EINSTEIN METRICS ON BUNDLES OVER QUATERNIONIC-KÄHLER MANIFOLDS

by

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ABSTRACT

EINSTEIN METRICS ON BUNDLES OVER QUATERNIONIC-KÄHLER MANIFOLDS

In this thesis, we investigate the Einstein metrics on a rank-three bundle defined over a $4n$-dimensional quaternionic Kähler base manifold. Defining Riemannian metrics and pseudo-Riemannian metrics of signature $(4n, 3)$ on the bundle, we find the connection forms and calculate their curvatures. Then, we obtain the Ricci curvatures by contracting the curvatures. Finally, we examine the case of Ricci curvature zero and compare our results with the results in the literature.

ÖZET

KUATERNİYONİK-KÄHLER MANİFOLDLARI ÜZERİNDEKİ DEMETLER ÜSTÜNDE EINSTEIN METRİKLERİ

Bu tezde, $4n$-boyutlu bir kuaterniyonik Kähler baz manifoldu üzerinde tanımlı, rankı üç olan bir demet üstünde Einstein metriklerini inceliyoruz. Demet üstünde Riemann metrikleri ve işareti $(4n, 3)$ olan sözde-Riemann metrikleri tanımlayarak, konneksiyon formları buluyoruz ve onların eğriliklerini hesaplıyoruz. Daha sonra eğrilikleri büzüp, Ricci eğriliklerini elde ediyoruz. Son olarak, Ricci eğriliğinin sıfır olması durumunu inceliyoruz ve sonuçlarımızı literatürdeki sonuçlarla karşılaştırıyoruz.

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LIST OF SYMBOLS

$c_1(M)$	The first Chern class of M
$\mathbb{C}P^n$	Complex projective space of dimension n
$End(TM)$	Endomorphisms of TM
$\mathbb{H}$	The division ring of quaternions
h^*	Quaternionic conjugate of h
$O(n)$	Orthogonal group
$SO(n)$	Special orthogonal group
$\mathfrak{so}(n)$	Lie algebra of the special orthogonal group
$SO(p,q)$	Indefinite special orthogonal group of signature (p,q)
$\mathfrak{so}(p,q)$	Lie algebra of $SO(p,q)$
TM	Tangent bundle of M
T_pM	Tangent space of M at p
trA	Trace of A
ϵ_{ijk}	Levi-Civita symbol
$\Lambda^1\Big(\mathfrak{so}(n)\Big)$	1-forms with values in the Lie algebra $\mathfrak{so}(n)$
Λ^2T^*M	Alternating 2-forms on M
π^*ds^2	The pull-back of ds^2 by the projection map π

1. INTRODUCTION

In this thesis, we investigate a class of Einstein metrics on the total space $\widehat{M}$ of a rank-three sub-bundle Q locally spanned by 3 almost complex structures, where the base space is a $4n$-dimensional quaternionic-Kähler manifold M.

An Einstein metric is a Riemannian metric g whose Ricci curvature, denoted by Ric, is constant. In other words, the Einstein equation $Ric = \Lambda g$ is satisfied ($\Lambda \in \mathbb{R}$).

The early examples of complete Einstein metrics were found among homogeneous spaces (see [1]). Recall that a homogeneous space is a Riemannian manifold that admits a transitive action of the isometry group (see chapter 7 of [1]). Early examples of non-homogeneous Einstein manifolds are provided by Yau [2] by his solution to the Calabi conjecture. Yau proved the existence of $Ric = 0$ Kähler metrics on complex manifolds with $c_1(M) = 0$. Later, Aubin [3] proved the existence of a Kähler-Einstein metric on every compact complex manifold M with $c_1(M) < 0$. Also, some of the first non-homogeneous examples of Einstein metrics are due to Page [4] and Calabi [5]. Page gave the first example of a compact inhomogeneous metric, with the topology of a non-trivial S^2 bundle over S^2 (or the connected sum $\mathbb{C}P^2\#-\mathbb{C}P^2$). The generalization of this method was given by Bergery in [6]. Calabi introduced metrics on the total space of the canonical bundle over a Kähler-Einstein manifold. This construction yielded non-compact examples of Ricci-flat Kähler metrics. Another similar work was given in [7] by Sakane and he constructed non-homogeneous Kähler-Einstein metrics on $\mathbb{C}P^1$ bundles over Kähler-Einstein manifolds. Some further examples of non-homogeneous metrics were obtained from S^2 bundles over Kähler-Einstein manifolds by Page and Pope in [8].

The construction of non-homogeneous Einstein metrics on bundles is a well-known method. The idea is to consider a bundle $\pi : V \to M$ such that both M and V have some special curvature properties and define a metric on the total space of V depend-

ing on one or more functions. These special curvature properties of the bundle and the base manifold reduce the Einstein equation to an ordinary differential equation of the functions defining the metric, that can be solved. For instance, using this technique, Page and Pope investigated Einstein metrics on S^4 and R^4 fiber bundles over a quaternionic-Kähler base manifold in [9]. They gave numerical arguments for the existence of complete compact non-homogeneous Einstein spaces with positive Ricci curvature and complete inhomogeneous non-compact spaces with non-positive Ricci curvature. Also, in [10, 11], the authors considered a quaternionic-Kähler manifold M and defined a Riemannian metric on the total space of a rank-three bundle. Moreover, when M is 4-dimensional, they were able to find complete Ricci-flat metrics. We point out that these metrics have holonomy G_2. On the other hand, in [12], the authors considered $(4n + 4)$-dimensional metrics on bundles over quaternionic-Kähler manifolds and investigated Einstein equations with positive Λ. They also gave numerical arguments for inhomogeneous compact Einstein spaces.

Using the bundle method mentioned above, we aim to find Einstein solutions to a metric Ansatz on the total space of a vector bundle. Our base space is a quaternionic-Kähler manifold. A quaternionic-Kähler manifold is a $4n$-dimensional Riemannian manifold that admits three locally defined almost complex structures and a metric that is compatible with the almost complex structures such that the Levi-Civita connection preserves the subspace (which is denoted by Q) generated by these almost complex structures. By Myers theorem [13], every complete Einstein manifold with $\Lambda > 0$ is compact. Since we search for Einstein metrics on a non-compact total space of a vector bundle, we consider only the cases $\Lambda \leq 0$. It is known that a quaternionic Kähler manifold is Einstein (see [1] for the details). The scalar curvature s, of a quaternionic-Kähler manifold, is either positive or negative. The case $s = 0$ is excluded as such manifolds have different geometric properties (these manifolds are locally HyperKähler). The special features of quaternionic-Kähler manifolds, along with the special curvature of the rank-three bundle, allow us to simplify the curvature computations on the total space.

The Riemannian metrics we consider, which are defined on the $(4n+3)$-dimensional total space $\widehat{M}$ of Q, are of the form

$$\widehat{g} = f(r)^2\Big(\sum_{i=1}^{3}(\alpha^i)^2\Big) + g(r)^2\pi^* ds^2, \tag{1.1}$$

where α^i, $1 \leq i \leq 3$, are 1-forms on the bundle Q, ds^2 is the metric on the $4n$-dimensional quaternionic-Kähler base space M, and f and g are two functions of the radial parameter r. These types of Riemannian metrics were also studied in [11] by Bryant and Salamon to find G_2 holonomy metrics in 7-dimensions. A metric with G_2 holonomy is always Ricci-flat. In this paper, the authors defined a G_2 3-form on the total space of Q, and the condition of that 3-form being parallel is an ordinary differential equation in f and g. The authors gave new Ricci-flat complete Einstein metrics, where the base manifold M has positive Einstein constant Λ. Their method is achievable only for $n = 1$, where the total space has 7 dimensions. Because a G_2 structure exists only in 7 dimensions. The purpose of this thesis is to compute the Ricci curvature and try to solve the equation $Ric = 0$. Our method is more general than the method used in [11], even in the $n = 1$ case, since the authors of [11] solve for parallel G_2 3-forms, giving G_2 holonomy metrics, while we solve for more general Ricci-flat metrics.

Throughout this thesis, in addition to the Riemannian metrics, we consider the pseudo-Riemannian metrics of signature $(4n, 3)$ on the total space defined by a similar formula. The 1-forms α^i $(i = 1, 2, 3)$ we take into consideration, are the same 1-forms as given in [10]. But our method is different. The Riemannian metric introduced in [10] has 3 different functions of the radial parameter while our Riemannian metric has 2 functions, f and g, of the radial parameter. Also, our method for finding the curvature of the metric $\widehat{g}$ of $\widehat{M}$ is different. Our approach consists of finding the connection forms and the curvatures directly. Their method involves an additional parameter for the metric, and they impose a hypersurface condition and use Gauss-Codazzi equations.

We find the connection 1-forms and calculate the curvatures for both the Riemannian and pseudo-Riemannian cases. Then, we obtain the Ricci curvatures and discuss the Ricci-flat condition. This condition reduces to a non-linear ordinary differential equation of f and g. Finally, we find a relation between Ricci-flat conditions of Riemannian and pseudo-Riemannian cases. In each n, a solution to $Ric = 0$, where the base manifold has Einstein constant Λ, gives a pseudo-Riemannian solution over a base manifold with Einstein constant $-\Lambda$ and vice versa. For the case $n = 1$, where the total space has 7-dimensions and $\Lambda > 0$, we solve Ricci-flat equations and obtain a Ricci-flat complete Riemannian metric that was given in [11]. The same functions are also a solution to the Ricci-flat equations for the pseudo-Riemannian metric of the signature $(4, 3)$ when $\Lambda < 0$.

2. PRELIMINARIES

In this section, we give some basic notions that will be used throughout this thesis. We assume that all manifolds are connected. We refer to [14, 15] for the proofs and details.

Definition 2.1. *Let M be a smooth manifold. A pseudo-Riemannian metric g is a $(0,2)$ tensor field which is*

i) symmetric, i.e., $g_p(X,Y) = g_p(Y,X)$ for each X,$Y \in T_pM$,
ii) non-degenerate, i.e., if $g_p(X,Y) = 0$ for each $Y \in T_pM$, then $X = 0$.

If, in addition to i), it is positive definite, i.e., $0 \leq g_p(X,X)$ for all $X \in T_pM$ with $g_p(X,X) = 0$ if and only if $X = 0$, then it is called a Riemannian metric.

It is possible to define Riemannian metrics on every manifold. A smooth manifold M of dimension n is called a (pseudo-) Riemannian manifold if there is a (pseudo-) Riemannian metric on M. A (pseudo-) Riemannian manifold is denoted by (M,g). In local coordinates $\{x^i\}$, we can write $g = g_{ij}dx^i \otimes dx^j$, where $g_{ij} = g(\partial_i, \partial_j)$ for $1 \leq i,j \leq n$. It can be briefly written by $g = g_{ij}dx^i dx^j$. If the matrix (g_{ij}) has p positive and q negative eigenvalues, then the metric is said to be of signature (p,q). When $q = 0$ and the metric is positive definite, then g is a Riemannian metric.

Definition 2.2. *Let M be a smooth manifold and $\Gamma(TM)$ denote the set of smooth sections on the tangent bundle of M. A linear connection on M is a bi-linear map $\nabla : \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ with*

i) $\nabla_{fX}(Y) = f\nabla_X(Y)$,
ii) $\nabla_X(fY) = f\nabla_X(Y) + X(f)Y$,

where $X, Y \in \Gamma(TM)$ and f is a smooth function on M.

Theorem 2.1. *Let (M, g) be a (pseudo-)Riemannian manifold. There exists a unique connection that is torsion-free, i.e., $\nabla_X Y - \nabla_Y X - [X, Y] = 0$, and metric-compatible, that is $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$.*

The preceding theorem is known as the Fundamental Theorem of (pseudo-) Riemannian Geometry. The theorem follows from the Kozsul formula. We refer to [14] for the details and the proof.

Definition 2.3. *For a (pseudo-) Riemannian manifold, the unique connection that is compatible with the (pseudo-) Riemannian metric and is torsion-free is called the Levi-Civita connection.*

For the rest of this thesis, we will assume that the induced connection is the Levi-Civita connection.

Definition 2.4. *Let M be a smooth manifold. Riemannian curvature tensor is defined to be the map $R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$ with*

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.1}$$

where $X, Y, Z \in \Gamma(TM)$.

Let M be an n-dimensional smooth manifold. Denote by $\{e_i\}$ a local frame of TM and by $\{e^i\}$ the co-frame given by $e^i(e_j) = \delta_{ij}$, where $1 \leq i, j \leq n$. As our manifolds will be (pseudo-) Riemannian, we will always use orthonormal frames and co-frames. An orthonormal frame $\{e_i\}$ satisfies $g(e_i, e_j) = \pm\delta_{ij}$. Now, since it is linear in all components, the Riemannian curvature tensor is determined by

$$R(e_c, e_d)e_b := \sum_{a=1}^{n} R^a_{bcd} e_a. \tag{2.2}$$

Let

$$\nabla_{e_k} e_j = \sum_{i=1}^{n} \omega_{ij}(e_k) e_i, \tag{2.3}$$

where ω_{ij} is called the connection 1-form. The curvature 2-form Ω_{ij} of the connection with respect to the given frame is given by

$$\Omega_{ij} = d\omega_{ij} + \sum_{k=1}^{n} \omega_{ik} \wedge \omega_{kj}. \tag{2.4}$$

The Riemannian curvature R and the curvature 2-form Ω are related by

$$\Omega_{ab} = \frac{1}{2} R^{a}_{bcd} e^{c} \wedge e^{d}. \tag{2.5}$$

The following will be used to determine the connection 1-form. The proof for the Riemannian case can be found in [15,16]. The argument presented in the proof is valid for both the Riemannian and pseudo-Riemannian metrics.

Theorem 2.2. *Let (M, g) be an n-dimensional Riemannian manifold with a local orthonormal co-frame $\{e^i\}$, $1 \leq i \leq n$. The connection 1-form ω_{ij} is uniquely determined by*

i) $\omega \in \Lambda^1\big(\mathfrak{so}(n)\big)$,
ii) $de^i = -\sum_{j=1}^{n} \omega_{ij} \wedge e^j$,

where $\mathfrak{so}(n)$ denotes the Lie algebra of $SO(n)$. Moreover, if g is a pseudo-Riemannian metric of signature (p, q), then the theorem holds if i) is replaced by $\omega \in \Lambda^1\Big(\mathfrak{so}(p, q)\Big)$.

The equation given in *ii)* of Theorem 2.2 is known as Cartan's first structure equation. For simplicity, we will use a summation convention and write $de^i = -\omega_{ij} \wedge e^j$.

Cartan's second structure equation is (2.4). For the same purpose, we write it as

$$\Omega_{ij} = d\omega_{ij} + \omega_{ik} \wedge \omega_{kj}. \tag{2.6}$$

Definition 2.5. *Let (M, g) be an n-dimensional Riemannian manifold and let $\{e_i\}$, $1 \leq i \leq n$, be a local orthonormal frame for $\Gamma(TM)$. The Ricci curvature is defined by*

$$Ric(X, Y) = \sum_{i=1}^{n} g\Big(R(e_i, X)Y, e_i\Big), \tag{2.7}$$

where $X, Y \in \Gamma(TM)$. The scalar curvature is defined by

$$s = \sum_{i=1}^{n} Ric(e_i, e_i) = \sum_{i=1}^{n} Ric_{ii}. \tag{2.8}$$

Definition 2.6. *Let (M, g) be a Riemannian manifold. We call g Einstein if it satisfies the Einstein equation, that is,*

$$Ric = \Lambda g, \tag{2.9}$$

where Λ is a constant (it is known as Einstein constant). If $\Lambda = 0$, then g is said to be Ricci-flat.

Definition 2.7. *Let $\widehat{M}$ and M be two smooth manifolds. A smooth vector bundle consists of $(\widehat{M}, M, \pi)$, where $\pi : \widehat{M} \to M$ is a projection map such that at each $p \in M$,*

i) each fiber $\pi^{-1}(p)$ is isomorphic to an n-dimensional vector space V,
ii) there is a neighborhood U of p and a diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times V$,
iii) the restriction $\phi_U|_p : \pi^{-1}(p) \to \{p\} \times V$ is linear.

Here, $\widehat{M}$ is said to be the total space and M is said to be the base space. The dimension of V is called the rank of the bundle. If N is a submanifold of $\widehat{M}$ such that $N \cap \pi^{-1}(p)$ is a vector subspace of V, then $(N, M, \pi|_N)$ is said to be a sub-bundle of $(\widehat{M}, M, \pi)$.

Definition 2.8. *Let M be a manifold. An almost complex structure is a field of automorphisms $J : TM \to TM$ such that $J^2 = -I$, where I is the identity.*

3. QUATERNIONIC-KÄHLER MANIFOLDS

In this section, we give some basic notions about quaternionic-Kähler manifolds which will be used throughout this thesis. For further details, see [17, 18]. We denote the base manifold by M and the total space by $\widehat{M}$. For convention, we write out the connection and curvatures of the total space with a "$\widehat{\ }$" on it.

Let M be a smooth manifold and J_1, J_2 and J_3 be three almost complex structures, defined on the tangent bundle TM on a neighborhood of some point in M with the relation

$$J_1J_2 = J_3 = -J_2J_1,\ J_1^2 = J_2^2 = J_3^2 = -I. \tag{3.1}$$

An almost quaternionic structure is a 3-dimensional sub-bundle $Q \in End(TM)$ such that at each $p \in M$, there are local sections $\{J_1, J_2, J_3\}$ of Q satisfying the above identities. By taking J_1, J_2 and J_3 as imaginary parts of the quaternions, one can define a left action on T_pM as $q.Y = (q_1+q_2J_1+q_3J_2+q_4J_3).Y$, where $q = q_1+q_2i+q_3j+q_4k \in \mathbb{H}$ and $Y \in T_pM$. Hence, T_pM admits a structure of a left $\mathbb{H}$-module. Thus, a manifold M with an almost quaternionic structure always has $4n$ dimensions. Moreover, if we have a Riemannian metric g on M compatible with the almost complex structures of Q in the sense that $g(JX, JY) = g(X, Y)$, where $J \in Q$ is an almost complex structure, then it is called an almost quaternionic Hermitian structure. We note that, given a Riemannian metric g, we can always obtain a metric compatible with the almost complex structures in Q. For example, one can define

$$g'(V, W) = g(V, W) + g(J_1V, J_1W) + g(J_2V, J_2W) + g(J_3V, J_3W). \tag{3.2}$$

Let Λ^2T^*M denote the space of alternating 2-forms on M. An almost quaternionic Hermitian structure Q can be embedded in Λ^2T^*M. To see this, we consider the

association of 2-forms to the almost complex structures.

$$w^i(V,W) = g(J_iV,W), \tag{3.3}$$

where $J_i \in Q$, $1 \leq i \leq 3$. If J_1, J_2 and J_3 satisfy (3.1), then they form an orthonormal frame field for Q in the metric

$$< A, B >= \frac{1}{4n} tr(A^T B) \tag{3.4}$$

on $End(TM)$. Furthermore, we note that the 2-forms $\{w^1, w^2, w^3\}$ defined by (3.3) are skew-symmetric, that is,

$$w^i(Y,X) = g(J_iY,X) = g(-Y,J_iX) = -g(J_iX,Y) = -w^i(X,Y). \tag{3.5}$$

There might not be a globally defined almost complex structure which is a section of Q, or even any almost complex structure whatsoever. For example, this is the case for the quaternionic projective space $\mathbb{H}P^n = \mathbb{H}^{n+1} - \{0\}/\mathbb{H}^*$ which is the space of quaternionic lines through the origin in $\mathbb{H}^{n+1}$ with the right action of $\mathbb{H}$, where $\mathbb{H}^{n+1}$ is the quaternionic vector space.

The following 4-form

$$\Omega = \sum_{i=1}^{3} w^i \wedge w^i \tag{3.6}$$

is globally defined and does not depend on the choice of the frame field [19].

Definition 3.1. *A Riemannian manifold (of dimension $n > 1$) together with an almost quaternionic Hermitian structure (M, Q, g) is quaternionic-Kähler if $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection.*

Quaternionic-Kähler manifolds can also be equivalently described by their holonomy groups (see [18]). Let (M, g) be a connected n-dimensional Riemannian manifold. Parallel translation along piecewise smooth closed curves at some point $p \in M$ induces a group of transformations $P_p : T_pM \to T_pM$ which is called the holonomy group. The holonomy group of a linear connection is a Lie subgroup of $GL(T_pM)$ but not necessarily a closed Lie subgroup. Choosing a frame at $p \in M$, the holonomy can be identified with a Lie subgroup of $GL_n(\mathbb{R})$. This subgroup is defined up to conjugation. If the connection preserves a metric, then with a choice of orthonormal frame at p, the holonomy group of a manifold is a subgroup of $O(n)$. We can also consider only piecewise smooth loops at p which are trivial in the fundamental group $\pi_1(M)$. In this case, we obtain a subgroup called the restricted holonomy group. This subgroup is the connected component of the identity in the full holonomy (for the details, we refer to [20]). Moreover, if M is path-connected, then changing the basepoint will change the holonomy by conjugation. Thus, holonomy groups for different points on a path-connected manifold are isomorphic. Note that (restricted) holonomy consists of a Lie group and a representation. We will only consider the holonomy of the Levi-Civita connection, the (pseudo-) Riemannian holonomy group. The manifold is called locally irreducible if the restricted holonomy representation is irreducible. For locally irreducible and non-locally symmetric spaces, possible restricted (pseudo-) Riemannian holonomy groups were given by Berger (for more details, see [1, 16]).

Now, we focus on a particular holonomy group. We denote by $M_n(\mathbb{H})$ the collection of $n \times n$ matrices whose entries are quaternions and denote by $GL_n(\mathbb{H})$ invertible such matrices. The group of quaternionic unitary transformations is defined by

$$Sp(n) = \{A \in GL_n(\mathbb{H}) : A^*A = I\}, \tag{3.7}$$

where A^* denotes the conjugate transpose of the matrix A and I is the identity matrix. For $n = 1$, we have the group of unit quaternions

$$Sp(1) = \{x + yi + zj + tk \in \mathbb{H} : x^2 + y^2 + z^2 + t^2 = 1\}. \tag{3.8}$$

This is topologically the 3-sphere. If $Sp(n)$ acts on $\mathbb{H}^n$ from the left in a usual way and $Sp(1)$ acts on $\mathbb{H}^n$ from the right by conjugation, then we get a map $\rho : Sp(n)\times Sp(1) \to GL_n(\mathbb{H})$ with $\rho(X,h)(q) = Xqh^*$, where $q \in \mathbb{H}^n$, $(X,h) \in Sp(n)\times Sp(1)$ and the group $\rho(Sp(n)\times Sp(1))$ is denoted by $Sp(n)Sp(1)$. Note that (X,h) and $(-X,-h)$ give the same action (cf. [17]). So, we write

$$Sp(n)Sp(1) = Sp(n)\times Sp(1)/\{(-I,-1),(I,1)\} = Sp(n)\times Sp(1)/\mathbb{Z}_2. \tag{3.9}$$

$Sp(n)Sp(1)$ is a Lie group that appears as a possible holonomy group in Berger's list and ρ is the holonomy representation. A more detailed discussion about holonomy groups and the holonomy group $Sp(n)Sp(1)$ can be found in [16].

Definition 3.2. *A Riemannian manifold M (of dimension $n > 1$) is called quaternionic-Kähler if its holonomy group is $G.Sp(1)$ with $G \subset Sp(n)$ a Lie subgroup. If (M,g) is not locally symmetric, then $G = Sp(n)$ by Berger's classification.*

If $n = 1$, we have $Sp(1)Sp(1) = SO(4)$ (cf. [16]). Hence, any oriented Riemannian manifold has a holonomy group in $Sp(1)Sp(1)$. Given an oriented 4-dimensional Riemannian manifold M, the Hodge star operator $* : \Lambda^2 \to \Lambda^2$ is involutive. Thus, it has eigenvalues ± 1. The corresponding eigenspaces Λ^2_+M and Λ^2_-M split the space of 2-forms into 2 pieces, i.e, $\Lambda^2 = \Lambda^2_+M \oplus \Lambda^2_-M$.

Under $\Lambda^2 = \Lambda^2_+M \oplus \Lambda^2_-M$, the curvature operator $R : \Lambda^2 \to \Lambda^2$ has the matrix form given by

$$R = \left[\begin{array}{c|c} W_+ + \dfrac{s}{12} & Ric_0 \\ \hline Ric_0 & W_- + \dfrac{s}{12} \end{array}\right]. \tag{3.10}$$

In the 4-dimensional oriented Riemannian case, the Weyl curvature W is $W = W_+ + W_-$. Also $Ric_0 = Ric - \frac{s}{4}g$. If $Ric_0 = 0$, then M is Einstein and M is called self-dual if $W_- = 0$ (cf. [1]).

Definition 3.3. *A 4-dimensional Riemannian manifold M is said to be quaternionic-Kähler if it is self-dual Einstein.*

We now focus on Definition 3.1 since $\nabla\Omega = 0$ has some particular results. We follow [21], for the construction below, with some different conventions.

First, we observe that $\nabla\Omega = 0$ implies $\sum_{i=1}^{3} \nabla w^i \wedge w^i = 0$. That gives us

$$\nabla w^j = \sum_{i=1}^{3} \alpha_{ij} \otimes w^i, \tag{3.11}$$

where $\alpha_{ij} = -\alpha_{ji}$. The connection matrix $A = (\alpha_{ij})$ is the connection on Q induced by the Levi-Civita connection. Its curvature is

$$R = dA + A \wedge A. \tag{3.12}$$

Since $dw^i = \sum_{i=1}^{3} \alpha_{ij} \wedge w^i$, by taking the exterior derivative of this equation, we observe that

$$\begin{aligned} 0 &= \sum_{i=1}^{3} \Big(d\alpha_{ij} \wedge w^i - \alpha_{ij} \wedge dw^i \Big) \\ &= \sum_{i=1}^{3} d\alpha_{ij} \wedge w^i - \sum_{i,l=1}^{3} \alpha_{ij} \wedge \alpha_{li} \wedge w^l \end{aligned} \tag{3.13}$$

which yields

$$\sum_{i=1}^{3} \Big(- d\alpha_{ji} - \sum_{k=1}^{3} \alpha_{jk} \wedge \alpha_{ki} \Big) \wedge w^i = 0. \tag{3.14}$$

Combining this with equation (3.12), we get

$$\sum_{i=1}^{3} R_{ij} \wedge w^i = 0. \tag{3.15}$$

Therefore,

$$R_{ij} = \lambda \sum_{k=1}^{3} \epsilon_{ijk} w^k, \tag{3.16}$$

where λ is a constant, ϵ_{ijk} denotes the Levi-Civita symbol and $i, j, k \in \{1, 2, 3\}$. We note that ϵ_{ijk} is totally skew-symmetric in i, j, k with $\epsilon_{123} = 1$.

Moreover, the local frame $\{J_1, J_2, J_3\}$ gives us local coordinates (a_1, a_2, a_3) on the bundle Q. By using the connection, the vertical 1-forms are defined by

$$\alpha^i = da_i + \sum_{j=1}^{3} \alpha_{ij} a_j, \tag{3.17}$$

where $1 \leq i \leq 3$. From (3.12), (3.16) and (3.17), we find its exterior derivative as

$$\begin{aligned} d\alpha^i &= \sum_{j=1}^{3} \Big(d\alpha_{ij} a_j - \alpha_{ij} \wedge da_j \Big) \\ &= \sum_{j,l=1}^{3} \Big(R_{ij} - \alpha_{il} \wedge \alpha_{lj} \Big) a_j - \sum_{j=1}^{3} \Big(\alpha_{ij} \wedge \big(\alpha^j - \sum_{k=1}^{3} \alpha_{jk} a_k \big) \Big) \\ &= \sum_{j=1}^{3} \Big(R_{ij} a_j - \alpha_{ij} \wedge \alpha^j \Big) \\ &= \sum_{j=1}^{3} \Big(\lambda \big(\sum_{k=1}^{3} \epsilon_{ijk} w^k \big) a_j - \alpha_{ij} \wedge \alpha^j \Big). \end{aligned} \tag{3.18}$$

For simplicity, we write

$$d\alpha^i = \lambda \epsilon_{ijk} w^k a_j - \alpha_{ij} \wedge \alpha^j. \tag{3.19}$$

It turns out that we have $\lambda(n+2) = \Lambda$ for (3.16), where Λ is the Einstein constant of the quaternionic-Kähler manifold M. We remind the reader that a quaternionic-Kähler manifold (M, g) is Einstein, i.e., its Ricci curvature is a constant multiple of its metric. A proof of this fact can be found in [1]. We can see that the scalar curvature is a constant by taking the trace of the Ricci curvature. More precisely, it is equal to $4n\Lambda$. A quaternionic-Kähler manifold is called positive if its scalar curvature is positive. Likewise, it is called negative if its scalar curvature is negative. We exclude the case where the scalar curvature is zero as such manifolds are locally HyperKähler (see [1]) and their restricted holonomy is in $Sp(n)$. For each dimension, up to isometries, only finitely many positive complete quaternionic-Kähler manifolds exist (cf. [22]). Because quaternionic-Kähler manifolds are Einstein, by the Myers theorem, a complete positive quaternionic-Kähler manifold must be compact. Positive quaternionic-Kähler manifolds are also simply-connected (more details can be found in [23]). The only known examples of positive complete quaternionic-Kähler manifolds are symmetric spaces. These are also known as Wolf spaces (cf. [24]). It is conjectured by Lebrun and Salamon in [22] that all positive complete quaternionic-Kähler manifolds are symmetric spaces. This conjecture is verified for 4 and 8 dimensions. In the 4-dimensional case, a positive complete quaternionic-Kähler manifold is either S^4 (the 4-sphere) with the round metric or $\mathbb{C}P^2$ (the complex projective plane) with the Fubini-Study metric, for details see [25]. In contrast to the positive case, there exist infinitely many negative complete quaternionic-Kähler manifolds (cf. [26]).

Now, we choose an orthonormal frame for the $4n$-dimensional quaternionic-Kähler manifold M. With the vertical 1-forms defined by (3.17), we will define the Riemannian metrics and the pseudo-Riemannian metrics on the $(4n+3)$-dimensional total space of the rank-three bundle Q spanned by locally defined three almost complex structures J_1, J_2 and J_3.

4. METRICS ON BUNDLES

Let (M, ds^2) be a quaternionic-Kähler manifold and Q be the sub-bundle spanned by 3 almost complex structures. The Riemannian and pseudo-Riemannian metrics can be given on the total space of the bundle. Such metrics were studied by Gibbons, Page, and Pope in [10] and by Bryant and Salamon in [11]. We take

$$f^2((\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2) + g^2\pi^* ds^2, \tag{4.1}$$

where $\pi^* ds^2$ is the pull-back of the metric ds^2 defined on the base manifold M by the projection map π, and f and g are the functions of the radial parameter $r = a_1^2 + a_2^2 + a_3^2$, where (a_1, a_2, a_3) are the coordinate functions. By choosing an orthonormal frame on M, we define a metric on the $(4n+3)$-dimensional total space by

$$\widehat{g} = f(r)^2\Big((\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2\Big) + g(r)^2\Big((e^1)^2 + ... + (e^{4n})^2\Big) \tag{4.2}$$

and also a pseudo-Riemannian metric with signature $(4n, 3)$ by

$$\widehat{g} = -f(r)^2\Big((\alpha^1)^2 + (\alpha^2)^2 + (\alpha^3)^2\Big) + g(r)^2\Big((e^1)^2 + ... + (e^{4n})^2\Big). \tag{4.3}$$

We have the orthonormal co-frame in the $(4n+3)$-dimensional total space with respect to $\widehat{g}$.

$$\hat{e}^i = f(r)\alpha^i,\ i = 1, 2, 3;\ \hat{e}^\alpha = g(r)e^\alpha,\ \alpha = 1, ..., 4n. \tag{4.4}$$

Note that for $i = \alpha \in \{1, 2, 3\}$, $\hat{e}^i$ and $\hat{e}^\alpha$ denote different terms. For the remainder of this thesis, we shall use Latin letters for the frame and the co-frame of the fibers and Greek letters for the frame and the co-frame of the $4n$-dimensional base manifold. Occasionally, a, b, c, d will denote indices running over all frame and co-frame elements. Also, we will use a summation convention. Typically, repeated indices will be summed

over their range in the computations.

We compute the connection form $\hat{\omega}$ using Theorem 2.2, where i) and ii) uniquely specify $\hat{\omega}$ such that $\hat{\omega} \in \Lambda^1\big(\mathfrak{so}(4n+3)\big)$ for the Riemannian case and $\hat{\omega} \in \Lambda^1\big(\mathfrak{so}(4n,3)\big)$ for the pseudo-Riemannian case. In the latter case, $\hat{\omega}_{ij} = -\hat{\omega}_{ji}$, $\hat{\omega}_{\alpha\beta} = -\hat{\omega}_{\beta\alpha}$, and $\hat{\omega}_{\alpha i} = \hat{\omega}_{i\alpha}$. It can be expressed as

$$\hat{\omega} = \left[\begin{array}{c|c} A & B \\ \hline B^T & C \end{array}\right], \tag{4.5}$$

where $A^T = -A$ is a 3×3 matrix and $C^T = -C$ is a $4n \times 4n$ matrix.

To find the connection 1-forms of the total space, we need to compute the exterior derivatives of the co-frame (4.4). For this purpose, we first observe from (4.4) that

$$dr = 2a_1\alpha^1 + 2a_2\alpha^2 + 2a_3\alpha^3 = \frac{2}{f}\sum_{j=1}^{3} a_j\hat{e}^j. \tag{4.6}$$

It can be easily seen from (3.19) and (4.4) that

$$d\hat{e}^i = d(f\alpha^i) = df \wedge \alpha^i + f d\alpha^i = f'(r)dr \wedge \alpha^i + f(\lambda\epsilon_{ijk}a_j w^k - \alpha_{ij} \wedge \alpha^j). \tag{4.7}$$

Since $\alpha^i = \dfrac{1}{f}\hat{e}^i$ from (4.4), in terms of (4.6), we obtain

$$d\hat{e}^i = \frac{2f'}{f^2}\Big(\sum_{j=1}^{3} a_j\hat{e}^j \wedge \hat{e}^i\Big) + \lambda f\epsilon_{ijk}a_j w^k - \alpha_{ij} \wedge \hat{e}^j. \tag{4.8}$$

Similarly, it is clear from (4.4) that

$$d\hat{e}^\alpha = d\big(ge^\alpha\big) = g'(r)dr \wedge e^\alpha + g de^\alpha. \tag{4.9}$$

By means of Theorem 2.2, we can write $de^\alpha = -\omega_{\alpha\beta} \wedge e^\beta$, where $\omega_{\alpha\beta}$ denotes the

Levi-Civita connection on the quaternionic-Kähler base manifold M. Thus, from (4.4) and (4.6), we have

$$d\hat{e}^{\alpha} = \frac{2g'}{fg}\Big(\sum_{i=1}^{3} a_i \hat{e}^i \wedge \hat{e}^{\alpha}\Big) - \omega_{\alpha\beta} \wedge \hat{e}^{\beta}. \tag{4.10}$$

Note that each $\hat{\omega}_{ab}$ can be written by

$$\hat{\omega}_{ab} = \sum_{i=1}^{3} \hat{\omega}^i_{ab} \hat{e}^i + \sum_{\alpha=1}^{4n} \hat{\omega}^{\alpha}_{ab} \hat{e}^{\alpha} \tag{4.11}$$

since (4.4) is a co-frame. By the summation convention, we simply write

$$\hat{\omega}_{ab} = \hat{\omega}^i_{ab} \hat{e}^i + \hat{\omega}^{\alpha}_{ab} \hat{e}^{\alpha}. \tag{4.12}$$

Also, we note that w^k are alternating two forms, associated with the almost complex structures J_k, defined on the tangent bundle of $4n$-dimensional quaternionic-Kähler base manifold. In terms of the co-frame (4.4), we can write

$$w^k = w^k_{\alpha\beta} e^{\alpha} \wedge e^{\beta} = \frac{1}{g^2} w^k_{\alpha\beta} \hat{e}^{\alpha} \wedge \hat{e}^{\beta}. \tag{4.13}$$

Similarly,

$$\alpha_{ij} = \alpha^{\beta}_{ij} e^{\beta} = \frac{1}{g} \alpha^{\beta}_{ij} \hat{e}^{\beta}. \tag{4.14}$$

Next, we separate the components of the connection form to the co-frame as

$$d\hat{e}^i = -\hat{\omega}^k_{ij} \hat{e}^k \wedge \hat{e}^j - \hat{\omega}^{\beta}_{ij} \hat{e}^{\beta} \wedge \hat{e}^j - \hat{\omega}^k_{i\alpha} \hat{e}^k \wedge \hat{e}^{\alpha} - \hat{\omega}^{\beta}_{i\alpha} \hat{e}^{\beta} \wedge \hat{e}^{\alpha}, \tag{4.15}$$

$$d\hat{e}^{\alpha} = -\hat{\omega}^k_{\alpha j} \hat{e}^k \wedge \hat{e}^j - \hat{\omega}^{\beta}_{\alpha j} \hat{e}^{\beta} \wedge \hat{e}^j - \hat{\omega}^k_{\alpha\beta} \hat{e}^k \wedge \hat{e}^{\beta} - \hat{\omega}^{\gamma}_{\alpha\beta} \hat{e}^{\gamma} \wedge \hat{e}^{\beta}. \tag{4.16}$$

Comparing these expressions with (4.8) and (4.10), we obtain some relations.

i) $-\hat{\omega}^k_{ij}\hat{e}^k \wedge \hat{e}^j = \frac{2f'}{f^2}a_j\hat{e}^j \wedge \hat{e}^i$,

ii) $-\hat{\omega}^\beta_{ij}\hat{e}^\beta \wedge \hat{e}^j - \hat{\omega}^k_{i\alpha}\hat{e}^k \wedge \hat{e}^\alpha = -\alpha_{ij} \wedge \hat{e}^j$,

iii) $-\hat{\omega}^\beta_{i\alpha}\hat{e}^\beta \wedge \hat{e}^\alpha = f\lambda a_j\epsilon_{ijk}w^k$,

iv) $-\hat{\omega}^k_{\alpha j}\hat{e}^k \wedge \hat{e}^j = 0$,

v) $\frac{2g'}{fg}a_i\hat{e}^i \wedge \hat{e}^\alpha = -\hat{\omega}^\beta_{\alpha j}\hat{e}^\beta \wedge \hat{e}^j - \hat{\omega}^k_{\alpha\beta}\hat{e}^k \wedge \hat{e}^\beta$,

vi) $-\hat{\omega}^\gamma_{\alpha\beta}\hat{e}^\gamma \wedge \hat{e}^\beta = -\omega_{\alpha\beta} \wedge \hat{e}^\beta$.

These yield

$$\hat{\omega}^k_{\alpha j} = \hat{\omega}^j_{\alpha k}, \ \ \Big(\hat{\omega}^\alpha_{ij} - \hat{\omega}^j_{i\alpha}\Big)\hat{e}^\alpha = \alpha_{ij}, \ \ \hat{\omega}_{ij} = \frac{2f'}{f^2}\Big(a_j\hat{e}^i - a_i\hat{e}^j\Big) + \hat{\omega}^\beta_{ij}\hat{e}^\beta. \tag{4.17}$$

Since $\hat{\omega}^\alpha_{ij}$ and α_{ij} are skew-symmetric, $\hat{\omega}^j_{i\alpha}$ is also skew-symmetric in i and j. Moreover, considering the first identity in (4.17), we deduce that $\hat{\omega}^j_{i\alpha} = 0$. This is true for both the Riemannian and pseudo-Riemannian metric. Thus, we have $\hat{\omega}^\alpha_{ij}\hat{e}^\alpha = \alpha_{ij}$. Therefore,

$$\hat{\omega}_{ij} = \frac{2f'}{f^2}\Big(a_j\hat{e}^i - a_i\hat{e}^j\Big) + \alpha_{ij}. \tag{4.18}$$

Next, since $\hat{\omega}^j_{i\alpha} = 0$, only base manifold co-frame components of $\hat{\omega}_{i\alpha}$ exist. By comparing the terms in **v)** and the directions of the co-frame, we can easily conclude that

$$\hat{\omega}^\alpha_{\alpha j} = \frac{2g'}{fg}a_j \ \text{ and } \ \hat{\omega}^\beta_{\alpha j} = \hat{\omega}^j_{\alpha\beta}, \tag{4.19}$$

where $\alpha \neq \beta$. Likewise, by **iii)**, we have $\hat{\omega}^\beta_{i\alpha} = \frac{f\lambda}{g^2}\epsilon_{ijk}a_jw^k_{\alpha\beta}$. Thus, we write the components of $\hat{\omega}_{\alpha i}$ as

$$\hat{\omega}_{\alpha i} = -\frac{f\lambda}{g^2}\epsilon_{ijk}a_jw^k_{\alpha\beta}\hat{e}^\beta + \frac{2g'}{fg}a_i\hat{e}^\alpha. \tag{4.20}$$

For the pseudo-Riemannian case, we have

$$\hat{\omega}_{\alpha i} = \frac{f\lambda}{g^2}\epsilon_{ijk}a_jw^k_{\alpha\beta}\hat{e}^\beta + \frac{2g'}{fg}a_i\hat{e}^\alpha. \tag{4.21}$$

Finally, by **vi)** and (4.19), we observe that

$$\hat{\omega}_{\alpha\beta} - \omega_{\alpha\beta} = -\frac{f\lambda}{g^2}\epsilon_{ijk}a_j w^k_{\alpha\beta}\hat{e}^i. \tag{4.22}$$

In the case of the pseudo-Riemannian metric, we deduce that

$$\hat{\omega}_{\alpha\beta} - \omega_{\alpha\beta} = \frac{f\lambda}{g^2}\epsilon_{ijk}a_j w^k_{\alpha\beta}\hat{e}^i. \tag{4.23}$$

Now, we have all components of the connection form $\hat{\omega}$ of the Riemannian metric and pseudo-Riemannian metric.

4.1. Curvature Calculations

For simplifications, we choose an orthonormal frame $\{e_\alpha\}$ on the quaternionic-Kähler base manifold M such that $\omega_{\alpha\beta}(p) = 0$ for a fixed point $p \in M$. Similarly, we choose $\{J_1, J_2, J_3\}$ such that $\alpha_{ij}(p) = 0$. Recall that w^1, w^2 and w^3 are defined by (3.3). The following lemma helps us to simplify the curvature calculations.

Lemma 4.1. *Let $\{e_\beta\}$ be an orthonormal frame on the $4n$-dimensional quaternionic-Kähler manifold M, where the connection 1-form $\omega_{\alpha\beta}$ vanishes at a fixed point $p \in M$, and $\{J_1, J_2, J_3\}$ is a frame such that $\alpha_{ij}(p) = 0$. Then,*

$$d\big(w^k_{\alpha\beta}e^\beta\big)_p = \nabla\big(w^k_{\alpha\beta}e^\beta\big)_p = d\big(w^k_{\alpha\beta}\big)_p = 0, \tag{4.24}$$

where d denotes the exterior derivative and ∇ is the Levi-Civita connection.

Proof. Note that by (2.3) and (3.11), we have

$$\nabla e_\beta = \omega_{\alpha\beta} \otimes e_\alpha \text{ and } \nabla w^k = \alpha_{ik} \otimes w^i. \tag{4.25}$$

Let

$$\psi^k_\alpha := w^k_{\alpha\beta} e^\beta = \frac{1}{2} w^k(e_\alpha, -). \tag{4.26}$$

By using (2.3) and (3.11), we derive

$$\begin{aligned} \nabla(w^k_{\alpha\beta} e^\beta) &= \frac{1}{2}\alpha_{ik} \otimes w^i(e_\alpha, -) + \frac{1}{2}\omega_{\beta\alpha} \otimes w^k(e_\beta, -) \\ &= \alpha_{ik} \otimes \psi^i_\alpha + \omega_{\beta\alpha} \otimes \psi^k_\beta \end{aligned} \tag{4.27}$$

and

$$d\psi^k_\alpha = \alpha_{ik} \wedge \psi^i_\alpha + \omega_{\beta\alpha} \wedge \psi^k_\beta. \tag{4.28}$$

We have

$$dw^k_{\alpha\beta} = \nabla w^k_{\alpha\beta} = \frac{1}{2}\nabla w^k(e_\alpha, e_\beta) + \frac{1}{2} w^k(\nabla e_\alpha, e_\beta) + \frac{1}{2} w^k(e_\alpha, \nabla e_\beta) \tag{4.29}$$

which gives

$$dw^k_{\alpha\beta} = \frac{1}{2}\alpha_{ik} w^k(e_\alpha, e_\beta) + \frac{1}{2}\omega_{\gamma\alpha} w^k(e_\gamma, e_\beta) + \frac{1}{2}\omega_{\gamma\beta} w^k(e_\alpha, e_\gamma). \tag{4.30}$$

It follows from the hypothesis that

$$d\big(w^k_{\alpha\beta} e^\beta\big)_p = \nabla\big(w^k_{\alpha\beta} e^\beta\big)_p = d\big(w^k_{\alpha\beta}\big)_p = 0. \tag{4.31}$$

□

Note that under the assumption in Lemma 4.1, at point $p \in M$, the equation (3.12) becomes $R = dA$. Moreover, it follows from (3.17) that $da_i = \alpha^i$. By the assumption of Lemma 4.1, it follows from (2.4) that $\Omega_{\alpha\beta} = d\omega_{\alpha\beta}$ for the base manifold

M at $p \in M$. We use Cartan's second structure equation to find the curvatures (see [15]). The curvature 2-form $\widehat{\Omega}$ of the total space can be calculated by

$$\widehat{\Omega} = d\hat{\omega} + \hat{\omega} \wedge \hat{\omega}, \tag{4.32}$$

more precisely in the following matrix form

$$\widehat{\Omega}_{ab} = d\hat{\omega}_{ab} + \sum_{k=1}^{3} \hat{\omega}_{ak} \wedge \hat{\omega}_{kb} + \sum_{\alpha=1}^{4n} \hat{\omega}_{a\alpha} \wedge \hat{\omega}_{\alpha b}. \tag{4.33}$$

Let $\widehat{\Omega}$ denote the curvature of the $(4n+3)$-dimensional total space $\widehat{M}$ and Ω the curvature of the $4n$-dimensional quaternionic-Kähler base manifold M. From (4.33), we have

$$\widehat{\Omega}_{ij} = d\hat{\omega}_{ij} + \sum_{k=1}^{3} \hat{\omega}_{ik} \wedge \hat{\omega}_{kj} + \sum_{\alpha=1}^{4n} \hat{\omega}_{i\alpha} \wedge \hat{\omega}_{\alpha j}, \tag{4.34}$$

where $1 \leq i, j \leq 3$.

In order to find $\widehat{\Omega}_{ij}$, we need to calculate $d\hat{\omega}_{ij}$, $\hat{\omega}_{ik} \wedge \hat{\omega}_{kj}$ and $\hat{\omega}_{i\alpha} \wedge \hat{\omega}_{\alpha j}$. For this purpose, we first take the exterior derivative of (4.18) as

$$\begin{aligned} d\hat{\omega}_{ij} &= d\Big(\frac{2f'}{f^2}\big(a_j\hat{e}^i - a_i\hat{e}^j\big) + \alpha_{ij}\Big) \\ &= 2\Big(\frac{f'}{f^2}\Big)' dr \wedge \Big(a_j\hat{e}^i - a_i\hat{e}^j\Big) + \frac{2f'}{f^2}\Big(da_j \wedge \hat{e}^i - da_i \wedge \hat{e}^j\Big) \\ &\quad + \frac{2f'}{f^2}\Big(a_j d\hat{e}^i - a_i d\hat{e}^j\Big) + d\alpha_{ij}. \end{aligned} \tag{4.35}$$

Then, at point $p \in M$, from (3.12), (3.16), (3.17), (4.6) and (4.8), it follows for both the Riemannian and pseudo-Riemannian cases that

$$\begin{aligned} d\hat{\omega}_{ij} = &\Big(\frac{4f''}{f^3} - \frac{4(f')^2}{f^4}\Big)a_j a_k \hat{e}^k \wedge \hat{e}^i + \Big(\frac{-4f''}{f^3} + \frac{4(f')^2}{f^4}\Big)a_i a_k \hat{e}^k \wedge \hat{e}^j \\ &- \frac{4f'}{f^3}\hat{e}^i \wedge \hat{e}^j + \frac{2\lambda f'}{f}\Big(\epsilon_{ilk} a_j a_l w^k - \epsilon_{jlk} a_i a_l w^k\Big) + d\alpha_{ij}. \end{aligned} \tag{4.36}$$

By (4.18), we deduce for both the Riemannian and pseudo-Riemannian cases that

$$\begin{aligned}\hat{\omega}_{ik}\wedge\hat{\omega}_{kj} &= \frac{4(f')^2}{f^4}\Big(a_k\hat{e}^i - a_i\hat{e}^k\Big)\wedge\Big(a_j\hat{e}^k - a_k\hat{e}^j\Big)\\ &= \frac{4(f')^2}{f^4}\Big(a_ka_j\hat{e}^i\wedge\hat{e}^k - a_ka_k\hat{e}^i\wedge\hat{e}^j + a_ka_i\hat{e}^k\wedge\hat{e}^j\Big).\end{aligned}\tag{4.37}$$

Moreover, by (4.20), we get for the Riemannian case that

$$\hat{\omega}_{i\beta}\wedge\hat{\omega}_{\beta j} = \Big(\frac{\lambda f}{g^2}\epsilon_{ilk}a_lw^k_{\beta\gamma}\hat{e}^\gamma - \frac{2g'}{fg}a_i\hat{e}^\beta\Big)\wedge\Big(-\frac{\lambda f}{g^2}\epsilon_{jpq}a_pw^q_{\beta\delta}\hat{e}^\delta + \frac{2g'}{fg}a_j\hat{e}^\beta\Big)\tag{4.38}$$

which gives

$$\begin{aligned}\hat{\omega}_{i\beta}\wedge\hat{\omega}_{\beta j} = &-\frac{\lambda^2f^2}{g^4}\epsilon_{ilk}\epsilon_{jpq}a_la_pw^k_{\beta\gamma}w^q_{\beta\delta}\hat{e}^\gamma\wedge\hat{e}^\delta + \frac{2\lambda g'}{g^3}a_j\epsilon_{ilk}a_lw^k_{\beta\gamma}\hat{e}^\gamma\wedge\hat{e}^\beta\\ &+\frac{2\lambda g'}{g^3}a_i\epsilon_{jpq}a_pw^q_{\beta\delta}\hat{e}^\beta\wedge\hat{e}^\delta.\end{aligned}\tag{4.39}$$

For the pseudo-Riemannian metric of the signature $(4n,3)$, we obtain from (4.21) that

$$\begin{aligned}\hat{\omega}_{i\beta}\wedge\hat{\omega}_{\beta j} = &\frac{\lambda^2f^2}{g^4}\epsilon_{ilk}\epsilon_{jpq}a_la_pw^k_{\beta\gamma}w^q_{\beta\delta}\hat{e}^\gamma\wedge\hat{e}^\delta + \frac{2\lambda g'}{g^3}a_j\epsilon_{ilk}a_lw^k_{\beta\gamma}\hat{e}^\gamma\wedge\hat{e}^\beta\\ &+\frac{2\lambda g'}{g^3}a_i\epsilon_{jpq}a_pw^q_{\beta\delta}\hat{e}^\beta\wedge\hat{e}^\delta.\end{aligned}\tag{4.40}$$

By adding (4.36), (4.37), (4.39) and taking into consideration (3.12) and (3.16) with Lemma 4.1, we get the curvature $\widehat{\Omega}_{ij}$ for the Riemannian case as

$$\begin{aligned}\widehat{\Omega}_{ij} = &\Big(\frac{4f''}{f^3} - \frac{8(f')^2}{f^4}\Big)a_ja_k\hat{e}^k\wedge\hat{e}^i + \Big(\frac{-4f''}{f^3} + \frac{8(f')^2}{f^4}\Big)a_ia_k\hat{e}^k\wedge\hat{e}^j\\ &-\Big(\frac{4f'}{f^3} + \frac{4(f')^2}{f^4}r\Big)\hat{e}^i\wedge\hat{e}^j - \frac{\lambda^2f^2}{g^4}\epsilon_{ilk}\epsilon_{jpq}a_la_pw^k_{\beta\gamma}w^q_{\beta\delta}\hat{e}^\gamma\wedge\hat{e}^\delta\\ &+2\lambda\Big(\frac{f'}{f} - \frac{g'}{g}\Big)\epsilon_{ilk}a_ja_lw^k - 2\lambda\Big(\frac{f'}{f} - \frac{g'}{g}\Big)\epsilon_{jlk}a_ia_lw^k + \lambda\epsilon_{ijk}w^k.\end{aligned}\tag{4.41}$$

On the other hand, considering (3.12), (3.16), (4.36), (4.37), (4.40) and Lemma 4.1,

for the pseudo-Riemannian metric, we find the curvature $\widehat{\Omega}_{ij}$ as

$$\begin{aligned}\widehat{\Omega}_{ij} &= \Big(\frac{4f''}{f^3} - \frac{8(f')^2}{f^4}\Big)a_j a_k \hat{e}^k \wedge \hat{e}^i + \Big(\frac{-4f''}{f^3} + \frac{8(f')^2}{f^4}\Big)a_i a_k \hat{e}^k \wedge \hat{e}^j \\ &\quad - \Big(\frac{4f'}{f^3} + \frac{4(f')^2}{f^4}r\Big)\hat{e}^i \wedge \hat{e}^j + \frac{\lambda^2 f^2}{g^4}\epsilon_{ilk}\epsilon_{jpq}a_l a_p w^k_{\beta\gamma} w^q_{\beta\delta}\hat{e}^\gamma \wedge \hat{e}^\delta \\ &\quad + 2\lambda\Big(\frac{f'}{f} - \frac{g'}{g}\Big)\epsilon_{ilk}a_j a_l w^k - 2\lambda\Big(\frac{f'}{f} - \frac{g'}{g}\Big)\epsilon_{jlk}a_i a_l w^k + \lambda\epsilon_{ijk}w^k. \end{aligned} \tag{4.42}$$

From (4.33), we have

$$\widehat{\Omega}_{\alpha i} = d\hat{\omega}_{\alpha i} + \sum_{k=1}^{3}\hat{\omega}_{\alpha k}\wedge\hat{\omega}_{ki} + \sum_{\beta=1}^{4n}\hat{\omega}_{\alpha\beta}\wedge\hat{\omega}_{\beta i}. \tag{4.43}$$

We begin to calculate it by taking the exterior derivative of (4.20) for the Riemannian case as

$$\begin{aligned} d\hat{\omega}_{\alpha i} &= d\Big(-\frac{f\lambda}{g^2}\epsilon_{ijk}a_j w^k_{\alpha\beta}\hat{e}^\beta\Big) + d\Big(\frac{2g'}{fg}a_i\hat{e}^\alpha\Big) \\ &= -\epsilon_{ijk}\Big(\frac{\lambda f}{g^2}\Big)' dr\wedge a_j w^k_{\alpha\beta}\hat{e}^\beta - \epsilon_{ijk}\frac{\lambda f}{g^2}d\big(a_j\big)\wedge w^k_{\alpha\beta}\hat{e}^\beta - \frac{\lambda f}{g^2}\epsilon_{ijk}a_j d\Big(w^k_{\alpha\beta}\hat{e}^\beta\Big) \\ &\quad + \Big(\frac{2g'}{fg}\Big)' dr\wedge a_i\hat{e}^\alpha + \frac{2g'}{fg}d\big(a_i\big)\hat{e}^\alpha + \frac{2g'}{fg}a_i d\big(\hat{e}^\alpha\big). \end{aligned} \tag{4.44}$$

In terms of Lemma 4.1 at the fixed point $p\in M$, by (4.6) and (4.10), we deduce for the Riemannian case that

$$\begin{aligned} d\hat{\omega}_{\alpha i} &= -2\lambda\Big(\frac{f'}{fg^2} - \frac{2g'}{g^3}\Big)\epsilon_{ijk}a_p a_j w^k_{\alpha\beta}\hat{e}^p\wedge\hat{e}^\beta - \frac{\lambda}{g^2}\epsilon_{ijk}w^k_{\alpha\beta}\hat{e}^j\wedge\hat{e}^\beta + \frac{2g'}{f^2g}\hat{e}^i\wedge\hat{e}^\alpha \\ &\quad + \Big(\frac{4g''}{f^2g} - \frac{4f'g'}{f^3g}\Big)a_j a_i\hat{e}^j\wedge\hat{e}^\alpha. \end{aligned} \tag{4.45}$$

In the pseudo-Riemannian case, we find the exterior derivative of (4.21) as

$$\begin{aligned} d\hat{\omega}_{\alpha i} &= +2\lambda\Big(\frac{f'}{fg^2} - \frac{2g'}{g^3}\Big)\epsilon_{ijk}a_p a_j w^k_{\alpha\beta}\hat{e}^p\wedge\hat{e}^\beta + \frac{\lambda}{g^2}\epsilon_{ijk}w^k_{\alpha\beta}\hat{e}^j\wedge\hat{e}^\beta + \frac{2g'}{f^2g}\hat{e}^i\wedge\hat{e}^\alpha \\ &\quad + \Big(\frac{4g''}{f^2g} - \frac{4f'g'}{f^3g}\Big)a_j a_i\hat{e}^j\wedge\hat{e}^\alpha. \end{aligned} \tag{4.46}$$

For the Riemannian case, we find the wedge product of (4.18) and (4.20) at $p \in M$.

$$\hat{\omega}_{\alpha j} \wedge \hat{\omega}_{ji} = \Big(-\frac{f\lambda}{g^2}\epsilon_{jpq}a_p w^q_{\alpha\beta}\hat{e}^\beta + \frac{2g'}{fg}a_j\hat{e}^\alpha\Big) \wedge \frac{2f'}{f^2}\Big(a_i\hat{e}^j - a_j\hat{e}^i\Big). \tag{4.47}$$

A simple computation yields

$$\hat{\omega}_{\alpha j} \wedge \hat{\omega}_{ji} = -\frac{2\lambda f'}{fg^2}a_i a_p\epsilon_{jpq}w^q_{\alpha\beta}\hat{e}^\beta \wedge \hat{e}^j - \frac{4f'g'}{f^3g}r\hat{e}^\alpha \wedge \hat{e}^i + \frac{4f'g'}{f^3g}a_i a_j\hat{e}^\alpha \wedge \hat{e}^j. \tag{4.48}$$

On the other hand, for the pseudo-Riemannian metric, at $p \in M$, we find by (4.18) and (4.21) that

$$\hat{\omega}_{\alpha j} \wedge \hat{\omega}_{ji} = \frac{2\lambda f'}{fg^2}a_i a_p\epsilon_{jpq}w^q_{\alpha\beta}\hat{e}^\beta \wedge \hat{e}^j - \frac{4f'g'}{f^3g}r\hat{e}^\alpha \wedge \hat{e}^i + \frac{4f'g'}{f^3g}a_i a_j\hat{e}^\alpha \wedge \hat{e}^j. \tag{4.49}$$

Further, by (4.20), (4.22) and Lemma 4.1, we have for the Riemannian case that

$$\begin{aligned}\hat{\omega}_{\alpha\beta} \wedge \hat{\omega}_{\beta i} &= \Big(-\frac{\lambda f}{g^2}\epsilon_{pqk}a_q w^k_{\alpha\beta}\hat{e}^p\Big) \wedge \Big(-\frac{\lambda f}{g^2}\epsilon_{ijl}a_j w^l_{\beta\gamma}\hat{e}^\gamma + \frac{2g'}{fg}a_i\hat{e}^\beta\Big) \\ &= \frac{\lambda^2 f^2}{g^4}\epsilon_{pqk}\epsilon_{ijl}a_q a_j w^k_{\alpha\beta}w^l_{\beta\gamma}\hat{e}^p \wedge \hat{e}^\gamma - \frac{2\lambda g'}{g^3}\epsilon_{pqk}a_q a_i w^k_{\alpha\beta}\hat{e}^p \wedge \hat{e}^\beta. \end{aligned}\tag{4.50}$$

At $p \in M$, taking the wedge of (4.21) and (4.23), for the pseudo-Riemannian case, we get

$$\hat{\omega}_{\alpha\beta} \wedge \hat{\omega}_{\beta i} = \frac{\lambda^2 f^2}{g^4}\epsilon_{pqk}\epsilon_{ijl}a_q a_j w^k_{\alpha\beta}w^l_{\beta\gamma}\hat{e}^p \wedge \hat{e}^\gamma + \frac{2\lambda g'}{g^3}\epsilon_{pqk}a_q a_i w^k_{\alpha\beta}\hat{e}^p \wedge \hat{e}^\beta. \tag{4.51}$$

From (4.45), (4.48) and (4.50), it follows for the Riemannian case that

$$\begin{aligned}\widehat{\Omega}_{\alpha i} = &-2\lambda\Big(\frac{f'}{fg^2} - \frac{2g'}{g^3}\Big)\epsilon_{ijk}a_p a_j w^k_{\alpha\beta}\hat{e}^p \wedge \hat{e}^\beta - \frac{\lambda}{g^2}\epsilon_{ijk}w^k_{\alpha\beta}\hat{e}^j \wedge \hat{e}^\beta \\ &+ \Big(\frac{4g''}{f^2g} - \frac{8f'g'}{f^3g}\Big)a_j a_i\hat{e}^j \wedge \hat{e}^\alpha - \Big(\frac{2g'}{f^2g} + \frac{4f'g'}{f^3g}r\Big)\hat{e}^\alpha \wedge \hat{e}^i \\ &- \frac{2\lambda f'}{fg^2}a_i a_p\epsilon_{jpq}w^q_{\alpha\beta}\hat{e}^\beta \wedge \hat{e}^j - \frac{2\lambda g'}{g^3}\epsilon_{pqk}a_q a_i w^k_{\alpha\beta}\hat{e}^p \wedge \hat{e}^\beta \\ &+ \frac{\lambda^2 f^2}{g^4}\epsilon_{pqk}\epsilon_{ijl}a_q a_j w^k_{\alpha\beta}w^l_{\beta\gamma}\hat{e}^p \wedge \hat{e}^\gamma. \end{aligned}\tag{4.52}$$

Moreover, for the pseudo-Riemannian case, we deduce from (4.46), (4.49) and (4.51) that

$$\begin{aligned}\widehat{\Omega}_{\alpha i} &= 2\lambda\Big(\frac{f'}{fg^2}-\frac{2g'}{g^3}\Big)\epsilon_{ijk}a_pa_jw^k_{\alpha\beta}\hat{e}^p\wedge\hat{e}^\beta+\frac{\lambda}{g^2}\epsilon_{ijk}w^k_{\alpha\beta}\hat{e}^j\wedge\hat{e}^\beta\\ &\quad+\Big(\frac{4g''}{f^2g}-\frac{8f'g'}{f^3g}\Big)a_ja_i\hat{e}^j\wedge\hat{e}^\alpha-\Big(\frac{2g'}{f^2g}+\frac{4f'g'}{f^3g}r\Big)\hat{e}^\alpha\wedge\hat{e}^i\\ &\quad+\frac{2\lambda f'}{fg^2}a_ia_p\epsilon_{jpq}w^q_{\alpha\beta}\hat{e}^\beta\wedge\hat{e}^j+\frac{2\lambda g'}{g^3}\epsilon_{pqk}a_qa_iw^k_{\alpha\beta}\hat{e}^p\wedge\hat{e}^\beta\\ &\quad+\frac{\lambda^2f^2}{g^4}\epsilon_{pqk}\epsilon_{ijl}a_qa_jw^k_{\alpha\beta}w^l_{\beta\gamma}\hat{e}^p\wedge\hat{e}^\gamma.\end{aligned}\tag{4.53}$$

From (4.33), we have

$$\widehat{\Omega}_{\alpha\beta}=d\hat{\omega}_{\alpha\beta}+\sum_{i=1}^{3}\hat{\omega}_{\alpha i}\wedge\hat{\omega}_{i\beta}+\sum_{\gamma=1}^{4n}\hat{\omega}_{\alpha\gamma}\wedge\hat{\omega}_{\gamma\beta}.\tag{4.54}$$

We note that since $\omega_{\alpha\beta}(p)=0$ by our assumption, we have $\Omega_{\alpha\beta}=d\omega_{\alpha\beta}$ at the point $p\in M$.

By considering (4.22) and taking its exterior derivative, for the Riemannian case, it follows from (4.6), (4.8) and Lemma 4.1 that

$$\begin{aligned}d\hat{\omega}_{\alpha\beta}-d\omega_{\alpha\beta}&=-\Big(\frac{f\lambda}{g^2}\Big)'dr\wedge\epsilon_{ijk}a_jw^k_{\alpha\beta}\hat{e}^i-\frac{f\lambda}{g^2}\epsilon_{ijk}da_j\wedge w^k_{\alpha\beta}\hat{e}^i-\frac{f\lambda}{g^2}\epsilon_{ijk}a_jw^k_{\alpha\beta}d\hat{e}^i\\ &=-2\lambda\Big(\frac{f'}{fg^2}-\frac{2g'}{g^3}\Big)\epsilon_{ijk}a_pa_jw^k_{\alpha\beta}\hat{e}^p\wedge\hat{e}^i-\frac{\lambda}{g^2}\epsilon_{ijk}w^k_{\alpha\beta}\hat{e}^j\wedge\hat{e}^i\\ &\quad-\frac{2\lambda f'}{fg^2}\epsilon_{ijk}a_ja_lw^k_{\alpha\beta}\hat{e}^l\wedge\hat{e}^i-\frac{\lambda f^2}{g^4}\epsilon_{ijk}\epsilon_{ins}a_ja_nw^k_{\alpha\beta}w^s_{\gamma\theta}\hat{e}^\gamma\wedge\hat{e}^\theta.\end{aligned}\tag{4.55}$$

For the pseudo-Riemannian case, it follows from (4.6), (4.8), (4.23) and Lemma 4.1 that

$$\begin{aligned}d\hat{\omega}_{\alpha\beta}-d\omega_{\alpha\beta}&=2\lambda\Big(\frac{f'}{fg^2}-\frac{2g'}{g^3}\Big)\epsilon_{ijk}a_pa_jw^k_{\alpha\beta}\hat{e}^p\wedge\hat{e}^i+\frac{\lambda}{g^2}\epsilon_{ijk}w^k_{\alpha\beta}\hat{e}^j\wedge\hat{e}^i\\ &\quad+\frac{2\lambda f'}{fg^2}\epsilon_{ijk}a_ja_lw^k_{\alpha\beta}\hat{e}^l\wedge\hat{e}^i+\frac{\lambda f^2}{g^4}\epsilon_{ijk}\epsilon_{ins}a_ja_nw^k_{\alpha\beta}w^s_{\gamma\theta}\hat{e}^\gamma\wedge\hat{e}^\theta.\end{aligned}\tag{4.56}$$

Next, by (4.22) and Lemma 4.1, we deduce for the Riemannian case that

$$\begin{aligned}\hat{\omega}_{\alpha\gamma}\wedge\hat{\omega}_{\gamma\beta} &= \Big(-\frac{\lambda f}{g^2}\epsilon_{ijk}a_j w^k_{\alpha\gamma}\hat{e}^i\Big)\wedge\Big(-\frac{\lambda f}{g^2}\epsilon_{lpq}a_p w^q_{\gamma\beta}\hat{e}^l\Big)\\ &= \frac{\lambda^2 f^2}{g^4}\epsilon_{ijk}\epsilon_{lpq}a_j a_p w^k_{\alpha\gamma}w^q_{\gamma\beta}\hat{e}^i\wedge\hat{e}^l. \end{aligned}\tag{4.57}$$

In the pseudo-Riemannian case, from (4.23), both $\hat{\omega}_{\alpha\gamma}$ and $\hat{\omega}_{\gamma\beta}$ have the opposite sign to the Riemannian case. Thus, we end up with the same result for the wedge product above. So, (4.57) is also true for the pseudo-Riemannian case. Furthermore, by (4.20), we obtain for the Riemannian case that

$$\begin{aligned}\hat{\omega}_{\alpha i}\wedge\hat{\omega}_{i\beta} &= \Big(-\frac{f\lambda}{g^2}\epsilon_{ijk}a_j w^k_{\alpha\gamma}\hat{e}^\gamma+\frac{2g'}{fg}a_i\hat{e}^\alpha\Big)\wedge\Big(\frac{f\lambda}{g^2}\epsilon_{ipq}a_p w^q_{\beta\delta}\hat{e}^\delta-\frac{2g'}{fg}a_i\hat{e}^\beta\Big)\\ &= \frac{2\lambda g'}{g^3}\Big(a_i a_j\epsilon_{ijk}w^k_{\alpha\gamma}\hat{e}^\gamma\wedge\hat{e}^\beta+\epsilon_{ipq}a_i a_p w^q_{\beta\delta}\hat{e}^\alpha\wedge\hat{e}^\delta\Big)\\ &\quad-\frac{f^2\lambda^2}{g^4}\epsilon_{ijk}\epsilon_{ipq}a_j a_p w^k_{\alpha\gamma}w^q_{\beta\delta}\hat{e}^\gamma\wedge\hat{e}^\delta-\frac{4(g')^2 r}{f^2g^2}\hat{e}^\alpha\wedge\hat{e}^\beta\\ &= -\frac{f^2\lambda^2}{g^4}\epsilon_{ijk}\epsilon_{ipq}a_j a_p w^k_{\alpha\gamma}w^q_{\beta\delta}\hat{e}^\gamma\wedge\hat{e}^\delta-\frac{4(g')^2 r}{f^2g^2}\hat{e}^\alpha\wedge\hat{e}^\beta. \end{aligned}\tag{4.58}$$

For the pseudo-Riemannian case, it follows from (4.21) that

$$\begin{aligned}\hat{\omega}_{\alpha i}\wedge\hat{\omega}_{i\beta} &= \frac{f^2\lambda^2}{g^4}\epsilon_{ijk}\epsilon_{ipq}a_j a_p w^k_{\alpha\gamma}w^q_{\beta\delta}\hat{e}^\gamma\wedge\hat{e}^\delta+\frac{4(g')^2 r}{f^2g^2}\hat{e}^\alpha\wedge\hat{e}^\beta\\ &\quad+\frac{2\lambda g'}{g^3}\Big(a_i a_j\epsilon_{ijk}w^k_{\alpha\gamma}\hat{e}^\gamma\wedge\hat{e}^\beta+\epsilon_{ipq}a_i a_p w^q_{\beta\delta}\hat{e}^\alpha\wedge\hat{e}^\delta\Big)\\ &= \frac{f^2\lambda^2}{g^4}\epsilon_{ijk}\epsilon_{ipq}a_j a_p w^k_{\alpha\gamma}w^q_{\beta\delta}\hat{e}^\gamma\wedge\hat{e}^\delta+\frac{4(g')^2 r}{f^2g^2}\hat{e}^\alpha\wedge\hat{e}^\beta. \end{aligned}\tag{4.59}$$

Notice that the term

$$a_i a_j\epsilon_{ijk}w^k_{\alpha\gamma}\hat{e}^\gamma\wedge\hat{e}^\beta+\epsilon_{ipq}a_i a_p w^q_{\beta\delta}\hat{e}^\alpha\wedge\hat{e}^\delta \tag{4.60}$$

in (4.58) and (4.59) vanishes since a_i is contracted twice with the skew-symmetric ϵ_{ijk}.

From (4.55), (4.57) and (4.58), we get for the Riemannian case that

$$\begin{aligned}\widehat{\Omega}_{\alpha\beta} - \Omega_{\alpha\beta} = &-2\lambda\Big(\frac{2f'}{fg^2} - \frac{2g'}{g^3}\Big)\epsilon_{ijk}a_p a_j w^k_{\alpha\beta}\hat{e}^p \wedge \hat{e}^i - \frac{\lambda}{g^2}\epsilon_{ijk}w^k_{\alpha\beta}\hat{e}^j \wedge \hat{e}^i \\ &+ \frac{\lambda^2 f^2}{g^4}\epsilon_{ijk}\epsilon_{lpq}a_j a_p w^k_{\alpha\gamma}w^q_{\gamma\beta}\hat{e}^i \wedge \hat{e}^l - \frac{f^2\lambda^2}{g^4}\epsilon_{ijk}\epsilon_{ipq}a_j a_p w^k_{\alpha\gamma}w^q_{\beta\delta}\hat{e}^\gamma \wedge \hat{e}^\delta \\ &- \frac{4(g')^2 r}{f^2 g^2}\hat{e}^\alpha \wedge \hat{e}^\beta - \frac{\lambda^2 f^2}{g^4}\epsilon_{ijk}\epsilon_{ins}a_j a_n w^k_{\alpha\beta}w^s_{\gamma\theta}\hat{e}^\gamma \wedge \hat{e}^\theta. \end{aligned} \tag{4.61}$$

For the pseudo-Riemannian case, we deduce from (4.56), (4.57) and (4.59) that

$$\begin{aligned}\widehat{\Omega}_{\alpha\beta} - \Omega_{\alpha\beta} = &\,2\lambda\Big(\frac{2f'}{fg^2} - \frac{2g'}{g^3}\Big)\epsilon_{ijk}a_p a_j w^k_{\alpha\beta}\hat{e}^p \wedge \hat{e}^i + \frac{\lambda}{g^2}\epsilon_{ijk}w^k_{\alpha\beta}\hat{e}^j \wedge \hat{e}^i \\ &+ \frac{\lambda^2 f^2}{g^4}\epsilon_{ijk}\epsilon_{lpq}a_j a_p w^k_{\alpha\gamma}w^q_{\gamma\beta}\hat{e}^i \wedge \hat{e}^l + \frac{f^2\lambda^2}{g^4}\epsilon_{ijk}\epsilon_{ipq}a_j a_p w^k_{\alpha\gamma}w^q_{\beta\delta}\hat{e}^\gamma \wedge \hat{e}^\delta \\ &+ \frac{4(g')^2 r}{f^2 g^2}\hat{e}^\alpha \wedge \hat{e}^\beta + \frac{\lambda^2 f^2}{g^4}\epsilon_{ijk}\epsilon_{ins}a_j a_n w^k_{\alpha\beta}w^s_{\gamma\theta}\hat{e}^\gamma \wedge \hat{e}^\theta. \end{aligned} \tag{4.62}$$

4.2. Ricci Curvature

We determine the Ricci curvature of the (pseudo-) Riemannian metric by contracting the curvature 2-forms. Let $\{\hat{e}_i\}$ (respectively, $\{\hat{e}_\alpha\}$) denote the orthonormal frame dual to $\{\hat{e}^i\}$ (respectively, $\{\hat{e}^\alpha\}$) as given in (4.4). We find the Ricci curvature as

$$Ric_{ab} := Ric\big(\hat{e}_a, \hat{e}_b\big) = \sum_{k=1}^{3} g\Big(R\big(\hat{e}_k, \hat{e}_a\big)\hat{e}_b, \hat{e}_k\Big) + \sum_{\alpha=1}^{4n} g\Big(R\big(\hat{e}_\alpha, \hat{e}_a\big)\hat{e}_b, \hat{e}_\alpha\Big). \tag{4.63}$$

In terms of wedge products, we have $\Omega_{ab} = \frac{1}{2}R^a_{bcd}\hat{e}^c \wedge \hat{e}^d$. Hence,

$$Ric_{ab} = \sum_{k=1}^{3} \Omega_{kb}(\hat{e}_k, \hat{e}_a) + \sum_{\alpha=1}^{4n} \Omega_{\alpha b}(\hat{e}_\alpha, \hat{e}_a). \tag{4.64}$$

For the Riemannian metric (4.2), taking into account (4.41), (4.52) and (4.61), we find the Ricci curvatures for the total space as

$$Ric_{ij} = \Big(-\frac{4f''}{f^3} + \frac{8(f')^2}{f^4} - \frac{16ng''}{f^2g} + \frac{32nf'g'}{f^3g} - \frac{n\lambda^2 f^2}{g^4}\Big)a_i a_j, \tag{4.65}$$

where $i \neq j$. Moreover, we obtain

$$\begin{aligned} Ric_{ii} = &\Big(-\frac{4f''}{f^3} + \frac{8(f')^2}{f^4} + \frac{32nf'g'}{f^3g} - \frac{16ng''}{f^2g} - \frac{n\lambda^2 f^2}{g^4}\Big)a_i^2 \\ &+ \Big(-\frac{4f''}{f^3} - \frac{16nf'g'}{f^3g} + \frac{n\lambda^2 f^2}{g^4}\Big)r - \frac{8f'}{f^3} - \frac{8ng'}{f^2g}. \end{aligned} \tag{4.66}$$

Furthermore, for the base manifold indices, we deduce

$$Ric_{\alpha\alpha} = \Big(-\frac{\lambda^2 f^2}{g^4} - \frac{4g''}{f^2g} - \frac{4f'g'}{f^3g} - \frac{4(4n-1)(g')^2}{f^2g^2}\Big)r + \frac{(n+2)\lambda}{g^2} - \frac{6g'}{f^2g}. \tag{4.67}$$

If we consider the pseudo-Riemannian metric given in (4.3) and take into account (4.42), (4.53) and (4.62), we see that Ric_{ij} and Ric_{ii} are the same as the Riemannian metric. However, for the pseudo-Riemannian metric, we obtain

$$Ric_{\alpha\alpha} = \Big(\frac{\lambda^2 f^2}{g^4} + \frac{4g''}{f^2g} + \frac{4f'g'}{f^3g} + \frac{4(4n-1)(g')^2}{f^2g^2}\Big)r + \frac{(n+2)\lambda}{g^2} + \frac{6g'}{f^2g}. \tag{4.68}$$

On the other hand, we have the following for both the Riemannian and pseudo-Riemannian metrics that

$$Ric_{i\alpha} = 0 \text{ and } Ric_{\alpha\beta} = 0, \text{ where } \alpha \neq \beta. \tag{4.69}$$

For the Riemannian metric, the Einstein equation $Ric = \widehat{\Lambda}\,\widehat{g}$ for the total space gives

$$-\frac{4f''}{f^3} + \frac{8(f')^2}{f^4} - \frac{16ng''}{f^2g} + \frac{32nf'g'}{f^3g} - \frac{n\lambda^2 f^2}{g^4} = 0 \tag{4.70}$$

with

$$\Big(-\frac{4f''}{f^3}-\frac{16nf'g'}{f^3g}+\frac{n\lambda^2f^2}{g^4}\Big)r-\frac{8f'}{f^3}-\frac{8ng'}{f^2g}=\widehat{\Lambda} \tag{4.71}$$

and

$$\Big(-\frac{\lambda^2f^2}{g^4}-\frac{4g''}{f^2g}-\frac{4f'g'}{f^3g}-\frac{4(4n-1)(g')^2}{f^2g^2}\Big)r+\frac{(n+2)\lambda}{g^2}-\frac{6g'}{f^2g}=\widehat{\Lambda}. \tag{4.72}$$

In the case of the pseudo-Riemannian metric, the latter becomes

$$\Big(\frac{\lambda^2f^2}{g^4}+\frac{4g''}{f^2g}+\frac{4f'g'}{f^3g}+\frac{4(4n-1)(g')^2}{f^2g^2}\Big)r+\frac{(n+2)\lambda}{g^2}+\frac{6g'}{f^2g}=\widehat{\Lambda}. \tag{4.73}$$

We note that $\widehat{g}_{ij}=0$, where $i\neq j$. Therefore, (4.70) holds true for the pseudo-Riemannian case. Also, we have $\widehat{g}_{ii}=-1$ $(i=1,2,3)$ for the pseudo-Riemannian metric. Thus, in this situation the right-hand side of (4.71) is $-\widehat{\Lambda}$.

If we set $\widehat{\Lambda}=0$, we get the Ricci-flat condition for both cases. In terms of the Einstein equations, we deduce the following proposition.

Proposition 4.2. *Let f and g be the functions of the radial parameter r. Then, f and g define a Ricci-flat Riemannian metric on the bundle Q over a quaternionic Kähler manifold of Einstein constant $\Lambda=(n+2)\lambda$ if and only if f and g define a Ricci-flat metric of signature $(4n,3)$ over a quaternionic Kähler manifold of Einstein constant $-\Lambda$.*

Moreover, we observe that the Ricci-flat equations obey some properties. If $\big(f,g\big)$ is a solution for the zero Ricci case, then $\big(cf,cg\big)$ is also a solution for $c>0$. Moreover, if $\big(f(r),g(r)\big)$ is a solution, then $\big(cf(c^2r),g(c^2r)\big)$ is also a solution.

It can be easily verified that for $n=1$ and $\lambda=1$, $f=(1+r)^{-\frac{1}{4}}$ and $g=\sqrt{2}(1+r)^{\frac{1}{4}}$ give a Ricci-flat metric. This solution was found in [11] and it gives two complete Ricci-flat solutions. We note that in 4 dimensions, there exist only 2 complete self-dual

Einstein spaces with positive scalar curvature. These are S^4 and $\mathbb{C}P^2$ (see [25] for more details). Also, by Proposition 4.2, for $n = 1$, the same functions give Ricci-flat solutions for the pseudo-Riemannian metric of signature $(4, 3)$ on the bundle over any self-dual Einstein manifold of negative scalar curvature. Unlike positive scalar curvature case, there exist infinitely many self-dual Einstein manifolds with negative scalar curvature, see [26].

The Ricci-flat equations for the Riemannian case can be expressed by the constraint equation

$$\frac{2n(n+2)\lambda}{g^2} - \frac{4(f')^2}{f^4}r - \frac{32nf'g'}{f^3g}r - \frac{n\lambda^2 f^2}{g^4}r - \frac{16ng'}{f^2g} - \frac{8n(4n-1)(g')^2}{f^2g^2}r - \frac{4f'}{f^3} = 0, \tag{4.74}$$

where

$$f'' = -\frac{4nf'g'}{g} + \frac{n\lambda^2 f^5}{4g^4} - \frac{2f'}{r} - \frac{2nfg'}{gr} \tag{4.75}$$

and

$$g'' = \frac{(n+2)f^2\lambda}{4gr} - \frac{3g'}{2r} - \frac{\lambda^2 f^4}{4g^3} - \frac{f'g'}{f} - \frac{(4n-1)(g')^2}{g}. \tag{4.76}$$

For the pseudo-Riemannian case, f'' is the same. On the other hand, the constraint equation is

$$-\frac{2n(n+2)\lambda}{g^2} - \frac{4(f')^2}{f^4}r - \frac{32nf'g'}{f^3g}r - \frac{n\lambda^2 f^2}{g^4}r - \frac{16ng'}{f^2g} - \frac{8n(4n-1)(g')^2}{f^2g^2}r - \frac{4f'}{f^3} = 0 \tag{4.77}$$

with

$$g'' = \frac{(n+2)f^2\lambda}{4gr} - \frac{3g'}{2r} - \frac{\lambda^2 f^4}{4g^3} - \frac{f'g'}{f} - \frac{(4n-1)(g')^2}{g}. \tag{4.78}$$

This is a second-order ordinary differential equation system. By local existence and uniqueness theorem, there exist solutions for some small intervals of r.

5. CONCLUSION

In this thesis, using a well-known bundle method, we investigated Riemannian and pseudo-Riemannian Einstein metrics on the total space of a bundle with rank-three, where the base manifold is a $4n$-dimensional quaternionic-Kähler manifold. We were able to calculate the curvatures, and with the aid of special curvature properties of the bundle and the quaternionic-Kähler manifold, we expressed the Einstein equation as a relatively simple ordinary differential equation of two functions for the Riemannian and pseudo-Riemannian metrics and solved it for the case of 4-dimensions, where the base manifold has positive Einstein constant in the Riemannian case and negative Einstein constant in the pseudo-Riemannian case. We emphasize that the Einstein metrics on bundles over quaternionic-Kähler manifolds were studied before in [10, 11]. We note that, in these papers, the authors focused on the same bundle. They defined an additional radial parameter and calculated the curvatures. They gave the Einstein equation in terms of 3 functions of a radial parameter. Our method of calculating the curvatures was different and our Einstein equations were given in terms of 2 functions of a radial parameter. Bryant and Salamon [11] also considered these metrics but in the context of finding G_2 holonomy metrics in 7 dimensions. Their method involves defining a 3-form on the bundle and solving for the condition for which the 3-form is parallel. This method is only valid in 7-dimensional space. Our method, which is valid in $4n + 3$ dimensions, was to compute the curvatures and try to solve for the $Ric = 0$ case, in particular. In addition to the Riemannian metrics, we considered the pseudo-Riemannian metrics of signature $(4n, 3)$ and investigated the conditions for the Einstein equations as well.

REFERENCES

1. Besse, A. L., *Einstein Manifolds*, Springer-Verlag, Heidelberg, 1987.

2. Yau, S. T., "Calabi's Conjecture and Some New Results in Algebraic Geometry", *Proceedings of the National Academy of Sciences*, Vol. 74, No. 5, pp. 1798–1799, 1977.

3. Aubin, T., "Équations du type Monge-Ampère sur les Variétés Kählériennes Compactes", *Bulletin des Sciences Mathématiques (2)*, Vol. 102, No. 1, pp. 63–95, 1978.

4. Page, D. N., "A Compact Rotating Gravitational Instanton", *Physics Letters B*, Vol. 79, No. 3, pp. 235–238, 1978.

5. Calabi, E., "Métriques Kählériennes et Fibrés Holomorphes", *Annales scientifiques de l'École Normale Supérieure (4)*, Vol. 12, No. 2, pp. 269–294, 1979.

6. Bergery, L. B., *Sur de Nouvelles Varietes Riemanniennes D'Einstein*, Institut Elie Cartan, Nancy, 1982.

7. Sakane, Y., "Examples of Compact Einstein Kähler Manifolds with Positive Ricci Tensor", *Osaka Journal of Mathematics*, Vol. 23, No. 3, pp. 585–616, 1986.

8. Page, D. N. and C. N. Pope, "Inhomogeneous Einstein Metrics on Complex Line Bundles", *Classical Quantum Gravity*, Vol. 4, No. 2, pp. 213–225, 1987.

9. Page, D. N. and C. N. Pope, "Einstein Metrics on Quaternionic Line Bundles", *Classical Quantum Gravity*, Vol. 3, No. 2, pp. 249–259, 1986.

10. Gibbons, G. W., D. N. Page and C. N. Pope, "Einstein Metrics on S^3, R^3 and R^4 Bundles", *Communications in Mathematical Physics*, Vol. 127, No. 3, pp. 529–553, 1990.

11. Bryant, R. L. and S. M. Salamon, "On the Construction of Some Complete Metrics with Exceptional Holonomy", *Duke Mathematical Journal*, Vol. 58, No. 3, pp. 829–850, 1989.

12. Hiragane, M., Y. Yasui and H. Ishihara, "Compact Einstein Spaces Based on Quaternionic Kähler Manifolds", *Classical and Quantum Gravity*, Vol. 20, No. 18, pp. 3933–3950, 2003.

13. Myers, S. B., "Riemannian Manifolds with Positive Mean Curvature", *Duke Mathematical Journal*, Vol. 8, No. 2, pp. 401–404, 1941.

14. Lee, J. M., *Introduction to Riemannian Manifolds*, Springer-Verlag, New York, 1997.

15. Sternberg, S., *Curvature in Mathematics and Physics*, Dover Publications, New York, 2012.

16. Salamon, S., *Riemannian Geometry and Holonomy Groups*, Longman Scientific & Technical, Harlow, 1989.

17. Swann, A. F., *HyperKähler and Quaternionic Kähler Geometry*, Ph.D. Thesis, Oriel College, Oxford, 1990.

18. Thurman, J., *Quaternionic Geometry and Special Holonomy*, Ph.D. Thesis, Stony Brook University, 2018.

19. Kraines, V. Y., "Topology of Quaternionic Manifolds", *Bulletin of American Mathematical Society*, Vol. 71, No. 3, pp. 526–527, 1965.

20. Kobayashi, S. and K. Nomizu, *Foundations of Differential Geometry*, Volumes I & II, Interscience Publishers, New York, 1969.

21. Galicki, K. and H. B. Lawson, "Quaternionic Reductions and Quaternionic Orb-

ifolds", *Mathematische Annalen*, Vol. 282, No. 1, pp. 1–21, 1988.

22. LeBrun, C. and S. Salamon, "Strong Rigidity of Positive quaternion-Kähler Manifolds", *Inventiones Mathematicae*, Vol. 118, No. 1, pp. 109–132, 1994.

23. Salamon, S., "Quaternionic Kähler Manifolds", *Inventiones Mathematicae*, Vol. 67, No. 1, pp. 143–171, 1982.

24. Wolf, J. A., "Complex Homogeneous Contact Manifolds and Quaternionic Symmetric Spaces", *Journal of Mathematics and Mechanics*, Vol. 14, No. 6, pp. 1033–1047, 1965.

25. Hitchin, N. J., "Kählerian Twistor Spaces", *Proceedings of London Mathematical Society (3)*, Vol. 43, No. 1, pp. 133–150, 1981.

26. Lebrun, C., "On Complete Quaternionic-Kähler Manifolds", *Duke Mathematical Journal*, Vol. 63, No. 3, pp. 723–743, 1991.