k times the magnitude of z.

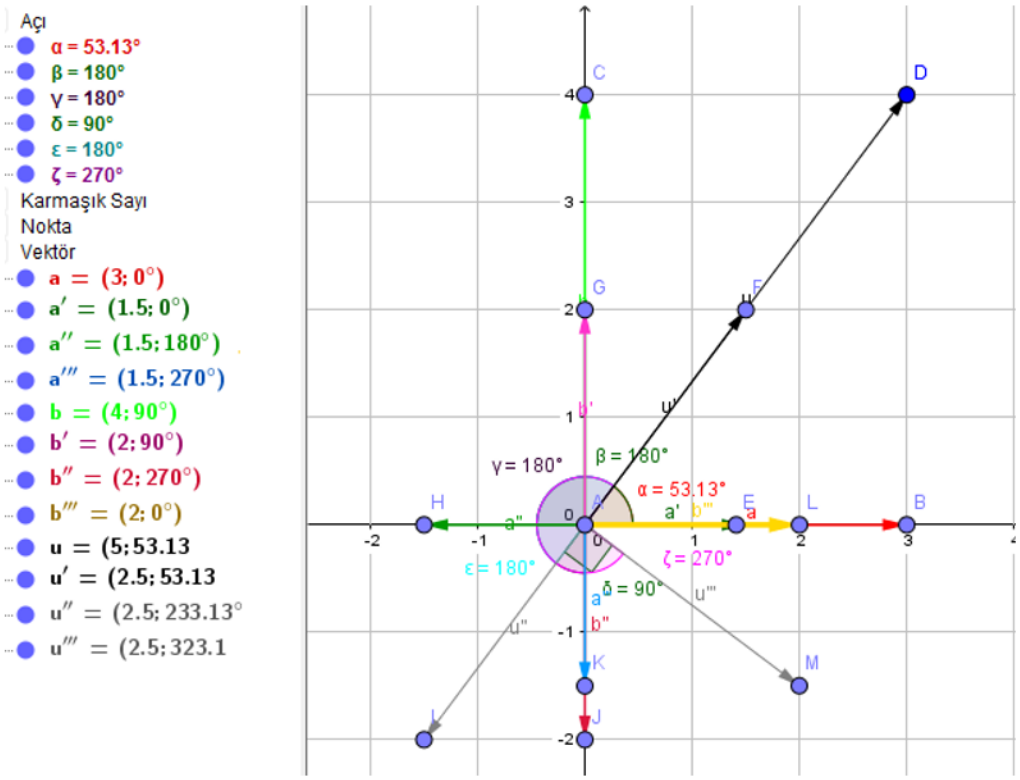


Figure A.20. Articulation of the Understanding 20.

When we multiply z = a + bi with another complex number, z' = c + di, we think that we multiply a + bi with c and di one by one and get two vectors [c.(a + bi) and di.(a + bi)].Then using vector summation [c.(a + bi) + di.(a + bi) = [(a + bi).(c + di)]] we get the vector of z.z' = (a + bi).(c + di).

For instance (see Figure A.21), when we multiply u = 3 + 4i with z = 2 + 2i, firstly we find the projection vectors a'= 6 + 0i and b' = 0 + 8i of the vector u'= 2.(3 + 4i) = 6 + 8i by multiplying the magnitudes of the projection vectors a = 3 + 0i and b = 0 + 4i by 2. Then, in order to find u"= 2i.(3 + 4i) = - 8 + 6i, we rotate a' and b' vectors in 90° counterclockwise and we find a"= 0 + 6i and b" = - 8 + 0i vectors. Then when we sum these vectors a" and b", we find u". In order to find the w = (3 + 4i).(2 + 2i) = - 2 + 14i, firstly we move the u' vector to the final point of the vector u" = - 8 + 6i (that is named as v) and find the vector with the help of vector summation.

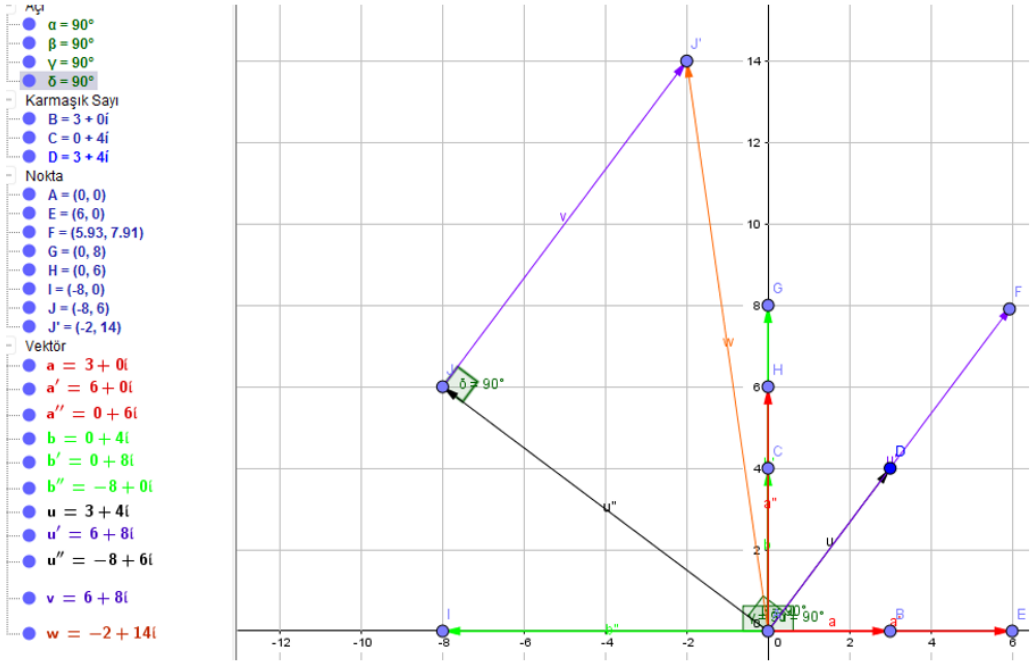


Figure A.21. Articulation of the Understanding 21.

For example (see Figure A.22), when we multiply u = 3 + 4i with z = $^1/_2$ + $^1/_2$i, firstly we find the projection vectors a'= 1.5 + 0i and b' = 0 + 2i of the vector u'= $^1/_2$.(3 + 4i) = 1.5 + 2i by multiplying the magnitudes of the projection vectors a = 3 + 0i and b = 0 + 4i by $^1/_2$. Then, in order to find u"= $^1/_2$i.(3 + 4i) = - 2 + 1.5i, we rotate a' and b' vectors in 90° counterclockwise and we find a"= 0 + 1.5i and b" = - 2 + 0i vectors. Then when we sum these vectors a" and b", we find u". In order to find the c = (3 + 4i).($^1/_2$ + $^1/_2$ i) = - 0.5 + 3.5 i, firstly we move the u' vector to the final point of the vector u" = - 2 + 1.5i and find the vector c with the help of vector summation.

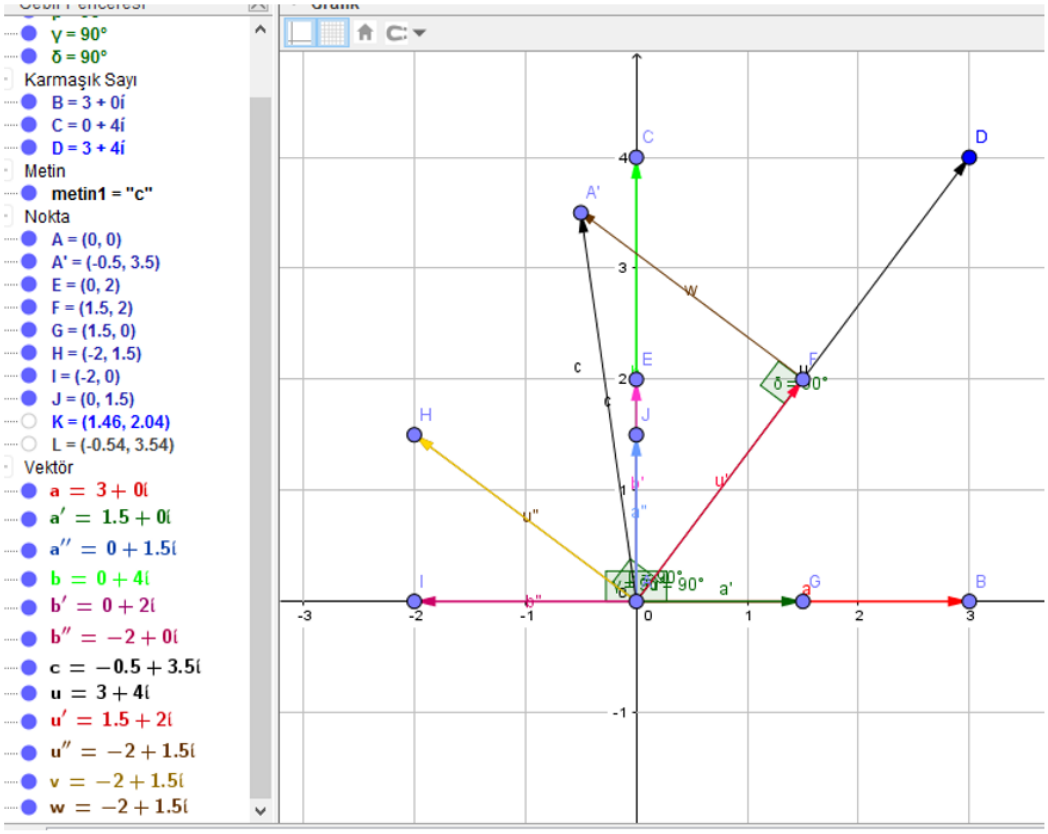


Figure A.22. Articulation of the Understanding 22.

As another example (see Figure A.23), when we multiply u = 3 + 4i with z = ½ - ½i, firstly we find the projection vectors a'= 1.5 + 0i and b' = 0 + 2i of the vector u'= ½.(3 + 4i) = 1.5 + 2i by multiplying the magnitudes of the projection vectors a = 3 + 0i and b = 0 + 4i by ½. Then, in order to find u"= - ½i. (3 + 4i) = 2 - 1.5i, firstly we think that we multiply u by ½ and we find the projection vectors a'= 1.5 + 0i and b' = 0 + 2i of the vector u'= ½.(3 + 4i) = 1.5 + 2i. After that we reverse the direction of the each projection vectors of $\vec{a}'$ = 1.5 + 0i and $\vec{b}'$ = 0 + 2i and move their starting points to the origin. Thus we obtain $\vec{a}$" = - 1.5 + 0i and $\vec{b}$"= 0 - 2i vectors. Then we think that we multiply u" = - ½u with i and this means that we rotate each projection vector, $\vec{a}$" = - 1.5 + 0i and $\vec{b}$"'= 0 - 2i by 90° in the counterclockwise without any change in their magnitude. Thus, we just carry the vector a" on the real axis onto the imaginary axis and get the new vector a"' = 0 - 1.5i. Similarly, we just carry the vector b' on the imaginary axis onto the real axis and get the vector b"' = 2 + 0i. Thus we obtain the complex number that u"' = 2

- 1.5i by making vector summation. We could also say that multiplication of u= 3 + 4i with -½ i means that cutting in half the magnitude of u, reverse the direction and 90° counterclockwise rotation of the half vector to get u". In order to find the v = (3 + 4i).(½ - ½ i) = 3.5 + 0.5 i, firstly we move the u' vector to the final point of the vector u" = 3.5 + 0.5i and find the vector c with the help of vector summation.

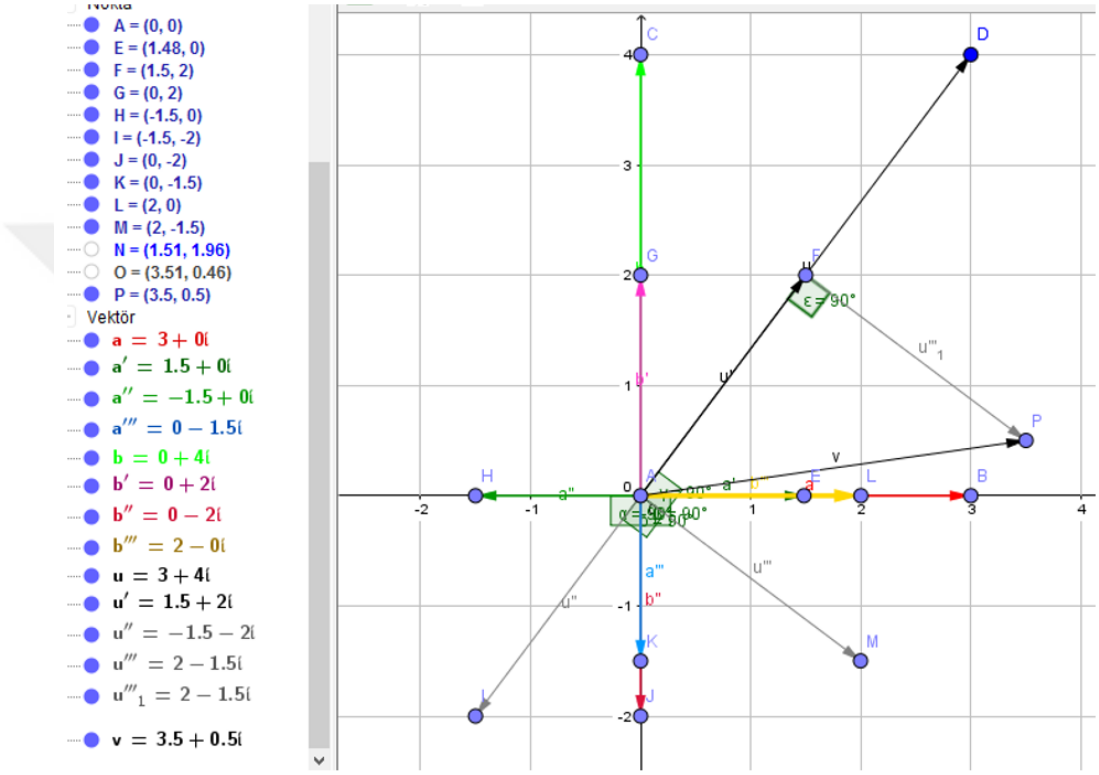


Figure A.23. Articulation of the Understanding 23.

When we multiply z = a + bi with another complex number, z' = c + di where c - 0 and d ≠ 0, as we said above we think that we multiply a + bi with c and di one by one and get two vectors [c.(a + bi) and di.(a + bi)]. Then using vector summation [c.(a + bi) + di.(a + bi) = [(a + bi).(c + di)]] we get the vector of z.z' = (a + bi).(c + di). In such a situation, the cases for the real part and the imaginary part of the multiplier complex number gain importance, since we consider that we multiply a + bi with c and di one by one, we treat c.(a + bi) as we said before according to the value of c, which are - 1 < c, c = - 1, - 1 < c < 0, 0 > c > 1, c = 1 and c > 1. Similarly, we treat di.(a + bi) as we said before according to the value of d, which are

- 1 < d, d = - 1, - 1 < d < 0, 0 > d > 1, d = 1 and d > 1. If we want to relate vector operations to complex number operations in polar coordinates, we take into consideration the changes in magnitude and angle of the complex number when it is multiplied with a real number and an imaginary number separately. Then we carry the projection vector di.(a + bi) into the end of the projection vector c.(a + bi) in order to get the vector that c.(a + bi) + di.(a + bi) = (a + bi).(c + di). When the situation is thus, the projection vectors form a right triangle, whose hypotenuse is the complex number (a + bi).(c + di). If c and d are both positive real numbers, then they cause a θ° counterclockwise rotation, because when c and d are both positive real numbers (see Figure A.24), c.(a + bi) vector has the same direction with a + bi and has a magnitude, which is equal to c times the magnitude of a + bi. However, di.(a + bi) vector rotates 90° from the vector a + bi in the counterclockwise and has a magnitude, which is equal to d times the magnitude of a + bi. Since we search for these two projection vectors' summation c.(a + bi) + di.(a + bi), we focus on the acute angle between them, which is 90° in the counterclockwise. In order to sum the vectors geometrically, we should carry the di.(a + bi) vector to the end of the c.(a + bi) vector without changing the angle between them. At the end of the carrying operation, a right triangle, whose legs are the vectors c.(a + bi) and di.(a + bi) and hypotenuse is c.(a + bi) + di.(a + bi) = (a + bi).(c + di), is formed. Since c.(a + bi) has the same direction with a + bi, the acute angle between the c.(a + bi) vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di), is the angle between a + bi and (a + bi).(c + di) at the same time. In other words, the rotation between a + bi and (a + bi).(c + di) is equal to the amount of the acute angle between the c.(a + bi) vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di). In order to calculate this acute angle, we could utilize the trigonometric proportions in a right triangle. At this time, the tangent of this acute angle is equal to the proportion of the magnitude of the vector di.(a + bi) to the magnitude of the vector c.(a + bi). Since the magnitude of the vector di.(a + bi) is d times the magnitude of a + bi and the magnitude of the vector c.(a + bi) is equal to c times the magnitude of a + bi (i.e, $||a + bi||$), then they are rated, the proportion is equal to $\frac{d}{c}$. Thus,the tangent of this acute angle is equal to $\frac{d}{c}$ and since d and c are positive real numbers, the proportion is also positive and this means the acute angle requires a counterclockwise rotation (see

Figure A.25, $\tan \varepsilon = \frac{d}{c} = 1$,so $\varepsilon = 45^o$). If we take into consideration the magnitude of the vector that is c.(a + bi) + di.(a + bi) = (a + bi).(c + di), we use the Pythagorean Theorem. According to theorem, when we take the square of the magnitude of the legs of the right triangle, we reach the magnitude of the hypotenuse. At that point, when we take the square of the magnitude of c.(a + bi) and di.(a + bi), which are equal to $c^2.(||a + bi||^2)$ and $d^2.(||a + bi||^2)$, respectively. When we sum $c^2.(||a + bi||^2)$ and $d^2.(||a + bi||^2)$, we obtain $c^2.(||a + bi||^2) + d^2.(||a + bi||^2) = (c^2 + d^2).(||a + bi||^2)$ and this is the square of the magnitude of the magnitude of (a + bi).(c + di). When we take the square root of $(c^2 + d^2).(||a + bi||^2)$, we reach the magnitude of (a + bi).(c + di), which is equal to $\sqrt{(c^2 + d^2)\left(||a + bi||\right)^2} = \sqrt{(c^2 + d^2)}||a + bi||$, i.e., di times the magnitude of a + bi, where the magnitude of c + di ($||c + di||$) is equal to $\sqrt{(c^2 + d^2)}$). Consequently, when we multiplyz = a + bi with another complex number, z' = c + di where c > 0 and d > 0, then we dilate the magnitude of z by the magnitude of z' and also rotate the dilated vector by θ° in the counterclockwise. So, the magnitude of (a + bi).(c + di) is equal to $|z|.|z'|$ and the angle of (a + bi).(c + di) is equal to the angle of (a + bi) plus the acute angle between the c.(a + bi) vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di).(see Figure A.24, the angle - of the vector (3 + 4i).(2 + 2i) is equal to $53{,}13^o + 45^o = 98.13^o$).

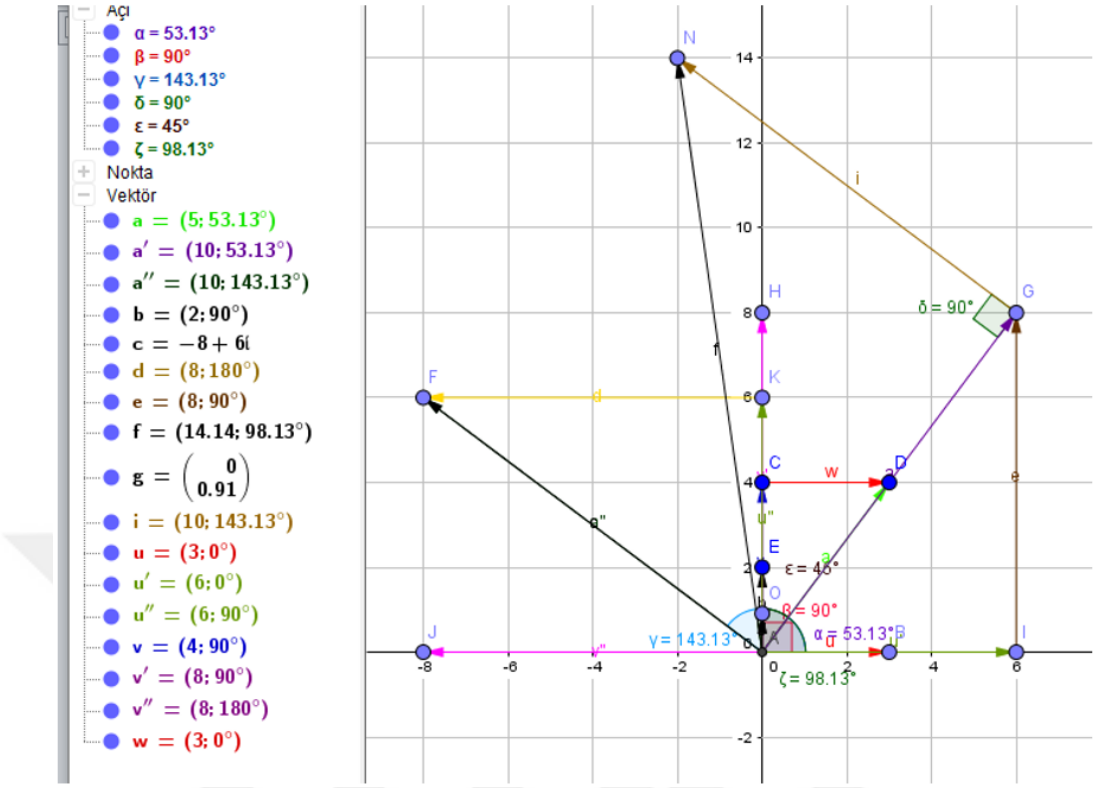


Figure A.24. Articulation of the Understanding 24.

While the projection vectors form a right triangle, whose hypotenuse is the complex number (a + bi).(c + di), if c is a positive real number and d is a negative real number (see Figure A.25), c.(a + bi) vector has the same direction with a + bi and has a magnitude, which is equal to c times the magnitude of a + bi. However, di.(a + bi) vector rotates 270° (180° counterclockwise rotation is because of multiplication with a negative real number and then 90° counterclockwise rotation is because of multiplication with i) from the vectora + bi in the counterclockwise and has a magnitude, which is equal to $|d|$ times the magnitude of a + bi. Since we search for these two projection vectors' summation c.(a + bi) + di.(a + bi), we focus on the acute angle between them, which is 270° in the counterclockwise or in other words, 90° in the clockwise. In order to sum the vectors geometrically, we should carry the di.(a + bi) vector to the end of thec.(a + bi) vector without changing the angle between them. At the end of the carrying operation, a right triangle, whose legs are the vectors c.(a + bi) and di.(a + bi) and hypotenuse is c.(a + bi) + di.(a + bi) = (a + bi).(c + di), is formed. Since c.(a + bi) has the same direction with a + bi, the acute angle between the c.(a + bi)

vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di), is the angle between a + bi and (a + bi).(c + di) at the same time. In other words, the rotation between a + bi and (a + bi).(c + di) is equal to the amount of the acute angle between the c.(a + bi) vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di). In order to calculate this acute angle, we could utilize the trigonometric proportions in a right triangle. At this time, the tangent of this acute angle is equal to the proportion of the magnitude of the vector di.(a + bi) to the magnitude of the vector c.(a + bi).Since the magnitude of the vector di.(a + bi) is d times the magnitude of a + bi and the magnitude of the vector c.(a + bi) is equal to c times the magnitude of a + bi (i.e, $|a + bi|$), then they are rated, the proportion is equal to $\frac{|d|}{|c|}$, (see Figure A.26, tan $\zeta = \frac{1/2}{1/2}$, so $\zeta = 45^o$). Thus,the tangent of this acute angle is equal to $\frac{|d|}{|c|}$, and since d is a negative real number and c is a positive real number, the acute angle requires a clockwise rotation. If we take into consideration the magnitude of the vector that is c.(a + bi) + di.(a + bi) = (a + bi).(c + di), we use the Pythagorean Theorem. According to theorem, when we take the square of the magnitude of the legs of the right triangle, we reach the magnitude of the hypotenuse. At that point, when we take the square of the magnitude of c.(a + bi) and di.(a + bi), which are equal to c^2. ($||a+bi||^2$) and ($|d|^2$. ($||a+bi||^2$), respectively. When we sum c^2. ($||a+bi||^2$) and ($|d|^2$). ($||a+bi||^2$), we obtain c^2. ($||a+bi||^2$) + ($|d|^2$). ($||a+bi||^2$)= (c^2 + d^2). ($||a+bi||^2$) and this is the square of the magnitude of the magnitude of (a + bi).(c + di). When we take the square root of (c^2 + d^2). ($||a+bi||^2$), we reach the magnitude of (a + bi).(c + di), which is equal to $\sqrt{(c^2+d^2)\left(||a+bi||\right)^2} = \sqrt{(c^2+d^2)}||a+bi||$ $|a+bi|$, i.e., the magnitude of c + di times the magnitude of a + bi, where the magnitude of c + di ($||c+di||$) is equal to $\sqrt{(c^2+d^2)}$. Consequently, when we multiply z = a + bi with another complex number, z' = c + di where c > 0 and d < 0, then we dilate the magnitude of z by the magnitude of z' and also rotate the dilated vector θ^o in the clockwise. So, the magnitude of (a + bi).(c + di) is equal to $|z|.|z'|$ and the angle of (a + bi).(c + di) is equal to the angle of (a + bi) minus the acute angle between the c.(a + bi) vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di).(see Figure A.25, the angle - of the vector (3 + 4i).(2 + 2i) is equal to $53{,}13^o$ - 45° = 8.13^o).

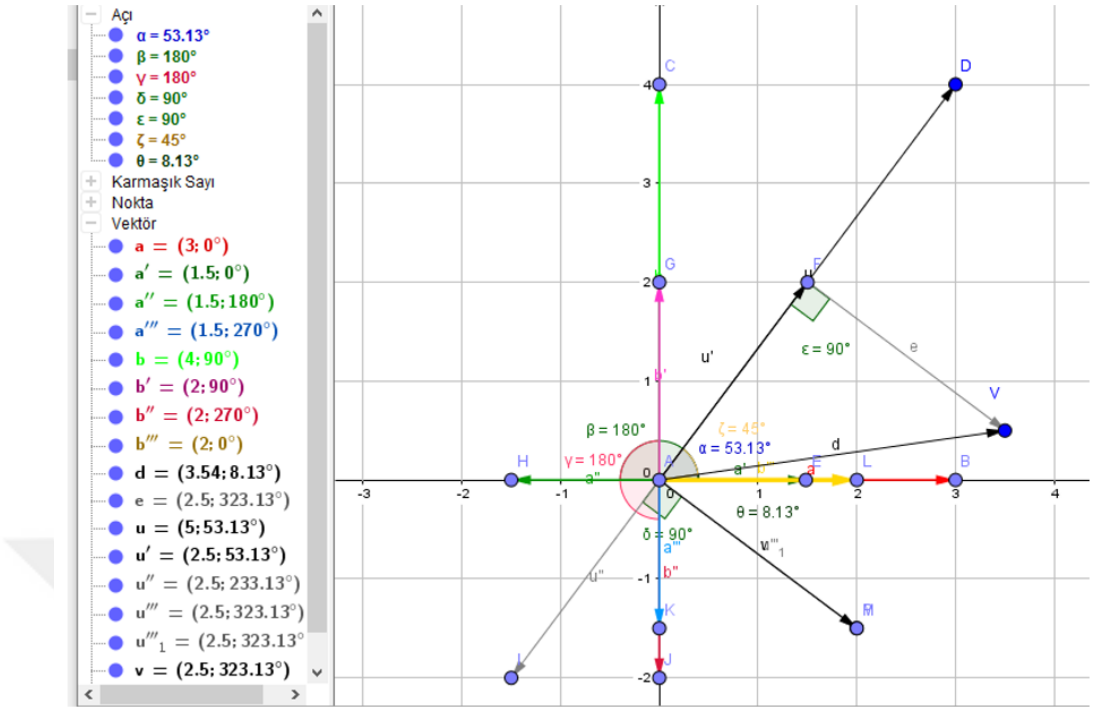


Figure A.25. Articulation of the Understanding 25.

Similarly, if c is a negative real number and d is a positive real number, When c is a negative real number and d is a positive real number (see Figure A.26), c.(a + bi) vector has the reverse direction with a + bi, i. e., it rotates 180° from the vectora + bi in the counterclockwise and has a magnitude, which is equal to $|c|$ times the magnitude of a + bi. On the other hand, since d is a positive real number, d. (a + bi) vector does not rotate from the vectora + bi and has a magnitude, which is equal to dtimes the magnitude of a + bi. Then, we multiply d.(a + bi) by i and so, di.(a + bi) vector rotates 90° from the vectord .(a + bi) in the counterclockwise. Since we search for these two projection vectors' summation c.(a + bi) + di.(a + bi), we focus on the acute angle between them, which is 90° in the counterclockwise. In order to sum the vectors geometrically, we should carry the di.(a + bi) vector to the end of the c.(a + bi) vector without changing the angle between them. At the end of the carrying operation, a right triangle, whose legs are the vectors c.(a + bi) and di.(a + bi) and hypotenuse is c.(a + bi) + di.(a + bi) = (a + bi).(c + di), is formed. Since c.(a + bi) has the reverse direction with a + bi, the acute angle between the c.(a + bi) vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di),

is 180° more from the angle between a + bi and (a + bi).(c + di). In other words, the c.(a + bi) vector is 180° further to the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di) when it is compared with the vector a + bi. In order to calculate this acute angle,we could utilize the trigonometric proportions in a right triangle. At this time, the tangent of this acute angle is equal to the proportion of the magnitude of the vector di.(a + bi) to the magnitude of the vector c.(a + bi). Since the magnitude of the vector di.(a + bi) is d times the magnitude of a + bi (i.e., $||a + bi||$) and the magnitude of the vector c.(a + bi) is equal to $|c|$ times the magnitude of a + bi, then they are rated, the proportion is equal to $\frac{d}{|c|}$. Thus,the tangent of this acute angle is equal to $\frac{d}{|c|}$ and since c is a negative real number and d is a positive real number, this means the acute angle requires a clockwise rotation (see Figure A.26), tan $\beta = \frac{2}{2}$, so $\beta = 45^o$. If we take into consideration the magnitude of the vector that is c.(a + bi) + di.(a + bi) = (a + bi).(c + di), we use the Pythagorean Theorem. According to theorem, when we take the square of the magnitude of the legs of the right triangle, we reach the magnitude of the hypotenuse. At that point, when we take the square of the magnitude of c.(a + bi) and di.(a + bi), which are equal to $(|c|^2).(||a + bi||^2)$ and $d^2.(||a + bi||^2)$, respectively. When we sum $(|c|^2).(||a + bi||^2)$ and $d^2.(||a + bi||^2)$, we obtain $(|c|^2).(||a + bi||^2) + d^2.(||a + bi||^2) = (c^2 + d^2).(||a + bi||^2)$ and this is the square of the magnitude of the magnitude of (a + bi).(c + di). When we take the square root of $(c^2 + d^2).(||a + bi||^2)$, we reach the magnitude of (a + bi).(c + di), which is equal to $\sqrt{(c^2 + d^2)\left(||a + bi||\right)^2} = \sqrt{(c^2 + d^2)}||a + bi||$, i.e., the magnitude of c + di (i.e., $||c + di||$) times the magnitude of a + bi, where the magnitude of c + di is equal to $\sqrt{(c^2 + d^2)}$. Consequently, when we multiplyz = a + bi with another complex number, z' = c + di where c < 0 and d >0, then we dilate the magnitude of z by the magnitude of z' and also rotate the dilated vector 180° - θ^o in the counterclockwise. So, the magnitude of (a + bi).(c + di) is equal to $|z|.|z'|$ and the angle of (a + bi).(c + di) is equal to the sum of the angle of (a + bi) with 180° minus the acute angle between the c.(a + bi) vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di).(see Figure A.26, the angle - of the vector (3 + 4i).(- 2 + 2i) is equal to $53.13^o + (180^o - 45^o) = 53.13^o + 135^o = 188.13^o$).

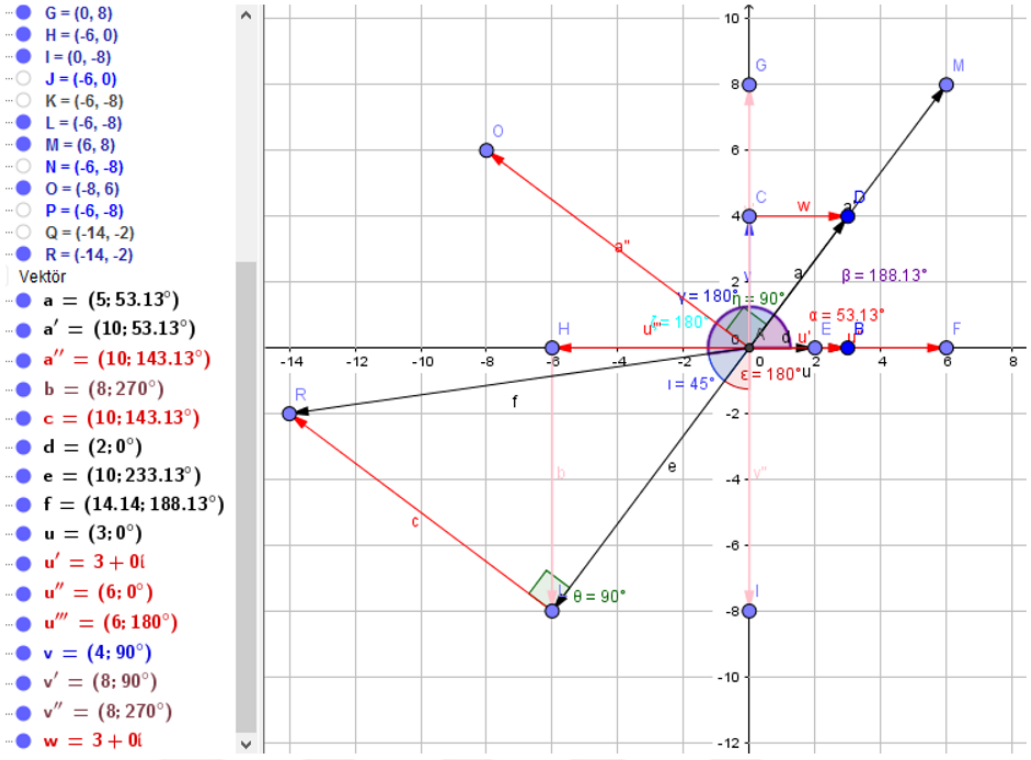


Figure A.26. Articulation of the Understanding 26.

Moreover, if c is a positive real number and d is a negative real number, then they cause a counterclockwise rotation (see Figure A.27), c.(a + bi) vector has the reverse direction with a + bi, i. e., it rotates 180° from the vectora + bi in the counterclockwise and has a magnitude, which is equal to $|c|$ times the magnitude of a + bi. On the other hand, since d is also a negative real number, d.(a + bi) vector rotates 180° from the vector a + bi in the counterclockwise and has a magnitude, which is equal to $|d|$ times the magnitude of a + bi. Then, we multiply d.(a + bi) by i and so, di.(a + bi) vector rotates 90° from the vector d.(a + bi) in the counterclockwise.Therefore, di.(a + bi) vector rotates totally $180^o + 90^o = 270^o$ in the counterclockwise. Since we search for these two projection vectors' summation c.(a + bi) + di.(a + bi), we focus on the acute angle between them, which is 90° in the counterclockwise. In order to sum the vectors geometrically, we should carry the di.(a + bi) vector to the end of the c.(a + bi) vector without changing the angle between them. At the end of the carrying operation, a right triangle, whose legs are the vectors c.(a + bi) and di.(a + bi) and hypotenuse is c.(a + bi) + di.(a + bi) = (a + bi).(c + di), is formed. Since c.(a + bi) has the reverse

direction with a + bi, the acute angle between the c.(a + bi) vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di), is 180° more from the angle between a + bi and (a + bi).(c + di). In other words, the c.(a + bi) vector is 180° further to the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di) when it is compared with the vector a + bi. In order to calculate this acute angle,we could utilize the trigonometric proportions in a right triangle. At this time, the tangent of this acute angle is equal to the proportion of the magnitude of the vector di.(a + bi) to the magnitude of the vector c.(a + bi).Since the magnitude of the vector di.(a + bi) is $|d|$ times the magnitude of a + bi (i.e, $||a+bi||$) and the magnitude of the vector c.(a + bi) is equal to $|c|$ times the magnitude of a + bi, then they are rated, the proportion is equal to $\frac{||d||}{||c||}$. Thus, the tangent of this acute angle is equal to $\frac{||d||}{||c||}$ and since c and d are negative real numbers, their ratio is positive and this means the acute angle requires a counterclockwise rotation (see Figure A.27, $\tan \eta = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$, so $\eta = 45^o$. If we take into consideration the magnitude of the vector that is c.(a + bi) + di.(a + bi) = (a + bi).(c + di), we use the Pythagorean Theorem.

According to theorem, when we take the square of the magnitude of the legs of the right triangle, we reach the magnitude of the hypotenuse. At that point, when we take the square of the magnitude of c.(a + bi) and di.(a + bi), which are equal to $(|c|^2).(||a + bi||^2)$ and $(|d|^2).(||a + bi||^2)$, respectively. When we sum $(|c|^2).(||a + bi||^2)$ and $(|d|^2).(||a + bi||^2)$, we obtain $(|c|^2).(||a + bi||^2) + (c^2.|d|^2).(||a + bi||^2)$= $(c^2 + d^2).(||a + bi||^2)$ and this is the square of the magnitude of the magnitude of (a + bi).(c + di). When we take the square root of $(c^2 + d^2).(||a + bi||^2)$, we reach the magnitude of (a + bi).(c + di), which is equal to $\sqrt{(c^2 + d^2)\left(||a + bi||\right)^2} = \sqrt{(c^2 + d^2)}||a + bi||$, i.e., the magnitude of c + di (i.e.,$||c + di||$) times the magnitude of a + bi, where the magnitude of c + di is equal to $\sqrt{(c^2 + d^2)}$.

Consequently, when we multiplyz = a + bi with another complex number, z' = c + di where c < 0 and d <0, then we dilate the magnitude of z by the magnitude of z' and also rotate the dilated vector $180^o + 90^o$ in the counterclockwise. So, the magnitude of (a + bi).(c + di) is equal to $|z|.|z'|$ and the angle of (a + bi).(c + di) is equal to the sum of the angle of (a + bi) with 180° plus the acute angle between the

c.(a + bi) vector leg of the triangle and the hypotenuse, c.(a + bi) + di.(a + bi) = (a + bi).(c + di).(see Figure A.27, the angle ı of the vector (3 + 4i).(- 2 + 2i) is equal to $53.13^o + (180^o + 45^o) = 53.13^o + 225^o = 178.13^o$).

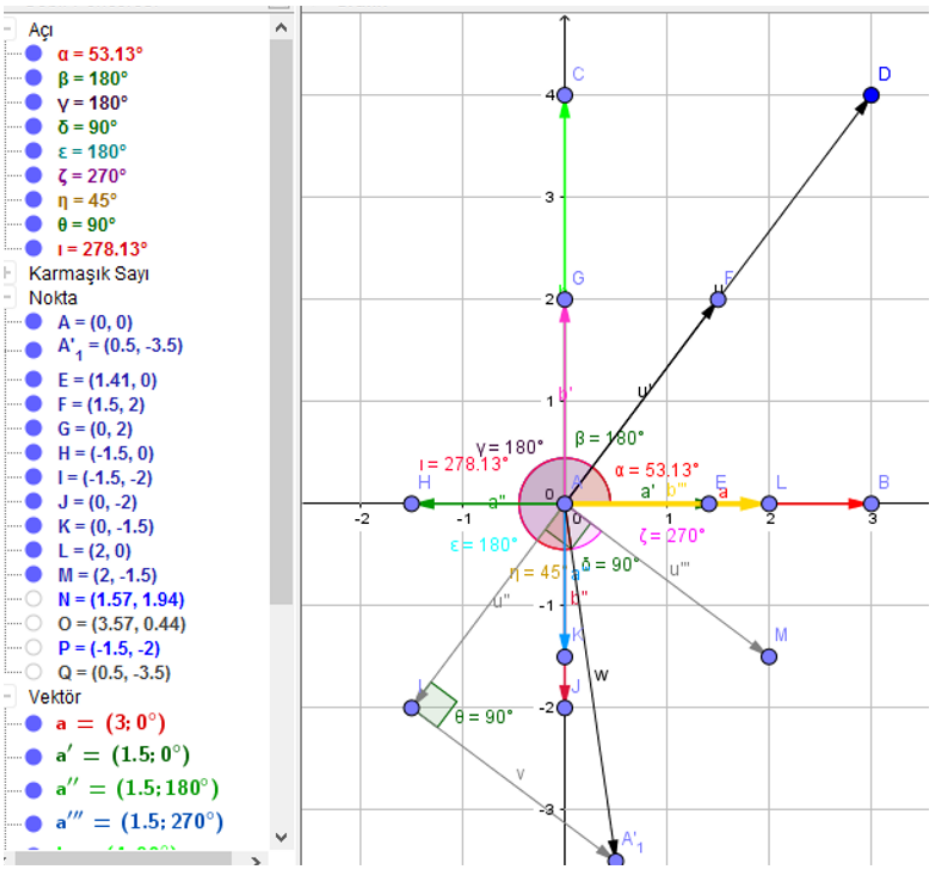


Figure A.27. Articulation of the Understanding 27.

APPENDIX B: PRE- AND POST-ASSESSMENT QUESTIONS

(1) What is a complex number? Define it please.

(2) Jason says that he thinks of the number 2 + 6i in terms of two different parts; the '2' and '6i'. Sharilyn, however, says that she thinks of 2 + 6i as a single number, '2 + 6i' rather than in terms of two different parts. Do you think about 2 + 6i like Jason does, like Sharilyn does, or a different way? Please explain your reasoning.

(3) For any complex number z = a + bi;
   (a) What do a and b refer to algebraically?
   (b) What do a and b refer geometrically?

(4) What is a vector? Define it please.

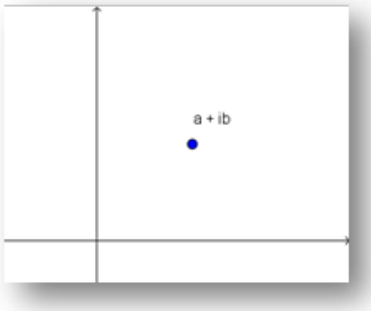
(5) Could complex numbers be described as vectors? Why or why not? Please explain your reasoning.

(6) How could you represent complex numbers in polar coordinates? Please explain your reasoning. Could you give examples?

(7) Any complex number z = a + bi could also be represented as (r, θ). In this case,
   (a) What do r and θ refer to algebraically?
   (b) What do r and θ refer geometrically?

(8) Let z_1 = a + ib and z_2 = x + iy be two complex numbers.
   (a) What is the algebraic meaning of the addition of the complex number a + bi with another complex number x + yi? Please explain your reasoning.
   (b) What is the geometric meaning of the addition of the complex number a + bi with another complex number x + yi? Please explain your reasoning.

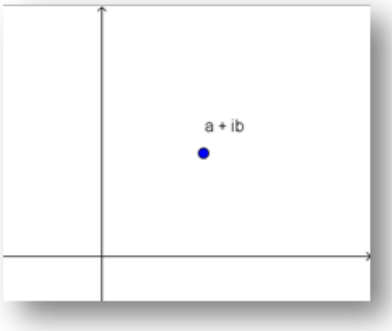
(9) b) What is the geometric meaning of the multiplication of the complex number a + bi with another complex number x + yi? Please explain your reasoning.



a) What is the algebraic meaning of the multiplication of the complex number a + bi with another complex number x + yi? Please explain your reasoning. Could you give examples?

Figure B.1. Pre- and Post-Assessment Questions 1.

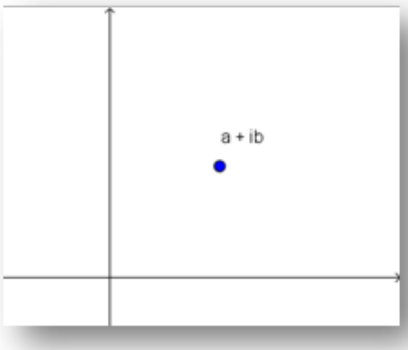
(10) b)What is the geometric meaning of the multiplication of the complex number a + bi with a real number? Please explain your reasoning.



a) What is thealgebraic meaning of the multiplication of the complex number a + bi with a real number? Please explain your reasoning. Could you give examples?

Figure B.2. Pre- and Post Assessment Questions 2.

(11) b) What is the geometric meaning of the multiplication of the complex number a + bi



a) What is the algebraic meaning of the multiplication of the complex number a + bi
   1) with the imaginary number "i"?
   2) with an imaginary number "yi"?

Please explain your reasoning. Could you give examples?

Figure B.3. Pre- and Post Assessment Questions 3.

(a) with the imaginary number "i"?

(b) with an imaginary number "yi"?

Please explain your reasoning. Could you give examples?

APPENDIX C: PRE-INTERVIEW QUESTIONS

(1) Could you define complex numbers?

(2) What is i here?

(3) How could you define z geometrically?

(4) Could you explain the algebraic and geometric meaning of the complex number a + bi with the complex number x + yi? Please explain your reasoning.

(5) How could you explain the multiplication of the complex number a + bi with the complex number x + yi in polar form? And what is θ? Could you write θ, the angle of the multiplication vector in terms of the angle of z_1 and z_2? Do you know why r is the multiplication of the lenghts of z_1 and z_2?

(6) When you think of the multiplication of complex numbers, can you explain the relation between the Cartesian and polar form of multiplication of complex numbers?

(7) Could you explain the algebraic and geometric meaning of the multiplication of the complex number a + bi with a real number c? What are the changing and unchanging properties of these four cases?

(8) Could you explain the algebraic and geometric meaning of the multiplication of the complex number a + bi with imaginary number di? What are the changing and unchanging properties of these four cases?

APPENDIX D: POST-INTERVIEW QUESTIONS

(1) Could you show and explain the multiplication of $z_1 = a + ib$ and $z_2 = x + iy$ in Cartesian form? What does Cartesian representation mean?

(2) Could you show and explain the multiplication of $z_1 = a + ib$ and $z_2 = x + iy$ in vector form? What does vector representation mean?

(3) Could you show and explain the multiplication of $z_1 = a + ib$ and $z_2 = x + iy$ in polar form? What does polar representation mean?

(4) Could you explain the relation between the multiplication of $z_1 = a + ib$ and $z_2 = x + iy$ in Cartesian form, vector form and polar form? How are they relevant to each other?

(5) Why do you need to express it in terms of theta one and theta two? Why tan ? is equal to $\frac{y}{x}$?

(6) Why are you writing in terms of z_1 and z_2?

(7) I want to ask you a question. You always look for tangents. Why do you always look for tangents?

(8) Could you show me where is theta one plus theta two exactly?

(9) Where is theta one? Where is theta two?

(10) How can you describe its magnitude?

(11) You said it doesn't matter which one I separate, a + bi or x + yi. Why it does not matter?

(12) Well, when summing them up, which ones of them you sum up with each other and -why? Why do you multiply a and bi with x plus yi part by part? You did it like this before, did not you? You did it by separating x plus yi to its parts for the rest. Why?

APPENDIX E: HYPOTHETICAL TEACHING SEQUENCE

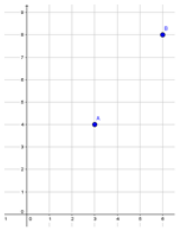
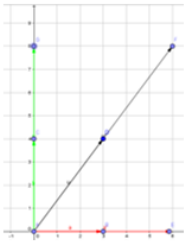
The Instruction Plan for Developing the Multiplication of Complex Numbers	
Teaching Session I (60 - 75 Minutes)	
Researcher's Questions	**Participant's Activity**
The researcher asks that: 1) What are the algebraic and geometrical meanings of multiplication of the complex number z = 3 + 4i with the real number 2? If the participant have a difficulty to explain the geometrical meaning, the researcher will direct the following questions to participant: • Does the magnitude of z change? If yes, how? If no, why? Please explain. • What about the direction of z? Does the direction change? If yes, how? If no, why? Please explain. The above questions probably direct the participant to think of the vector form of complex numbers.	1) The expected answer to the question that z' = 6 + 8i, real and imaginary part of z will be multiplied by 2. In order to explain the multiplication geometrically they will use the complex plane. Firstly, they will mark the point (3, 4) and then by looking at the resulting complex number z'= 6 + 8i, they will mark the point (6, 8) on the plane (see Figure 1). • The expected answer to the question that the magnitude of z will increase and the direction of z does not change (see Figure 2).  Figure 1  Figure 2

Figure E.1. Hypothetical Teaching Sequence 1.

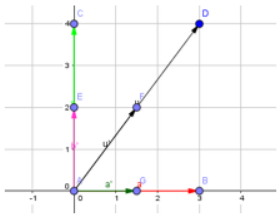
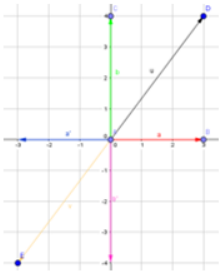
2) When you consider the all real numbers, if we multiply z = 3 + 4i with any real number rather than 2, does the magnitude of z still increase? • Is it possible to decrease the magnitude of z by multiplying a real number? If yes, could you give examples? If no, please explain why you think so? • Could you change the direction of the real number z = 3 + 4i by multiplying a real number? If yes, could you give examples? If no, please explain why you think so?	2) The expected answer to the question that "Yes, it increases, since we are multiplying." • The expected answer to the question that "When we multiply the complex number z = 3 + 4i with ½ (see Figure 3), we get z' = $^{3}/_{2}$+ 2i and the magnitude of z decreases, since we split each projection vector of the vector z into two equal parts and so the magnitude of the vector z." • The expected answer to that question that "When we multiply the complex number z = 3 + 4i with a negative real number such as -1 (see Figure 4), we get z' = - 3 - 4i and the direction of z reverses, since we multiply each projection vector of the vector z with -1, this does not change their magnitude, but will change their direction in the opposite way and so their vector summation also results with a vector z' which has the same magnitude with z, but has the opposite direction with z."  Figure 3  Figure 4

Figure E.2. Hypothetical Teaching Sequence 2.

3) Could you categorize the real numbers according to their impact on the magnitude and direction of the complex numbers that they multiplied with? • Which real numbers increase the magnitude of the complex numbers that they multiplied with? Please give examples? • Which real numbers decrease the magnitude of the complex numbers that they multiplied with? Please give examples? • Which real numbers change the direction of the complex numbers that they multiplied with? Please give examples?	3) The expected answer to the question that "Yes, I can categorize the real numbers." • The expected answer to the question that "For instance as in the Figure 2, when we multiply with the complex number z with 2, the magnitude of z increases, since the magnitude of the projection vectors on the real and imaginary axis doubled and so their vectorial summation results with a vector whose magnitude is two times bigger than the vector z. Thus, I can say that when we multiply a complex number with the real numbers which are bigger than 1, the magnitude of the complex number increases". • The expected answer to the question that "For instance as in the Figure 3, when we multiply with the complex number z with ½, the magnitude of z decreases, since the magnitude of the projection vectors on the real and imaginary axis cuts in half and so their vectorial summation results with a vector whose magnitude is half of the magnitude of the vector z. Thus, I can say that when we multiply a complex number with the real numbers which are between 0 and 1, the magnitude of the complex number decreases." • The expected answer to the question that "For instance as in the Figure 4, when we multiply with the complex number z with -1, the magnitude of z does not change, since the magnitude of the projection vectors on the real and imaginary axis does not change and so their vectorial summation results with a vector whose magnitude is the same with the vector z, but whose direction is opposite to the direction of z, since the direction of the each projection vector reverse. Thus, I can say that when we multiply a complex number with the real numbers which are less than 0, the direction of the complex number change in the reverse way."

Figure E.3. Hypothetical Teaching Sequence 3.

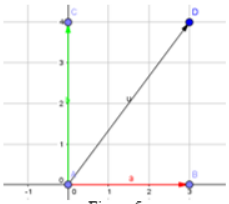
• Which real numbers do not change the direction of the complex numbers that they multiplied with? Please give examples?	• The expected answer to the question that "For instance (see Figure 5), when we multiply with the complex number z with 1, the direction of z does not change, since the direction of the projection vectors on the real and imaginary axis does not change and so their vectorial summation results with a vector whose direction is the same with the vector z. Thus, I can say that when we multiply a complex number with the real numbers which are bigger than 0, the direction of the complex number does not change."  Figure 5

Figure E.4. Hypothetical Teaching Sequence 4.

Teaching Session II (60 – 75 Minutes)	
Researcher's Questions	**Participant's Activity**
The researcher asks that: 1) What are the algebraic and geometrical meanings of multiplication of the complex number z = 3 + 4i with the imaginary number i?	1) The expected answer to the question that "When we multiply the complex number z with i, we get z' = z .i = 3i + 4i² = 3i + 4.(-1) = - 4 + 3i algebraically. When we think geometrical meaning of this multiplication, the real component of z is carried on the imaginary axis and the imaginary component of z is carried on negative part of the real axis. Thus, we can say that multiplication with i rotate the components of the vector z in the 90° counterclockwise and so the summation of these rotated components results with the complex number z' which is equal to rotation of z in the 90° counterclockwise (see Figure 6)."
• Does the magnitude of z change? If yes, how? If no, why? Please explain.	• The expected answer to that question that "The magnitude of z does not change, since the magnitude of the projection vectors on the real and imaginary axis does not change and so their vectorial summation results with a vector whose magnitude is the same with the vector z."
• What about the direction of z? Does the direction change? If yes, how? If no, why? Please explain.	• The expected answer to that question that "The direction of z changes in the 90° counterclockwise, since we rotate each projection vector of the vector z by 90° counterclockwise, this does not change their magnitude, but will change their direction in the 90° counterclockwise and so their vectorial sum also results with a vector z' which has the same magnitude with z, but has the direction which is rotated 90° in the counterclockwise from z."

Figure E.5. Hypothetical Teaching Sequence 5.

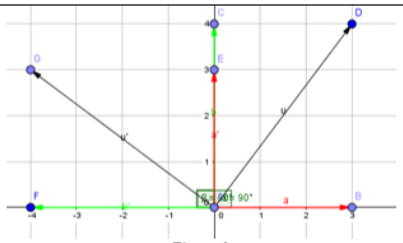
	 Figure 6
2) What are the algebraic and geometrical meanings of multiplication of the complex number z = 3 + 4i with the imaginary number 2i?	2) The expected answer to that question that "When we multiply z with 2i, algebraically we calculate z' = z. 2i = (3 + 4i). 2i = 6i + 8i² = 6i + 8. (-1) = -8 + 6i, i. e. each component of z is multiplied with 2i. However, if we think the geometrical meaning of this calculation, we double the magnitude of each one of the projection vectors of z and then rotate each one of the new projection vectors in 90° counterclockwise. Then, we sum the new projection vectors and get z' (see Figure 7)."
• Does the magnitude of z change? If yes, how? If no, why? Please explain.	• The expected answer to that question that "Yes, the magnitude of z increases due to multiplication with real coefficient 2 of i."
• What about the direction of z? Does the direction change? If yes, how? If no, why? Please explain.	• The expected answer to that question that "Yes, the direction of z changes by rotating 90° from the direction of z in the counterclockwise due to multiplication with the imaginary number i."

Figure E.6. Hypothetical Teaching Sequence 6.

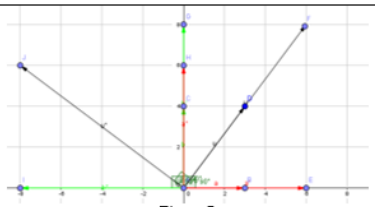
• Is there any difference between the multiplication of z with i and the multiplication of z with 2i? If yes, how? If no, why? Please explain.	 Figure 7 • The expected answer to that question that "Yes, the magnitude of z doubles due to multiplication with real coefficient 2 of i during the multiplication of z with 2i, but when we multiply z with i, the magnitude of z does not change due to the real coefficient of i is 1. The direction of z changes by rotating 90° from the direction of z in the counterclockwise due to multiplication with the imaginary number i both in the multiplication of z with i and the multiplication of z with 2i."
3) When you consider the all real coefficients of i, if we multiply z = 3 + 4i with any real coefficient of i rather than 2, does the magnitude of z still increase? • Is it possible to decrease the magnitude of z by multiplying any real coefficient of i? If yes, could you give examples? If no, please explain why you think so?	3) The expected answer to the question that "No, it does not have to increase. For instance when we multiply z with i, the magnitude of z does not change." • The expected answer to that question that "Yes, when we multiply z with an imaginary number, the real coefficient of the number changes the magnitude. So if we take into consideration the multiplication of z with the real number between 0 and 1 such that ½, the magnitude of z decreased. Thus, if we multiply z with an imaginary number whose real coefficient is between 0 and 1, then the magnitude will decrease (see Figure 8)."

Figure E.7. Hypothetical Teaching Sequence 7.

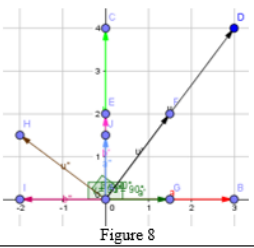
• Could you change the direction of the real number z = 3 + 4i in the opposite way by multiplying any real coefficient of i? If yes, could you give examples? If no, please explain why you think so?	• The expected answer to that question that "Yes, when we multiply z with a negative real number, the direction of z reversed. So if we take into consideration the multiplication of z with an imaginary number whose real coefficient is less than 0, the direction of each projection vectors of z reverses and so their vectorial sum results with a new vector opposite to z. Then, 90° rotation from the direction of the new vector in the counterclockwise realizes due to multiplication with the imaginary number i (see Figure 9)."  Figure 8  Figure 9

Figure E.8. Hypothetical Teaching Sequence 8.

4) Could you categorize the real coefficients of i according to their impact on the magnitude and direction of the complex numbers that they multiplied with? • Which real coefficients of i increase the magnitude of the complex numbers that they multiplied with? Please give examples? • Which real coefficients of i decrease the magnitude of the complex numbers that they multiplied with? Please give examples? * What about negative real coefficients of i? Could they decrease the magnitude of the complex numbers that they multiplied with? Please give examples?	4) The expected answer to the question that "Yes, I can categorize." • The expected answer to the question that "The real coefficients of i which are bigger than 1 (see Figure 7) and the real coefficients of i which are less than -1 (see Figure 9) increase the magnitude of the complex numbers that they multiplied with." • The expected answer to the question that "The real coefficients of i which are between 0 and 1 (see Figure 7) decreases the magnitude of the complex numbers that they multiplied with." * The expected answer to the question that "If we think the positive real coefficients between 0 and 1, the negative real coefficients of i which are between -1 and 0 (see Figure 10) will also decreases the magnitude of the complex numbers that they multiplied with."

Figure E.9. Hypothetical Teaching Sequence 9.

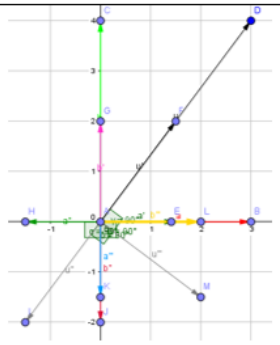
• Which real coefficients of i change the direction of the complex numbers that they multiplied with? Please give examples? • Which real coefficients of i do not change the direction of the complex numbers that they multiplied with? Please give examples?	 Figure 10 • The expected answer to the question that "The real coefficients of i which are less than 0 (see Figure 9 and Figure 10) decreases the magnitude of the complex numbers that they multiplied with. " • The expected answer to the question that "The real coefficients of i which are bigger than 0 (see Figure 7 and Figure 8) decreases the magnitude of the complex numbers that they multiplied with."

Figure E.10. Hypothetical Teaching Sequence 10.

Teaching Session III (60 – 75 Minutes)	
Researcher's Questions	**Participant's Activity**
The researcher asks that: 1) What are the similarities and differences between the multiplication of the complex number z = 3 + 4i with 2 and the multiplication of the complex number z = 3 + 4i with 2i? • If we firstly multiply z with i and then multiply it with 2, is there any difference in result? If yes, how? If no, why? Please explain.	1) The expected answer to that question that "In both of two multiplication, we multiply the complex number z with 2, i. e. we double the magnitude of each one of the projection vectors of z without making any change in their directions and then sum the vectors on the complex plane to get the new vector z' = 6 + 8i. But, since there is also the imaginary number i in the multiplication of the complex number z = 3 + 4i with 2i, we then rotate the z' in 90° counterclockwise." • The expected answer to that question that "Firstly multiplication with i and then multiplication with 2 do not cause any difference in the result. At that point, we just rotate the projection vectors of z in in 90° counterclockwise first, then we double their magnitudes and lastly we sum these vectors using vectorial sum and get again the vector z' = -8 + 6i."
2) As you remember, in the 1st step, you multiplied the complex number z = 3 + 4i with 2 and then in the 5th step, you multiplied the complex number z = 3 + 4i with 2i. Now, if you think about the multiplication of z = 3 + 4i with the complex number w = 2 + 2i, what are the algebraic and geometrical meanings of multiplication of the complex number z = 3 + 4i with the complex number w = 2 + 2i?	2) The expected answer to that question that "Algebraically, we do the calculation that $z.w = (3+4i).(2+2i) = 6 + 6i + 8i + 8i^2 = 6 + 6i + 8i + 8 (-1)$ $= 6 + 6i + 8i - 8$ $= -2 + 14i$ This means we multiply 3 and 4i with 2 and 2i one by one and then sum real parts and imaginary parts. If we look at this multiplication from another perspective, when we multiply z with w, we multiply first z with 2 and then we multiply first z with 2i and we sum the results, i.e., 2.(3 + 4i) + 2i.(3 + 4i). Geometrically, I could mark the new point -2 + 14i on complex plane."

Figure E.11. Hypothetical Teaching Sequence 11.

* Why did you write in this way: 2.(3 + 4i) + 2i.(3 + 4i)? How did you reason? What is the geometrical meaning of the calculation that 2.(3 + 4i) + 2i.(3 + 4i)? Please explain. The supporting questions will be these: →Does the magnitude of z change? If yes, how? If no, why? Please explain. →What about the direction of z? Does the direction change? If yes, how? If no, why? Please explain. →What are the similarities and differences between the multiplication of the complex number z = 3 + 4i with 2, the multiplication of the complex number z = 3 + 4i with 2i and the multiplication of the complex number z = 3 + 4i with w = 2 + 2i? Please explain.	* The expected answer to that question that "Since in the 1st step, I multiplied the complex number z = 3 + 4i with 2 and then in the 5th step, I multiplied the complex number z = 3 + 4i with 2i and in both of those I think that I split the vector z into real and imaginary components, multiplied each component with the real number and then made vectorial sum between two components by accepting "+" sign between real component (i.e., 3) and imaginary component (i. e., 4i) as the vectorial sum. In the case of z.w, I choose to see "+" sign between real component of w (i.e., 2) and imaginary component of w (i. e., 2i) as the vectorial sum and to act z as a vector. So in the multiplication with 2, I double the magnitude of z and in the multiplication with 2i, I double the magnitude of z first and then rotate it 90° in the counterclockwise. After that I sum these two new vectors to get z.w using vectorial sum." →The expected answer to that questions that "The magnitude of z increases, since during the multiplication of 2 and 2i, the magnitude of z doubles and so their vectorial sum also results with a vector which has a bigger magnitude than z." →The expected answer to that questions that "The direction of z also changes because of multiplication with i in the part of multiplication of 2i. Multiplication of z with 2 does not change its direction, since 2 is a positive real number, but during the multiplication with 2i process, we first multiply z with 2 and then we rotate obtained vector 90° in the counterclockwise. When we sum the vectors z.2 and z.2i using vectorial sum, the direction of z will changes in the counterclockwise." →The expected answer to that questions that "The similarities between those three multiplications, the magnitude of z increases and the "+" sign refers to vectorial sum of real and imaginary components. The differences between those three multiplications, the direction of z changes in the multiplication with 2i and 2 + 2i and in the multiplication with 2 + 2i, z is assumed firstly as a whole vector without separated it real and imaginary components and the complex number w is seen as a vectorial sum of two vectors 2 and 2i, then during the multiplication of z and w, firstly z is multiplied with 2 as in the in the 1st step and then multiplied with 2i as in the 5th step then the resulting vectors are summed according to vectorial sum."

Figure E.12. Hypothetical Teaching Sequence 12.

3) If you make connections between your reasoning in the 3rd and the 7th steps, could you categorize the real and imaginary parts of the complex number w's according to their impact on the magnitude and direction of the complex number z's that they multiplied with? Please explain your reasoning by giving examples.	3) The expected answer to that question that "If we look at the impact of w on the magnitude of z during the multiplication of z and w, (i)when the real and imaginary parts of w are bigger than 1, then the magnitude of z increases like multiplication of 3+ 4i with 2+ 2i (see Figure 11), (ii) when the real and imaginary parts of w are between 0 and 1(or -1 and 0) like multiplication of 3+ 4i with ½ - ½ i (see Figure 12), then the magnitude of z decreases, (iii) when the real part of w is bigger than 1(or less than -1) and imaginary part of w is between 0 and 1 (or -1 and 0) or when the real part of w is between 0 and 1 (or -1 and 0) and imaginary part of w is bigger than 1 (or less than -1), then the magnitude of z increases or decreases according to proportion between increase and decrease in the magnitudes. On the other hand, when the real and imaginary parts of w is more than 0, the direction changes due to the multiplication with i, but when the real and imaginary parts of w is less than 0, the directions reverse and then the vector which is obtained at the end of the multiplication of z with the imaginary part of w rotates 90° in the counterclockwise, so the summation vector of the new vectors rotates in the counterclockwise. Also, when the real part of w is bigger than 0 and imaginary part of w is less than 0, their vectorial sum results with a vector which is rotated from z in the clockwise. However, when the real part of w is less than 0 and imaginary part of w is bigger than 0, then their vectorial sum results with a vector which is rotated from z in the counterclockwise."

Figure E.13. Hypothetical Teaching Sequence 13.

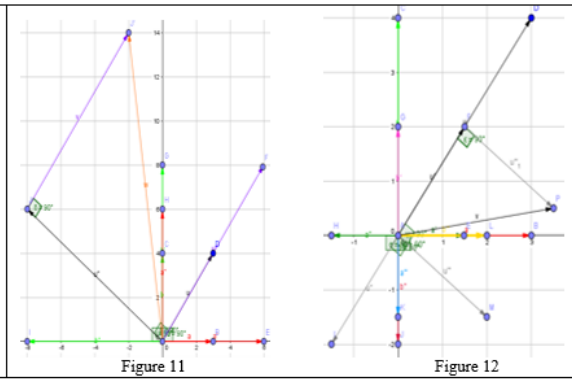
	 Figure 11 Figure 12

Figure E.14. Hypothetical Teaching Sequence 14.

Teaching Session IV (60 – 75 Minutes)	
Researcher's Questions	**Participant's Activity**
The researcher asks that: Let's think about the polar form $(\|z\|, \theta)$ of complex number z = 3 + 4i, where $\|z\|$ is the magnitude of z and θ is the angle of z thereafter. 1) As you remember, in the 3rd step you categorized the real numbers according to their impact on the magnitude and direction of the complex numbers that they multiplied with. Now, please take into consideration your categorization and explain if you think about the polar representation of the complex numbers, what are the algebraic and geometrical meanings in polar form of multiplication of the complex number z = 3 + 4i with any real number? Please give examples for each one of the 4 categories.	1) The expected answer to that question that "When we multiply a complex number z = x + yi by a positive real number, which is bigger than 1, we increase its position vector's magnitude (e. g.$\|z\|$) and the magnitude of the new vector will be the amount of the real number times the magnitude of the position vector (e. g. 3 + 4i times 2, namely 2.$\|z\|$). However, when we multiply a complex number z = x + yi by a positive real number, which is smaller than 1, we decrease its position vector's magnitude and the magnitude of the new vector will be the amount of the real number times the magnitude of the position vector (e. g. 3 + 4i times $\frac{1}{2}$, namely $\frac{1}{2}$. $\|z\|$). When we multiply a complex number z = x + yi by a negative real number, which is smaller than -1 (e. g. 3 + 4i times -2), we increase its position vector's magnitude and the magnitude of the new vector will be the absolute value of the real number times the magnitude of the position vector (e. g., 2 . $\|z\|$). Then we reverse the direction of the new vector and move its starting point to the origin. However, when we multiply a complex number z = x + yi by a real number, which is between -1 and 0 (e. g. 3 + 4i times $-\frac{1}{2}$), we decrease its position vector's magnitude and the magnitude of the new vector will be the absolute value of the real number times the magnitude of the position vector(e. g., $\frac{1}{2}$. $\|z\|$). Then we reverse the direction of the new vector and move its starting point to the origin. At that point, reversing the direction causes 180° rotation of the vectors in the counterclockwise. Thus, vectors rotate 180° in the counterclockwise from their location (e. g., their angles θ which are measured in the counterclockwise) and so their new angle equal to $180° + \theta$"

Figure E.15. Hypothetical Teaching Sequence 15.

2) As you remember, in the 7th step you categorized the real coefficients of i according to their impact on the magnitude and direction of the complex numbers that they multiplied with. Now, please take into consideration your categorization and explain if you think about the polar representation of the complex numbers, what are the algebraic and geometrical meanings in polar form of multiplication of the complex number z = 3 + 4i with any real coefficient of i? Please give examples for each one of the 4 categories.	2) The expected answer to the question that "When we multiply a complex number z = x + yi by a positive real number, which is bigger than 1, times I (e. g. 3 + 4i times 2i), we increase its position vector's magnitude and the magnitude of the new vector will be the amount of the real number times the magnitude of the position vector. So, the magnitude of the new vector is 2 . $\|z\|$. Then we rotate the new vector 90° counterclockwise and the angle of the new vector is 90° + θ. However, when we multiply a complex number z = x + yi by a positive real number, which is smaller than 1, times I (e. g. 3 + 4i times ½ i), we decrease its position vector's magnitude and the magnitude of the new vector will be the amount of the real number times the magnitude of the position vector, namely $\frac{1}{2}$. $\|z\|$. Then we rotate the new vector 90° counterclockwise and again the angle of the new vector is 90° + θ. When we multiply a complex number z = x + yi by a negative real number, which is smaller than -1, times i (e. g. 3 + 4i times -2i), we increase its position vector's magnitude and the magnitude of the new vector will be the absolute value of the real number times the magnitude of the position vector, namely $\frac{1}{2}$. $\|z\|$. Then, we reverse the direction of the new vector and move its starting point to the origin. Thus, we rotate 180° in the counterclockwise from their location and new angle is equal to 180° + θ. Then, we rotate the new vector 90° in the counterclockwise, because of multiplication with i. Thus, the vector rotates 90° in the counterclockwise from its last position of 180° + θ and its final angle is equal to 180° + θ + 90° = 270° + θ. However, when we multiply a complex number z = x + yi by a real number, which is between -1 and 0 times i (e. g. 3 + 4i times $-\frac{1}{2}$ i), we decrease its position vector's magnitude and the magnitude of the new vector will be the absolute value of the real number times the magnitude of the position vector, namely $\frac{1}{2}$. $\|z\|$. Then, we reverse the direction of the new vector and move its starting point to the origin. Thus, we rotate 180° in the counterclockwise from their location and new angle is equal to 180° + θ. Then we rotate the new vector 90° in the counterclockwise, because of multiplication with i. Thus, the vector rotates 90° in the counterclockwise from its last position of 180° + θ and its final angle is equal to 180° + θ + 90° = 270° + θ."

Figure E.16. Hypothetical Teaching Sequence 16.

3) As you remember, in the 10th step you categorized the real and imaginary parts of the complex numbers w's according to their impact on the magnitude and direction of the complex numbers z's that they multiplied with. Now, please take into consideration your categorization and explain if you think about the polar representation of the complex numbers, what are the algebraic and geometrical meanings in polar form of multiplication of the complex number z = 3 + 4i with any complex number a + bi? Please give examples for each one of the combination of 8 categories.	3) The expected answer to the question that "In the 10th step, I made such a categorization: (i) When the real and imaginary parts of w are bigger than 1, then the magnitude of z increases as in multiplication of z = 3 + 4i with w =2 + 2i (see Figure 11) and this increase is equal to the common effect of increase in multiplication with 2 and 2i, which is the magnitude of the complex number w = 2 + 2i, since at the end of the vectorial sum, we form a right triangle whose components are the vectors 2.(3 + 4i) and 2i.(3 + 4i) and if we take the vector 3 + 4i as the unit of our multiplication, the magnitude of resulting vector z.w = (3 + 4i).(2 + 2i) will be equal to the vector w times the unit vector z. So, when we think the magnitudes, the magnitudes of 2.(3 + 4i) and 2i.(3 + 4i) are equal to the magnitudes of the vectors 2 and 2i, which are both 2 times the magnitude of the unit vector, $\|z\|$. Thus, the magnitude of resulting vector z.w = (3 + 4i).(2 + 2i) will be equal to $\|w\| = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$ times the magnitude of z, $\|z\| = \sqrt{3^2 + 4^2} = \sqrt{5}$ according to Pythagorean Theorem and since $\frac{1}{\sqrt{2}}$ is less than 1, the magnitude of z will decrease in final. If we think about the angles, we form a right triangle whose components are the vectors 2.(3 + 4i) and 2i.(3 + 4i), and the angle of the vector 2.(3 + 4i) is equal to the angle of z, which is θ and the angle of the vector 2i.(3 + 4i) is equal 90° + θ. When we add these vectors end to end, there will be a counter clockwise rotation in the end and this rotation will be equal to the $\theta + \alpha$, where α is equal to the angle between final vector and the vector 2.(3 + 4i), which has the same starting point with the final vector, (ii) When the real and imaginary parts of w are between 0 and 1(or -1 and 0) as in the multiplication of 3+ 4i with ½ - ½ i (see Figure 12), then the magnitude of z decreases and this decrease is equal to the common effect of increase in multiplication with ½ and ½ i, which is the magnitude of the complex number ½ - ½ i. So, when we think the magnitudes, the magnitudes of ½ .(3 + 4i) and - ½ i.(3 + 4i) are equal to the magnitudes of the vectors ½ and - ½ i, which are both ½ times the magnitude of the unit vector $\|z\|$. Thus, the magnitude of resulting vector z.w = (3 + 4i).(½ - ½ i) will be equal to $\|w\| = \sqrt{(½)^2 + (½)^2} = \sqrt{½} = \frac{1}{\sqrt{2}}$ times the magnitude

Figure E.17. Hypothetical Teaching Sequence 17.

	of z, $\|z\| = \sqrt{3^2 + 4^2} = \sqrt{5}$ according to Pythagorean Theorem and since $\frac{1}{\sqrt{5}}$ is less than 1, the magnitude of z will decrease in final. If we think about the angles, we form a right triangle whose components are the vectors ½ .(3 + 4i) and - ½ i.(3 + 4i), and the angle of the vector ½ .(3 + 4i) is equal to the angle of z, which is θ and the angle of the vector – ½ i.(3 + 4i) is equal $270° + \theta$. When we add these vectors end to end, there will be a clockwise rotation in the end and this rotation will be equal to the $\theta - \alpha$, where α is equal to the angle between final vector and the vector ½ i.(3 + 4i), which has the same starting point with the final vector, (iii) When the real part of w is bigger than 1(or less than -1) and imaginary part of w is between 0 and 1 (or -1 and 0) or when the real part of w is between 0 and 1 (or -1 and 0) and imaginary part of w is bigger than 1 (or less than -1), then the magnitude of z increases, since one of the real and imaginary parts is bigger than 1,and this increase is equal to the common effect of increase in multiplication with real and imaginary parts, which is the magnitude of the complex number w. On the other hand, when the real and imaginary parts of w is more than 0, the direction changes due to the multiplication with i, as in the multiplication (3 + 4i) (2 + 2i), so the final angle is equal to $\theta + \alpha$. →When the real and imaginary parts of w is less than 0 (e. g. -2 – 2i), the angle of the vector -2.(3 + 4i) is equal to $180° + \theta$ and the angle of the vector -2i.(3 + 4i) is equal $270° + \theta$. When we add these vectors end to end, there will be a counter clockwise rotation in the end and this rotation will be equal to the $180° + \theta + \alpha$, where α is equal to the angle between final vector and the vector -2.(3 + 4i), which has the same starting point with the final vector. Thus, the directions reverse and then the vector which is obtained at the end of the multiplication of z with the imaginary part of w rotates 90º in the counterclockwise, so the vectorial sum of the new vectors rotates in the counterclockwise. →When the real part of w is bigger than 0 and imaginary part of w is less than 0 (e. g., 2 - 2i), the angle of the vector 2.(3 + 4i) is equal to the angle of z, which is θ and the angle of the vector -2i.(3 + 4i) is equal $270° + \theta$. When we add these vectors

Figure E.18. Hypothetical Teaching Sequence 18.

	end to end, there will be a clockwise rotation in the end and this rotation will be equal to the $\theta - \alpha$, where α is equal to the angle between final vector and the vector 2.(3 + 4i), which has the same starting point with the final vector.

Figure E.19. Hypothetical Teaching Sequence 19.