MAXIMALLY SYMMETRIC COSMOLOGIES

by

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MAXIMALLY SYMMETRIC COSMOLOGIES

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ABSTRACT

MAXIMALLY SYMMETRIC COSMOLOGIES

In this thesis we will give a review of maximally symmetric spacetimes. There are only three of these spacetimes namely, Minkowski, de Sitter and anti-de Sitter spacetimes represented with zero, positive and negative cosmological constant respectively. We will obtain metrics in various forms for each of these manifolds using our mathematical freedom of choosing different coordinate patches. This will lead us to different cosmological models for the same manifold.

ÖZET

MAKSİMAL SİMETRİK KOZMOLOJİLER

Bu tezde maksimal simetrik uzay-zamanları inceleyeceğiz. Üç tane olan bu uzay-zamanlar Minkowski, de Sitter ve anti-de Sitter uzay-zamanları olup sırasıyla sıfır, pozitif kozmolojik sabit ve negatif kozmolojik sabit ile temsil edilirler. Bu uzay-zamanların farklı bölgelerini farklı koordinat sistemleriyle kaplayarak değişik fiziksel kozmolojik modeller elde edilir.

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LIST OF SYMBOLS

$a(t)$	Time dependent scale factor
e^i	Orthonormal basis 1-form
f^i_{jk}	Structure constant
$g_{\mu\nu}$	Metric tensor
$G_{\mu\nu}$	Einstein tensor
G	Gravitational constant
$H(t)$	Hubble constant
$i^{\pm}$	Past (Future) timelike infinity
i^0	Spacelike infinity
$\mathcal{I}^{\pm}$	Past (Future) null infinity
$R^{\rho}_{\mu\nu\sigma}$	Riemann tensor
$R_{\mu\nu}$	Ricci tensor
R	Ricci scalar
$T_{\mu\nu}$	Energy-momentum tensor
u^{μ}	Four-velocity
$\Gamma^{\rho}_{\mu\nu}$	Christoffel symbol
Λ	Cosmological constant
ξ^{μ}	Killing vector
ω^i_j	Connection 1-form
Ω^i_j	Curvature 2-form
Ω	Conformal factor
∇_{μ}	Covariant derivative

LIST OF ACRONYMS/ABBREVIATIONS

AdS	Anti-de Sitter
CMB	Cosmic Microwave Background
CFT	Conformal Field Theory
dS	de Sitter
FLRW	Friedmann-Lemaitre-Robertson-Walker
ΛCDM	Λ-Cold Dark Matter
$SO(N)$	Special orthogonal group in N dimensions

1. INTRODUCTION

Cosmology is one of the most studied branches of physics. Throughout the time there have been made many models to explain the structure and evolution of the cosmos we live in. As is true in every branch of physics observations either rule out or support certain models. For example, Edwin Hubble's discovery of redshift [1] eliminated the static models and together with the discovery of the cosmic microwave background (CMB) [2] favored the Friedmann-Lemaitre-Robertson-Walker (FLRW) model whose spatial sections are scaled with a time dependent factor and the timelike coordinate is taken as the cosmological time. The accelerated expansion of the universe from observations of distance–redshift relation for Type Ia supernovae [3,4] implies a positive cosmological constant Λ and gives rise to the current standard cosmology which is called Lambda Cold Dark Matter (ΛCDM) model.

Of course it is not possible to prove a model wrong by observations if it has a complicated structure. At this point, symmetries as we love them come to our aid. It is easier to rule out a model with higher number of symmetries. Minkowski spacetime which has the maximum number of symmetries describes a non-expanding flat universe and is therefore ruled out. There are two other maximally symmetric spacetimes, namely de Sitter (dS) and anti-de Sitter (AdS) that are described by a positive and a negative cosmological constant respectively. None of these three spacetimes fully support the observations as they correspond to vacuum solutions of Einstein's field equations and are classified as $\Lambda = 0$, $\Lambda > 0$ and $\Lambda < 0$ models.

The outline of this thesis is as follows: In chapter 2 we set the mathematical formulation of symmetries in general relativity and give a description of maximally symmetric spacetimes. We briefly give a group theoretical approach. Then we give a construction of these spacetimes in two ways: One by guessing the form of a metric and two by embedding it into a higher dimensional spacetime. Then we describe FLRW metrics which have maximal symmetry only in spatial sections.

Next we will investigate the three maximally symmetric spacetimes with different coordinate patches. For each coordinate system we choose on the manifold we will obtain different metrics. These metrics consequently give distinct descriptions for cosmology. Some of these metrics will be identified as cosmological in the sense that they give a corresponding case of the FLRW models with different scale factors and spatial sections. Some of other cosmological models will not have the timelike coordinate as the cosmological time, can be static or even anisotropic. We will conclude with a discussion of these models.

2. SYMMETRIES

In this chapter we will be dealing with the mathematical grounds of maximally symmetric space(time)s that are used widely as cosmological models. For this we will start with a Riemannian space equipped with the metric $g_{\mu\nu}(x)$ and find its symmetry transformations without any knowledge of the attached coordinate system. To achieve this we will start with describing symmetries in a covariant way, independent of any particular coordinate system.

2.1. Killing Vectors

Under a given coordinate transformation $x \to x'$, the metric tensor $g_{\mu\nu}(x)$ at a point x is transformed as

$$g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \tag{2.1}$$

following the covariance of the interval ds^2. If we demand the metric to be form-invariant, i.e. $g'_{\mu\nu}(x') = g_{\mu\nu}(x')$ we obtain the symmetry transformation

$$g_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}(x) \tag{2.2}$$

or equivalently

$$g_{\mu\nu}(x) = \frac{\partial x'^\rho}{\partial x^\mu}\frac{\partial x'^\sigma}{\partial x^\nu} g_{\rho\sigma}(x'). \tag{2.3}$$

The coordinate transformation $x \to x'$ is a symmetry of the metric called isometry if it satisfies Equation 2.3.

Consider an infinitesimal coordinate transformation

$$x'^\mu = x^\mu + \varepsilon\xi^\mu(x) \tag{2.4}$$

with $|\varepsilon| \ll 1$ under which Equation 2.3 in the first order of ε becomes

$$\begin{aligned} g_{\mu\nu}(x) &= \left[\partial_\mu \left(x^\rho + \varepsilon\xi^\rho(x)\right)\right]\left[\partial_\nu \left(x^\sigma + \varepsilon\xi^\sigma(x)\right)\right] g_{\rho\sigma}(x + \varepsilon\xi(x)) \\ &= \left[\delta^\rho_\mu + \varepsilon\partial_\mu\xi^\rho(x)\right]\left[\delta^\sigma_\nu + \varepsilon\partial_\nu\xi^\sigma(x)\right]\left[g_{\rho\sigma}(x) + \varepsilon\xi^\lambda(x)\partial_\lambda g_{\mu\nu}(x)\right] \\ &= g_{\mu\nu}(x) + \varepsilon\left[\partial_\mu\xi^\rho(x)g_{\rho\nu}(x) + \partial_\nu\xi^\sigma(x)g_{\mu\sigma}(x) + \xi^\lambda(x)\partial_\lambda g_{\mu\nu}(x)\right]. \end{aligned} \tag{2.5}$$

This transformation is an isometry of the metric if the vector field $\xi^\mu(x)$ satisfies the condition

$$0 = \partial_\mu\xi^\rho(x)g_{\rho\nu}(x) + \partial_\nu\xi^\sigma(x)g_{\mu\sigma}(x) + \xi^\lambda(x)\partial_\lambda g_{\mu\nu}(x). \tag{2.6}$$

This equation is known as the Killing equation and $\xi^\mu(x)$ is known as the Killing vector field. First rewriting Equation 2.6 with total derivatives we can write this in terms of the covariant components $\xi_\rho \equiv g_{\mu\rho}\xi^\mu$:

$$\begin{aligned} 0 &= \partial_\mu(\xi^\rho g_{\rho\nu}) - \xi^\rho\partial_\mu g_{\rho\nu} + \partial_\nu(\xi^\sigma g_{\mu\sigma}) - \xi^\sigma\partial_\nu g_{\mu\sigma} + \xi^\lambda(x)\partial_\lambda g_{\mu\nu} \\ &= \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \xi^\rho(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\mu\rho} - \partial_\rho g_{\mu\nu}) \\ &= \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - g_{\sigma\rho}\xi^\rho g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \\ &= \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - 2\xi_\sigma\Gamma^\sigma_{\mu\nu} \end{aligned} \tag{2.7}$$

where we used the definition of the Christoffel symbols,

$$\Gamma^\sigma_{\mu\nu} = \frac{1}{2}g^{\sigma\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu}) \tag{2.8}$$

which are symmetric in their lower two indices. We can also write Equation 2.7 in a more compact way in terms of the covariant derivative as

$$\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\sigma_{\mu\nu}V_\sigma. \tag{2.9}$$

Using the notation $\nabla_\mu V_\nu = V_{\nu;\mu}$ Equation 2.7 becomes,

$$0 = \xi_{\nu;\mu} + \xi_{\mu;\nu} \tag{2.10}$$

So we can identify all infinitesimal isometries of a given metric $g_{\mu\nu}(x)$ by finding all Killing vectors that satisfies Equations 2.6 or 2.10.

Now we consider the commutation of two covariant derivatives,

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]V_\sigma &= \nabla_\mu(\nabla_\nu V_\sigma) - \nabla_\nu(\nabla_\mu V_\sigma) \\
&= \partial_\mu(\partial_\nu V_\sigma - \Gamma^\rho_{\nu\sigma}V_\rho) - \Gamma^\gamma_{\mu\nu}(\partial_\gamma V_\sigma - \Gamma^\rho_{\gamma\sigma}V_\rho) - \Gamma^\gamma_{\mu\sigma}(\partial_\nu V_\gamma - \Gamma^\rho_{\nu\gamma}V_\rho) \\
&- \partial_\nu(\partial_\mu V_\sigma - \Gamma^\rho_{\mu\sigma}V_\rho) + \Gamma^\gamma_{\nu\mu}(\partial_\gamma V_\sigma - \Gamma^\rho_{\gamma\sigma}V_\rho) + \Gamma^\gamma_{\nu\sigma}(\partial_\mu V_\gamma - \Gamma^\rho_{\mu\gamma}V_\rho) \\
&= (\partial_\nu\Gamma^\rho_{\mu\sigma} - \partial_\nu\Gamma^\rho_{\nu\sigma})V_\rho + \Gamma^\gamma_{\mu\sigma}\Gamma^\rho_{\nu\gamma}V_\rho - \Gamma^\gamma_{\nu\sigma}\Gamma^\rho_{\mu\gamma}V_\rho \\
&= -R^\rho_{\sigma\mu\nu}V_\rho
\end{aligned} \tag{2.11}$$

where we used the definition of the Riemann curvature tensor in the last step

$$R^\rho_{\sigma\mu\nu} = \partial_\mu\Gamma^\rho_{\nu\sigma} - \partial_\nu\Gamma^\rho_{\mu\sigma} + \Gamma^\gamma_{\nu\sigma}\Gamma^\rho_{\mu\gamma} - \Gamma^\gamma_{\mu\sigma}\Gamma^\rho_{\nu\gamma} \tag{2.12}$$

that satisfies with $R^\lambda_{\sigma\mu\nu} = g^{\lambda\rho}R_{\rho\sigma\mu\nu}$

$$\begin{aligned}
R_{\rho\sigma\mu\nu} &= -R_{\sigma\rho\mu\nu} \\
R_{\rho\sigma\mu\nu} &= -R_{\rho\sigma\nu\mu} \\
R_{\rho\sigma\mu\nu} &= R_{\mu\nu\rho\sigma}.
\end{aligned} \tag{2.13}$$

If we take the cyclic permutations of the relation given in Equation 2.11

$$\begin{aligned}
V_{\sigma;\nu\mu} - V_{\sigma;\mu\nu} &= -R^\rho_{\sigma\mu\nu}V_\rho \\
V_{\nu;\mu\sigma} - V_{\nu;\sigma\mu} &= -R^\rho_{\nu\sigma\mu}V_\rho \\
V_{\mu;\sigma\nu} - V_{\mu;\nu\sigma} &= -R^\rho_{\mu\nu\sigma}V_\rho
\end{aligned} \tag{2.14}$$

and sum them up, we have

$$V_{\sigma;\nu\mu} - V_{\sigma;\mu\nu} + V_{\nu;\mu\sigma} - V_{\nu;\sigma\mu} + V_{\mu;\sigma\nu} - V_{\mu;\nu\sigma} = 0 \tag{2.15}$$

where we used the Bianchi identity

$$R^{\rho}_{\sigma\mu\nu} + R^{\rho}_{\mu\nu\sigma} + R^{\rho}_{\nu\sigma\mu} = 0 \tag{2.16}$$

for the left hand side. If $V_\rho = \xi_\rho$ is a Killing vector satisfying Equation 2.10, which leads to the relation,

$$\xi_{\nu;\mu\sigma} = -\xi_{\mu;\nu\sigma}. \tag{2.17}$$

Then we can rewrite Equation 2.15 for a Killing vector as

$$\xi_{\sigma;\nu\mu} - \xi_{\sigma;\mu\nu} - \xi_{\mu;\nu\sigma} = 0 \tag{2.18}$$

and Equation 2.11 becomes

$$\xi_{\mu;\nu\sigma} = R^{\rho}_{\sigma\nu\mu}\xi_\rho \tag{2.19}$$

where we used the antisymmetry property of the Riemann tensor in the last two indices.

We can find the second and higher derivatives from Equation 2.19, once ξ_μ and $\xi_{\mu;\nu}$ at some point x_0 is given. This allows us to expand any particular Killing vector $\xi_\mu(x)$ of a given metric as Taylor series in the neighborhood of the point x_0 given $\xi_\mu(x_0)$ and $\xi_{\mu;\nu}(x_0)$ as

$$\xi_\rho(x) = A^{\sigma}_{\rho}(x;x_0)\xi_\sigma(x_0) + B^{\sigma\gamma}_{\rho}(x;x_0)\xi_{\sigma;\gamma}(x_0). \tag{2.20}$$

The functions A^{σ}_{ρ} and $B^{\sigma\gamma}_{\rho}$ depend on the metric and on x_0 with $B^{\sigma\gamma}_{\rho} = -B^{\gamma\sigma}_{\rho}$, and are the same for all Killing vectors. We see from Equation 2.20 that in N dimensions, there can be N independent quantities of $\xi_{\rho}(x_0)$ and $N(N-1)/2$ independent quantities of $\xi_{\rho;\sigma}(x_0)$ giving a maximum number of linearly independent Killing vectors $\xi_{\rho}(x)$ as $N(N+1)/2$.

2.2. Maximally Symmetric Spacetimes

A Riemannian space with a metric $g_{\mu\nu}(x)$ is homogeneous if we can carry any point x_0 to another point x in its neighborhood with an infinitesimal isometry transformation given in Equation 2.4. This amounts to saying that in N dimensions a given metric has N linearly independent translational Killing vectors at the point x_0. This is stated in [5] as choosing a set of N Killing vectors $\xi^{(\mu)}_{\rho}(x; x_0)$ with

$$\xi^{(\mu)}_{\rho}(x_0; x_0) = \delta^{\mu}_{\rho}. \tag{2.21}$$

A Riemannian space with a metric $g_{\mu\nu}(x)$ is isotropic about a point x_0 if there exist infinitesimal isometry transformations given in Equation 2.4 that leave the point x_0 fixed. This means that at the point x_0 all Killing vectors $\xi_{\rho}(x_0)$ vanish. However their first derivatives, $\xi_{\rho;\sigma}(x_0)$ take all possible values respecting the antisymmetry condition given in Equation 2.10. This is stated in [5] as choosing a set of $N(N-1)/2$ linearly independent Killing vectors $\xi^{(\mu\nu)}_{\rho}(x; x_0)$ with

$$\xi^{(\mu\nu)}_{\rho}(x; x_0) \equiv -\xi^{(\nu\mu)}_{\rho}(x; x_0) \tag{2.22}$$

$$\xi^{(\mu\nu)}_{\rho}(x_0; x_0) \equiv 0 \tag{2.23}$$

$$\xi^{(\mu\nu)}_{\rho;\sigma}(x_0; x_0) \equiv \left[\partial_{\sigma}\xi^{(\mu\nu)}_{\rho}(x; x_0)\right]_{x=x_0} \equiv \delta^{\mu}_{\rho}\delta^{\nu}_{\sigma} - \delta^{\mu}_{\sigma}\delta^{\nu}_{\rho}. \tag{2.24}$$

A one good non-trivial example to understand these is S^2 with the metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$ where the orthonormal basis 1-forms are $e^1 = d\theta$, $e^2 = \sin\theta d\phi$ and

orthonormal basis vectors are $\vec{e}_1 = \partial_\theta$ and $\vec{e}_2 = (1/\sin\theta)\partial_\phi$ satisfying $< e^a, \vec{e}_b >= \delta^a_b$. Solving for Equation 2.6 gives three Killing vectors as

$$\begin{aligned} \xi_{(1)} &= \sin\theta\, \vec{e}_2 \\ \xi_{(2)} &= \sin\phi\, \vec{e}_1 + \cos\phi\cos\theta\, \vec{e}_2 \\ \xi_{(3)} &= \cos\phi\, \vec{e}_1 - \sin\phi\cos\theta\, \vec{e}_2. \end{aligned} \tag{2.25}$$

Choosing an easy fixed point at $\theta = \pi/2$, $\phi = \pi/2$ gives two translations $\xi_{(1)} = \vec{e}_2$, $\xi_{(2)} = \vec{e}_1$ and $\xi_{(3)} = 0$. For the isotropy about a point we can choose $\theta = 0$ and leave ϕ a constant since it has no meaning as a coordinate at the north pole. In this case we have $\xi_{(1)} = 0$, $\xi_{(2)} = \sin\phi\, \vec{e}_1 + \cos\phi\, \vec{e}_2$ and $\xi_{(3)} = \cos\phi\, \vec{e}_1 - \sin\phi\, \vec{e}_2$ where $\xi_{(2)}$ and $\xi_{(3)}$ together give a rotation of orthonormal vector basis.

A Riemannian space with a metric $g_{\mu\nu}(x)$ is maximally symmetric if in N dimensions has N linearly independent $\xi^{(\mu)}_\rho(x; x_0)$ and $N(N-1)/2$ linearly independent $\xi^{(\mu\nu)}_\rho(x; x_0)$ giving the maximum number of $N(N+1)/2$ Killing vectors. This means that a maximally symmetric space is a homogeneous space that is isotropic about some point x_0. We arrive at the same conclusion if a space is isotropic about every point since it implies homogeneity.

Now we will exploit a very special feature of maximally symmetric spaces: They are constant curvature spaces. Moreover, for a given constant curvature K and the signature of a metric there corresponds a unique maximally symmetric space. We will conclude with given two maximally symmetric spaces with the same K and the signature there exists a coordinate transformation from one metric to the other. We start with considering a general expression for the commutation of two covariant derivatives of a tensor

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]T_{\rho\sigma} &= \nabla_\mu(\nabla_\nu T_{\rho\sigma}) - \nabla_\nu(\nabla_\mu T_{\rho\sigma}) \\
&= \nabla_\mu(\partial_\nu T_{\rho\sigma} - \Gamma^\gamma_{\nu\sigma}T_{\rho\gamma} - \Gamma^\gamma_{\rho\nu}T_{\gamma\sigma}) - \nabla_\nu(\partial_\mu T_{\rho\sigma} - \Gamma^\gamma_{\mu\sigma}T_{\rho\gamma} - \Gamma^\gamma_{\rho\mu}T_{\gamma\sigma}) \\
&= \partial_\mu(\partial_\nu T_{\rho\sigma} - \Gamma^\gamma_{\nu\sigma}T_{\rho\gamma} - \Gamma^\gamma_{\rho\nu}T_{\gamma\sigma}) - \Gamma^\lambda_{\mu\nu}(\partial_\lambda T_{\rho\sigma} - \Gamma^\gamma_{\lambda\sigma}T_{\rho\gamma} - \Gamma^\gamma_{\rho\lambda}T_{\gamma\sigma}) \\
&\quad - \Gamma^\lambda_{\mu\rho}(\partial_\nu T_{\lambda\sigma} - \Gamma^\gamma_{\nu\sigma}T_{\lambda\gamma} - \Gamma^\gamma_{\lambda\nu}T_{\gamma\sigma}) - \Gamma^\lambda_{\mu\sigma}(\partial_\nu T_{\rho\lambda} - \Gamma^\gamma_{\nu\lambda}T_{\rho\gamma} - \Gamma^\gamma_{\rho\nu}T_{\gamma\lambda}) \\
&\quad - (\mu \leftrightarrow \nu) \\
&= -\partial_\mu\Gamma^\gamma_{\nu\sigma}T_{\rho\gamma} - \Gamma^\gamma_{\nu\sigma}\partial_\mu T_{\rho\gamma} - \partial_\mu\Gamma^\gamma_{\rho\nu}T_{\gamma\sigma} - \Gamma^\gamma_{\rho\nu}\partial_\mu T_{\gamma\sigma} - \Gamma^\lambda_{\mu\rho}\partial_\nu T_{\lambda\sigma} \\
&\quad + \Gamma^\lambda_{\mu\rho}\Gamma^\gamma_{\nu\sigma}T_{\lambda\gamma} + \Gamma^\lambda_{\mu\rho}\Gamma^\gamma_{\lambda\nu}T_{\gamma\sigma} - \Gamma^\lambda_{\mu\sigma}\partial_\nu T_{\rho\lambda} + \Gamma^\lambda_{\mu\sigma}\Gamma^\gamma_{\nu\lambda}T_{\rho\gamma} + \Gamma^\lambda_{\mu\sigma}\Gamma^\gamma_{\rho\nu}T_{\gamma\lambda} \\
&\quad - (\mu \leftrightarrow \nu) \\
&= (\partial_\nu\Gamma^\gamma_{\mu\sigma} - \partial_\mu\Gamma^\gamma_{\nu\sigma} + \Gamma^\lambda_{\mu\sigma}\Gamma^\gamma_{\nu\lambda} - \Gamma^\lambda_{\nu\sigma}\Gamma^\gamma_{\mu\lambda})T_{\rho\gamma} \\
&\quad + (\partial_\nu\Gamma^\gamma_{\mu\rho} - \partial_\mu\Gamma^\gamma_{\nu\rho} + \Gamma^\lambda_{\mu\rho}\Gamma^\gamma_{\nu\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\gamma_{\mu\lambda})T_{\gamma\sigma} \\
&= R^\gamma_{\sigma\nu\mu}T_{\rho\gamma} + R^\gamma_{\rho\nu\mu}T_{\gamma\sigma}
\end{aligned} \tag{2.26}
$$

where the rest of the terms are vanished under a relabeling of the dummy indices. Let us define this tensor as $T_{\rho\sigma} = \xi_{\rho;\sigma}$ which is antisymmetric due to condition given in Equation 2.10. Making use of Equation 2.19

$$
\begin{aligned}
[\nabla_\mu, \nabla_\nu]\nabla_\sigma\xi_\rho &= \nabla_\mu(R^\gamma_{\nu\sigma\rho}\xi_\gamma) - \nabla_\nu(R^\gamma_{\mu\sigma\rho}\xi_\gamma) \\
&= \nabla_\mu R^\gamma_{\nu\sigma\rho}\xi_\gamma + R^\gamma_{\nu\sigma\rho}\nabla_\mu\xi_\gamma - \nabla_\nu R^\gamma_{\mu\sigma\rho}\xi_\gamma - R^\gamma_{\mu\sigma\rho}\nabla_\nu\xi_\gamma
\end{aligned} \tag{2.27}
$$

which is true only for a Killing vector. From equating 2.26 and 2.27

$$
\begin{aligned}
R^\gamma_{\sigma\nu\mu}\nabla_\gamma\xi_\rho + R^\gamma_{\rho\nu\mu}\nabla_\sigma\xi_\gamma &= \nabla_\mu R^\gamma_{\nu\sigma\rho}\xi_\gamma + R^\gamma_{\nu\sigma\rho}\nabla_\mu\xi_\gamma - \nabla_\nu R^\gamma_{\mu\sigma\rho}\xi_\gamma - R^\gamma_{\mu\sigma\rho}\nabla_\nu\xi_\gamma \\
-R^\gamma_{\sigma\nu\mu}\nabla_\rho\xi_\gamma + R^\gamma_{\rho\nu\mu}\nabla_\sigma\xi_\gamma &= (\nabla_\mu R^\gamma_{\nu\sigma\rho} - \nabla_\nu R^\gamma_{\mu\sigma\rho})\xi_\gamma + R^\gamma_{\nu\sigma\rho}\nabla_\mu\xi_\gamma - R^\gamma_{\mu\sigma\rho}\nabla_\nu\xi_\gamma \\
(\nabla_\nu R^\gamma_{\mu\sigma\rho} - \nabla_\mu R^\gamma_{\nu\sigma\rho})\xi_\gamma &= (R^\gamma_{\nu\sigma\rho}\delta^\lambda_\mu - R^\gamma_{\mu\sigma\rho}\delta^\lambda_\nu + R^\gamma_{\sigma\nu\mu}\delta^\lambda_\rho - R^\gamma_{\rho\nu\mu}\delta^\lambda_\sigma)\nabla_\lambda\xi_\gamma
\end{aligned} \tag{2.28}
$$

which relates ξ_γ to $\xi_{\gamma;\lambda}$ without any information of the metric. As we discussed above for a homogeneous space ξ_γ can take all possible values at a certain point where $\nabla_\lambda\xi_\gamma$ vanishes. Since there are N linearly independent ξ_γ, its coefficient satisfies $R^\gamma_{\mu\sigma\rho;\nu} =$

$R^{\gamma}_{\nu\sigma\rho;\mu}$. For an isotropic space at a fixed point ξ_γ vanishes therefore from Equation 2.10 the term in parenthesis in Equation 2.28 must be symmetric in λ and γ,

$$R^{\gamma}_{\nu\sigma\rho}\delta^{\lambda}_{\mu} - R^{\gamma}_{\mu\sigma\rho}\delta^{\lambda}_{\nu} + R^{\gamma}_{\sigma\nu\mu}\delta^{\lambda}_{\rho} - R^{\gamma}_{\rho\nu\mu}\delta^{\lambda}_{\sigma} = R^{\lambda}_{\nu\sigma\rho}\delta^{\gamma}_{\mu} - R^{\lambda}_{\mu\sigma\rho}\delta^{\gamma}_{\nu} + R^{\lambda}_{\sigma\nu\mu}\delta^{\gamma}_{\rho} - R^{\lambda}_{\rho\nu\mu}\delta^{\gamma}_{\sigma} \tag{2.29}$$

which are both satisfied for a maximally symmetric space. Now we will derive a formula for the curvature constant. Contracting λ and σ in Equation 2.29 gives

$$\begin{aligned} R^{\gamma}_{\nu\sigma\rho}\delta^{\sigma}_{\mu} - R^{\gamma}_{\mu\sigma\rho}\delta^{\sigma}_{\nu} + R^{\gamma}_{\sigma\nu\mu}\delta^{\sigma}_{\rho} - NR^{\gamma}_{\rho\nu\mu} &= R^{\sigma}_{\nu\sigma\rho}\delta^{\gamma}_{\mu} - R^{\sigma}_{\mu\sigma\rho}\delta^{\gamma}_{\nu} + R^{\sigma}_{\sigma\nu\mu}\delta^{\gamma}_{\rho} - R^{\lambda}_{\rho\nu\mu}\delta^{\gamma}_{\lambda} \\ R^{\gamma}_{\nu\mu\rho} - R^{\gamma}_{\mu\nu\rho} + R^{\gamma}_{\rho\nu\mu} - NR^{\gamma}_{\rho\nu\mu} + R^{\gamma}_{\rho\nu\mu} &= R^{\sigma}_{\nu\sigma\rho}\delta^{\gamma}_{\mu} - R^{\sigma}_{\mu\sigma\rho}\delta^{\gamma}_{\nu} \end{aligned} \tag{2.30}$$

where $R^{\sigma}_{\sigma\nu\mu}$ vanishes, $R^{\sigma}_{\mu\sigma\rho} = R_{\mu\rho}$ is the Ricci tensor, and in N dimensions $\delta^{\sigma}_{\sigma} = N$. Using the Bianchi identity we can replace

$$R^{\gamma}_{\nu\mu\rho} + R^{\gamma}_{\mu\rho\nu} = -R^{\gamma}_{\rho\nu\mu} \tag{2.31}$$

and so we have

$$\begin{aligned} -R^{\gamma}_{\rho\nu\mu} + R^{\gamma}_{\rho\nu\mu} - NR^{\gamma}_{\rho\nu\mu} + R^{\gamma}_{\rho\nu\mu} &= R_{\nu\rho}\delta^{\gamma}_{\mu} - R_{\mu\rho}\delta^{\gamma}_{\nu} \\ (N-1)R^{\gamma}_{\rho\nu\mu} &= R_{\mu\rho}\delta^{\gamma}_{\nu} - R_{\nu\rho}\delta^{\gamma}_{\mu}. \end{aligned} \tag{2.32}$$

We can lower the indices with the metric as

$$\begin{aligned} (N-1)R^{\gamma}_{\rho\nu\mu}g_{\gamma\lambda} &= R_{\mu\rho}\delta^{\gamma}_{\nu}g_{\gamma\lambda} - R_{\nu\rho}\delta^{\gamma}_{\mu}g_{\gamma\lambda} \\ (N-1)R_{\lambda\rho\nu\mu} &= R_{\mu\rho}g_{\nu\lambda} - R_{\nu\rho}g_{\mu\lambda}. \end{aligned} \tag{2.33}$$

For the right hand side we use the antisymmetry property of the Riemann tensor in λ and ρ

$$R_{\mu\rho}g_{\nu\lambda} - R_{\nu\rho}g_{\mu\lambda} = -R_{\mu\lambda}g_{\nu\rho} + R_{\nu\lambda}g_{\mu\rho}. \tag{2.34}$$

Contracting ν with λ gives us the relation

$$\begin{aligned} R_{\mu\rho}g_{\nu\lambda}g^{\nu\lambda} - R_{\nu\rho}g_{\mu\lambda}g^{\nu\lambda} &= -R_{\mu\lambda}g_{\nu\rho}g^{\nu\lambda} + R_{\nu\lambda}g_{\mu\rho}g^{\nu\lambda} \\ NR_{\mu\rho} - R_{\nu\rho}\delta^{\nu}_{\mu} &= -R_{\mu\lambda}\delta^{\lambda}_{\rho} + Rg_{\mu\rho} \\ R_{\mu\rho} &= \frac{R}{N}g_{\mu\rho} \end{aligned} \tag{2.35}$$

recalling that $g^{\nu\lambda}R_{\nu\lambda} = R$ is the Ricci scalar. We can insert this relation into Equation 2.33 and find

$$R_{\lambda\rho\nu\mu} = \frac{R}{N(N-1)}(g_{\mu\rho}g_{\nu\lambda} - g_{\nu\rho}g_{\mu\lambda}). \tag{2.36}$$

Now we will find the coordinate dependence of the Ricci scalar R. For this we will use the second Bianchi identity

$$R_{\lambda\rho\nu\mu;\delta} + R_{\lambda\rho\delta\nu;\mu} + R_{\lambda\rho\mu\delta;\nu} = 0. \tag{2.37}$$

Contracting with first $g^{\lambda\nu}$ and then with $g^{\rho\mu}$ noting that the covariant derivative of the metric is zero gives

$$\begin{aligned} (g^{\lambda\nu}R_{\lambda\rho\nu\mu})_{;\delta} + (g^{\lambda\nu}R_{\lambda\rho\delta\nu})_{;\mu} + (g^{\lambda\nu}R_{\lambda\rho\mu\delta})_{;\nu} &= 0 \\ R_{\rho\mu;\delta} - R_{\rho\delta;\mu} + R^{\nu}_{\rho\mu\delta;\nu} &= 0 \\ (g^{\rho\mu}R_{\rho\mu})_{;\delta} - (g^{\rho\mu}R_{\rho\delta})_{;\mu} + (g^{\rho\mu}R^{\nu}_{\rho\mu\delta})_{;\nu} &= 0 \\ R_{;\delta} - R^{\mu}_{\delta;\mu} - R^{\nu}_{\delta;\nu} &= 0 \\ \left(\frac{1}{2}\delta^{\nu}_{\delta}R - R^{\nu}_{\delta}\right)_{;\nu} &= 0 \\ \left(\frac{1}{2}\delta^{\nu}_{\delta}R - \frac{1}{N}\delta^{\nu}_{\delta}R\right)_{;\nu} &= 0 \\ \left(\frac{1}{2} - \frac{1}{N}\right)R_{;\nu} &= 0 \end{aligned} \tag{2.38}$$

where we used Equation 2.35. Since R is a scalar, the covariant derivative becomes the partial derivative that shows R is independent of coordinates. So any maximally symmetric space with more than two dimensions, Equations 2.35 and 2.36 hold everywhere

and has constant R. With this we arrive at our first conclusion that we can define a curvature constant K such that

$$R = N(N-1)K. \tag{2.39}$$

Then for a maximally symmetric space we can write the Ricci tensor as

$$R_{\mu\rho} = K(N-1)g_{\mu\rho} \tag{2.40}$$

and the Riemann tensor as

$$R_{\lambda\rho\nu\mu} = K(g_{\mu\rho}g_{\nu\lambda} - g_{\nu\rho}g_{\mu\lambda}). \tag{2.41}$$

If we take two constant curvature spaces $g_{\mu\nu}(x)$ and $g'_{\mu\nu}(x')$ with the same K and the same signature of the metric, then these two maximally symmetric spaces are equivalent up to a coordinate transformation given in Equation 2.1. This tells us that to all possible values of K there exists a unique maximally symmetric space.

2.2.1. Coset Space Description of Maximally Symmetric Spaces

If G is a Lie group and H is a subgroup of G the coset space $M = G/H$ is defined as a manifold on which G acts transitively and H leaves any chosen point invariant [6]. The most well known example is S^2 on which G is $SO(3)$ and H is $SO(2)$, the invariance group of a fixed point usually taken as the north pole. Whereas G acts non-linearly on the coset space, H acts linearly. If we consider the tangent spaces $\mathcal{G}, \mathcal{H}, \mathcal{M}$ of the manifolds G, H and $M = G/H$ for $\mathcal{G}$ and $\mathcal{H}$ we obtain a vector space structure whereas $\mathcal{G}$ and $\mathcal{H}$ are Lie algebras because of the fact that G and H are Lie groups. The projection of the group multiplication to $\mathcal{M}$ is more complicated. We have a direct sum decomposition $\mathcal{G} = \mathcal{H} \oplus \mathcal{M}$ if $\mathcal{G}$ is semisimple with

$$[\mathcal{H}, \mathcal{H}] \subset \mathcal{H} \qquad [\mathcal{H}, \mathcal{M}] \subset \mathcal{M} \qquad [\mathcal{M}, \mathcal{M}] \subset \mathcal{G}. \tag{2.42}$$

The first commutator just shows that $\mathcal{H}$ is a subalgebra. G/H is called a reductive homogeneous space if we have the second commutator. All semisimple Lie algebras are reductive. If the third commutator gives $[\mathcal{M},\mathcal{M}] \subset \mathcal{H}$ plus reductivity makes G/H a symmetric space [7]. All maximally symmetric spaces are symmetric spaces.

2.2.1.1. Calculation of Curvature on G/H. Consider $\mathcal{G}$ and choose an orthonormal basis in $\mathcal{G}$. For $dim(H) = d_H$ and $dim(M) = d_M$ this basis are given by θ_a where $a = 1, \ldots d_H$ and μ_i where $i = 1, \ldots, d_M$.

Construct elements of M by $m = exp(x^i \mu_i)$ where x^i is a coordinate chart on M. Consider the Maurer-Cartan 1-form $m^{-1}dm$, since $m(x) \in G$ this belongs to $\mathcal{G}$ but can be separated into parts in $\mathcal{H}$ and $\mathcal{M}$ such that $(m^{-1}dm)_{\mathcal{M}} = \mu_i e^i_j dx^j$ [6]. Choosing $e^i = e^i_j dx^j$ as the orthonormal basis 1-forms on $\mathcal{M}$ one straightforwardly calculates the connection and curvature in terms of the coordinates x^i explicitly by using Cartan's equations of structure

$$\begin{aligned} 0 &= de^i + \omega^i_j \wedge e^j \\ \Omega^i_j &= d\omega^i_j + \omega^i_k \wedge \omega^k_j. \end{aligned} \tag{2.43}$$

This formulation smoothly leads to the components of the curvature in the orthonormal basis. The commutation relations of $\mathcal{G}$ give

$$\begin{aligned} [\lambda_a, \lambda_b] &= f^c_{ab}\lambda_c \\ [\lambda_a, \lambda_i] &= f^j_{ai}\lambda_j \\ [\lambda_i, \lambda_j] &= f^a_{ij}\lambda_a + f^k_{ij}\lambda_k \end{aligned} \tag{2.44}$$

and the Riemann tensor is calculated as [8]

$$R^i_{jkl} = \frac{1}{2} f^i_{ja} f^a_{kl} - \frac{1}{2} f^i_{jm} f^m_{kl} + \frac{1}{4} \left(f^i_{km} f^m_{jl} - f^i_{lm} f^m_{jk} \right). \tag{2.45}$$

2.2.2. Conservation Laws

We have seen the Killing vectors as the generators of infinitesimal transformations of a given metric giving isometries. As we learn in our classical mechanical course, for a system under some set of transformations Noether's theorem states that there corresponds a physical conserved quantity for each symmetry of the system. Let us consider a geodesic $x^\mu(\tau)$, τ being the proper time that parameterizes the geodesic of a given metric and we consider a tangent vector $u^\mu(\tau) = \dot{x}^\mu = \frac{dx^\mu}{d\tau}$ which is the four-velocity on that geodesic. If $\xi(x)$ is a Killing vector field of the spacetime than the quantity $\xi_\mu(x^\mu)u^\mu$ is conserved along the geodesic [9]. We can prove this by taking the covariant derivative of this object along the geodesic

$$u^\nu \nabla_\nu \left(\xi_\mu u^\mu\right) = u^\nu u^\mu \nabla_\nu \xi_\mu + \xi_\mu \left(u^\nu \nabla_\nu u^\mu\right). \tag{2.46}$$

We showed in Equation 2.10 the antisymmetry of $\nabla_\nu \xi_\mu$ which means the first term is zero since $u^\nu u^\mu$ is symmetric under a relabeling of the indices. For the second term we consider the geodesic equation

$$\frac{d^2x^\alpha}{ds^2} + \Gamma^\alpha_{\nu\rho}\frac{dx^\nu}{ds}\frac{dx^\rho}{ds} = 0. \tag{2.47}$$

If we use the four-velocity along the geodesic this equation becomes

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\nu\rho}u^\nu u^\rho = 0. \tag{2.48}$$

Since $u^\mu = u^\mu(x(s))$ then we have

$$\frac{du^\mu(x(s))}{ds} = \frac{dx^\alpha}{ds}\frac{\partial u^\mu}{\partial x^\alpha} = u^\alpha \frac{\partial u^\mu}{\partial x^\alpha} \tag{2.49}$$

which then gives

$$u^\alpha \left(\frac{\partial u^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta}u^\beta\right) = 0. \tag{2.50}$$

The term in the parenthesis is the covariant derivative of u^μ so that we have

$$u^\alpha \nabla_\alpha u^\mu = 0 \tag{2.51}$$

and so $\xi_\mu u^\mu = \xi_\mu \dot{x}^\mu$ is the conserved charge.

2.3. Constructing Maximally Symmetric Spaces

In this section we will solve for a spatial metric in 3-dimensions. We will assume that the maximally symmetric space has $SO(3)$ symmetry with r-dependent functions $f(r)$ and $g(r)$. We will make use of it for the FLRW models in the following section. A note here is although we take here $N = 3$ we showed the requirements for maximally symmetric spaces in the previous section for N dimensions.

The space interval is

$$ds^2 = f^2(r)dr^2 + g^2(r)(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.52}$$

with basis 1-forms

$$\begin{aligned} e^1 &= f(r)dr \to dr = \frac{1}{f}e^1 \\ e^2 &= g(r)d\theta \to d\theta = \frac{1}{g}e^2 \\ e^3 &= g(r)\sin\theta d\phi \to d\phi = \frac{1}{g\sin\theta}e^3 \end{aligned} \tag{2.53}$$

and we take the differentials

$$\begin{aligned} de^1 &= 0 \\ de^2 &= g'dr \wedge d\theta = \frac{g'}{fg}e^1 \wedge e^2 \\ de^3 &= g'\sin\theta dr \wedge d\phi + g\cos\theta d\theta \wedge d\phi = \frac{g'}{fg}e^1 \wedge e^3 + \frac{\cot\theta}{g}e^2 \wedge e^3. \end{aligned} \tag{2.54}$$

Using Cartan's first structure equation

$$de^i + \omega^i_j \wedge e^j = 0 \tag{2.55}$$

we can find the connection 1-forms $\omega^i_j = \omega^i_{jk} e^k$ for each i, j component separately keeping in mind $\omega^i_j = -\omega^j_i$.

$$\begin{aligned} 0 &= de^1 + \omega^1_2 \wedge e^2 + \omega^1_3 \wedge e^3 \\ &= \omega^1_{21} e^1 \wedge e^2 + \omega^1_{23} e^3 \wedge e^2 + \omega^1_{31} e^1 \wedge e^3 + \omega^1_{32} e^2 \wedge e^3 \end{aligned} \tag{2.56}$$

which gives

$$\omega^1_{21} = 0 \qquad \omega^1_{31} = 0 \qquad \omega^1_{32} = \omega^1_{23} \tag{2.57}$$

$$\begin{aligned} 0 &= de^2 + \omega^2_1 \wedge e^1 + \omega^2_3 \wedge e^3 \\ &= \frac{g'}{fg} e^1 \wedge e^2 + \omega^2_{12} e^2 \wedge e^1 + \omega^2_{13} e^3 \wedge e^1 + \omega^2_{31} e^1 \wedge e^3 + \omega^2_{32} e^2 \wedge e^3 \end{aligned} \tag{2.58}$$

gives

$$\omega^2_{12} = \frac{g'}{fg} \qquad \omega^2_{32} = 0 \qquad \omega^2_{13} = \omega^2_{31} \tag{2.59}$$

$$\begin{aligned} 0 &= de^3 + \omega^3_1 \wedge e^1 + \omega^3_2 \wedge e^2 \\ &= \frac{g'}{fg} e^1 \wedge e^3 + \frac{\cot\theta}{g} e^2 \wedge e^3 + \omega^3_{12} e^2 \wedge e^1 + \omega^3_{13} e^3 \wedge e^1 + \omega^3_{21} e^1 \wedge e^2 + \omega^3_{23} e^3 \wedge e^2 \end{aligned} \tag{2.60}$$

gives

$$\omega^3_{13} = \frac{g'}{fg} \qquad \omega^3_{23} = \frac{\cot\theta}{g} \qquad \omega^3_{12} = \omega^3_{21}. \tag{2.61}$$

Noting the antisymmetry in the first two indices $\omega_{ijk} = -\omega_{jik}$ and using $\omega^i_j = \omega^i_{jk} e^k$ we find the non-zero connection one-forms as

$$\begin{aligned} \omega^2_1 &= \frac{g'}{fg} e^2 \\ \omega^3_1 &= \frac{g'}{fg} e^3 \\ \omega^3_2 &= \frac{\cot\theta}{g} e^3. \end{aligned} \tag{2.62}$$

Now we're moving on to calculating curvature 2-forms from Cartan's second structure equation

$$\Omega^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j. \tag{2.63}$$

$$\begin{aligned} \Omega^1_2 &= d\omega^1_2 + \omega^1_3 \wedge \omega^3_2 \\ &= d\left(-\frac{g'}{fg} e^2\right) + \left(-\frac{g'}{fg} e^3\right) \wedge \left(\frac{\cot\theta}{g} e^3\right) \\ &= \left(\frac{g'(f'g + fg') - g''fg}{f^3 g^2}\right) e^1 \wedge e^2 - \frac{g'}{f^2 g^2} e^1 \wedge e^2 \\ &= \left(\frac{f'g'}{f^3 g} - \frac{g''}{f^2 g}\right) e^1 \wedge e^2 \end{aligned} \tag{2.64}$$

$$\begin{aligned} \Omega^1_3 &= d\omega^1_3 + \omega^1_2 \wedge \omega^2_3 \\ &= d\left(-\frac{g'}{fg} e^3\right) + \left(-\frac{g'}{fg} e^2\right) \wedge \left(-\frac{\cot\theta}{g} e^3\right) \\ &= \left(\frac{g'(f'g + fg') - g''fg}{f^3 g^2}\right) e^1 \wedge e^3 - \frac{g'}{fg}\left(\frac{g'}{fg} e^1 \wedge e^3 + \frac{\cot\theta}{g} e^2 \wedge e^3\right) + \frac{g' \cot\theta}{fg^2} e^2 \wedge e^3 \\ &= \left(\frac{f'g'}{f^3 g} - \frac{g''}{f^2 g}\right) e^1 \wedge e^3 \end{aligned} \tag{2.65}$$

$$\begin{aligned}\Omega_3^2 &= d\omega_3^2 + \omega_1^2 \wedge \omega_3^1 \\ &= d\left(-\frac{\cot\theta}{g}e^3\right) + \left(\frac{g'}{fg}e^2\right) \wedge \left(-\frac{g'}{fg}e^3\right) \\ &= \frac{g'\cot\theta}{fg^2}e^1 \wedge e^3 + \frac{\csc^2\theta}{g^2}e^2 \wedge e^3 - \frac{\cot\theta}{g}\left(\frac{g'}{fg}e^1 \wedge e^3 + \frac{\cot\theta}{g}e^2 \wedge e^3\right) - \frac{g'^2}{f^2g^2}e^2 \wedge e^3 \\ &= \frac{1}{g^2}\left(1 - \frac{g'^2}{f^2}\right)e^2 \wedge e^3\end{aligned} \tag{2.66}$$

We can extract the Riemann curvature tensor using $\Omega_j^i = \frac{1}{2}R^i_{jkl}e^k \wedge e^l$

$$\begin{aligned}R^1_{212} &= \frac{f'g'}{f^3g} - \frac{g''}{f^2g} \\ R^1_{313} &= \frac{f'g'}{f^3g} - \frac{g''}{f^2g} \\ R^2_{323} &= \frac{1}{g^2}\left(1 - \frac{g'^2}{f^2}\right)\end{aligned} \tag{2.67}$$

and we find the Ricci tensor with the contraction $R_{jk} = R^i_{jik}$

$$\begin{aligned}R_{11} &= R^2_{121} + R^3_{131} = 2\frac{f'g'}{f^3g} - 2\frac{g''}{f^2g} \\ R_{22} &= R^1_{212} + R^3_{232} = \frac{f'g'}{f^3g} - \frac{g''}{f^2g} + \frac{1}{g^2}\left(1 - \frac{g'^2}{f^2}\right) \\ R_{33} &= R^1_{313} + R^2_{323} = \frac{f'g'}{f^3g} - \frac{g''}{f^2g} + \frac{1}{g^2}\left(1 - \frac{g'^2}{f^2}\right).\end{aligned} \tag{2.68}$$

For this metric to be maximally symmetric it must satisfy the requirement

$$R_{\mu\nu} = K(N-1)g_{\mu\nu}. \tag{2.69}$$

Since we are in $N = 3$ and using the metric $g_{\mu\nu} = \delta_{\mu\nu}$ this condition becomes

$$R_{\mu\nu} = 2K\delta_{\mu\nu} \tag{2.70}$$

and help us solve for $f(r)$ and $g(r)$, giving two equations

$$\begin{aligned} \frac{f'g'}{f^3 g} - \frac{g''}{f^2 g} &= K \\ 1 - \frac{g'^2}{f^2} &= Kg^2. \end{aligned} \tag{2.71}$$

From the second one we have $g' = f\sqrt{1 - Kg^2}$ and the derivative with respect to r gives

$$g'' = f'\sqrt{1 - Kg^2} - \frac{Kfg}{\sqrt{1 - Kg^2}}. \tag{2.72}$$

Implementing g' and g'' into the first condition gives

$$\begin{aligned} \frac{f'\sqrt{1 - Kg^2}}{f^2 g} - \frac{1}{f^2 g}\left(f'\sqrt{1 - Kg^2} - \frac{Kfg}{\sqrt{1 - Kg^2}}\right) &= K \\ \frac{1}{\sqrt{1 - Kg^2}} &= f \end{aligned} \tag{2.73}$$

and from this equation $g'(r) = 1$. Then $g(r) = r$ gives

$$f(r) = \frac{1}{\sqrt{1 - Kr^2}}. \tag{2.74}$$

So the maximally symmetric spatial metric becomes

$$ds^2 = \frac{1}{1 - Kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.75}$$

At this point it is useful to define the curvature constant in terms of a specified length L, as $K = k/L^2$ so that k only takes values -1, 0 or 1. Then with a rescaling of the radial coordinate, $\tilde{r} = r/L$

$$\frac{1}{1 - Kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) = L^2\left(\frac{1}{1 - k\tilde{r}^2}d\tilde{r}^2 + \tilde{r}^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{2.76}$$

so the overall L^2 can be absorbed into a scale factor. We immediately see that $k = 0$ case is the Euclidean 3-space.

If $k = 1$ the metric is

$$ds^2 = \frac{1}{1 - \frac{r^2}{L^2}} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2.77}$$

We can show that the singularity at $r = L$ is only a coordinate singularity by a re-parameterization of the radial coordinate as

$$\frac{1}{1 - \frac{r^2}{L^2}} dr^2 = d\chi^2 \tag{2.78}$$

which is satisfied with $r = L\sin\chi$. We see that the final metric becomes that of 3-sphere S^3,

$$ds^2 = d\chi^2 + L^2\sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2) = d\Omega_3^2. \tag{2.79}$$

If $k = -1$ the metric is

$$ds^2 = \frac{1}{1 + \frac{r^2}{L^2}} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.80}$$

Although we don't have a coordinate singularity in this case it is useful to change the radial coordinate as

$$\frac{1}{1 + \frac{r^2}{L^2}} dr^2 = d\chi^2 \tag{2.81}$$

which in this case is satisfied with $r = L\sinh\chi$ and the final metric now becomes the 3-hyperboloid H^3,

$$ds^2 = d\chi^2 + L^2\sinh^2\chi(d\theta^2 + \sin^2\theta d\phi^2) = d\tilde{\Omega}_3^2. \tag{2.82}$$

Note that these results can be generalized to N dimensions and if we put them all together

$$ds^2 = \frac{1}{1-k\frac{r^2}{L^2}}dr^2 + r^2 d\Omega_{N-1}^2 = d\chi^2 + L^2 S_k^2(\chi) d\Omega_{N-1}^2 \tag{2.83}$$

where

$$S_k(\chi) = \begin{cases} \sin\chi & k=+1 \\ \chi & k=0 \\ \sinh\chi & k=-1. \end{cases} \tag{2.84}$$

One last thing we can do for this metric is to go to so called isotropic coordinates where we write the metric with an overall conformal factor. We will make a re-parameterization of the radial coordinate again but this will be different then we did above. That is we want to solve

$$\frac{1}{1-k\frac{r^2}{L^2}}dr^2 + r^2 d\Omega_{N-1}^2 = g^2(\eta)\left(d\eta^2 + \eta^2 d\Omega_{N-1}^2\right) \tag{2.85}$$

for η and $g(\eta)$. The above equality gives us

$$\begin{aligned} r^2 &= g^2(\eta)\eta^2 \\ \frac{1}{1-k\frac{r^2}{L^2}}dr^2 &= g^2(\eta)d\eta^2 \end{aligned} \tag{2.86}$$

and together reduce to

$$\int \frac{dr}{r\sqrt{1-k\frac{r^2}{L^2}}} = \int \frac{d\eta}{\eta} \tag{2.87}$$

where we take the positive root. The right hand side gives $\ln\eta + \text{cst}$ and for the left hand side with $\sqrt{k}r = L\sin u$ and so $\sqrt{k}dr = L\cos u du$ which gives

$$\int \csc u du \tag{2.88}$$

and if we multiply both the nominator and denominator by $v = \csc u + \cot u$ noting $dv = -(\cot u \csc u + \csc^2 u)du$

$$\begin{aligned} -\int \frac{dv}{v} &= -\ln v + c \\ &= -\ln(\csc u + \cot u) + c. \end{aligned}$$

Using $\sin u = \sqrt{k}r/L$ and so $\cos u = \sqrt{1 - k(r^2/L^2)}$ this becomes

$$\int \frac{dr}{r\sqrt{1 - k\frac{r^2}{L^2}}} = \ln\left(\frac{r/L}{1 + \sqrt{1 - k\frac{r^2}{L^2}}}\right) \tag{2.89}$$

where we factored out $\ln\sqrt{k}$ and absorbed it into the constant and then set to zero to give $\eta = 0$ when $r = 0$. We can now find $g(\eta)$ from the relation $r = g(\eta)\eta$,

$$\begin{aligned} \frac{r}{L} - \eta &= \eta\sqrt{1 - k\frac{r^2}{L^2}} \\ \frac{r^2}{L^2} - 2\eta\frac{r}{L} &= -k\eta^2\frac{r^2}{L^2} \\ r &= \frac{2}{(1 + k\eta^2)}\eta \\ g(\eta) &= \frac{2}{(1 + k\eta^2)} \end{aligned} \tag{2.90}$$

replacing $\eta \to \eta/2$ together with $g(\eta)$ gives

$$ds^2 = \left(1 + \frac{k}{4}\eta^2\right)^{-2}\left(d\eta^2 + \eta^2 d\Omega_{N-1}^2\right). \tag{2.91}$$

This metric actually is the stereographic projection of S^{N-1} for $k = +1$ and H^{N-1} for $k = -1$. Since we have a flat metric with a conformal factor in isotropic coordinates

we see that maximally symmetric spaces are conformally flat for which the Weyl (or conformal tensor) is zero [9].

2.3.1. Constructing Maximally Symmetric Spacetimes

Another useful construction is the embedding of a N dimensional hypersurface (for example S^2) into a $N+1$ dimensional flat $\mathbb{R}^{(p,q)}$ space(time) (for example $\mathbb{R}^3$). This embedding induces a metric on the higher dimensional space(time) preserving the isometries of a given hypersurface [10]. We could use this construction in the first part just for spaces as well where embedding space has the Euclidean signature. Here we use this construction for maximally symmetric spacetimes, namely, de Sitter and anti-de Sitter spacetimes.

We start with the defining equation of a N dimensional hypersurface, a hyperboloid in $N+1$ dimensional embedding space, given with

$$\eta_{ab}x^a x^b = \eta_{\mu\nu}x^\mu x^\nu + k(x^N)^2 = kL^2 \tag{2.92}$$

where the indices a, b run from 0 to N and η_{ab} is the Minkowski metric, $\mathbb{R}^{(N,1)}$ for de Sitter and $\mathbb{R}^{(N-1,2)}$ for anti-de Sitter. Also k is -1 or 1 giving anti-de Sitter or de Sitter spacetimes respectively. We can rewrite this as

$$\sqrt{k}x^N = \pm\sqrt{kL^2 - \eta_{\mu\nu}x^\mu x^\nu} \tag{2.93}$$

noting that the indices μ, ν run from 0 to $N-1$. Taking the differential gives

$$\sqrt{k}dx^N = \mp\frac{\eta_{\mu\nu}x^\mu dx^\nu}{(kL^2 - \eta_{\alpha\beta}x^\alpha x^\beta)^{1/2}}. \tag{2.94}$$

We can write the $N+1$ dimensional flat spacetime respecting the signature of the defining equation as

$$\begin{aligned} ds^2 = \eta_{ab}dx^a dx^b &= \eta_{\mu\nu}dx^\mu dx^\nu + kdx^N dx^N \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + \frac{(\eta_{\lambda\tau}x^\lambda dx^\tau)(\eta_{\rho\sigma}x^\rho dx^\sigma)}{kL^2 - \eta_{\alpha\beta}x^\alpha x^\beta} \\ &= \eta_{\mu\nu}dx^\mu dx^\nu + \frac{\eta_{\lambda\tau}\eta_{\rho\sigma}x^\lambda x^\rho \delta^\tau_\mu \delta^\sigma_\nu}{kL^2 - \eta_{\alpha\beta}x^\alpha x^\beta}dx^\mu dx^\nu \\ &= \left(\eta_{\mu\nu} + \frac{x_\mu x_\nu}{kL^2 - \eta_{\alpha\beta}x^\alpha x^\beta}\right)dx^\mu dx^\nu. \end{aligned} \tag{2.95}$$

Hence components of the metric are given by

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{x_\mu x_\nu}{\eta_{\alpha\beta}x^\alpha x^\beta - kL^2}. \tag{2.96}$$

We can find the inverse of the metric starting with

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} - \frac{\eta_{\rho\mu}x^\rho \eta_{\sigma\nu}x^\sigma}{\eta_{\alpha\beta}x^\alpha x^\beta - kL^2} \\ &= \eta_{\mu\nu}\left(1 - \frac{\eta^{\mu\nu}\eta_{\rho\mu}x^\rho \eta_{\sigma\nu}x^\sigma}{\eta_{\alpha\beta}x^\alpha x^\beta - kL^2}\right) \end{aligned} \tag{2.97}$$

and so

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\nu}\left(\frac{\eta_{\alpha\beta}x^\alpha x^\beta - kL^2 - \eta^{\mu\nu}\eta_{\rho\mu}x^\rho \eta_{\sigma\nu}x^\sigma}{\eta_{\alpha\beta}x^\alpha x^\beta - kL^2}\right)^{-1} \\ &= \eta^{\mu\nu}\left(\frac{\eta_{\alpha\beta}x^\alpha x^\beta - kL^2 - \eta_{\sigma\rho}x^\rho x^\sigma}{\eta_{\alpha\beta}x^\alpha x^\beta - kL^2}\right)^{-1} \\ &= \eta^{\mu\nu}\left(1 - \frac{\eta_{\alpha\beta}x^\alpha x^\beta}{kL^2}\right). \end{aligned} \tag{2.98}$$

From here we can calculate the connections, Ricci tensor, Riemann tensor and Ricci scalar. We won't do this explicitly since we have already derived the form of Ricci tensor, Riemann curvature tensor and the scalar curvature for maximally symmetric spacetimes.

Now we consider the vacuum solution of Einstein equation with a cosmological constant, Λ.

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{2.99}$$

Contraction with $g^{\mu\nu}$ gives the curvature scale in terms of the cosmological constant.

$$R = \frac{2N}{N-2}\Lambda \tag{2.100}$$

Comparing this with the maximal symmetry requirement 2.39 in terms of the constant K as,

$$K = \frac{2}{(N-1)(N-2)}\Lambda = \frac{k}{L^2} \tag{2.101}$$

where we used the previous definition $K = kL^{-2}$. For $N = 4$, this gives

$$L^2 = k\frac{3}{\Lambda} \tag{2.102}$$

where we took $\Lambda > 0$ and so for maximally symmetric spacetimes that follow Einstein's vacuum equations $k = 0, 1, -1$ gives, respectively, Minkowski, de Sitter and anti-de Sitter spacetimes and their natural length scales are given with L^2.

2.4. FLRW Universe

FLRW model of the universe admits two conditions. First of these follows from the 'Cosmological Principle' which states that at large scales there are no special points (homogeneity) in the universe. For a N dimensional space(time) this condition gives us N translational Killing vectors. Another statement is that at large scales, there are no preferred directions (isotropy) in the universe and so gives $N(N-1)/2$ rotational Killing vectors. The second condition follows from the observation on the expansion of the universe. This observation tells us that the metric has some time dependence

and we cannot have a time translational symmetry. Hence in this model there is no timelike Killing vector therefore we consider a maximally symmetric space and not a spacetime. The metric we already constructed for a maximally symmetric space in the previous section will be our spatial slice for each point in time. That is in $3+1$ dimensions the metric is,

$$ds^2 = -dt^2 + a^2(t)\left(\frac{1}{1-Kr^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right) \tag{2.103}$$

where $a(t)$ is an unknown time-dependent function called the cosmic scale factor which can be determined by the Einstein's field equations.

For simplicity we will denote $f^2(r) = 1/(1-Kr^2)$. Introducing the basis 1-forms

$$\begin{aligned} e^0 &= idt \to dt = -ie^0 \\ e^1 &= a(t)f(r)dr \to dr = \frac{1}{af}e^1 \\ e^2 &= a(t)rd\theta \to d\theta = \frac{1}{ar}e^2 \\ e^3 &= a(t)r\sin\theta d\phi \to d\phi = \frac{1}{ar\sin\theta}e^3 \end{aligned} \tag{2.104}$$

and

$$\begin{aligned} de^0 &= 0 \\ de^1 &= \dot{a}fdt \wedge dr = -i\frac{\dot{a}}{a}e^0 \wedge e^1 \\ de^2 &= \dot{a}rdt \wedge d\theta + adr \wedge d\theta = -i\frac{\dot{a}}{a}e^0 \wedge e^2 + \frac{1}{afr}e^1 \wedge e^2 \\ de^3 &= \dot{a}r\sin\theta dt \wedge d\phi + a\sin\theta dr \wedge d\phi + ar\cos\theta d\theta \wedge d\phi \\ &= -i\frac{\dot{a}}{a}e^0 \wedge e^3 + \frac{1}{afr}e^1 \wedge e^3 + \frac{\cot\theta}{ar}e^2 \wedge e^3. \end{aligned} \tag{2.105}$$

We will use Cartan's first structure equation for each component,

$$\begin{aligned} 0 &= de^0 + \omega^0_1 \wedge e^1 + \omega^0_2 \wedge e^2 + \omega^0_3 \wedge e^3 \\ &= \omega^0_{10} e^0 \wedge e^1 + \omega^0_{12} e^2 \wedge e^1 + \omega^0_{13} e^3 \wedge e^1 + \omega^0_{20} e^0 \wedge e^2 + \omega^0_{21} e^1 \wedge e^2 \\ &+ \omega^0_{23} e^3 \wedge e^2 + \omega^0_{30} e^0 \wedge e^3 + \omega^0_{31} e^1 \wedge e^3 + \omega^0_{32} e^2 \wedge e^3 \end{aligned} \tag{2.106}$$

which gives

$$\omega^0_{j0} = 0 \quad \omega^0_{21} = \omega^0_{12} \quad \omega^0_{31} = \omega^0_{13} \quad \omega^0_{23} = \omega^0_{32} \tag{2.107}$$

where $j = 1, 2, 3$. For the r coordinate

$$\begin{aligned} 0 &= de^1 + \omega^1_0 \wedge e^0 + \omega^1_2 \wedge e^2 + \omega^1_3 \wedge e^3 \\ &= -i\frac{\dot{a}}{a} e^0 \wedge e^1 + \omega^1_{01} e^1 \wedge e^0 + \omega^1_{02} e^2 \wedge e^0 + \omega^1_{03} e^3 \wedge e^0 + \omega^1_{20} e^0 \wedge e^2 \\ &+ \omega^1_{21} e^1 \wedge e^2 + \omega^1_{23} e^3 \wedge e^2 + \omega^1_{30} e^0 \wedge e^3 + \omega^1_{31} e^1 \wedge e^3 + \omega^1_{32} e^2 \wedge e^3 \end{aligned} \tag{2.108}$$

which gives

$$\omega^1_{01} = -i\frac{\dot{a}}{a} \quad \omega^1_{21} = 0 \quad \omega^1_{31} = 0 \quad \omega^1_{32} = \omega^1_{23} \quad \omega^1_{02} = \omega^1_{20} \quad \omega^1_{03} = \omega^1_{30}. \tag{2.109}$$

For the θ coordinate

$$\begin{aligned} 0 &= de^2 + \omega^2_0 \wedge e^0 + \omega^2_1 \wedge e^1 + \omega^2_3 \wedge e^3 \\ &= -i\frac{\dot{a}}{a} e^0 \wedge e^2 + \frac{1}{afr} e^1 \wedge e^2 + \omega^2_{01} e^1 \wedge e^0 + \omega^2_{02} e^2 \wedge e^0 + \omega^2_{03} e^3 \wedge e^0 + \omega^2_{10} e^0 \wedge e^1 \\ &+ \omega^2_{12} e^2 \wedge e^1 + \omega^2_{13} e^3 \wedge e^1 + \omega^2_{30} e^0 \wedge e^3 + \omega^2_{31} e^1 \wedge e^3 + \omega^2_{32} e^2 \wedge e^3 \end{aligned} \tag{2.110}$$

that gives

$$\omega^2_{02} = -i\frac{\dot{a}}{a} \quad \omega^2_{12} = \frac{1}{afr} \quad \omega^2_{32} = 0 \quad \omega^2_{01} = \omega^2_{10} \quad \omega^2_{03} = \omega^2_{30} \quad \omega^2_{13} = \omega^2_{31}. \tag{2.111}$$

For the ϕ component

$$\begin{aligned}0 &= de^3 + \omega^3_0 \wedge e^0 + \omega^3_1 \wedge e^1 + \omega^3_2 \wedge e^2 \\ &= -i\frac{\dot{a}}{a}e^0 \wedge e^3 + \frac{1}{afr}e^1 \wedge e^3 + \frac{\cot\theta}{ar}e^2 \wedge e^3 + \omega^3_{01}e^1 \wedge e^0 + \omega^3_{02}e^2 \wedge e^0 + \omega^3_{03}e^3 \wedge e^0 \\ &+ \omega^3_{10}e^0 \wedge e^1 + \omega^3_{12}e^2 \wedge e^1 + \omega^3_{13}e^3 \wedge e^1 + \omega^3_{20}e^0 \wedge e^2 + \omega^3_{21}e^1 \wedge e^2 + \omega^3_{23}e^3 \wedge e^2\end{aligned} \tag{2.112}$$

which gives

$$\omega^3_{03} = -i\frac{\dot{a}}{a} \quad \omega^3_{13} = \frac{1}{afr} \quad \omega^3_{23} = \frac{\cot\theta}{ar} \quad \omega^3_{01} = \omega^3_{10} \quad \omega^3_{02} = \omega^3_{20} \quad \omega^3_{12} = \omega^3_{21}. \tag{2.113}$$

Noting the antisymmetry in the first two indices $\omega_{abc} = -\omega_{bac}$ and using $\omega^a_b = \omega^a_{bc}e^c$ we find the non-zero connection 1-forms as

$$\begin{aligned}\omega^j_0 &= -i\frac{\dot{a}}{a}e^j \\ \omega^2_1 &= \frac{1}{afr}e^2 \\ \omega^3_1 &= \frac{1}{afr}e^3 \\ \omega^3_2 &= \frac{\cot\theta}{ar}e^3.\end{aligned} \tag{2.114}$$

Now we will find the curvature 2-forms from Cartan's second structure equation $\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b$ for each index:

$$\begin{aligned}\Omega^0_1 &= d\omega^0_1 + \omega^0_2 \wedge \omega^2_1 + \omega^0_3 \wedge \omega^3_1 \\ &= d\left(i\frac{\dot{a}}{a}e^1\right) \\ &= \frac{\ddot{a}a - \dot{a}^2}{a^2}e^0 \wedge e^1 + \frac{\dot{a}^2}{a^2}e^0 \wedge e^1 \\ &= \frac{\ddot{a}}{a}e^0 \wedge e^1\end{aligned} \tag{2.115}$$

$$\begin{aligned}\Omega_2^0 &= d\omega_2^0 + \omega_1^0 \wedge \omega_2^1 + \omega_3^0 \wedge \omega_1^3 \\ &= d\left(i\frac{\dot{a}}{a}e^2\right) + \left(i\frac{\dot{a}}{a}e^1\right) \wedge \left(-\frac{1}{afr}e^2\right) \\ &= \frac{\ddot{a}a - \dot{a}^2}{a^2}e^0 \wedge e^2 + \frac{\dot{a}^2}{a^2}e^0 \wedge e^2 + i\frac{\dot{a}}{a^2fr}e^1 \wedge e^2 - i\frac{\dot{a}}{a^2fr}e^1 \wedge e^2 \\ &= \frac{\ddot{a}}{a}e^0 \wedge e^2\end{aligned} \tag{2.116}$$

$$\begin{aligned}\Omega_3^0 &= d\omega_3^0 + \omega_1^0 \wedge \omega_3^1 + \omega_2^0 \wedge \omega_3^2 \\ &= d\left(i\frac{\dot{a}}{a}e^3\right) + \left(i\frac{\dot{a}}{a}e^1\right) \wedge \left(-\frac{1}{afr}e^3\right) + \left(i\frac{\dot{a}}{a}e^2\right) \wedge \left(-\frac{\cot\theta}{ar}e^3\right) \\ &= \frac{\ddot{a}a - \dot{a}^2}{a^2}e^0 \wedge e^3 + i\frac{\dot{a}}{a}\left(-i\frac{\dot{a}}{a}e^0 \wedge e^3 + \frac{1}{afr}e^1 \wedge e^3 + \frac{\cot\theta}{ar}e^2 \wedge e^3\right) \\ &\quad - i\frac{\dot{a}}{a^2fr}e^1 \wedge e^3 - i\frac{\dot{a}\cot\theta}{a^2r}e^2 \wedge e^3 \\ &= \frac{\ddot{a}}{a}e^0 \wedge e^3\end{aligned} \tag{2.117}$$

$$\begin{aligned}\Omega_2^1 &= d\omega_2^1 + \omega_0^1 \wedge \omega_2^0 + \omega_3^1 \wedge \omega_2^3 \\ &= d\left(-\frac{1}{afr}e^2\right) + \left(-i\frac{\dot{a}}{a}e^1\right) \wedge \left(i\frac{\dot{a}}{a}e^2\right) \\ &= -i\frac{\dot{a}}{a^2fr}e^0 \wedge e^2 + \frac{f + rf'}{a^2f^3r^2}e^1 \wedge e^2 - \frac{1}{afr}\left(-i\frac{\dot{a}}{a}e^0 \wedge e^2 + \frac{1}{afr}e^1 \wedge e^2\right) \\ &= \left(\frac{\dot{a}^2}{a^2} + \frac{f'}{a^2f^3r}\right)e^1 \wedge e^2 \\ &= \left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right)e^1 \wedge e^2\end{aligned} \tag{2.118}$$

$$\begin{aligned}
\Omega^1_3 &= d\omega^1_3 + \omega^1_0 \wedge \omega^0_3 + \omega^1_2 \wedge \omega^2_3 \\
&= d\left(-\frac{1}{afr}e^3\right) + \left(-i\frac{\dot{a}}{a}e^1\right) \wedge \left(i\frac{\dot{a}}{a}e^3\right) + \left(-\frac{1}{afr}e^2\right) \wedge \left(-\frac{\cot\theta}{ar}e^3\right) \\
&= -i\frac{\dot{a}}{a^2fr}e^0 \wedge e^3 + \frac{f + rf'}{a^2f^3r^2}e^1 \wedge e^3 - \frac{1}{afr}\left(-i\frac{\dot{a}}{a}e^0 \wedge e^3 + \frac{1}{afr}e^1 \wedge e^3 + \frac{\cot\theta}{ar}e^2 \wedge e^3\right) \\
&\qquad + \frac{\dot{a}^2}{a^2}e^1 \wedge e^3 + \frac{\cot\theta}{a^2fr^2}e^2 \wedge e^3 \\
&= \left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right)e^1 \wedge e^3
\end{aligned} \tag{2.119}$$

$$\begin{aligned}
\Omega^2_3 &= d\omega^2_3 + \omega^2_0 \wedge \omega^0_3 + \omega^2_1 \wedge \omega^1_3 \\
&= d\left(-\frac{\cot\theta}{ar}e^3\right) + \left(-i\frac{\dot{a}}{a}e^2\right) \wedge \left(i\frac{\dot{a}}{a}e^3\right) + \left(\frac{1}{afr}e^2\right) \wedge \left(-\frac{1}{afr}e^3\right) \\
&= -i\frac{\dot{a}\cot\theta}{a^2r}e^0 \wedge e^3 + \frac{\cot\theta}{a^2fr^2}e^1 \wedge e^3 + \frac{\csc^2\theta}{a^2r^2}e^2 \wedge e^3 \\
&\qquad - \frac{\cot\theta}{ar}\left(-i\frac{\dot{a}}{a}e^0 \wedge e^3 + \frac{1}{afr}e^1 \wedge e^3 + \frac{\cot\theta}{ar}e^2 \wedge e^3\right) + \frac{\dot{a}^2}{a^2}e^2 \wedge e^3 - \frac{1}{a^2f^2r^2}e^2 \wedge e^3 \\
&= \left(\frac{\dot{a}^2}{a^2} + \frac{1}{a^2r^2}\left(1 - \frac{1}{f^2}\right)\right)e^2 \wedge e^3 \\
&= \left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right)e^2 \wedge e^3.
\end{aligned} \tag{2.120}$$

We extract the Riemann curvature tensor using $\Omega^a_b = \frac{1}{2}R^a_{bcd}e^c \wedge e^d$ as

$$\begin{aligned}
R^0_{i0i} &= \frac{\ddot{a}}{a} \\
R^i_{jij} &= \frac{\dot{a}^2}{a^2} + \frac{K}{a^2} \qquad i \neq j.
\end{aligned} \tag{2.121}$$

We find the Ricci tensor with the contraction $R_{ab} = R^c_{acb}$

$$\begin{aligned}
R_{00} &= 3\frac{\ddot{a}}{a} \\
R_{ij} &= \left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 2\frac{K}{a^2}\right)\delta_{ij}
\end{aligned} \tag{2.122}$$

and the Ricci scalar

$$R = \delta^{ab} R_{ab} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right). \tag{2.123}$$

Now we can move on to the Einstein tensor

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = G_{\mu\nu} \tag{2.124}$$

and so we get the non-zero components in the orthonormal frame

$$\begin{aligned} G_{00} &= 3\frac{\ddot{a}}{a} - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) = -3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) \\ G_{ij} &= \left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) - 3\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right)\right]\delta_{ij} = -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right)\delta_{ij} \end{aligned} \tag{2.125}$$

which are related to components in the more familiar coordinate frame with $G_{\mu\nu} = e^a_\mu e^b_\nu G_{ab}$ where $\mu, \nu = 0, \ldots, 3$. Using the non-zero frame fields,

$$e^0_0 = i \qquad e^1_1 = a(t) f(r) \qquad e^2_2 = a(t) r \qquad e^3_3 = a(t) r \sin\theta \tag{2.126}$$

we find the Einstein tensor in coordinate frame as

$$\begin{aligned} G_{00} &= 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) \\ G_{ij} &= -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) g_{ij} \end{aligned} \tag{2.127}$$

where $i, j = 1, 2, 3$. Next we solve Einstein's field equations

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} \tag{2.128}$$

where the Einstein tensor $G_{\mu\nu}$ gives the spacetime curvature, Λ is the cosmological constant and the energy-momentum tensor $T_{\mu\nu}$ gives the matter content of the universe with $\kappa = 8\pi G$, G being Newton's gravitational constant. Here we consider non-zero cosmological constant for a generalization of the solution.

We will start with the form of the energy-momentum tensor. The above result for Einstein tensor dictates us the perfect fluid model, since off-diagonal interaction terms are zero, and the components are as

$$T_{00} = \rho(t) \qquad T_{ij} = p(t)\delta_{ij} \tag{2.129}$$

where the energy density ρ and the pressure p of the fluid are functions only of time because of the homogeneity [11]. This tensor is usually written in the covariant form as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}. \tag{2.130}$$

Here u_μ is the four-velocity of the fluid in comoving coordinates where the fluid is at rest, i.e. $u_\mu = (1, 0, 0, 0)$. In the orthonormal frame we can write the energy-momentum tensor with $T_{ab} = e^\mu_a e^\nu_b T_{\mu\nu}$ which gives $T_{ab} = -(\rho + p)u_a u_b + p\delta_{ab}$. We did not stress on it but Equation 2.38 shows us that the covariant derivative of the Einstein tensor is zero. This can be seen by the change $\delta^\nu_\mu \to g^\nu_\mu$,

$$\nabla_\nu \left(\frac{1}{2} g^\nu_\mu R - R^\nu_\mu \right) = 0 \tag{2.131}$$

therefore it must be compatible with the energy-momentum tensor, using $T^{\nu\mu} = g^{\mu\lambda} T^\nu_\lambda$

$$\begin{aligned} \nabla_\nu T^{\nu\mu} &= 0 \\ \nabla_\nu \left((\rho + p)u^\nu u^\mu + p g^{\nu\mu} \right) &= 0 \end{aligned} \tag{2.132}$$

from which we obtain four equations for the matter content.

So we could either use the orthonormal frame with the above energy-momentum tensor or the coordinate frame by putting Equations 2.127 and 2.130 into Einstein

equation given in Equation 2.128 which gives

$$\begin{aligned} 3\left(\frac{\dot{a}^2}{a^2}+\frac{K}{a^2}\right)-\Lambda &= \kappa\rho \\ -\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{K}{a^2}\right)+\Lambda &= \kappa p. \end{aligned} \tag{2.133}$$

These equations are called the Friedmann equations. Putting the first one into the second gives

$$\frac{\ddot{a}}{a}=\frac{\Lambda}{3}-\frac{\kappa}{6}(\rho+3p). \tag{2.134}$$

If we take the time derivative of the first one

$$\frac{\dot{a}}{a}\left(\frac{\ddot{a}}{a}-\frac{\dot{a}^2}{a^2}-\frac{K}{a^2}\right)=\frac{\kappa}{6}\dot{\rho} \tag{2.135}$$

and insert Equations 2.133 and 2.134 into this we obtain the equation of continuity of the matter content.

$$\dot{\rho}+3\frac{\dot{a}}{a}(\rho+p)=0 \tag{2.136}$$

The total energy of an expanding three dimensional space with volume $V=da^3$ where d is a constant and pressure p is $U=\rho V$. Taking the time derivative $dU/dt=(d\rho/dt)V+\rho(dV/dt)$ gives from the above equation $dU/dt=-p(dV/dt)$ so that the change in the total energy is $dU=-pdV$ which is the adiabatic expansion equation of thermodynamics [12].

3. MINKOWSKI SPACETIME

Minkowski spacetime is the simplest of the three maximally symmetric spacetimes. It does not contain any matter and corresponds to the $\Lambda = 0$ case. Therefore the curvature tensor is zero. In 4 dimensions the number of symmetries a 3+1 dimensional maximally symmetric spacetime admits is 10 which is the same number as the isometries of the Minkowski spacetime in 3+1 dimensions, namely Poincare group consisting of 4 translations 3 spatial rotations and 3 boosts. For the coset picture we say that Minkowski spacetime is a homogeneous space of Poincare symmetries in which a fixed point has Lorentz symmetries, or $\mathbb{R}^{1,3} \rtimes SO(3,1)/SO(3,1)$ where $\rtimes$ denotes the semi-direct product.

We start with the good old flat metric in cartesian coordinates

$$ds^2 = -dt^2 + d\vec{x}^2 \tag{3.1}$$

with $-\infty < t, \vec{x} < \infty$. Translational Killing vectors of this metric are trivial $\xi_\mu = \partial_\mu$ where $\mu = 0, \ldots, 3$. From Equation 2.10 where covariant derivative in this case is replaced by ordinary derivative we write $g_{\mu\rho}\partial_\nu \xi^\rho + g_{\nu\sigma}\partial_\mu \xi^\sigma = 0$ from which we obtain

$$\partial_\mu \xi^\mu = 0 \qquad \partial_0 \xi^i - \partial_i \xi^0 = 0 \qquad \partial_i \xi^j + \partial_j \xi^i = 0. \tag{3.2}$$

These give 4 translations, 3 boosts and 3 rotations

$$\xi_\mu = \partial_\mu \qquad \xi_{i0} = x_i \partial_0 + x_0 \partial_i \qquad \xi_{ij} = x_i \partial_j - x_j \partial_i \tag{3.3}$$

which is the maximum number in $N = 4$. If we write boosts and rotations in the compact form as $\xi_{\mu\nu} = \eta_{\mu\rho}x^\rho\partial_\nu - \eta_{\nu\sigma}x^\sigma\partial_\mu$ they satisfy the Poincare algebra

$$\begin{aligned}
[\xi_\mu, \xi_\nu] &= 0 \\
[\xi_{\mu\nu}, \xi_\rho] &= \left[\eta_{\mu\alpha}x^\alpha\partial_\nu - \eta_{\nu\beta}x^\beta\partial_\mu, \partial_\rho\right] \\
&= \eta_{\mu\alpha}\left[x^\alpha, \partial_\rho\right]\partial_\nu - \eta_{\nu\beta}\left[x^\beta, \partial_\rho\right]\partial_\mu\,, \qquad [x^\alpha, \partial_\beta] = -\delta^\alpha_\beta \\
&= \eta_{\nu\rho}\xi_\mu - \eta_{\mu\rho}\xi_\nu \\
[\xi_{\mu\nu}, \xi_{\rho\sigma}] &= \left[\eta_{\mu\alpha}x^\alpha\partial_\nu - \eta_{\nu\beta}x^\beta\partial_\mu, \eta_{\rho\gamma}x^\gamma\partial_\sigma - \eta_{\sigma\tau}x^\tau\partial_\rho\right] \\
&= \eta_{\mu\alpha}\eta_{\rho\gamma}\left(x^\alpha\left[\partial_\nu, x^\gamma\right]\partial_\sigma + x^\gamma[x^\alpha, \partial_\sigma]\partial_\nu\right) \\
&- \eta_{\mu\alpha}\eta_{\sigma\tau}\left(x^\alpha\left[\partial_\nu, x^\tau\right]\partial_\rho + x^\tau[x^\alpha, \partial_\rho]\partial_\nu\right) \\
&- \eta_{\nu\beta}\eta_{\rho\gamma}\left(x^\beta\left[\partial_\mu, x^\gamma\right]\partial_\sigma + x^\gamma[x^\beta, \partial_\sigma]\partial_\mu\right) \\
&+ \eta_{\nu\beta}\eta_{\sigma\tau}\left(x^\beta\left[\partial_\mu, x^\tau\right]\partial_\rho + x^\tau[x^\beta, \partial_\rho]\partial_\mu\right) \\
&= \eta_{\mu\sigma}\xi_{\nu\rho} + \eta_{\nu\rho}\xi_{\mu\sigma} - \eta_{\mu\rho}\xi_{\nu\sigma} - \eta_{\nu\sigma}\xi_{\mu\rho}
\end{aligned} \tag{3.4}$$

which can be generalized to N dimensions.

To understand the structure of infinities in Minkowski spacetime we will build the Penrose diagram described in Appendix A where we essentially bring all infinities to a finite region and preserve the causal relations [13]. We can start with going to spherical coordinates where two angles θ and ϕ are already compact. The metric reads

$$ds^2 = -dt^2 + dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{3.5}$$

with $0 \leq r < \infty$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$ where $\phi = 0$ and $\phi = 2\pi$ are identified. Since curvature is already zero $r = 0$, $r \to \infty$ and $\theta = 0, \pi$ are only coordinate singularities. Next we go to a special choice of coordinates, null coordinates with

$$u = t - r \qquad v = t + r \tag{3.6}$$

which gives

$$t = \frac{v+u}{2} \qquad r = \frac{v-u}{2} \tag{3.7}$$

noting that $r \geq 0$ gives $v \geq u$. The metric in these coordinates is

$$\begin{aligned} ds^2 &= -\left(\frac{dv+du}{2}\right)^2 + \left(\frac{dv-du}{2}\right)^2 + \left(\frac{v-u}{2}\right)^2 d\Omega_2^2 \\ &= -dvdu + \left(\frac{v-u}{2}\right)^2 d\Omega_2^2 \end{aligned} \tag{3.8}$$

with $-\infty < u, v < \infty$. These coordinates provide us a grid of constant u and v rotated 45° from r, t which is followed by light rays making the causal relations apparent. At each point of the grid of constant u, v sits a sphere of radius $(v-u)/2$. We can bring the infinities of null coordinates to a finite region through compactification

$$p = \tan^{-1} v \qquad q = \tan^{-1} u \tag{3.9}$$

which makes $du = \sec^2 q dq$ and $dv = \sec^2 p dp$. Then the metric reads

$$\begin{aligned} ds^2 &= -\sec^2 p \sec^2 q dp dq + \frac{1}{4}\left(\tan p - \tan q\right)^2 d\Omega_2^2 \\ &= \sec^2 p \sec^2 q \left(-dpdq + \frac{\sin^2(p-q)}{4} d\Omega_2^2\right) \end{aligned} \tag{3.10}$$

where $-\pi/2 < p, q < \pi/2$ and $v \geq u$ gives $p \geq q$. Now we reverse things and make the transformation

$$T = p + q \qquad X = p - q \quad \rightarrow \quad p = \frac{1}{2}T + X \qquad q = \frac{1}{2}T - X \tag{3.11}$$

where $p \geq q$ translates to $X \geq 0$ and ranges are

$$\begin{aligned} -\pi < T + X < \pi \\ -\pi < T - X < \pi \end{aligned} \tag{3.12}$$

and the metric becomes

$$ds^2 = \frac{1}{4}\sec^2\left(\frac{T+X}{2}\right)\sec^2\left(\frac{T-X}{2}\right)\left(-dT^2 - dX^2 + \sin^2 X d\Omega_2^2\right). \tag{3.13}$$

Now that we have the metric in the form $ds^2 = \Omega^2 d\tilde{s}^2$ and all ranges are now compact, we can draw the Penrose diagram. Considering Equation 3.12 together with $X \geq 0$ we obtain the region as shown in Figure 3.1. In this diagram $X = 0$ represents $r = 0$. Past and future null infinities $\mathcal{I}^-, \mathcal{I}^+$ are represented as $T - X = -\pi$ and $T + X = \pi$ respectively. Radial geodesics start and end at $X = \pi$, $T = 0$, so the spacelike infinity i^0 is a point which becomes S^1 if we add the ϕ coordinate, and S^2 if we also add the θ coordinate. Past and future timelike infinities i^- and i^+ are represented as $X = 0$, $T = -\pi$ and $X = 0$, $T = \pi$.

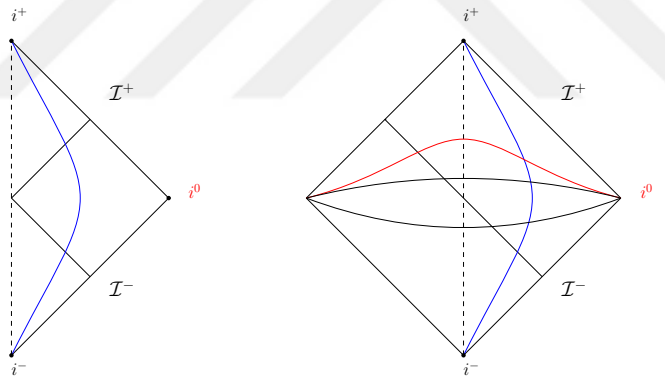

Figure 3.1. (a) Penrose diagram of the Minkowski spacetime. Null geodesics originate from $\mathcal{I}^-$ traveling straight at 45° from the vertical and end at $\mathcal{I}^+$. Blue curve is a radial geodesic. (b) When we add the ϕ coordinate. Red curve is a spacelike geodesic.

3.1. Milne Universe

When E. A. Milne published his paper in 1933 [14] on the expanding model he only considered the kinematic aspects. He used special relativity without gravitation or any accelerated frames in curved spacetime. By generalizing Maxwell-Boltzmann velocity distribution of a cluster of particles with varying speeds from 0 to c and considering an

"extended" version of special relativity (covariance of the distribution function under Lorentz transformations), he modeled the linearly expanding universe.

Here we refer to the Milne universe as the corresponding case of an FLRW universe with a linear expansion. It is the case of an empty universe $\rho = p = 0$ also with $\Lambda = 0$ so that there is no gravitation. From the Friedmann equations,

$$\dot{a}^2 = -k \tag{3.14}$$

and we have either $k = 0$ corresponding to a Minkowski spacetime with a constant scale factor or $k = -1$ that gives $a(t) = t$, starting from the $t = 0$, a universe linearly expanding with time or $a(t) = t_1 - t$, a universe linearly contracting with time ending with a big-crunch at $t = t_1$. Note that we are using the units with $c = 1$ and this means that the universe is expanding with speed of light. In the comoving coordinates for $k = -1$ and $a(t) = t$ the metric in Equation 2.82 with $L = 1$ is,

$$ds^2 = -dt^2 + t^2(d\chi^2 + \sinh^2\chi d\Omega_2^2). \tag{3.15}$$

This is the metric of the well known Milne universe. Although it is an empty universe with zero cosmological constant, it is an open space that is linearly expanding with the cosmological time. However this picture is just an illusion since with a re-parameterization we can show that the Milne universe is equivalent to the Minkowski spacetime, namely with the transformations,

$$\tau = t\cosh\chi \qquad \rho = t\sinh\chi \tag{3.16}$$

satisfying the hyperboloid constraint, $\tau^2 - \rho^2 = t^2$. It follows then

$$d\tau = \cosh\chi dt + t\sinh\chi d\chi \tag{3.17}$$

$$d\rho = \sinh\chi dt + t\cosh\chi d\chi \tag{3.18}$$

and we show the equivalence

$$-d\tau^2 + d\rho^2 + \rho^2 d\Omega_2^2 = -dt^2 + t^2 d\chi^2 + t^2 \sinh^2 d\Omega_2^2. \tag{3.19}$$

We can find the constant t surfaces in the stationary (ρ, τ) coordinate system for a test particle in the Milne universe. Using the constraint equation and the transformations given in Equation 3.16 we see that constant t hypersurfaces are hyperboloids in (ρ, τ) Minkowski coordinate system which are represented as hyperbolas in Figure 3.2. For $t = 0$, we have $\tau = \pm\rho$ at $\chi \to \infty$. Curves of constant χ are represented as straight lines. So although Milne has a negative spatial curvature, its spacetime curvature is

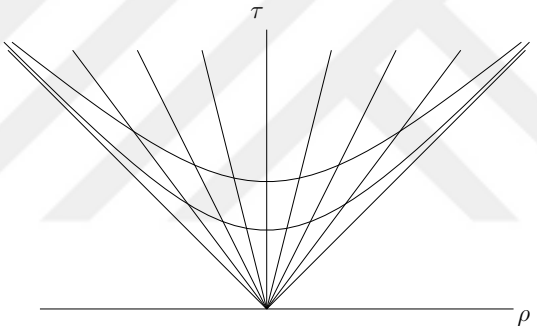

Figure 3.2. The constant proper time (simultaneity) and constant comoving distance (world lines) curves of the expanding (t, χ) Milne frame in the future light cone of the (τ, ρ) (stationary) Minkowski frame of a test particle.

zero. We showed the equivalence of the Milne and Minkowski metrics however Milne, expanding linearly in time, allows us to calculate a non-zero cosmological redshift unlike Minkowski. However we can compare the Doppler redshift due to a moving source of radiation. So let's take a test particle at $\rho = 0$ that we will call the observer in the (τ, ρ) inertial frame and a source moving with the Hubble flow in the (t, χ) frame and is emitting electromagnetic waves. The source is moving with the constant velocity v_ρ,

$$v_\rho = d\rho/d\tau = \tanh \chi \tag{3.20}$$

with proper time

$$dt = \sqrt{1 - \left(\frac{d\rho}{d\tau}\right)^2}\, d\tau \tag{3.21}$$

equal to cosmic time t, in the (τ, ρ) frame.

The apparent time dilation or the Doppler redshift is due to an inertial source moving with velocity v emitting electromagnetic waves [5]. The proper time interval between two successive wave fronts is $dt = \Delta t$. In the interval between the two wave fronts emitted, the source also has a displacement of $v d\tau$. So the total time period of two successive wave fronts received by the observer is

$$\Delta\tau = d\tau + v_\rho d\tau = (1 + v_\rho) d\tau = (1 + v_\rho) \left(\sqrt{1 - v_\rho^2}\right)^{-1} dt. \tag{3.22}$$

Then the ratio of the emission frequency to the observed frequency is

$$\frac{\nu_e}{\nu_o} = \frac{\Delta\tau}{\Delta t} = (1 + v_\rho) \left(\sqrt{1 - v_\rho^2}\right)^{-1} = (1 + v_\rho) \frac{d\tau}{dt}. \tag{3.23}$$

From the time dilation we insert the $dt/d\tau$ evaluated at the source frame,

$$\frac{\nu_e}{\nu_o} = (1 + v_\rho) \frac{d\tau}{dt} = (1 + v_\rho) \left(\sqrt{1 - \left(\frac{d\rho}{d\tau}\right)^2}\right)^{-1} = (1 + \tanh\chi)(\cosh\chi) = e^\chi \tag{3.24}$$

giving us the redshift,

$$1 + z \equiv \frac{\nu_e}{\nu_o} = e^\chi. \tag{3.25}$$

Now let us calculate the cosmological redshift in the (t, χ). The emitted wave front will follow the null geodesic, $ds = 0$,

$$ds^2 = 0 = -dt^2 + t^2 d\chi^2 \tag{3.26}$$

where we only consider the radial motion, $d\Omega_2 = 0$.

$$\int_{t_e}^{t_o} \frac{dt'}{t'} = \int_0^{\chi} d\chi' \tag{3.27}$$

which gives

$$\chi = \ln\left(\frac{t_0}{t_e}\right). \tag{3.28}$$

Using the cosmological redshift formula [5] with $a(t) = t$, and using the above result, we find

$$1 + z = \frac{a(t_o)}{a(t_e)} = \frac{t_o}{t_e} = e^{\chi} \tag{3.29}$$

which is in agreement with the Doppler redshift. We do not wish to do these calculations for other cosmological models since the method used here is speculative for us to apply to spacetimes with non-zero curvature tensor. This subject is covered in detailed in [15].

3.2. Rindler Coordinates

In its original form Rindler considered a uniformly accelerating frame in the $x > |t|$ region of the Minkowski space in cartesian coordinates giving an analogy with Kruskal space [16]. Here we consider the four dimensional Minkowski metric in spherical coordinates

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega_2^2 \tag{3.30}$$

and consider the coordinates

$$t = \rho \sinh \tau \qquad r = \rho \cosh \tau \tag{3.31}$$

satisfying the hyperbola constraint $r^2 - t^2 = \rho^2$ where $-\infty < \tau < \infty$ and since $0 \leq r < \infty$, and for $\tau = 0$ $r = \rho$, $0 \leq \rho < \infty$. From $(r-t)(r+t) \geq 0$ and $(r-t) = \rho e^{-\tau} \geq 0$ these coordinates cover the region $r > |t|$. The metric in these coordinates is

$$ds^2 = -\rho^2 d\tau^2 + d\rho^2 + \rho^2 \cosh^2 \tau d\Omega_2^2 \tag{3.32}$$

where $\rho = 0$ is just a coordinate singularity since curvature tensor is zero identically. This metric can be interpreted as an expanding isotropic but non-homogeneous cosmology where τ is a timelike coordinate which cannot be called a cosmological time. Constant ρ sections are three dimensional de Sitter spaces which we will see in the following section. In terms of the null coordinates we constructed above, this metric describes

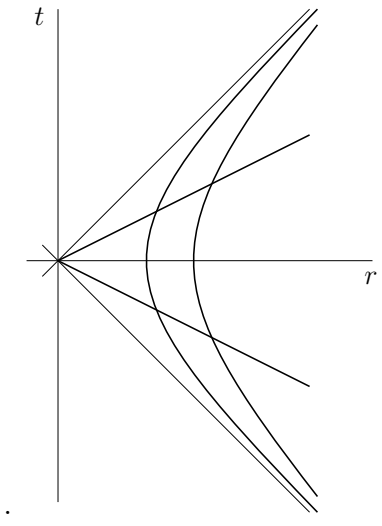

Figure 3.3. Rindler patch outside the light cone. Constant τ lines are straight lines and hyperbolas describe the observers with uniform acceleration

the region outside the light cones. Constant τ lines are straight lines (we suppress θ and ϕ coordinates) going through the origin. Constant ρ lines describe the observers with radial uniform acceleration (same is in x direction for example if we consider carte-

sian coordinates) as shown in Figure 3.3. The proper time for an observer in its rest frame is $dt_p = \rho d\tau$. The four velocity $u^\mu = dx^\mu/dt_p = (1/\rho)dx^\mu/d\tau$ gives $u^0 = \cosh\tau$ and $u^1 = \sinh\tau$ satisfying $\eta_{\mu\nu}u^\mu u^\nu = -1$. The acceleration $a^\mu = (1/\rho)du^\mu/d\tau$ gives $a^0 = (1/\rho)\sinh\tau$ and $a^1 = (1/\rho)\cosh\tau$ that gives $\eta_{\mu\nu}a^\mu a^\nu = \rho^{-2}$ which is a constant. So in Rindler coordinates we see a part of Minkowski space in which ρ =constant lines correspond to observers with constant acceleration in the radial direction.

If we contrast with Milne, the proper time is $dt_p = dt$ in the rest frame of the observer where χ, θ and ϕ are constant. The four velocity $u^\mu = dx^\mu/dt$ gives $u^0 = \cosh\chi$ and $u^1 = \sinh\chi$ satisfying $\eta_{\mu\nu}u^\mu u^\nu = -1$ and they are constant as expected.

4. de SITTER SPACETIME

In this chapter we will review the general structure and some coordinate patches on de Sitter spacetime. It is known for the exponential expansion factor and is considered to be the future case of our universe. It is the unique vacuum solution of Einstein's equations with a positive cosmological constant Λ. The curvature scalar is given with Equation 2.100 which in four dimensions gives $R = 4\Lambda$.

The defining equation of de Sitter spacetime in N dimensions is the hyperboloid

$$-(x^0)^2 + (x^i)^2 + (x^N)^2 = L^2 \tag{4.1}$$

where $i = 1, \ldots, N-1$. The length scale in terms of the cosmological constant can be given as $L^2 = (N-1)(N-2)/2\Lambda$ which can be seen from Equation 2.101. The defining equation is invariant under the group $SO(N,1)$. We can realize this space by embedding it into $\mathbb{R}^{N,1}$ as

$$ds^2 = -(dx^0)^2 + \sum_{i=1}^{N-1}(dx^i)^2 + (dx^N)^2 \tag{4.2}$$

which is done in section 1. The isometry group of this metric is $SO(N,1)$ and its dimension is $N(N+1)/2$ which is the maximum number of Killing vectors. Therefore de Sitter spacetime is a maximally symmetric spacetime. For a fixed point $SO(N-1,1)$ is the symmetry group. This allows us to write the N dimensional de Sitter spacetime as a coset space, $SO(N,1)/SO(N-1,1)$.

From now on our approach will basically be about choosing appropriate coordinates satisfying the defining equation, embedding them into the $R^{(N,1)}$ spacetime and analyze the picture that we have.

4.1. Global Coordinates

The defining equation for fixed x^0 is,

$$(x^i)^2 + (x^N)^2 = L^2 + (x^0)^2 \tag{4.3}$$

where the left hand side is a positive constant and so giving rise to the topology $\mathbb{R} \times S^{(N-1)}$. The general solution to Equation 4.3 is given with

$$x^0 = L\sinh(\tau/L) \qquad x^i = L\omega^i \cosh(\tau/L) \tag{4.4}$$

providing ω^i with $i = 1, \ldots, N$ is a unit vector on S^{N-1} satisfying

$$\sum_{i=1}^{N}(\omega^i)^2 = 1 \qquad \sum_{i=1}^{N}\omega^i d\omega^i = 0 \qquad \sum_{i=1}^{N}(d\omega^i)^2 = d\Omega^2_{N-1}. \tag{4.5}$$

The metric becomes

$$\begin{aligned} ds^2 &= -(dx^0)^2 + (dx^i)^2 + (dx^N)^2 \\ &= -\left(\cosh(\tau/L)d\tau\right)^2 + \sum_{i=1}^{N}\left(\omega^i \sinh(\tau/L)d\tau + L\cosh(\tau/L)d\omega^i\right)^2 \\ ds^2 &= -d\tau^2 + L^2\cosh^2(\tau/L)d\Omega^2_{N-1} \end{aligned} \tag{4.6}$$

where $-\infty < \tau < \infty$. So for fixed τ, spatial sections are $S^{(N-1)}$ spheres with radius $L\cosh(\tau/L)$ which are infinitely large for $\tau = -\infty$, that shrink and reach to a minimum at $\tau = 0$ then grow and become infinitely large again as $\tau = \infty$. This metric is also known as the closed slicing of de Sitter space. The apparent symmetries of this metric is of S^{N-1} dimensional sphere namely the $SO(N)$ group with $N(N-1)/2$ Killing vectors. Time translation is not a symmetry of this metric since it depends on τ. Therefore ∂_τ is not a Killing vector.

We can rewrite this metric with a conformal factor as

$$ds^2 = L^2 \cosh^2(\tau/L) \left(-\frac{d\tau^2}{L^2 \cosh^2(\tau/L)} + d\Omega_{N-1}^2 \right) = g^2(\eta) \left(-d\eta^2 + d\Omega_{N-1}^2 \right) \tag{4.7}$$

where η is the conformal time. This gives us the equation

$$d\eta = \frac{d\tau}{L \cosh(\tau/L)} \tag{4.8}$$

which has the solution

$$\tan(\eta/2L) = \tanh(\tau/2L) \tag{4.9}$$

and so the conformal factor is

$$\begin{aligned} \tan^2(\eta/2L) &= \tanh^2(\tau/2L) \\ \frac{1 - \cos(\eta/L)}{1 + \cos(\eta/L)} &= 1 - \frac{2}{1 + \cosh(\tau/L)} \\ \frac{1}{\cos(\eta/L)} &= \cosh(\tau/L). \end{aligned} \tag{4.10}$$

The conformal metric is

$$ds^2 = \frac{L^2}{\cos^2(\eta/L)} \left(-d\eta^2 + d\Omega_{N-1}^2 \right) \tag{4.11}$$

with $-\pi/2 < \eta/L < \pi/2$. Figure 4.1 shows the Penrose diagram for the metric $d\tilde{s}^2 = \cosh^2(\eta/L)/L^2 ds^2$ in which we suppress S^{N-2} spheres of radius $\sin\psi$ that exist at every point in the diagram. So we consider the metric

$$d\tilde{s}^2 = -d\eta^2 + d\psi^2 \tag{4.12}$$

with $0 \leq \psi \leq \pi$. Since ranges are of the same size we have a square diagram for de Sitter space (except the $N = 2$ case for which we have $0 \leq \psi \leq 2\pi$ where 0 and 2π are identified since we have circles instead of spheres and ψ is double the size of η)

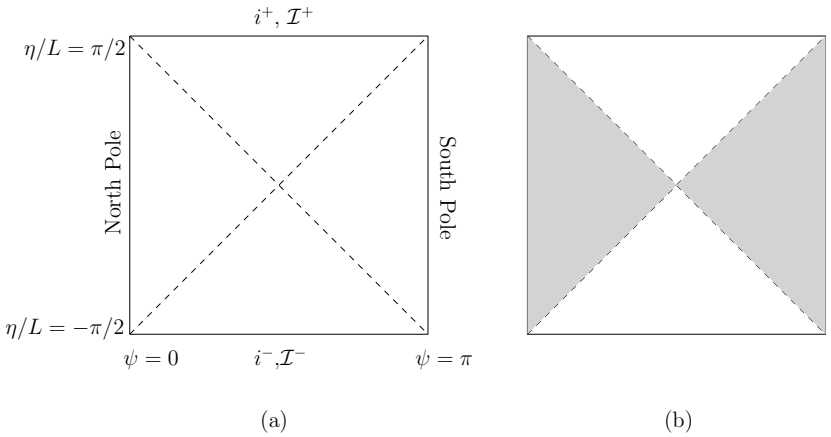

Figure 4.1. (a) Penrose Diagram for the global de Sitter spacetime. (b) Fully accesible regions.

in which the light travels at 45°. $\psi = 0$ and $\psi = \pi$ are the north and south poles of the $N-1$ dimensional sphere respectively. Future and past timelike infinities (i^+ and i^-) as well as future and past null infinities ($\mathcal{I}^+$, $\mathcal{I}^-$) are at the spacelike boundaries $\eta/L = \pm\pi/2$. This results in the existence of both particle and event horizons in de Sitter space [13]. If we consider a light signal sent from an observer sitting at the north pole it takes infinite amount of time until it reaches to south pole where there is $\mathcal{I}^+$. This creates a region with a particle horizon where this observer cannot send signals to or cannot influence. Since the same goes for an observer sitting at the south pole this creates another region with an event horizon where an observer at the north pole cannot receive any signal from or cannot be influenced. As can be seen from Figure 4.1 the intersection of these regions gives two causally disconnected regions for two different observers one sitting at the north pole and the other in south pole, that are fully accessible to them and is called the causal diamond.

4.2. In terms of FLRW

Now consider the Friedmann equations given in Equation 2.133 with $T_{\mu\nu} = 0$

$$\begin{aligned} 3\left(\frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) - \Lambda = 0 \\ -\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{K}{a^2}\right) + \Lambda = 0. \end{aligned} \tag{4.13}$$

From the first equation we have

$$\dot{a}^2 - L^{-2}a^2 = -K \tag{4.14}$$

where $L^{-2} = \Lambda/3$ for $N = 4$ and from $K = kL^{-2}$, k takes values as 0, -1 or 1. For $k = -1$ we have

$$\dot{a}^2 - L^{-2}a^2 = L^{-2} \tag{4.15}$$

satisfied with

$$a(t) = \sinh \frac{t}{L} \tag{4.16}$$

and so for this case of $k = -1$ the metric is

$$ds^2 = -dt^2 + \sinh^2(t/L) \left(\frac{dr^2}{1 + \frac{r^2}{L^2}} + r^2 d\Omega_{N-2}^2 \right). \tag{4.17}$$

The spatial part of the metric can be written as the metric of unit H^{N-1} provided $r = L \sinh \chi$ as we did in the Milne universe and becomes

$$ds^2 = -dt^2 + L^2 \sinh^2(t/L) \left(d\chi^2 + \sinh^2 \chi d\Omega_{N-2}^2 \right) = -dt^2 + L^2 \sinh^2(t/L) d\tilde{\Omega}_{N-1}^2. \tag{4.18}$$

The Hubble constant is

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{L} \coth \frac{t}{L}. \tag{4.19}$$

The $k = 0$ case is satisfied with

$$a(t) = e^{t/L} \tag{4.20}$$

and the metric is

$$ds^2 = -dt^2 + e^{2t/L} \sum_{i}^{N-1} (dx^i)^2 \tag{4.21}$$

with the spatial part being Minkowski space. The Hubble constant is

$$H(t) = \frac{1}{L}. \tag{4.22}$$

The $k = +1$ case is satisfied with

$$a(t) = \cosh \frac{t}{L} \tag{4.23}$$

and the metric is

$$ds^2 = -dt^2 + \cosh^2(t/L) \left(\frac{dr^2}{1 - \frac{r^2}{L^2}} + r^2 d\Omega_{N-2}^2 \right). \tag{4.24}$$

The spatial part can be written as the metric of unit S^{N-1} provided $r = L \sin \psi$ similar to the above case and we obtain the metric

$$ds^2 = -dt^2 + L^2 \cosh^2(t/L) \left(d\psi^2 + \sin^2 \psi d\Omega_{N-2}^2 \right) = -dt^2 + L^2 \cosh^2(t/L) d\Omega_{N-1}^2 \tag{4.25}$$

with the Hubble constant

$$H(t) = \frac{1}{L} \tanh \frac{t}{L}. \tag{4.26}$$

Now the question is can we find parameterizations such that the defining equation given in Equation 4.1 is satisfied? That is can we find what is called slicing of the de Sitter space for each cases of FLRW? The answer is yes and we already did show one of them. If we compare Equation 4.4 with the metric given in Equation 4.25, with the identification $L^2 = 3/\Lambda$ global coordinates give the closed slicing of de Sitter space. Next we will show the other two and some other parameterizations.

4.3. Planar Coordinates

These coordinates are called the flat slicing of N dimensional de Sitter space and they correspond to the flat $k = 0$ case of the FLRW solutions. To see this we start by rewriting the defining equation given in Equation 4.1 as

$$-(x^0)^2 + (x^N)^2 = L^2 - (x^i)^2 \tag{4.27}$$

where the x^i coordinates describe the $N-2$ dimensional sphere of radius $z^i e^{t/L}$ [17]. This gives

$$(x^1)^2 + \cdots + (x^{N-1})^2 = (z^i)^2 e^{2t/L} \tag{4.28}$$

and the left hand side describes the 2-dimensional hyperbola of radius $\sqrt{L^2 - (z^i)^2 e^{2t/L}}$.

$$-(x^0)^2 + (x^N)^2 = L^2 - (z^i)^2 e^{2t/L}. \tag{4.29}$$

These can be separated as

$$\begin{aligned} x^N + x^0 &= L e^{t/L} \\ x^N - x^0 &= L e^{-t/L} - \frac{r^2}{L} e^{t/L} \\ x^i &= z^i e^{t/L} \end{aligned} \tag{4.30}$$

where $r^2 = \sum_{i=1}^{N-1} (z^i)^2$ and makes it clear from $x^N + x^0 \geq 0$ that these coordinates cover half of the de Sitter space. The coordinate parameterizations can be made as

$$\begin{aligned} x^0 &= L \sinh(t/L) + \frac{r^2}{2L} e^t / L \\ x^i &= z^i e^{t/L} \qquad i = 1, \ldots, N-1 \\ x^N &= L \cosh(t/L) - \frac{r^2}{2L} e^{t/L} \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} dx^0 &= \left(\cosh(t/L) + \frac{r^2}{2L^2} e^t/L\right) dt \\ dx^i &= e^{t/L} dz^i + \frac{z^i}{L} e^{t/L} dt \\ dx^N &= \left(\sinh(t/L) - \frac{r^2}{2L^2} e^t/L\right) dt. \end{aligned} \tag{4.32}$$

The embedding gives the metric

$$\begin{aligned} ds^2 &= -(dx^0)^2 + \cdots + (dx^N)^2 \\ &= -\left(\cosh^2(t/L) + \frac{r^4}{4L^4} e^{2t/L} + \frac{r^2}{L^2}\cosh(t/L) e^{t/L}\right) dt^2 + e^{2t/L}(dz^i)^2 + \frac{r^2}{L^2} e^{2t/L} dt^2 \\ &\quad + \left(\sinh^2(t/L) + \frac{r^4}{4L^4} e^{2t/L} - \frac{r^2}{L^2}\sinh(t/L) e^{t/L}\right) dt^2 \\ ds^2 &= -dt^2 + e^{2t/L} \sum_{i=1}^{N-1} \left(dz^i\right)^2 \end{aligned} \tag{4.33}$$

and we get Equation 4.21 with ranges $-\infty < z^i, t < \infty$. Constant t surfaces are $N-1$ dimensional flat planes in which $N-1$ translations and $(N-1)(N-2)/2$ rotational symmetries of $\mathbb{R}^{N-1}$ giving a total number $N(N-1)/2$ of symmetries are manifest. Time translation is not a symmetry of the metric by itself. However if we consider infinitesimal translations in time $t \to t + \varepsilon$ with $x^i \to x^i - \varepsilon x^i$ up to a first order in ε this is a symmetry of the metric [18].

We can write the metric given in Equation 4.33 conformally flat if we define $u = -Le^{-t/L}$ that gives

$$ds^2 = \frac{L^2}{u^2}\left(-du^2 + \sum_{i=1}^{N-1} \left(dz^i\right)^2\right) \tag{4.34}$$

with $-\infty < u < 0$, a boundary at $u = 0$.

4.4. Hyperbolic Coordinates

These coordinates are called the open slicing since they cover the N dimensional de Sitter space with negative curvature spaces and correspond to the $k = -1$ case of the FLRW universe. To see this we start by rewriting the defining Equation 4.1 as

$$-(x^0)^2 + (x^i)^2 = L^2 - (x^N)^2 \tag{4.35}$$

where for $x^N > L$ the right hand side is negative and for constant x^N we have hypersurfaces H^{N-1}. This equation is satisfied with the parameterizations

$$\begin{aligned} x^0 &= L \sinh(\tau/L) \cosh\psi \\ x^N &= L \cosh(\tau/L) \\ x^i &= L\omega^i \sinh(\tau/L) \sinh\psi \end{aligned} \tag{4.36}$$

where ω^i follow Equation 4.5 for S^{N-2} as

$$\sum_{i=1}^{N-1} (\omega^i)^2 = 1 \qquad \sum_{i=1}^{N-1} (d\omega^i)^2 = d\Omega_{N-2}^2. \tag{4.37}$$

We continue

$$\begin{aligned} dx^0 &= \cosh(\tau/L) \cosh\psi d\tau + L \sinh(\tau/L) \sinh\psi d\psi \\ dx^N &= \sinh(\tau/L) d\tau \\ dx^i &= L \sinh(\tau/L) \sinh\psi d\omega^i + \omega^i \cosh(\tau/L) \sinh\psi d\tau + L\omega^i \sinh(\tau/L) \cosh\psi d\psi \end{aligned} \tag{4.38}$$

and the metric is then

$$\begin{aligned}
ds^2 = &-\cosh^2(\tau/L)\cosh^2\psi d\tau^2 - L^2\sinh^2(\tau/L)\sinh^2\psi d\psi^2 \\
&- 2L\cosh(\tau/L)\sinh(\tau/L)\cosh\psi\sinh\psi d\tau d\psi + \sinh^2(\tau/L)d\tau^2 \\
&+ L^2\sinh^2(\tau/L)\sinh^2\psi(d\omega^i)^2 + \cosh^2(\tau/L)\sinh^2\psi d\tau^2 \\
&+ L^2\sinh^2(\tau/L)\cosh^2\psi d\psi^2 + 2L\sinh(\tau/L)\cosh(\tau/L)\sinh\psi\cosh\psi d\tau d\psi \\
= &-d\tau^2 + L^2\sinh^2(\tau/L)\left(d\psi^2 + \sinh^2\psi d\Omega_{N-2}^2\right) \\
ds^2 = &-d\tau^2 + L^2\sinh^2(\tau/L)d\tilde{\Omega}_{N-1}^2
\end{aligned} \tag{4.39}$$

which is the metric given in Equation 4.18. It covers only the part where $x^N > L$. The apparent symmetries of this metric is that of H^{N-1}, that is the group $SO(N-1,1)$ with $N(N-1)/2$ Killing vectors. Time translations are not symmetries of this metric.

4.5. dS-ception

By dS-ception we mean that we can obtain a N dimensional de Sitter spacetime iteratively by a $N-1$ dimensional de Sitter spacetime. Here we will write the coordinate parameterizations which again do not cover de Sitter globally. They cover only the part left out from the hyperbolic coordinates, the region where $x^N < L$ and as a result the sign of the right hand side of Equation 4.35 is positive.

Using the coordinates

$$\begin{aligned}
x^0 &= L\sinh(\tau/L)\sin\psi \\
x^i &= L\omega^i\cosh(\tau/L)\sin\psi \qquad i = 1,\ldots,N-1 \\
x^N &= L\cos\psi
\end{aligned} \tag{4.40}$$

we have

$$\begin{aligned} dx^0 &= \cosh(\tau/L)\sin\psi d\tau + L\sinh(\tau/L)\cos\psi d\psi \\ dx^i &= L\cosh(\tau/L)\sin\psi d\omega^i + \omega^i\sinh(\tau/L)\sin\psi d\tau + L\omega^i\cosh(\tau/L)\cos\psi d\psi \\ dx^N &= -L\sin\psi d\psi. \end{aligned} \tag{4.41}$$

The metric reads

$$\begin{aligned} ds^2 =& -\cosh^2(\tau/L)\sin^2\psi d\tau^2 - L^2\sinh^2(\tau/L)\cos^2\psi d\psi^2 + L^2\cosh^2(\tau/L)\sin^2\psi d\Omega_{N-2}^2 \\ &+ \sinh^2(\tau/L)\sin^2\psi d\tau^2 + L^2\cosh^2(\tau/L)\cos^2\psi d\psi^2 + L^2\sin^2\psi d\psi^2 \\ ds^2 =& L^2 d\psi^2 + \sin^2\psi\left(-d\tau^2 + L^2\cosh^2(\tau/L)d\Omega_{N-2}^2\right) \end{aligned} \tag{4.42}$$

where constant ψ slices are $(N-1)$ dimensional global de Sitter spaces with the isometry group $SO(N-1,1)$ with $N(N-1)/2$ Killing vectors.

4.6. Static Coordinates

Now consider Equation 4.27 again and this time the x^i coordinates form the $N-1$ dimensional sphere of radius r, independent of time

$$-(x^0)^2 + (x^N)^2 = L^2 - r^2 \tag{4.43}$$

which is satisfied with the coordinates

$$\begin{aligned} x^0 &= L\sqrt{1-\frac{r^2}{L^2}}\sinh(t/L) \\ x^i &= r\omega^i \\ x^N &= L\sqrt{1-\frac{r^2}{L^2}}\cosh(t/L) \end{aligned} \tag{4.44}$$

where ω^i are the same as given in Equation 4.37. The first thing to notice is that for $r^2 > L^2$ the defining equation flips the sign and corresponds to another region, so

$r = L$ is not a boundary but a horizon. From $(x^N)^2 - (x^0)^2 = (x^N - x^0)(x^N + x^0) > 0$ we also have $x^N > x^0$ so these coordinates cover quarter of the hyperboloid. Taking the differentials

$$\begin{aligned} dx^0 &= \frac{r}{L}\left(1 - \frac{r^2}{L^2}\right)^{-1/2} \sinh(t/L)dr + \left(1 - \frac{r^2}{L^2}\right)^{1/2} \cosh(t/L)dt \\ dx^i &= \omega^i dr + r d\omega^i \\ dx^N &= \frac{r}{L}\left(1 - \frac{r^2}{L^2}\right)^{-1/2} \cosh(t/L)dr + \left(1 - \frac{r^2}{L^2}\right)^{1/2} \sinh(t/L)dt \end{aligned} \tag{4.45}$$

and the metric is then

$$\begin{aligned} ds^2 = &-\frac{r^2}{L^2}\left(1 - \frac{r^2}{L^2}\right)^{-1} \sinh^2(t/L)dr^2 - \left(1 - \frac{r^2}{L^2}\right)\cosh^2(t/L)dt^2 \\ &- \frac{2r}{L}\sinh(t/L)\cosh(t/L)drdt + dr^2 + r^2 d\Omega_{N-2}^2 \\ &+ \frac{r^2}{L^2}\left(1 - \frac{r^2}{L^2}\right)^{-1} \cosh^2(t/L)dr^2 + \left(1 - \frac{r^2}{L^2}\right)\sinh^2(t/L)dt^2 \\ &+ \frac{2r}{L}\sinh(t/L)\cosh(t/L)drdt \\ ds^2 = &-\left(1 - \frac{r^2}{L^2}\right)dt^2 + \left(1 - \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_{N-2}^2. \end{aligned} \tag{4.46}$$

For an observer sitting at $r = 0$ which corresponds to one of the poles of the global metric the static de Sitter has symmetries of S^{N-2}, the group $SO(N-1)$ and as the name implies has the time translation symmetry in the $0 \leq r < L$ region. This region is what we called the causal diamond before. It has an observer dependent horizon called the cosmological horizon at $r = L$ beyond which the metric changes sign and time translation symmetry no longer holds. In another words timelike Killing vector ∂_t is not global and becomes spacelike beyond the horizon.

4.6.1. Beyond the Horizon

Now consider the case where we pass the horizon, i.e. $r > L$. In this region the time coordinate t becomes spacelike denoted as R and the radial coordinate r becomes

timelike which is denoted as T here. That is

$$ds^2 = -\left(\frac{T^2}{L^2} - 1\right)^{-1} dT^2 + \left(\frac{T^2}{L^2} - 1\right) dR^2 + T^2 d\Omega_{N-2}^2. \tag{4.47}$$

Now let us define

$$d\tau = \left(\frac{T^2}{L^2} - 1\right)^{-1/2} dT \tag{4.48}$$

which is satisfied by

$$T = L\cosh(\tau/L). \tag{4.49}$$

We introduce the coordinates similar to those in Equation 4.44 as

$$\begin{aligned} x^0 &= L\sqrt{\frac{T^2}{L^2} - 1}\,\cosh(R/L) = L\sinh(\tau/L)\cosh\psi \\ x^i &= T\omega^i = L\cosh\psi\,\omega^i \\ x^N &= L\sqrt{\frac{T^2}{L^2} - 1}\,\sinh(R/L) = L\sinh(\tau/L)\sinh\psi \end{aligned} \tag{4.50}$$

and so

$$\begin{aligned} dx^0 &= \cosh(\tau/L)\cosh\psi d\tau + L\sinh(\tau/L)\sinh\psi \\ dx^i &= L\sinh\psi\omega^i d\psi + L\cosh\psi d\omega^i \\ dx^N &= \cosh(\tau/L)\sinh\psi d\tau + L\sinh(\tau/L)\cosh\psi d\psi. \end{aligned} \tag{4.51}$$

This gives us the metric

$$ds^2 = -d\tau^2 + L^2\sinh^2(\tau/L)d\psi^2 + L^2\cosh^2(\tau/L)d\Omega_{N-2}^2 \tag{4.52}$$

where spacelike-timelike exchange beyond the horizon is visible through the isometries as well. As discussed above there is no time translation symmetry beyond the horizon. As a FLRW cosmology this is not even isotropic since spacelike sections are either

$S^1 \times S^{N-2}$ or $\mathbb{R}^1 \times \mathbb{S}^{N-2}$ although spacetime has maximal symmetry. The trivially apparent symmetries are then $SO(2)$ with 1 Killing vector, and $SO(N-1)$ with $(N-1)(N-2)/2$ Killing vectors. Note that the rest of the Killing vectors which are not apparent ($2(N-1)$ of them) do not directly affect any features of the cosmology. Moreover in $N = 4$ dimensions at $\tau = 0$ spacelike sections are 2 dimensional and for $\tau > 0$ 3 dimensional. This metric strikingly demonstrates that the choice of the time variable in general relativity determines the physics.

4.6.2. Static de Sitter in Isotropic Coordinates

We will follow the same procedure as we did in the FLRW section. That is we will find the functions $f(\eta)$ and $g(\eta)$ satisfying

$$-\left(1-\frac{r^2}{L^2}\right)dt^2+\left(1-\frac{r^2}{L^2}\right)^{-1}dr^2+r^2d\Omega_{N-2}^2=-f^2(\eta)dt^2+g^2(\eta)\left(d\eta^2+\eta^2d\Omega_{N-2}^2\right). \tag{4.53}$$

Again by using $g(\eta) = r/\eta$ we solve the integral

$$\frac{1}{r}\left(1-\frac{r^2}{L^2}\right)^{-1/2}dr=\frac{d\eta}{\eta} \tag{4.54}$$

which is almost the same as in Equation 2.89 and the answer is with the substitution $k = 1$,

$$\frac{r}{L}=\frac{2\eta}{1+\eta^2}=g(\eta)\eta \tag{4.55}$$

we get

$$f^2(\eta)=\frac{\left(1-\eta^2\right)^2}{\left(1+\eta^2\right)^2}. \tag{4.56}$$

This gives the metric as

$$ds^2 = -\frac{\left(1-\frac{\eta^2}{4}\right)^2}{\left(1+\frac{\eta^2}{4}\right)^2}dt^2 + \frac{1}{\left(1+\frac{\eta^2}{4}\right)^2}\left(d\eta^2 + \eta^2 d\Omega_{N-2}^2\right) \tag{4.57}$$

where we took $\eta \to \eta/2$. Note that static de Sitter in isotropic form is horizon-free.

4.6.3. Eddington-Finkelstein Coordinates

These coordinates together with the Kruskal-Szekeres coordinates are used for Schwarzschild black hole metric [9]. They are adapted to radial null geodesics and describe for a central mass the outgoing or incoming light rays. Since we have the same form of the Schwarzschild metric in static de Sitter and we have a horizon at $r = L$ we can use these coordinates which will allow us together with the Kruskal-Szekeres coordinates to build the Penrose diagram for the static metric. Starting from Equation 4.46

$$ds^2 = \left(1-\frac{r^2}{L^2}\right)\left(-dt^2 + dr*^2\right) + r^2 d\Omega_{N-2}^2 \tag{4.58}$$

where $r*$ is the tortoise coordinate defined by

$$dr* = \left(1-\frac{r^2}{L^2}\right)^{-1} dr \tag{4.59}$$

and is solved by

$$\begin{aligned} r &= L\tanh x \\ dr &= L\,\mathrm{sech}^2\, x dx \\ dr* &= L\left(1-\tanh^2 x\right)^{-1}\mathrm{sech}^2\, x dx = L dx \\ r* &= Lx = L\tanh^{-1}(r/L) = \frac{L}{2}\ln\frac{1+r/L}{1-r/L}. \end{aligned} \tag{4.60}$$

Notice for $r*$ the horizon at $r = L$ is at infinity. Now we can introduce the light cone coordinates

$$\begin{aligned} u = t - r* \quad &\rightarrow \quad u = t - \frac{L}{2} \ln \frac{1 + r/L}{1 - r/L} \\ v = t + r* \quad &\rightarrow \quad v = t + \frac{L}{2} \ln \frac{1 + r/L}{1 - r/L} \end{aligned} \tag{4.61}$$

which are known as retarded and advanced coordinates. Here the horizon is at $u = -\infty$ or $v = \infty$. We obtain the metric for the advanced coordinate v as,

$$\begin{aligned} ds^2 &= \left(1 - \frac{r^2}{L^2}\right)\left(-dv^2 + 2dvdr*\right) + r^2 d\Omega_{N-2}^2 \\ &= -\left(1 - \frac{r^2}{L^2}\right) dv^2 + 2dvdr + r^2 d\Omega_{N-2}^2 \end{aligned} \tag{4.62}$$

and similarly for the retarded coordinate u,

$$ds^2 = -\left(1 - \frac{r^2}{L^2}\right) du^2 - 2dudr + r^2 d\Omega_{N-2}^2. \tag{4.63}$$

We can write t and $r*$ coordinates in terms of u and v as

$$t = \frac{v + u}{2} \qquad r* = \frac{v - u}{2} \tag{4.64}$$

and insert them into the static metric given in Equation 4.58 gives

$$\begin{aligned} ds^2 &= \left(1 - \tanh^2\left(\frac{r*}{L}\right)\right)\left(-dt^2 + dr*^2\right) + L^2 \tanh^2\left(\frac{r*}{L}\right) d\Omega_{N-2}^2 \\ &= \left(1 - \tanh^2\left(\frac{v-u}{2L}\right)\right)\left(-\frac{(dv+du)^2}{4} + \frac{(dv-du)^2}{4}\right) + L^2 \tanh^2\left(\frac{v-u}{2L}\right) d\Omega_{N-2} \\ ds^2 &= -\operatorname{sech}^2\left(\frac{v-u}{2L}\right) dvdu + L^2 \tanh^2\left(\frac{v-u}{2L}\right) d\Omega_{N-2}^2. \end{aligned} \tag{4.65}$$

4.6.4. Kruskal-Szekeres Coordinates

Let us first make some arrangements with the above parameterizations. Using Equation 4.60 we can write r coordinates in terms of u and v as

$$\frac{r}{L} = \frac{e^{r*/L} - e^{-r*/L}}{e^{r*/L} + e^{-r*/L}} = \frac{1 - e^{-2r*/L}}{1 + e^{-2r*/L}} = \frac{1 - e^{(u-v)/L}}{1 + e^{(u-v)/L}} \tag{4.66}$$

and

$$\begin{aligned} 1 - \frac{r^2}{L^2} = \operatorname{sech}^2\left(\frac{r*}{L}\right) = \frac{4}{\left(e^{r*/L} + e^{-r*/L}\right)^2} = \frac{4e^{-2r*/L}}{\left(1 + e^{-2r*/L}\right)^2} \\ 1 - \frac{r^2}{L^2} = \frac{4e^{(u-v)/L}}{(1 + e^{(u-v)/L})^2}. \end{aligned} \tag{4.67}$$

Now we can define new coordinates [19]

$$U = e^{u/L} \qquad V = -e^{-v/L} \tag{4.68}$$

which brings the horizon from $u = -\infty$, $v = \infty$ to $U = 0 = V$. These coordinates give us the above relations as

$$\frac{r}{L} = \frac{1 + UV}{1 - UV} \qquad \left(1 - \frac{r^2}{L^2}\right) = \frac{-4UV}{(1 - UV)^2} \tag{4.69}$$

and the differentials as

$$\begin{aligned} dU = \frac{1}{L} e^{u/L} du \qquad &\rightarrow \qquad du = Le^{-u/L} dU = L\frac{dU}{U} \\ dV = \frac{1}{L} e^{-v/L} dv \qquad &\rightarrow \qquad dv = Le^{v/L} dV = -L\frac{dV}{V}. \end{aligned} \tag{4.70}$$

The metric given in Equation 4.65 becomes

$$ds^2 = \frac{L^2}{(1 - UV)^2}\left(-4dUdV + (1 + UV)^2 d\Omega_{N-2}^2\right) \tag{4.71}$$

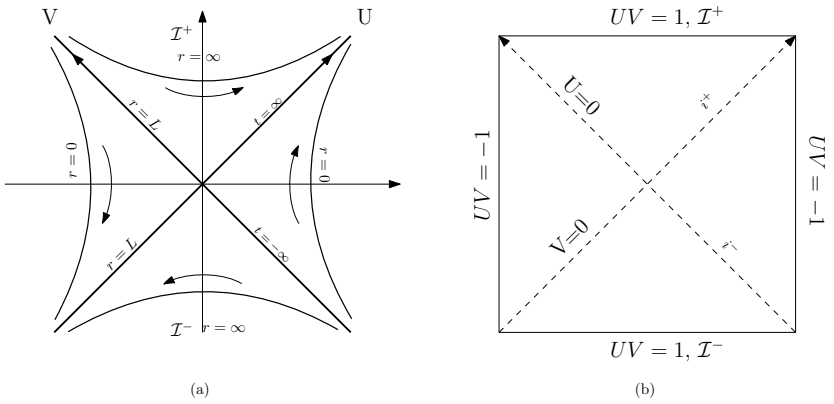

Figure 4.2. (a) Kruskal-Szekeres coordinates for de Sitter. (b) Penrose diagram

describes a spacetime where U and V axes are at 45°. We can see from the relation 4.69 that U-axis ($V = 0$) and V-axis ($U = 0$) correspond to the horizon at $r = L$. We get $r = 0$ when $UV = -1$ so they correspond to north and south poles of the global metric. $r \to \infty$ when $UV = 1$ corresponds to future and past null infinities $\mathcal{I}^+$ and $\mathcal{I}^-$. Constant r lines are represented as hyperbolas in the $U - V$ plane.

For the time coordinate using Equations 4.64 and 4.68 we obtain the relation

$$e^{2t/L} = -\frac{U}{V} \tag{4.72}$$

which shows that the past timelike infinity, i^- corresponds to $U = 0$ and the future timelike infinity, i^+ to $V = 0$. All these are shown in Figure 4.2 .

5. ANTI-de SITTER SPACETIME

In this chapter we will investigate the anti-de Sitter spacetime which is the unique vacuum solution of Einstein's equations with a negative cosmological constant Λ whose curvature scalar is $R = 4\Lambda$ with negative Λ, for $N = 4$. Although the observation of positive cosmological constant eliminates this spacetime as a cosmological model, AdS space has a big importance in theoretical physics thanks to Maldecena's famous work on AdS/CFT duality [20].

We start with the defining equation of N dimensional anti-de Sitter spacetime which is the hyperboloid

$$-(x^0)^2 + (x^i)^2 - (x^N)^2 = -L^2 \tag{5.1}$$

where $i = 1, \ldots, N-1$ and with two timelike coordinates. The length scale is $L^2 = (N-1)(N-2)/2\Lambda$. This equation is invariant under the group $SO(N-1,2)$. We can realize this hyperboloid by embedding it into $\mathbb{R}^{N-1,2}$ respecting the signature as

$$ds^2 = -(dx^0)^2 + \sum_{i=1}^{N-1}(dx^i)^2 - (dx^N)^2. \tag{5.2}$$

The isometry group of this metric is $SO(N-1,2)$ and its dimension is $N(N+1)/2$ which is the number of symmetries that a maximally symmetric space has. For a fixed point we have symmetries of the subgroup $SO(N-1,1)$. This allows us to write the N dimensional anti-de Sitter spacetime as a coset space, $SO(N-1,2)/SO(N-1,1)$.

We will give some but not all of the coordinates for anti-de Sitter spacetime that satisfies the defining equation given in Equation 5.1 and find the corresponding metrics via the embedding given in Equation 5.2.

5.1. Global Coordinates

The defining equation for fixed x^i is

$$(x^0)^2 + (x^N)^2 = L^2 + (x^i)^2 \tag{5.3}$$

which reveals the topology of anti-de Sitter spacetime as $S^1 \times \mathbb{R}^{N-1}$. These S^1 are the closed timelike curves as a result of two timelike coordinate passing through every point. The general solution to Equation 5.3 is given with

$$(x^0)^2 + (x^N)^2 = L^2 \cosh^2 \rho \qquad (x^i)^2 = L^2 \sinh^2 \rho \tag{5.4}$$

and is satisfied with the coordinates

$$\begin{aligned} x^0 &= L \cosh \rho \cos(\tau/L) \\ x^i &= L\omega^i \sinh \rho \quad i = 1, \ldots, N-1 \\ x^N &= L \cosh \rho \sin(\tau/L) \end{aligned} \tag{5.5}$$

where ω^i follow from Equation 4.5. Here two time coordinates forming S^1 are parameterized with an angle for which the multiples of $2\pi L$ are identified. We continue

$$\begin{aligned} dx^0 &= L \sinh \rho \cos(\tau/L) d\rho - \cosh \rho \sin(\tau/L) d\tau \\ dx^i &= L \sinh \rho d\omega^i + L\omega^i \cosh \rho d\rho \\ dx^N &= L \sinh \rho \sin(\tau/L) d\rho + \cosh \rho \cos(\tau/L) d\tau. \end{aligned} \tag{5.6}$$

Then the metric is

$$\begin{aligned} ds^2 &= -(dx^0)^2 + (dx^i)^2 - (dx^N)^2 \\ &= -\cosh^2 \rho d\tau^2 + L^2 \left(d\rho^2 + \sinh^2 \rho d\Omega_{N-2}^2 \right) \\ ds^2 &= -\cosh^2 \rho d\tau^2 + L^2 d\tilde{\Omega}_{N-1}^2 \end{aligned} \tag{5.7}$$

where $0 \leq \rho < \infty$. Spatial surfaces at constant τ are hyperbolic spaces H^{N-1} with negative curvature. So even though we made the embedding in $\mathbb{R}^{(N-1,2)}$ we ended up with only one time coordinate. Since the metric is independent of time, that is static, global anti-de Sitter has global timelike Killing vector ∂_τ unlike de Sitter. However it lacks the global spacelike Killing vector ∂_ρ.

Notice that when we look at the global metric given in Equation 5.7 the periodic nature of time coordinate is not evident. We can open up the timelike coordinate to extend $-\infty < \tau < \infty$ making infinite number of turns around the hyperboloid however this does not eliminate the identification. Instead we could consider the same metric but with the topology $\mathbb{R} \times \mathbb{R}^3$, as a new manifold without the periodic time [21]. This spacetime is known as the universal covering space of anti-de Sitter space.

An alternative parameterization which is similar to static de Sitter coordinates given in Equation 4.44 can be made if we consider

$$r = L \sinh \rho \tag{5.8}$$

and rewrite the coordinates from Equation 5.5 accordingly

$$\begin{aligned} x^0 &= L\sqrt{1 + \frac{r^2}{L^2}} \cos(\tau/L) \\ x^i &= r\omega^i \quad i = 1, \ldots, N-1 \\ x^N &= L\sqrt{1 + \frac{r^2}{L^2}} \sin(\tau/L) \end{aligned} \tag{5.9}$$

and so

$$\begin{aligned} dx^0 &= (r/L)(1 + r^2/L^2)^{-1/2} \cos(\tau/L) dr - (1 + r^2/L^2)^{1/2} \sin(\tau/L) d\tau \\ dx^i &= \omega^i dr + r d\omega^i \\ dx^N &= (r/L)(1 + r^2/L^2)^{-1/2} \sin(\tau/L) dr + (1 + r^2/L^2)^{1/2} \cos(\tau/L) d\tau. \end{aligned} \tag{5.10}$$

These coordinates give the metric

$$ds^2 = -\left(1 + \frac{r^2}{L^2}\right) d\tau^2 + \left(1 + \frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_{N-2}^2 \tag{5.11}$$

with $0 \leq r < \infty$ and $-\infty < \tau < \infty$ following the above discussion. This metric is global, static and unlike de Sitter, horizon-free. We can continue as what we did for the global de Sitter and obtain the conformal metric from Equation 5.7

$$ds^2 = \cosh^2 \rho \left(-d\tau^2 + L^2 \frac{d\rho^2}{\cosh^2 \rho} + L^2 \tanh^2 \rho d\Omega_{N-2}^2\right) \tag{5.12}$$

and define

$$d\eta = \frac{d\rho}{\cosh \rho} \tag{5.13}$$

which has the solution

$$\tan(\eta/2) = \tanh(\rho/2). \tag{5.14}$$

This gives

$$\cosh \rho = \frac{1}{\cos \eta} \tag{5.15}$$

with $0 \leq \eta < \pi/2$

$$\tanh \rho = \sin \eta. \tag{5.16}$$

The conformal metric is

$$ds^2 = \frac{1}{\cos^2 \eta} \left(-d\tau^2 + L^2 \left(d\eta^2 + \sin^2 \eta d\Omega_{N-2}^2\right)\right) = \frac{1}{\cos^2 \eta} \left(-d\tau^2 + L^2 d\Omega_{N-1}^2\right). \tag{5.17}$$

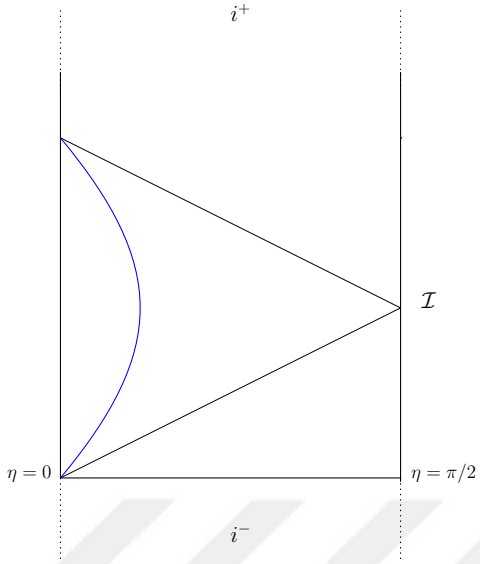

Figure 5.1. Penrose diagram of anti-de Sitter spacetime. The region inside the light Blue curve is a timelike geodesic.

So for fixed τ hypersurfaces are $N-1$ dimensional half-spheres with a spatial boundary at $\eta = \pi/2$ which corresponds to a boundary as $\rho, r \to \infty$ with a timelike character. These half-spheres are topologically equivalent to a cylinder for $N > 2$ that extends to infinity in time (in the universal covering space) which has S^{N-2} spheres of radius $L \sin \eta$ at each point.

Figure 5.1 shows the Penrose diagram of this conformal metric $d\tilde{s}^2 = \cos^2 \eta ds^2$ since it brings the infinity of the radial coordinate to a finite interval $0 \leq \eta < \pi/2$ with a boundary at $\pi/2$. Null geodesics make $\pm 45°$ to vertical. The past and future null infinities $\mathcal{I}^-, \mathcal{I}^+$ and the spatial infinity i^0 are at the timelike boundary $\eta = \pi/2$ and denoted as $\mathcal{I}$. At each point there is half-S^{N-2} with radius $L \sin \eta$ which are suppressed. For the time component we will consider an infinite strip for the universal covering space or we could use the closed timelike curves and identify the constant τ lines at 0 and 2π.

5.2. Isotropic Coordinates

We use the same procedure as we did before and start with the metric given in Equation 5.11 to find the functions $f(\psi)$ and $g(\psi)$

$$\begin{aligned} ds^2 &= -\left(1+\frac{r^2}{L^2}\right)d\tau^2 + \left(1+\frac{r^2}{L^2}\right)^{-1} dr^2 + r^2 d\Omega_{N-2}^2 \\ &= -f^2(\psi)d\tau^2 + g^2(\psi)\left(d\psi^2 + \psi^2 d\Omega_{N-2}^2\right). \end{aligned} \tag{5.18}$$

Comparing this with Equation 2.89 and setting $k = -1$ which describes the stereographic projection for H^{N-1} (remember it was $k = 1$ for de Sitter space) gives

$$g(\psi) = \frac{2L}{1-\psi^2} \qquad r = g(\psi)\psi \qquad f(\psi) = \frac{1+\psi^2}{1-\psi^2} \tag{5.19}$$

and so the anti-de Sitter space in isotropic coordinates is

$$ds^2 = -\frac{\left(1+\psi^2\right)^2}{\left(1-\psi^2\right)^2}d\tau^2 + \frac{4L^2}{\left(1-\psi^2\right)^2}\left(d\psi^2 + \psi^2 d\Omega_{N-2}^2\right) \tag{5.20}$$

where the range of ψ is $0 \leq \psi < 1$. This and the previous metrics we showed are all global where constant τ gives H^{N-1}.

5.3. In terms of FLRW

Let us consider the Friedmann equations given in Equation 2.133 for $N = 4$ with $T_{\mu\nu} = 0$ again. Since anti-de Sitter space is with the negative cosmological constant we set $L^{-2} = -|\Lambda|/3$ and $K = kL^{-2}$. Then from the first Friedmann equation we have

$$\dot{a}^2 + L^{-2}a^2 = -kL^{-2} \tag{5.21}$$

that gives solutions for each case of k as

$$\begin{aligned} k=0 \quad &\rightarrow \quad a(t)=e^{it/L} \\ k=1 \quad &\rightarrow \quad a(t)=i\sin(t/L) \\ k=-1 \quad &\rightarrow \quad a(t)=\sin(t/L) \end{aligned} \tag{5.22}$$

and the Hubble constant $H(t)=\dot{a}/a$ as

$$\begin{aligned} k=0 \quad &\rightarrow \quad H(t)=i/L \\ k=1 \quad &\rightarrow \quad H(t)=1/L\cot(t/L) \\ k=-1 \quad &\rightarrow \quad H(t)=1/L\cot(t/L). \end{aligned} \tag{5.23}$$

Notice that for $k=1$ the radial coordinate becomes a timelike coordinate and for $k=0$ the Hubble constant is imaginary. For $k=-1$ then we can write the metric whose spatial part is given with Equation 2.82 as

$$ds^2=-dt^2+L^2\sin^2(t/L)\left(d\chi^2+\sinh^2\chi d\Omega_2^2\right). \tag{5.24}$$

5.4. Hyperbolic Coordinates

Now consider the defining equation given in Equation 5.1 for fixed x^N which can be written as

$$-(x^0)^2+(x^i)^2=(x^N)^2-L^2. \tag{5.25}$$

If $|x^N|<L$ this equation describes the hyperboloid. Setting the parameters as

$$\begin{aligned} x^0 &= L\cosh\chi\sin(t/L) \\ x^i &= L\omega^i\sinh\chi\sin(t/L) \\ x^N &= L\cos(t/L) \end{aligned} \tag{5.26}$$

$$\begin{aligned} dx^0 &= L\sinh\chi\sin(t/L)d\chi + \cosh\chi\cos(t/L)dt \\ dx^i &= L\sinh\chi\sin(t/L)d\omega^i + L\omega^i\cosh\chi\sin(t/L)d\chi + \omega^i\sinh\chi\cos(t/L)dt \\ dx^N &= -\sin(t/L)dt \end{aligned} \tag{5.27}$$

gives us the metric

$$\begin{aligned} ds^2 &= -L^2\sinh^2\chi\sin^2(t/L)d\chi^2 - \cosh^2\chi\cos^2(t/L)dt^2 + L^2\sinh^2\chi\sin^2(t/L)d\Omega_{N-2}^2 \\ &\quad + L^2\cosh^2\chi\sin^2(t/L)d\chi^2 + \sinh^2\chi\cos^2(t/L)dt^2 - \sin^2(t/L)dt^2 \\ &= -dt^2 + L^2\sin^2(t/L)\left(d\chi^2 + \sinh^2\chi d\Omega_{N-2}^2\right) \\ ds^2 &= -dt^2 + L^2\sin^2(t/L)d\tilde{\Omega}_{N-1}^2 \end{aligned} \tag{5.28}$$

which is the FLRW solution. Note that we could choose $a(t) = \cos(t/L)$ as well and the coordinates accordingly [22].

5.5. dS in AdS

We continue from Equation 5.25 for the case $|x^N| > L$. We can parameterize this region as

$$\begin{aligned} x^0 &= L\sinh\chi\sinh(t/L) \\ x^i &= L\,\omega^i\sinh\chi\cosh(t/L) \\ x^N &= L\cosh\chi \end{aligned} \tag{5.29}$$

with

$$\begin{aligned} dx^0 &= L\cosh\chi\sinh(t/L)d\chi + \sinh\chi\cosh(t/L)dt \\ dx^i &= L\sinh\chi\cosh(t/L)d\omega^i + L\omega^i\cosh\chi\cosh(t/L)d\chi + \omega^i\sinh\chi\sinh(t/L)dt \\ dx^N &= L\sinh\chi d\chi. \end{aligned} \tag{5.30}$$

The metric is

$$\begin{aligned} ds^2 &= -L^2\cosh^2\chi\sinh^2(t/L)d\chi^2 - \sinh^2\chi\cosh^2(t/L)dt^2 \\ &\quad + L^2\sinh^2\chi\cosh^2(t/L)d\Omega_{N-2}^2 + L^2\cosh^2\chi\cosh^2(t/L)d\chi^2 \\ &\quad + \sinh^2\chi\sinh^2(t/L)dt^2 - L^2\sinh^2\chi d\chi^2 \\ &= L^2 d\chi^2 + \sinh\chi\left(-dt^2 + L^2\cosh^2(t/L)d\Omega_{N-2}^2\right). \end{aligned} \tag{5.31}$$

The metric inside the parenthesis is the $N-1$ dimensional global de Sitter space. So for constant χ we have $SO(N-1,1)$ symmetries.

5.6. Poincare Coordinates

Anti-de Sitter space in Poincare coordinates is very important because for N dimensional AdS space it gives a $N-1$ dimensional Minkowski space at the spatial boundary (or for any constant radial(like) coordinate). This is the metric used in AdS/CFT duality [20] because the isometry group of AdS_N, $SO(N-1,2)$ is also the Poincare group with conformal symmetries.

So we start with considering the coordinates given in [23] satisfying Equation 5.1,

$$\begin{aligned} x^0 &= L\frac{t}{w} \\ x^i &= L\frac{z^i}{w} \quad i = 1,\ldots,N-2 \\ x^+ &= \frac{1}{w}\left((z^i)^2 - t^2\right) + w \\ x^- &= L^2\frac{1}{w} \end{aligned} \tag{5.32}$$

where we use the light cone coordinates $x^\pm = x^N \pm x^{N-1}$. Here w is the generally used radial-like coordinate that divides the hyperboloid into two regions. From x^- we can see that $w > 0$ is the region $x^N > x^{N-1}$ and $w < 0$ is the region $x^N < x^{N-1}$. In this case the embedding metric becomes

$$ds^2 = -(dx^0)^2 + (dx^i)^2 + (dx^{N-1})^2 - (dx^N)^2 = -(dx^0)^2 + (dx^i)^2 + dx^-dx^+ \tag{5.33}$$

where

$$\begin{aligned} dx^0 &= L\frac{1}{w}dt - L\frac{t}{w^2}dw \\ dx^i &= L\frac{1}{w}dz^i - L\frac{z^i}{w^2}dw \\ dx^+ &= \left(-\frac{1}{w^2}\left((z^i)^2 - t^2\right) + 1\right)dw + \frac{2z^i}{w}dz^i - \frac{2t}{w}dt \\ dx^- &= -L^2\frac{1}{w^2}dw. \end{aligned} \tag{5.34}$$

This gives the metric as

$$\begin{aligned} ds^2 = &-L^2\frac{1}{w^2}dt^2 - L^2\frac{t^2}{w^2}dw^2 + 2L^2\frac{t}{w^3}dtdw + L^2\frac{1}{w^2}(dz^i)^2 + L^2\frac{(z^i)^2}{w^4}dw^2 \\ &- 2L^2\frac{z^i}{w^3}dz^i dw - L^2\frac{1}{w^2}\left(-\frac{1}{w^2}\left((z^i)^2 - t^2\right) + 1\right)dw^2 \\ &- 2L^2\frac{z^i}{w^3}dz^i dw + 2L^2\frac{t}{w^3}dtdw \\ ds^2 = &\, L^2\frac{1}{w^2}\left(-dt^2 + (dz^i)^2 + dw^2\right). \end{aligned} \tag{5.35}$$

So the metric in Poincare coordinates is conformally flat and $w = 0$ is the spatial boundary that cuts the hyperboloid into two regions and is not included in this parameterizations. For a fixed w we obtain the $N-1$ dimensional Minkowski metric.

An alternative form can be obtained with the definition $w = L^2/r$, $dw = (-w^2/L^2)dr$ gives

$$ds^2 = -\frac{r^2}{L^2}dt^2 + \frac{L^2}{r^2}dr^2 + \frac{r^2}{L^2}(dz^i)^2 \tag{5.36}$$

$w = 0$ boundary is $r \to \infty$ in this metric.

If we consider $w = Le^{u/L}$ and so $dw = e^{u/L}du$. The metric becomes

$$ds^2 = e^{-2u/L}\left(-dt^2 + (dz^i)^2\right) + du^2. \tag{5.37}$$

6. CONCLUSION

It is generally believed that the principle of general covariance indicates that physics is independent of the coordinates. Although this is true for spatial coordinates, coordinate transformations which involve both space and time need special care. Here we have indicated that a given spacetime manifold can lead to distinctly physically different cosmologies. The five distinct cosmological models of de Sitter spacetime three of them being given by FLRW spacetimes with $k = -1$ $a = \sinh(t/L)$, $k = 0$ $a = e^{(t/L)}$ and $k = -1$ $a = \cosh(t/L)$, the anisotropic model given in Equation 4.52 whose spacelike sections are $S^1 \times S^{N-2}$ and the static de Sitter metric which can perhaps be interpreted as a cosmology are physically distinct. These spacetimes are obtained by different coordinate systems on the same mathematical manifold $SO(N,1)/SO(N-1,1)$. Thus the same mathematical manifold is physically fertile and gives rise to five physically different cosmologies. The same can also be said for $R^{N-1,1}$ which is a flat pseudo-Riemannian N dimensional manifold of metric signature 2. This manifold gives rise to three different cosmologies described by the Minkowski spacetime, the Milne spacetime and the Rindler spacetime which have expanding spacelike sections.

If the formulation of general relativity with a cosmological constant is assumed to be fundamental then all maximally symmetric spacetimes correspond to an empty universe. If cosmological constant is assumed to be zero then the energy-momentum tensor of the FLRW de Sitter, anisotropic de Sitter and the static de Sitter spacetimes satisfy $p = -\rho$ for positive ρ. The static and FLRW anti-de Sitter spacetimes correspond to negative energy density and positive pressure. Minkowski, Milne and Rindler spacetimes are empty.

The fact that the time coordinate must be handled with care in general relativity is also indicated by the fact that in Minkowski space there are 10 symmetries of which 3 corresponds to rotations, 3 to Lorentz boosts and 4 to translations. In physics there are only conserved quantities obtained for a given physical system by choosing an action and using Noether's theorem to obtain energy conservation from

time translation symmetry, momentum conservation from space translation symmetry and angular momentum conservation from rotational symmetry. There is no conserved physical quantity corresponding to the Lorentz boosts. For a single free particle the conserved quantity obtained from invariance under Lorentz boosts maybe considered to be the initial position of a particle. However this is only for a single particle; for any interacting system of particles there is no such symmetry.

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APPENDIX A: PENROSE DIAGRAM

Penrose-Carter diagrams, or Penrose diagrams or as Penrose names "conformal diagrams" [24] are 2 dimensional pictures that helps us to get information about a spacetime described by a metric whose coordinates generally extend to infinity. We cannot obtain such diagrams simply by putting a finite boundary on the coordinates by hand because then we may not see the structure of geodesics and causal relations of observers clearly. Best option to avoid this is to write the metric in a way that null geodesics are straight lines at 45° then compactify the coordinates.

First we usually start with a coordinate transformation to write the spatial part of the metric in angular coordinates which are compact. From the non-compact time and radial coordinates our strategy is to form the so called light-cone coordinates as

$$u = t - r \qquad v = t + r \tag{A.1}$$

known as outgoing and incoming null coordinates which are still non-compact. We can compactify these coordinates by writing them as a continuous function like tan of some new coordinates. Then we can introduce some other coordinates where time and spatial parts separated. There is no unique way of doing this procedure but the general idea is to be able to write the metric as

$$\tilde{g}_{\mu\nu} = \Omega^{-2} g_{\mu\nu} \tag{A.2}$$

where Ω is the conformal factor and $\tilde{g}$ is the metric for the diagram. The key point is to keep track of the ranges of coordinates and the relation between them. When drawing the diagram we ignore the conformal factor since it only distorts the shape of geodesics but does not affect the null ones. On the diagram outgoing light lines reach to future null infinity $\mathcal{I}^+$ and incoming light rays originate from past null infinity $\mathcal{I}^-$. Time-like geodesics originate from past timelike infinity i^- and end at future timelike infinity i^-. Spacelike geodesics start and end at spacelike infinity i^0.