ISSN 1547-4771, Physics of Particles and Nuclei Letters, 2018, Vol. 15, No. 6, pp. 650–667. © Pleiades Publishing, Ltd., 2018.

PHYSICS OF SOLID STATE AND CONDENSED MATTER

Integrable Lattice Spin Models from Supersymmetric Dualities[1]

Ilmar Gahramanov[a, b, c, *] **and Shahriyar Jafarzade**[a, b, **]

[a] *Department of Physics, Mimar Sinan Fine Arts University, Istanbul, Bomonti 34380 Turkey*

[b] *Department of Mathematics, Khazar University, Baku, AZ1096 Azerbaijan*

[c] *Max Planck Institute for Gravitational Physics (Albert Einstein Institute), Potsdam, D-14476 Germany*

**e-mail: ilmar.gahramanov@msgsu.edu.tr*

***e-mail: shahriyar.jzade@gmail.com*

Received February 14, 2018

Abstract—Recently, there has been observed an interesting correspondence between supersymmetric quiver gauge theories with four supercharges and integrable lattice models of statistical mechanics such that the two-dimensional spin lattice is the quiver diagram, the partition function of the lattice model is the partition function of the gauge theory and the Yang–Baxter equation expresses the identity of partition functions for dual pairs. This correspondence is a powerful tool which enables us to generate new integrable models. The aim of the present paper is to give a short account on a progress in integrable lattice models which has been made due to the relationship with supersymmetric gauge theories and make clear notes on the special functions used by several authors.

Keywords: Yang–Baxter equation, star-triangle relation, integrable models, exactly solvable models, supersymmetry with four supercharge, supersymmetric duality, elliptic hypergeometric function, basic hypergeometric function, hyperbolichypergeometric function

DOI: 10.1134/S1547477118060079

CONTENTS

1. INTRODUCTION 650
2. A CRASH COURSE ON INTEGRABLE LATTICE MODELS 651
2.1. Lattice Models in Statistical Mechanics 651
2.2. Integrability of Models: Star-Triangle Relation 652
3. A CRASH COURSE ON SUPERSYMMETRIC DUALITIES AND EXACT RESULTS 653
3.1. Supersymmetric Duality 653
3.2. Quiver Gauge Theories 654
3.3. Partition Functions and Corresponding Solutions 654
4. INTEGRABILITY FROM DUALITY 655
4.1. $S^3/\mathbb{Z}_r \times S^1$ *Partition Function and Solution* 657
4.2. $S^3 \times S^1$ *Partition Function and Solution* 657
4.3. $S^2 \times S^1$ *Partition Function and Solution* 658
4.4. $S^2_b/\mathbb{Z}_r$ *Partition Function and Solution* 659
4.5. S^3_b *Partition Function and Solution* 659
4.6. S^2 *Partition Function and Solution* 660
4.7. $S^1 \times S^1$ *Partition Function and Solution* 661
5. SUMMARY AND DISCUSSION 661
APPENDIX 662
A. Notations 662
B. Theta Function 662
C. Elliptic Gamma Function 662
D. Hyperbolic Gamma Function and Its Extensions 663

1. INTRODUCTION

Integrability is a beautiful phenomenon which plays an important role in theoretical and mathematical physics. One of the key structural elements leading to quantum integrability is the Yang–Baxter equation [1–6]

$$\begin{aligned}&\mathbb{R}_{12}(u-v)\mathbb{R}_{13}(u)\mathbb{R}_{23}(v)\\&=\mathbb{R}_{23}(v)\mathbb{R}_{13}(u)\mathbb{R}_{12}(u-v),\end{aligned}\qquad(1.1)$$

[1] The article is published in the original.

where the operators $\mathbb{R}_{ik}(u)$ act in the tensor product of some three vector spaces $\mathbb{V}\otimes\mathbb{V}\otimes\mathbb{V}$ and depend on the spectral parameter u. The importance of the Yang–Baxter equation as a condition for integrability was noticed by Ludvig Faddeev who developed (with his Leningrad group [7–9]) a deep connection between integrability and other areas of mathematical physics. Nowadays the Yang-Baxter equation has a relation to quantum field theory, knot theory, string theory, statistical physics, conformal field theory etc.

In this work we consider quantum integrability of two-dimensional square lattice spin models of statistical mechanics with the pair interaction between neighboring spins. The most known example of such models is the two-dimensional Ising model [10] which was solved by Onsager [11]. Onsager also observed that the Boltzmann weights of the Ising model satisfies the star-triangle relation which is a special form [12] of the Yang–Baxter equation for integrable statistical models with spin variables living on sites of the lattice:

$$\sum_{\sigma_0}\mathcal{S}(\sigma_0)\mathcal{W}_{\eta-\alpha}(\sigma_i,\sigma_0)\mathcal{W}_{\eta-\beta}(\sigma_j,\sigma_0)\mathcal{W}_{\eta-\gamma}(\sigma_k,\sigma_0) = \mathcal{R}(\alpha,\beta,\gamma)\mathcal{W}_{\alpha}(\sigma_j,\sigma_k)\mathcal{W}_{\beta}(\sigma_i,\sigma_k)\mathcal{W}_{\gamma}(\sigma_j,\sigma_i), \quad (1.2)$$

where $\mathcal{W}$ and $\mathcal{S}$ stand for the Boltzmann weight functions of the model and $\eta=\alpha+\beta+\gamma$. The star-triangle relation appears as a condition for commuting transfer matrices what makes the model integrable [4, 13].

There have been many developments in the integrable lattice spin models since Onsager's solution. There are by now many solutions[2] of the star-triangle equation, i.e. Ising-like lattice models, most notable ones are the Fateev–Zamolodchikov model [15] (the case $N=2$ gives the Ising model), Kashiwara–Miwa model [16–18], chiral Potts model [19–21], Faddeev–Volkov model [22, 23], Bazhanov–Sergeev model [24] etc.

One of the most surprising developments in the field has appeared recently [25] coming from a different area of theoretical physics. It was observed a relationship between exact results in supersymmetric quiver gauge theories and exactly solvable two-dimensional lattice models in statistical mechanics [26–28]. In the gauge/YBE correspondence, as it is called[3], the integrability in statistical models is a direct consequence of supersymmetric duality. Roughly speaking, gauge/YBE correspondence relates the Yang-Baxter equation with the equality of partition functions for supersymmetric dual theories. This relationship has led to the construction of new exactly solvable models of statistical mechanics and we believe that much more are to be found.

[2] Classification of all solutions [6, 14] to the quantum Yang–Baxter equation still remainsan open problem.

[3] The name "gauge/YBE" was first used by Yamazaki in [26] probably in an analogue to the gauge/Bethe correspondence [39, 40].

In this work we try to present an elementary description of the gauge/YBE correspondence and to list solutions of the star-triangle relation found (or related) by this correspondence. Of course, it is impossible to give all details of this recent subject of research, therefore in some places these notes have a sketchy character. We hope to convince the reader, both with mainly integrability background and supersymmetry alike, that the subject has many interesting applications and new open problems.

The rest of the paper is organized in the following way:

• In Section 2 we review integrable lattice models and formulate the star-triangle relation (Yang–Baxter equation) for Ising-like lattice models of statistical mechanics.

• In Section 3 we give a very brief review of supersymmetric duality, exact results for partition functions and quiver notations.

• We present a survey of recent progress in integrable statistical models inspired by supersymmetric gauge theory computations in Section 4 and list all recently found solutions to the star-triangle relation.

• The paper concludes with comments on the recent status of the correspondence and briefly discusses some open problems in Section 5.

2. A CRASH COURSE ON INTEGRABLE LATTICE MODELS

The main players in the notes are solvable[4] lattice models and quiver gauge theories with four supercharges in two, three and four dimensions. In this section, we set up basic terminology about the exactly solvable lattice models of statistical mechanics. More details on the subject can be found in the book by Baxter [4] and in the review papers [12, 41–45]. The section will mainly follow the exposition in [33, 38, 41].

2.1. Lattice Models in Statistical Mechanics

An Ising-like model on a two-dimensional square lattice is defined as follows. At each site i there is a "spin" variable σ_i which takes some set of continuous or discrete values (or both as we will see later) in some range. Two adjacent spins i and j interact with an energy $\epsilon(\sigma_i,\sigma_j)$. The quantities of interest in statistical physics are statistical sums, such as the following partition function

$$Z=\sum_{\{\sigma\}}e^{-\frac{E(\sigma)}{k_BT}}, \quad (2.1)$$

[4] The terms "solvable" and "integrable" are the same in the context of this paper and will be used interchangeably.

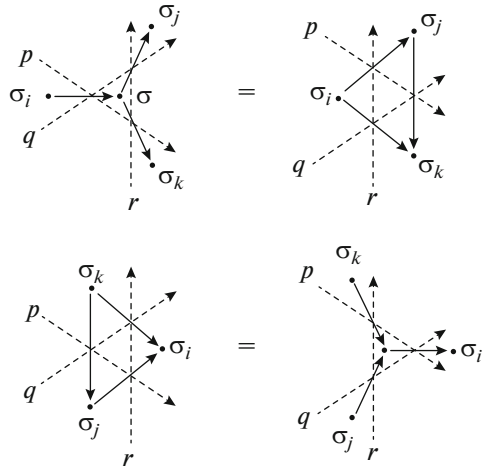


Fig. 1. Star-triangle relation.

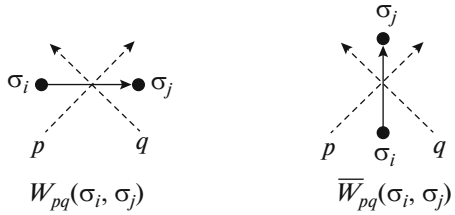


Fig. 2. Two types of Boltzmann weights.

where the summation runs over all values of spins; $E(\sigma), k_B$ and T are the energy of the system, Boltzmann constant and temperature respectively. Let

$$\mathcal{W}(\sigma_i,\sigma_j) = \exp\left(-\frac{\epsilon(\sigma_i,\sigma_j)}{k_B T}\right), \tag{2.2}$$

be the Boltzmann weight of the edge (i, j), with an interaction energy $\epsilon(\sigma_i,\sigma_j)$ between spins σ_i and σ_j. Then the partition function can be written in the following way

$$Z = \sum \prod_{\langle i,j\rangle} \mathcal{W}(\sigma_i,\sigma_j). \tag{2.3}$$

The "integrable model" means that one can evaluate the partition function (2.3) in the thermodinamic limit $N \to \infty$, where N is a number of sites of the lattice.

There exist two other types of models in two-dimensional statistical mechanics, the IRF model and the vertex model[5].

In the "interaction round a face model" (IRF) version of spin models four spins round a face of the lattice interact with each other. This interaction can be determined by the energy of face $\varepsilon(\sigma_1,\sigma_2,\sigma_3,\sigma_4)$ depending on four spins. The notable examples of IRF models are the hard hexagon model [50], the cyclic solid-on-solid model [51–55] and the restricted solid-on-solid model [56].

In the vertex model, spin variables are located on the edges of the spin lattice. In this case one associates the local Boltzmann weight with each vertex configuration, namely statistical weights depend on four spins surrounding each site. The most known examples of vertex models are the six-vertex model [57, 58], the eight-vertex model [3] and the nineteen-vertex model [59].

2.2. Integrability of Models: Star-Triangle Relation

The search for the Ising-like integrable models can be reduced to the problem of finding the Boltzmann weights that satisfy the so-called star-triangle relation[6]

$$\begin{aligned}
&\sum_{\sigma} \mathcal{S}(\sigma)\bar{\mathcal{W}}_{qr}(\sigma,\sigma_j)\mathcal{W}_{pr}(\sigma,\sigma_k)\bar{\mathcal{W}}_{pq}(\sigma_i,\sigma)\\
&= \mathcal{R}(p,q,r)\mathcal{W}_{pq}(\sigma_j,\sigma_k)\bar{\mathcal{W}}_{pr}(\sigma_i,\sigma_j)\mathcal{W}_{qr}(\sigma_i,\sigma_k),\\
&\sum_{\sigma} \mathcal{S}(\sigma)\bar{\mathcal{W}}_{pq}(\sigma,\sigma_i)\mathcal{W}_{pr}(\sigma_k,\sigma)\bar{\mathcal{W}}_{qr}(\sigma_j,\sigma)\\
&= \mathcal{R}(p,q,r)\mathcal{W}_{pq}(\sigma_k,\sigma_j)\bar{\mathcal{W}}_{pr}(\sigma_j,\sigma_i)\mathcal{W}_{qr}(\sigma_k,\sigma_i),
\end{aligned} \tag{2.4}$$

Graphical form of the star-triangle relation (2.4) where

• Summation is over all spin variables, $\mathcal{W}_{pq}(\sigma_i,\sigma_j)$ and $\bar{\mathcal{W}}_{pq}(\sigma_i,\sigma_j)$ are two different kinds of the Boltzmann weights describe the interaction between two spins;

• $\mathcal{S}(\sigma)$ is the rapidity-independent single-spin Boltzmann weight assigned for each spin σ on the lattice;

• $\mathcal{R}(p,q,r)$ is some factor depending on three rapidity variables and independent of the spins.

The star-triangle relation is a sufficient condition for the existence of an infinite set of commuting transfer matrices[7] and thereby the model can be exactly solved using the transfer matrix method [61]. Namely by defining the transfer matrices

$$\begin{aligned}
&(T_q)_{\sigma,\bar{\sigma}} = \prod_{i=1}^{L} \mathcal{W}_{pq}(\sigma_i,\bar{\sigma}_i)\bar{\mathcal{W}}_{pq}(\sigma_{i+1},\bar{\sigma}_i),\\
&\text{and}\quad (\bar{T}_r)_{\sigma,\bar{\sigma}} = \prod_{i=1}^{L} \bar{\mathcal{W}}_{pr}(\sigma_i,\bar{\sigma}_i)\mathcal{W}_{pr}(\sigma_i,\bar{\sigma}_{i+1}),
\end{aligned} \tag{2.5}$$

[5] As was mentioned above we will not be interested in the solutions of IRF and vertex-type Yang–Baxter equation and talk about them here only for completeness. We would like to point out that for some models in our list discussed in the next sections the IRF and vertex-type solutions are known. We will not discuss those solution and refer to the original papers [29, 30, 46–49].

[6] Note that in the literature "the star-triangle relation" often is used also for theIRF-type models, see, e.g. [60].

[7] Here we do not discuss a transfer matrix method and relation of the star-triangle equation to commutativity of transfer matrices, the interested reader can find details in many places, for instance, in [4, 44].

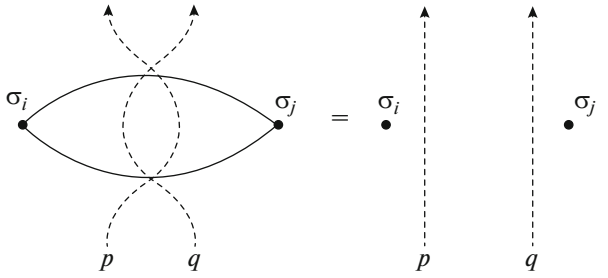


Fig. 3. Inversion relation (2.9).

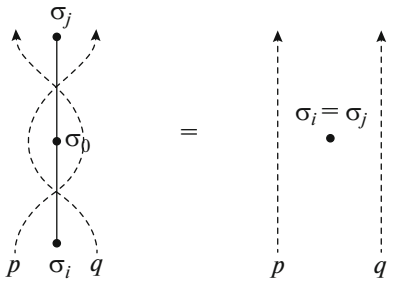


Fig. 4. Inversion relation (2.10).

with periodic boundary conditions that $\sigma_{L+1} = \sigma_1$ and $\bar{\sigma}_{L+1} = \bar{\sigma}_1$ one can prove that $T_q\bar{T}_r = T_r\bar{T}_q$. If one has such a family of commuting transfer matrices then a partition function (2.3) can be calculated exactly.

In all our examples here, the two types of Boltzmann weights $\mathcal{W}_{pq}$, $\bar{\mathcal{W}}_{pq}$, depend on rapidity variables only via their difference[8] $p - q$. Consequently the Boltzmann weights will be written in terms of the spectral variable $\alpha = p - q$, as

$$\begin{aligned} &\mathcal{W}_\alpha(\sigma_i,\sigma_j) := W_{pq}(\sigma_i,\sigma_j), \\ \text{and}\quad &\bar{\mathcal{W}}_\alpha(\sigma_i,\sigma_j) := \bar{\mathcal{W}}_{pq}(\sigma_i,\sigma_j). \end{aligned} \tag{2.6}$$

The two Boltzmann weights are also related by the crossing symmetry

$$\bar{\mathcal{W}}_\alpha(\sigma_i,\sigma_j) = \mathcal{W}_{\eta-\alpha}(\sigma_i,\sigma_j), \tag{2.7}$$

where $\eta > 0$ is a real valued, model dependent "crossing parameter". Thus all two-spin interactions in the lattice model may be described in terms of the single Boltzmann weight $\mathcal{W}_\alpha(\sigma_i,\sigma_j)$.

The Boltzmann weights considered here are spin reflection symmetric, i.e. unchanged by interchanging the spin variables σ_i and σ_j:

$$\mathcal{W}_\alpha(\sigma_i,\sigma_j) = \mathcal{W}_\alpha(\sigma_j,\sigma_i). \tag{2.8}$$

Considering all above facts for Botzmann weights pair of equations (2.4) are equivalent to (1.2).

The simple consequence of the star-triangle relation and initial condition gives the unitarity and inversion relations[9]

$$\mathcal{W}_\alpha(\sigma_i,\sigma_j)\mathcal{W}_{-\alpha}(\sigma_i,\sigma_j) = 1, \tag{2.9}$$

$$\begin{aligned} &\sum_{\sigma_0}\int \mathcal{S}(\sigma_0)\mathcal{W}_{\eta-\alpha}(\sigma_i,\sigma_0)\mathcal{W}_{\eta+\alpha}(\sigma_0,\sigma_j) \\ &= \frac{1}{\mathcal{S}(\sigma_i)}(\delta(x_i + x_j)\delta_{m_i,-m_j} + \delta(x_i - x_j)\delta_{m_i,m_j}). \end{aligned} \tag{2.10}$$

[8] The majority of lattice models of statistical mechanics satisfy this property, the most notable exception being the Chiral Potts model [21].

[9] Note that the inversion relation for the partition function may exist even for a model which is not integrable.

3. A CRASH COURSE ON SUPERSYMMETRIC DUALITIES AND EXACT RESULTS

Obviously, it is impossible to review supersymmetric dualities and exact results in supersymmetric gauge theories in a few pages. Our intention is to give some short description of the important keywords on the subject.

3.1. Supersymmetric Duality

In this section we very briefly remind some facts about supersymmetric duality. For more details, see e.g. [62–65].

About two decades ago Seiberg [66] and many others found a non-trivial quantum equivalence between different supersymmetric theories, called supersymmetric duality. To be more precise it was shown that two or more different theories may describe the same physics in the far infrared limit, i.e. an observer testing the low energy physics (or physics at long distances) cannot distinguish the dual theories[10].

The supersymmetric (Seiberg) duality was first constructed for four-dimensional $\mathcal{N} = 1$ gauge theory with a matter in the fundamental representation. Later many examples of dualities have been found with complicated matter content, different gauge and flavor groups in different dimensions. Today, supersymmetric duality has become a key tool for studying strongly coupled effects.

The basic example of supersymmetric duality [66] is an $SU(N_c)$ electric gauge theory with N_F flavors of quarks (Q, Q') which possesses a dual description in terms of N_f magnetic flavors of quarks (q, q') charged under $SU(N_F - N_c)$ gauge group[11] in the so-called conformal window $\frac{3}{2}N_c < N_F < 3N_c$. The field content of dual theories is summarized in the table below.

[10] It is worth mentioning that supersymmetric dual theories are not identical, but they give rise to the same physics at long distances.

[11] In this case the gauge singlets of the dual theory interact with the flavors via the superpotential term.

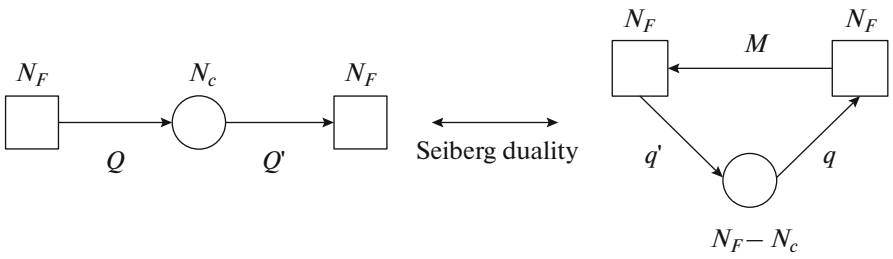


Fig. 5. Quiver diagram for the Seiberg duality.

These two theories flow to the same infrared fixed point.

	$SU(N_c)$	$SU(N_F)_L$	$SU(N_F)_R$
Q	f	f	1
Q'	$\bar{f}$	1	$\bar{f}$

Matter content of the electric theory

	$SU(N_F - N_c)$	$SU(N_F)_L$	$SU(N_F)_R$
q	f	$\bar{f}$	1
q'	$\bar{f}$	1	f
M	1	f	$\bar{f}$

Matter content of the magnetic theory; $(\bar{f})f$ denotes (anti) fundamental representation, and M represents the meson

The main point for us about the supersymmetric dualities is that the partition functions of dual theories are expected to be equal. In the context of gauge/YBE correspondence integrability on the statistical models' side is equivalent to the equality of partition functions of supersymmetric dual theories. It means that one may generate solutions to the Yang–Baxter equation by considering partition functions of suitable duality.

3.2. Quiver Gauge Theories

Here we briefly outline quiver notation of gauge theories which is a very useful tool for summarizing the group-theoretical data about a gauge theory in a compact way. Quiver gauge theories have been studied in physics more than forty years, initially, they were used in composite model building in the context of the Standard Model. For more details, see e.g. [67–69].

Supersymmetric gauge theories considered in the work are specified with gauge group G (in our examples we consider just $SU(2)$ group) and the matter fields transforming as chiral multiplets in a suitable representation. One can encode this information[12] in quiver diagrams using nodes for the gauge groups and edges for the matter multiplets.

[12]Note that the superpotential term of the theory is not encoded by the quiver diagram.

Consider a theory with the gauge group G as a direct product of simple groups G_i

$$G = G_1 \times G_2 \times \ldots \times G_n. \tag{3.1}$$

In the quiver diagram

• each node g_i corresponds to a vector multiplet in the adjoint representation of a gauge group G_i;

• each edge corresponds to the matter multiplet in the bifundamental representation. In general, one uses arrows between nodes. The arrow going from g_i to g_j corresponds to a chiral multiplet in the fundamental representation of g_i and the anti-fundamental representation of g_j.

The quiver diagram encoding the $SU(N_c)$ Seiberg duality from the previous section is described in Fig. 5.

In this work we deal with supersymmetric (with four supercharges) dualities of quiver gauge theories built from bifundamental matter, namely the matter content of gauge theories are represented as bifundamentals between gauge groups.

3.3. Partition Functions and Corresponding Solutions

In this section we collect the expressions for the matrix models associated to the lens index, supersymmetric index and squashed sphere partition functions in four, three and two dimensions. The localization technique enables us to calculate the partitionfunction of supersymmetric gauge theories with four supercharges on different manifolds exactly

$$Z = \frac{1}{|W|}\sum \int \frac{dz_i}{2\pi i z_i} \times \prod_{i=1}^{\text{rank}G} Z_{\text{gauge}}(z_i; m_i) \prod_{\Phi} Z_{\Phi}(z_i, t_a, ; m_i). \tag{3.2}$$

In the examples of the next sections we only discuss theories without the Chern–Simons and Fayet–Iliopoulos terms therefore the classical terms is absent in our expressions.

• The integral is performed over the Cartan subgroup of the gauge group. It is parameterized by the diagonal entries of the scalar z in the gauge group.

Table 1. Contributions of vector and chiral supermultiplets to the supersymmetric partition functions

Manifold	Vector Multiplet	Chiral Multiplet
$S^3/\mathbb{Z}_r \times S^1$	$\prod_\alpha \left(\Gamma_e(\alpha(z)^{\pm}; \pm\rho_j(m); p, q)\right)^{-1}$	$\prod_j \Gamma_e((pq)^{\frac{\Delta_j}{2}+\rho_j(m)+\phi_j(n)} \rho_j(z)\phi_j(a); p, q)$
$S^3 \times S^1$	$\prod_\alpha \left(\Gamma\left(\alpha(z)^{\pm}; p, q\right)\right)^{-1}$	$\prod_j \Gamma((pq)^{\frac{\Delta_j}{2}} \rho_j(z)\phi_j(a); p, q)$
$S^2 \times S^1$	$\prod_\alpha q^{-\frac{1}{2}\lvert\alpha(m)\rvert}\left(1-\alpha(z)^{\pm}q^{\frac{\lvert\alpha(m)\rvert}{2}}\right)$	$\prod_j \frac{(q^{1-\frac{\Delta_j}{2}+\frac{\lvert\rho_j(m)+\phi_j(n)\rvert}{2}} \rho_j(z)^{-1}\phi_j(a)^{-1}; q)_\infty}{(q^{\frac{\Delta_j}{2}+\frac{\lvert\rho_j(m)+\phi_j(n)\rvert}{2}} \rho_j(z)\phi_j(a); q)_\infty}$
$S^3_b\mathbb{Z}_r$	$\prod_\alpha \left(\hat{s}_{b,\alpha(m)}\left(i\frac{Q}{2} \pm \alpha(z)\right)\right)^{-1}$	$\prod_j \hat{s}_{b,-\rho_j(m)-\phi_j(n)}\left(i\frac{Q}{2}(1-\Delta_j) - \rho_j(z) - \phi_j(a)\right)$
S^3_b	$\prod_\alpha \left(\gamma^{(2)}\left(\alpha(z)^{\pm}; \omega_1, \omega_2\right)\right)^{-1}$	$\prod_j \gamma^{(2)}\left(\frac{(\omega_1+\omega_2)}{2}\Delta_j + \rho_j(z) + \phi_j(a); \omega_1, \omega_2\right)$
S^2	$e^{2\pi i\delta(m)}\prod_\alpha\left(\frac{\alpha(m)^2}{4} + \alpha(z)^2\right)$	$\prod_j \frac{\Gamma\left(\frac{\Delta_j}{2} - i\rho_j(z) - i\phi_j(a) - \frac{\rho_j(m)+\phi_j(n)}{2}\right)}{\Gamma\left(1-\frac{\Delta_j}{2} + i\rho_j(z) + i\phi_j(a) + \frac{\rho_j(m)+\phi_j(n)}{2}\right)}$
$S^1 \times S^1$	$\prod_\alpha \left(\Delta\left(\alpha^{\pm}(z); q, t\right)\right)^{-1}$	$\prod_j \Delta(t^{\frac{\Delta_j}{2}} \rho_j(z)\phi_j(a); q, t)$

• The factor of inverse $|W|$ represents the order of the Weyl group of the gauge group.

• The contribution of the vector multiplet is parameterized by the positive roots of the algebra. Actually, the Vandermonde determinant in the measure exactly cancels the part of the one-loop determinant of the vector multiplet.

• The jth chiral multiplet has an R charge Δ_j. Each chiral multiplet is in the corresponding representation of the gauge group G with weight $\rho_j(z)$ and in the corresponding representation of the flavor group F, with weight $\phi_j(a)$.

The observation of the correspondence between supersymmetric theories and solvable lattice models is based on the fact that in both fields appears same special functions of hypergeometric type [25]. For instance, the partition functions of supersymmetric theories on different manifolds can be expressed in terms of the following hypergeometric functions (for details see, e.g. [70–74])

• $S^3 \times S^1$, $S^3/\mathbb{Z}_r \times S^1$: elliptic hypergeometric integral

• $S^2 \times S^1$: basic hypergeometric integral

• S^3_b, $S^3_b/\mathbb{Z}_r$: hyperbolic hypergeometric integral

• S^2, $S^1 \times S^1$: ordinary hypergeometric integral

Similar integral identities appear in solvable lattice models (see, e.g. [24, 30, 75, 76]).

4. INTEGRABILITY FROM DUALITY

In this section, we summarize the present status of the gauge/YBE correspondence and list solutions to the star-triangle relation found (or related) via this correspondence.

A central phenomenon in the construction of integrable lattice models via the gauge/YBE correspondence is the existence of the corresponding Seiberg duality. Here we consider the special Seiberg duality for supersymmetric theories with four supercharges in different dimensions. All known solutions to the star-triangle relation found via the gauge/YBE correspondence results from the following duality [66]

• **Theory A**: $SU(2)$ gauge group with $N_f = 6$ flavors, chiral multiplets in the fundamental representation of the flavor group $SU(6)$ and in the fundamental representation of the gauge group.

• **Theory B**: without gauge degrees of freedom and the chiral fields (gauge-invariant "mesons") in the 15-dimensional totally antisymmetric tensor representation of the flavor group.

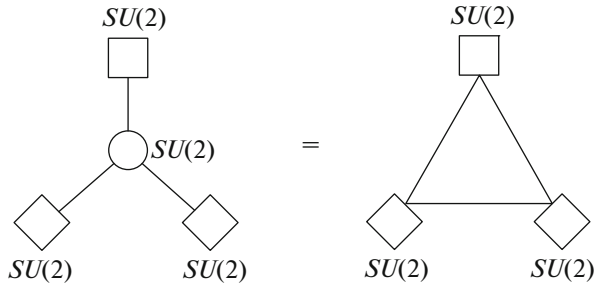


Fig. 6. Supersymmetric duality.

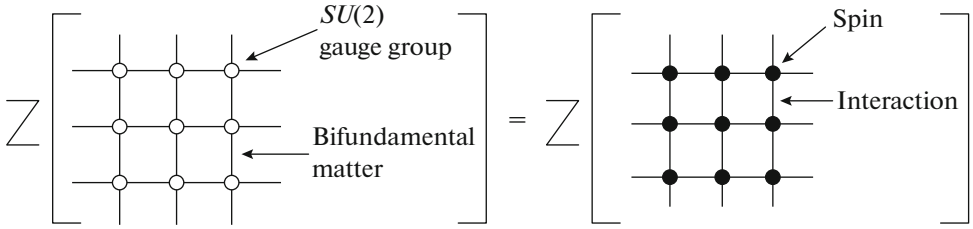


Fig. 7. Equivalence of the partition functions in the context of gauge/YBE correspondence: The left-hand side is the partition function of the supersymmetric quiver gauge theory, and the right side is the partition function of the integrable lattice model.

Note that we consider only the field content of the dual theories. Of course, this duality has different features in different dimensions, but such details are not crucial for our discussions. For instance, in order to get the right duality, one needs to specify the exact form of the superpotential [77, 78].

It turns out that identity of partition functions of dual theories can be written[13] in the form of the star-triangle relation

$$\sum_{\sigma_0}\int \mathcal{S}(\sigma_0)\mathcal{W}_{\eta-\alpha}(\sigma_i,\sigma_0)\mathcal{W}_{\eta-\beta}(\sigma_j,\sigma_0)\mathcal{W}_{\eta-\gamma}(\sigma_k,\sigma_0) = \mathcal{R}(\alpha,\beta,\gamma)\mathcal{W}_{\alpha}(\sigma_j,\sigma_k)\mathcal{W}_{\beta}(\sigma_i,\sigma_k\mathcal{W}_{\gamma}(\sigma_j,\sigma_i). \quad (4.1)$$

where $\eta = \alpha + \beta + \gamma$. In all models we discuss here the scalar factor R can be absorbed into the Boltzmann weights via the function $k(\alpha)$. This function plays important role when we evaluate partition function per site in thermodynamiclimit [61, 79]. In Fig. 7. we described the quiver diagram of the above duality which gives the star-triangle relation, $SU(2)$ gauge group lives on each node presented by circles and bifundamental matter on each edge, the boxes represent the flavor groups $SU(2)$.

It is not clear in gauge/YBE correspondence why the above duality is special, but it is the duality for which the correspondence takes place. We think that one can find solutions to the star-triangle relation only via this duality, all other supersymmetric dualities may give a solution to the star-star relation.

In the context of gauge/YBE correspondence the spin lattice models can be identified with the quiver gauge theory with SU(1.2) gauge groups on the sites of the lattice. Then the partition function of the corresponding integrable model is equivalent to the supersymmetric partition function of the corresponding supersymmetric quiver gauge theory. The contribution of chiral and vector multiplets to the supersymmetric partition function correspond to the nearest-neighbor Boltzmann weights and the self-interaction, respectively. In Fig. 7 we describe the correspondence of partition functions pictorially.

We would like to mention that the inversion relation (2.10) has an interesting counterpart on supersymmetry side of the gauge/YBE correspondence, namely, it is related to the chiral symmetry breaking of the corresponding supersymmetric gauge theory. Such relation can be derived in many different ways, from supersymmetric gauge theory side one can obtain inversion relation by an accurate limit of parameters in the partition functions of dual theories [80].

The Seiberg duality can be realized in the context of the brane language [81–83], namely one can obtain the duality by exchanging NS5-branes. Such construction gives an opportunity to obtain integrable lattice models directly using brane construction. We will not discuss this direction in the paper and refer the interested reader to the papers [26, 28, 35].

We finish this section by remarking that one can also construct an IRF-type integrable multispin

[13] One needs to break the flavor symmetry down to $SU(2)\times SU(2)\times SU(2)$.

model. For that, one needs to take $SU(N)$ gauge theory with $SU(N)\times SU(N)$ flavor symmetry. Then the identity of partition functions for dual theories can be written as the star-star relation for the IRF-type model. Unfortunately, it is quite complicated to prove analytically such integral identities [29, 37, 84–87].

Below we list all solutions to the star-triangle relation found (or related) by gauge/YBE correspondence, but we do not treat them in detail.

4.1. $S^3/\mathbb{Z}_r\times S^1$ Partition Function and Solution

The solution to the star-triangle relation was found[14] by Kels in [31] on the bases of the special case of the star-star relation found by Yamazaki[15] in [27].

The idea is that the sum-integral identity for the four-dimensional lens supersymmetric indices [88] of supersymmetric dual theories can be written as the star-triangle relation

$$\begin{aligned}&\frac{(p^r,p^r)_\infty(q^r,q^r)_\infty}{2}\\ &\times\sum_{m_0=0}^{r-1}\int_0^1 dz\,\frac{\prod_{i=1}^6\Gamma_e((a_i\pm z),(m_i\pm m_0);\sigma,\tau)}{\Gamma_e(\pm 2z,\pm 2m_0);\sigma,\tau}\\ &=\prod_{1\le i<j\le 6}\Gamma_e((a_i+a_j),(m_i+m_j);\sigma,\tau),\end{aligned}\tag{4.2}$$

with the balancing conditions $\sum_{i=1}^6 a_i=\sigma+\tau$ and $\sum_{i=1}^6 m_i=0$. The definition of the lens gamma function is given in Appendix C.

The lattice model has two spin variables on each site, discrete and continuous spin

$$\begin{aligned}&\sigma_j=(x_j,m_j),\quad\text{where}\quad 0\le x_j<2\pi,\\ &m_j=0,1,2,\ldots,\lfloor r/2\rfloor,\quad r\in\mathbb{Z}_{>0},\end{aligned}\tag{4.3}$$

where $\lfloor\ \rfloor$ stands for the floor function. The Boltzmann weight and self-interaction term of the model are [31]

$$\begin{aligned}&\mathcal{W}_\alpha(\sigma_i,\sigma_j)=\frac{e^{-2\alpha(|m_i-m_j|_\pm+|m_i+m_j|_\pm)/r}}{k(\alpha)}\\ &\times\frac{\Phi_{r,m_i-m_j}(x_i-x_j+i\alpha)\Phi_{r,m_i+m_j}(x_i+x_j+i\alpha)}{\Phi_{r,m_i-m_j}(x_i-x_j-i\alpha)\Phi_{r,m_i+m_j}(x_i+x_j-i\alpha)},\end{aligned}\tag{4.4}$$

$$\begin{aligned}&\mathcal{S}(\sigma_0)=\frac{\varepsilon_0}{\pi}(p^{2r};p^{2r})_\infty(q^{2r};q^{2r})_\infty e^{2\eta|2m_0|_\pm/r}\\ &\times\Phi_{r,-2m_0}(-2x_0-i\eta)\Phi_{r,2m_0}(2x_0-i\eta),\end{aligned}\tag{4.5}$$

where $k(\alpha)$ represents the partition function per edge and is defined as follows:

$$k(\alpha)=\exp\left(\sum_{n\ne 0}\frac{e^{4\alpha n}((pq)^{rn}-(pq)^{-rn})}{n((pq)^{2n}-(pq)^{-2n})(p^{rn}-p^{-rn})(q^{rn}-q^{-rn})}\right),\tag{4.6}$$

and

$$\varepsilon_0=\begin{cases}\dfrac{1}{2}, & \text{if}\quad m_0=0\quad\text{or}\quad [r-m_0]_r\\ 1, & \text{otherwise.}\end{cases}\tag{4.7}$$

Here we use the notation of [31] for the lens elliptic gamma function which is defined in the following way

$$\begin{aligned}\Phi_{r,m}(z)=&\prod_{j,k=0}^{\infty}\frac{1-e^{2iz}p^{-2[m]_r}(pq)^{2j+1}p^{2kr+2r}}{1-e^{-2iz}p^{2[m]_r}(pq)^{2j+1}p^{2kr}}\\ &\times\frac{1-e^{2iz}q^{-2[m]_r}(pq)^{2j+1}q^{2kr}}{1-e^{-2iz}q^{-2[m]_r}(pq)^{2j+1}q^{2kr+2r}},\end{aligned}\tag{4.8}$$

where $[m]_r\in\{0,1,2\ldots,r-1\}$ denotes m modulus r and $[m]_\pm:=[m]_r[-m]_r$.

We should mention that the solution in terms of lens gamma functions is the top level known solution to the star-triangle relation at this time which gives almost all known other models in the limiting case.

4.2. $S^3\times S^1$ Partition Function and Solution

The identity for the superconformal indices [89, 90] of the dual theories is the following elliptic beta integral [91]

$$\begin{aligned}&\frac{(q;q)_\infty(p;p)_\infty}{2}\oint\frac{dz}{2\pi iz}\frac{\prod_{i=1}^6\Gamma(a_iz^\pm;p,q)}{\Gamma(z^{\pm 2};p,q)}\\ &=\prod_{1\le i<j\le 6}\Gamma(a_ia_j;p,q),\end{aligned}\tag{4.9}$$

with the balancing condition $\prod_{i=1}^6 a_i=pq$. This identity was introduced and proven by Spiridonov in [92]. Later Bazhanov and Sergeev interpreted this identity as the star-triangle relation and introduced a new integrable spin lattice model [24]. Star-triangle relation in operator form appeared in [93] while proving of the

[14] Kels also gives an analytic proof of the integral identity in [31].

[15] Actually, Yamazaki constructed [29] the star-star relation for the dual theories with SU(N) gauge group and SU(N) SU(N) avor group.

integral Bailey lemma. In the corresponding integrable lattice model spin variables get continuous values

$$0 \le \sigma_i < 2\pi, \tag{4.10}$$

and the Boltzmann weights are expressed in terms of elliptic gamma functions

$$\mathcal{W}_\alpha(\sigma_i,\sigma_j) = \frac{1}{k(\alpha)}\Gamma(e^{\alpha-\eta\pm i\sigma_i\pm i\sigma_j}; p,q), \tag{4.11}$$

$$\mathcal{S}(\sigma_0) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi}\theta(e^{\pm 2i\sigma_0}; q), \tag{4.12}$$

where

$$k(\alpha) = \frac{\Gamma(e^{2\alpha}(pq)^2; p,q,(pq)^2)}{\Gamma(e^{2\alpha}pq; p,q,(pq)^2)}, \tag{4.13}$$

$$\Gamma(z;p,q,t) := \prod_{i,j,k=0}^{\infty}(1-z^{-1}p^{i+1}q^{j+1}t^{k+1}) \times (1-zp^iq^jt^k). \tag{4.14}$$

Here we use the notations of [25]. In order to keep track of notations of the paper [24] by Bazhanov one has to use the following form of the elliptic gamma function:

$$\Phi(x) = \Gamma(e^{-i(x-i\eta)}; p^2, q^2);$$

where $p = e^{i\pi\tau}$, $q = e^{i\pi\sigma}$, $\eta = -i\pi(\tau+\sigma)$, $\operatorname{Im}\tau > 0$ and $\operatorname{Im}\sigma > 0$. Then the Boltzmann weights for the Bazhanov–Sergeev model get the following form:

$$\mathcal{W}_\alpha(\sigma_i,\sigma_j) = k(\alpha)^{-1} \times \frac{\Phi(\sigma_i-\sigma_j+i\alpha)}{\Phi(\sigma_i-\sigma_j-i\alpha)}\frac{\Phi(\sigma_i+\sigma_j+i\alpha)}{\Phi(\sigma_i+\sigma_j-i\alpha)}, \tag{4.15}$$

$$S(\sigma_0) = \frac{e^{\eta/4}}{4\pi}\vartheta_1(\sigma_0|\tau)\vartheta_1(\sigma_0|\sigma), \tag{4.16}$$

where

$$k(\alpha) = \exp\left(\sum_{n\neq 0}\frac{e^{2\alpha n}}{n(p^n-p^{-n})(q^n-q^{-n})((pq)^n+(pq)^{-n})}\right). \tag{4.17}$$

Here $\vartheta_1(x|)$ is the Jacobi theta function defined in the Appendix B.

This solution of the star-triangle relation can be obtained from the lens supersymmetric index by taking $r = 1$. Note that in [84] the authors found a multi-spin generalization[16] of this model and constructed the star-star relation for it.

4.3. $S^2\times S^1$ Partition Function and Solution

The identity for $S^2\times S^1$ partition functions (three-dimensional supersymmetric indices [94, 95]) of dual theories[17] is the following q-beta hypergeometric sum-integral [100]

$$\sum_{m=-\infty}^{\infty}\oint\prod_{i=1}^{6}\frac{(q^{1+(m+n_i)/2}/a_iz;q)_\infty(q^{1+(n_i-m)/2}z/a_i;q)_\infty}{(q^{(m+n_i)/2}a_iz;q)_\infty(q^{(n_i-m)/2}a_i/z;q)_\infty}\frac{(1-q^mz^{\pm 2})}{q^mz^{6m}}\frac{dz}{2\pi iz} = \frac{2}{\prod_{i=1}^{6}a_i^{n_i}} \times \prod_{1\le i<j\le 6}\frac{(q^{1+(n_i+n_j)/2}/a_ia_j;q)_\infty}{(q^{(n_i+n_j)/2}a_ia_j;q)_\infty}, \tag{4.18}$$

with the balancing conditions $\prod_{i=1}^{6}a_i = q$ and $\sum_{i=1}^{6}n_i = 0$. This identity was studied in the context of supersymmetric dualities in [100], integrability in [30, 31], and from point of view of orthogonal polynomials in [101].

In the corresponding integrable model we again have discrete and continuous spin variables

$$\sigma_j = (x_j, m_j) \quad \text{where} \quad 0 \le x_j < 1 \quad \text{and} \quad m_j \in \mathbb{Z}. \tag{4.19}$$

The Boltzmann weights for the model are

$$\mathcal{W}_\alpha(\sigma_i,\sigma_j) = \frac{q^{-2i(x_im_i+x_jm_j)}}{k(\alpha)}\frac{(q^{1+(m_i+m_j)/2}q^{\eta-\alpha-i(x_i+x_j)};q)_\infty}{(q^{(m_i+m_j)/2}q^{\alpha-\eta+i(x_i+x_j)};q)_\infty} \times \frac{(q^{1+(m_j-m_i)/2}q^{\eta-\alpha+i(x_i-x_j)};q)_\infty}{(q^{(m_j-m_i)/2}q^{\alpha-\eta-i(x_i-x_j)};q)_\infty} \times \frac{(q^{1-(m_i+m_j)/2}q^{\eta-\alpha+i(x_i+x_j)};q)_\infty}{(q^{-(m_i+m_j)/2}q^{\alpha-\eta-i(x_i+x_j)};q)_\infty} \times \frac{(q^{1+(m_i-m_j)/2}q^{\eta-\alpha+i(x_j-x_i)};q)_\infty}{(q^{(m_i-m_j)/2}q^{\alpha-\eta-i(x_j-x_i)};q)_\infty}, \tag{4.20}$$

$$\mathcal{S}(\sigma_0) = \frac{1}{2\pi q^m}\frac{(q^{\pm 2x_0+m};q)_\infty}{(q^{\pm 2x_0+m+1};q)_\infty}, \tag{4.21}$$

where

$$k(\alpha) = \frac{(q^2e^{4\pi i\alpha}, qe^{-4\pi i\alpha};q,q^2)_\infty}{(qe^{4\pi i\alpha}, q^2e^{-4\pi i\alpha};q,q^2)_\infty}; \qquad (z,w;p,q)_\infty := \prod_{i,j=0}^{\infty}(1-zp^iq^j)(1-wp^iq^j). \tag{4.22}$$

[16] In terms of supersymmetric gauge theories one needsto consider the dual theories with $SU(N)$ gauge group and $SU(N)\times SU(N)$ flavor group.

[17] Note that here we presented the so-called generalized supersymmetric index [96]. The ordinary supersymmetric index with enhanced symmetry for the $N_f = 4$ case was considered in [97] (see also the cases with broken gauge group in [98, 99]).

The multi-spin generalization of this solution was constructed by authors in [86].

In [31] Kels uses slightly different notations. In his case the Boltzmann weight and self interaction terms are expressed as

$$\begin{aligned}\mathcal{W}_{\alpha}(\sigma_i.\sigma_j) &= \frac{e^{-2\alpha|m_i-m_j|-2\alpha|m_i+m_j|}}{k(\alpha)} \\ &\times \frac{Q(x_i-x_j+i\alpha,, m_i-m_j)}{Q(x_i-x_j-i\alpha,, m_i-m_j)} \\ &\times \frac{Q(x_i+x_j+i\alpha,, m_i+m_j)}{Q(x_i+x_j-i\alpha,, m_i+m_j)},\end{aligned} \tag{4.23}$$

$$\mathcal{S}(\sigma_0) = \frac{1}{2\pi} Q(2x_0 - i\eta, 2m_0) Q(-2x_0 - i\eta, -2m_0), \tag{4.24}$$

with

$$\begin{aligned}k(\alpha) &= \exp\left\{-\sum_{n\neq 0} \frac{e^{4\alpha n}}{n((q)^n - (q)^{-n})}\right\}; \\ Q(z;n) &= \frac{(e^{2iz}(p/q)^{-n}(pq)^{1+|n|};(pq)^2)_\infty}{(e^{-2iz}(p/q)^{n}(pq)^{1+|n|};(pq)^2)_\infty}.\end{aligned} \tag{4.25}$$

4.4. $S^3_b/\mathbb{Z}_r$ Partition Function and Solution

The sum-integral identity for the three dimensional lens partition function [88, 102, 103] of supersymmetric dual theories reads as

$$\begin{aligned}&\sum_{m_0=0}^{r-1} \int_{\mathbb{R}} \frac{dx_0}{r\sqrt{\omega_1\omega_2}} 2\sinh\frac{2\pi}{r\omega_1}(x_0 - i\omega_1 m_0)\sinh\frac{2\pi}{r\omega_2} \\ &\times (x_0 + i\omega_2 m_0)\prod_{i=1}^{6}\frac{\hat{s}_{b,-m_0-m_k}(x_0+x_k+iQ/2)}{\hat{s}_{b,-m_0+m_k}(x_0-x_k-iQ/2)} \\ &= \prod_{1\le i<j\le 6}\hat{s}_{b,-m_j-m_k}(x_j+x_k+iQ/2),\end{aligned} \tag{4.26}$$

with the balancing conditions $i\sum_{i=1}^{6} x_i = Q$ and $\sum_{i=1}^{6} m_i = 0$, where $Q = b + \frac{1}{b}$. Here we used the improved double sine function defined as

$$\hat{s}_{b,-m}(x) = e^{\frac{i\pi}{2r}([m](r-[m])-(r-1)m^2)}\varphi_{r,m}(x). \tag{4.27}$$

The function $\varphi_{r,m}(z)$ is generalization of the Faddeev's quantum dilogarithm

$$\begin{aligned}\varphi_{r,m}(z) &= \exp\left\{\int_0^\infty dx\left(\frac{iz}{\omega_1\omega_2 r x^2}\right.\right. \\ &- \frac{\sinh(2izx - \omega_1(r-2[m]x))}{2x\sinh(\omega_1 rx)\sinh((\omega_1+\omega_2)x)} \\ &\left.\left. - \frac{\sinh(2izx + \omega_2(r-2[m]x))}{2x\sinh(\omega_2 rx)\sinh((\omega_1+\omega_2)x)}\right)\right\}.\end{aligned} \tag{4.28}$$

The hyperbolic limit of the lens index solution was presented in [33] where the authors also give an interpretation of this solution in terms of supersymmetric gauge theory.

In this model there are continuous and discrete spin variables living on each site of the lattice

$$\begin{aligned}\sigma_j = (x_j, m_j) \quad &\text{where} \quad 0 \le x_j < \infty \\ &\text{and} \quad m_j = 0,1,2,\ldots,\lfloor r/2 \rfloor,\end{aligned} \tag{4.29}$$

and the Boltzmann weights are

$$\begin{aligned}\mathcal{W}_{\alpha}(\sigma_i,\sigma_j) &= \frac{1}{k(\alpha)} \\ &\times \frac{\varphi_{m_i+m_j}(x_i+x_j+i\alpha)\varphi_{m_i-m_j}(x_i-x_j+i\alpha)}{\varphi_{m_i+m_j}(x_i+x_j-i\alpha)\varphi_{m_i-m_j}(x_i-x_j-i\alpha)},\end{aligned} \tag{4.30}$$

$$\begin{aligned}\mathcal{S}(\sigma_0) &= \frac{4\varepsilon_0}{r\sqrt{\omega_1\omega_2}}\sinh\left(\frac{2\pi}{\omega_1 r}(x_0 - i\omega_1 m_0)\right) \\ &\times \sinh\left(\frac{2\pi}{\omega_2 r}(x_0 + i\omega_2 m_0)\right),\end{aligned} \tag{4.31}$$

where the normalization constant gets the following form

$$\begin{aligned}k(\alpha) = \exp\left\{\int_0^\infty dx\left(-\frac{\alpha}{\omega_1\omega_2 r x^2}\right.\right. \\ \left.\left. + \frac{\sinh(4\alpha x)\sinh(2r\eta x)}{2x\sinh(\omega_1 rx)\sinh(\omega_2 rx)\sinh(4\eta x)}\right)\right\}.\end{aligned} \tag{4.32}$$

For $r = 1$ the this solution reduces to the Spiridonov's generalization [25] of the Faddeev–Volkov model which we will consider in the next section.

4.5. S^3_b Partition Function and Solution

The hyperbolic hypergeometric integral identity for the three-dimensional squashed sphere partition function[18] of supersymmetric dual theories reads as

$$\begin{aligned}&\int_{-\infty}^{\infty}\frac{\prod_{j=1}^{6}\gamma^{(2)}(g_k \pm iz;\omega)}{\gamma^{(2)}(\pm 2iz;\omega)}dz \\ &= 2\sqrt{\omega_1\omega_2}\prod_{1\le j<k\le 6}\gamma^{(2)}(g_j+g_k;\omega),\end{aligned} \tag{4.33}$$

[18] Note that depending on different squashings of three-sphere and on choice of the preserved charges in the supersymmetric localization one can get different partition functions depending on values of squashing parameter b (see [104–106] for details). For the specific choice of the preserved supercharge the partition function on squashed three-sphere with $SU_l(2)\times U_r(1)$ isometry gives the $b = 1$ case [107] (the case $b = 1$ also corresponds to the round sphere partition function). The integral identity with general values of b is written for the partition functions of dualtheories on three-sphere with $U(1)\times U(1)$ isometry (also with $SU_l(2)\times U_r(1)$ [108]).

with the balancing condition $\sum_{j=1}^{6} g_j = \omega_1 + \omega_2 := 2\eta$. The definition of the hyperbolic gamma function is given in Appendix D.

This integral identity was introduced and studied by Stokman in [109]. It appears in the supersymmetric computations [73] where it was obtained via the reduction procedure of [74] from the elliptic beta integral. As the star-triangle relation this identity was considered by Spiridonov in [25]. It is a generalization of the Faddeev–Volkov model[19] [22, 111, 112]. In this integrable model spins get continuous values and the Boltzmann weights are

$$\mathcal{W}_\alpha(\sigma_i,\sigma_j) = \frac{1}{k(\alpha)}\gamma^{(2)}(\alpha-\eta\pm i\sigma_i\pm i\sigma_j;\omega), \quad (4.34)$$

$$\mathcal{S}(\sigma_0) = \frac{2}{\sqrt{\omega_1\omega_2}}\sinh\frac{2\pi\sigma_0}{\omega_1}\sinh\frac{2\pi\sigma_0}{\omega_2}, \quad (4.35)$$

where

$$k(\alpha) = \exp\left\{-\pi i\alpha^2 - \frac{\pi i}{24}(1-2(b+b^{-1})^2)\right\} \times \frac{(\tilde{q}e^{2\pi iu/b};\tilde{q}^2)_\infty}{(qe^{2\pi iub};q^2)_\infty}\prod_{j,k=0}^{\infty}\frac{1+e^{\pi iu/(b+b^{-1})}\tilde{p}^{j+1}\tilde{q}^{2k}}{1-e^{\pi iu/(b+b^{-1})}\tilde{p}^{j+1}\tilde{q}^{2k}}, \quad (4.36)$$

with $b=\omega_1$ and $b^{-1}=\omega_2$, $q=e^{2\pi ib^2}$, $\tilde{q}=e^{-2\pi i/b^2}$, and $\tilde{p}=e^{-\pi i/(1+b^2)}$.

This solution can also be expressed in a different way [41]

$$\mathcal{W}_\alpha(\sigma_i,\sigma_j) = \frac{1}{k(\alpha)}\frac{\Phi(\sigma_i-\sigma_j+i\alpha)}{\Phi(\sigma_i-\sigma_j-i\alpha)} \times \frac{\Phi(\sigma_i+\sigma_j+i\alpha)}{\Phi(\sigma_i+\sigma_j-i\alpha)}, \quad (4.37)$$

$$\mathcal{S}(\sigma_0) = 2\sinh(2\pi\eta\sigma_0)\sinh(2\pi\eta^{-1}\sigma_0), \quad (4.38)$$

with

$$k(\alpha) = \exp\left\{\frac{1}{8}\int_{PV}\frac{e^{4\alpha w}}{\sinh(\eta w)\sinh(\eta^{-1}w)\cosh((\eta+\eta^{-1})w)}\frac{dw}{w}\right\}, \quad (4.39)$$

where the other version of the hyperbolic gamma function is used

$$\Phi(z) = \exp\left\{\frac{1}{4}\int_{PV}\frac{e^{-2izw}}{\sinh(wb)\sinh(wb^{-1})}\frac{dw}{w}\right\}. \quad (4.40)$$

One can find the proof of the star-triangle relation in terms of Boltzmann weights (4.34)–(4.35) in many places, e.g. see [113]. The quasi-classical limit $b\to 0$ of the model was considered in [41]. It is possible to consider Euler's gamma function limit of this model which corresponds to the limit to vortex partition functions [114]. This limit results to the solution of the star-triangle relation with non-symmetric Boltzmann weights and continous spin [115].

4.6. S^2 Partition Function and Solution

The solution to the star-triangle relation was obtained by Kels in [76] (see also [31]). The interpretation in terms of supersymmetric sphere partition function [116, 117] and the star-star relation is given in [115].

The integral identity for sphere partition functions of dual theories reads as

$$\sum_{m\in\mathbb{Z}}\int_{-\infty}^{+\infty}\frac{dz}{2\pi}\frac{\Gamma(m\pm 2iz+1)}{\Gamma(m\pm 2iz)}\prod_{i=1}^{6}\frac{\Gamma\left(\frac{m+n_i}{2}+a_i+iz\right)}{\Gamma\left(1+\frac{m+n_i}{2}-a_i-iz\right)} \times \frac{\Gamma\left(\frac{m-n_i}{2}+a_i-iz\right)}{\Gamma\left(1+\frac{m-n_i}{2}-a_i+iz\right)} = \prod_{1\le i<j\le 6}\frac{\Gamma\left(a_i+a_j+\frac{n_i+n_j}{2}\right)}{\Gamma\left(1-a_i-a_j-\frac{n_i+n_j}{2}\right)}, \quad (4.41)$$

with the balancing conditions $\sum_{i=1}^{6}a_i = 1$ and $\sum_{i=1}^{6}n_i = 0$.

In the corresponding models spin variables get discrete and continuous values

$$\sigma_j = (x_j, m_j) \quad \text{where} \quad 0\le x_j<2\pi \quad \text{and} \quad m_j\in Z. \quad (4.42)$$

[19] Actually from the supersymmetric viewpoint the Faddeev–Volkov model [110] corresponds to above duality with broken gauge group.

The Boltzmann weight and self-interaction term for the model are

$$\mathcal{W}_{\alpha}(\sigma_i,\sigma_j)=\frac{\Gamma\left(\frac{1+\alpha}{2}\right)}{\Gamma\left(\frac{1-\alpha}{2}\right)}\frac{\Gamma\left(\frac{1-\alpha}{2}\pm\frac{i(x_i+x_j)-(m_i+m_j)}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\pm\frac{i(x_i-x_j)-(m_i-m_j)}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\pm\frac{i(x_i+x_j)+(m_i+m_j)}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\pm\frac{i(x_i-x_j)+(m_i-m_j)}{2}\right)}, \tag{4.43}$$

$$\mathcal{S}(\sigma_0)=\frac{1}{2\pi}\frac{\Gamma(m_0\pm 2ix_0+1)}{\Gamma(m_0\pm 2ix_0)}. \tag{4.44}$$

4.7. $S^1\times S^1$ Partition Function and Solution

By using two dimensional supersymmetric indices of dual theories [118] the following integral is obtained

$$\frac{1}{2}\left(\frac{(q;q)_\infty^2}{\theta(y;q)}\right)\oint\frac{dz}{2\pi iz}\frac{\prod_{i=1}^{6}\Delta(a_iz^{\pm1};q,y)}{\Delta(z^{\pm2};q,y)}=\prod_{1\le i<j\le 6}\Delta(a_ia_j;q,y), \tag{4.45}$$

with the balancing condition $\prod_{i=1}^{6}a_i=\frac{q}{y}$. Here we used the following ratios of the theta functions

$$\Delta(z;q,y):=\frac{\theta(zy,q)}{\theta(z,q)}. \tag{4.46}$$

This integral identity was written as the star-triangle relation in [46]. In this model spin variables get the cantionuos values $0\le\sigma_i<2\pi$. The Boltzmann weights and self-interaction term are

$$\mathcal{W}_{\alpha}(\sigma_i,\sigma_j)=\frac{\theta(e^{-\alpha-\eta\mp i(\sigma_i\pm\sigma_j)};q)}{\theta(e^{\alpha-\eta\pm i(\sigma_i\pm\sigma_j)};q)}, \tag{4.47}$$

$$\mathcal{S}(\sigma_0)=\frac{1}{4\pi}\left(\frac{(q;q)_\infty^2}{\theta(y;q)}\right)\frac{\theta(e^{\pm2i\sigma_0};q)}{\theta(e^{-2\eta\pm2i\sigma_0};q)}, \tag{4.48}$$

where the infinite products in the formula are defined in the second expression of (4.22).

Actually one can rewrite this solution in terms of Jacobi theta function . In that case the integral identity (4.45) will be the indices of dual theories defined for RR sector[20]. Then the Boltzmann weights have the following form

$$\mathcal{W}_{\alpha}(\sigma_i,\sigma_j)=\frac{\theta_1(e^{-\alpha-\eta\mp i(\sigma_i\pm\sigma_j)};q)}{\theta_1(e^{\alpha-\eta\pm i(\sigma_i\pm\sigma_j)};q)}, \tag{4.49}$$

$$\mathcal{S}(\sigma_0)=\frac{1}{4\pi}\left(\frac{\eta(q)^3}{i\theta_1(e^{-2\eta};q)}\right)\frac{\theta_1(e^{\pm2i\sigma_0};q)}{\theta_1(e^{-2\eta\pm2i\sigma_0};q)}. \tag{4.50}$$

[20] Because of spectral duality the index defined for RR sector [119, 120] is identical with the index defined for the NS-NS sector [118].

We would like to mention that the multi-spin case of the model was constructed in [29].

5. SUMMARY AND DISCUSSION

In this paper, we review integrable Ising-like square lattice models obtained (or related) from supersymmetric gauge theory computations.

In all models which we consider in the paper the Boltzmann weights depend only on the differences of the spin variables at the neighbor sites and rapidity variables at the ends of the edge. In principle, it should be possible to obtain a model without such symmetries from supersymmetric computations [115].

However, many key questions have not been answered yet. For example, given a supersymmetric duality with a gauge and matter multiplets in some representation of the gauge and flavor groups, what is the corresponding integrable lattice model? In our opinion, this is one of the most important questions posed by consideration of the subject. Actually, it is absolutely unclear whether gauge/YBE is generic or a feature of a few special dualities. We wish to elucidate the origin of our solution in the framework of the representation theory of quantum group.

The reader might wonder whether the gauge/YBE could be used for three-dimensional lattice models. Actually, there are a lot of attempts to extend the idea of integrability to three- [121–123] and higher-dimensional generalization [124, 125] of lattice models. The condition of commutativity for the transfer matrices in the three-dimensional case takes the form of the so-called tetrahedron equation by Zamolodchikov [126]. It would be interesting to extend the relationship between supersymmetric gauge theory computations and integrable models to higher dimensions and find a solution to the tetrahedron equation.

We remark that the reader might have expected that one can use integrability methods to study dualities and supersymmetric gauge theories via the correspondence. The closed form expressions for the partition function of the models discussed above may provide insight towards an understanding of supersymmetric quiver gauge theories and dualities for them. This is an important point and much work remains to be done in this direction.

Linear quiver gauge theories can be formulated using brane constructions, hence one can obtain solutions of the Yang–Baxter equation directly from the latter via corresponding topological quantum field

theories. We refer to the work by [28] for a discussion on this formulation.

The Seiberg duality [66] for quiver gauge theories corresponds to the cluster mutation [127] for cluster algebras. To be more precise, a cluster algebra defined by a quiver and different quivers are related by the so-called cluster mutation (it is also called cluster transformation) and it happens that the mutation action on quivers is exactly the same as Seiberg duality. It would be interesting to find a relation of integrable models to cluster algebras [34].

APPENDIX

A. Notations

For $z \in \mathbb{C}$, $|q| < 1$, we define the infinite q-product (also called q-Pochhammer symbol)

$$(z;q)_\infty := \prod_{k=0}^{\infty}(1 - zq^k). \tag{A.1}$$

Denote that we will use the following definition for theta and gamma functions (Euler, elliptic and hyperbolic) through the paper

$$\begin{aligned}(a,b;q)_\infty &:= (a;q)_\infty(b;q)_\infty;(az^{\pm 1};q)_\infty \\ &:= (az;q)_\infty(az^{-1};q)_\infty.\end{aligned} \tag{A.2}$$

B. Theta Function

The $\theta(z, p)$ is the theta function defined by

$$\theta(z;p) = \prod_{i=0}^{\infty}(1 - z^{-1}p^{i+1})(1 - zp^i). \tag{B.1}$$

It is related to the Jacobi theta functions. The first Jacobi theta function which is used in present paper can be expressed in terms of theta function

$$\begin{gathered}\theta_1(\tau|z) = -iq^{1/8}y^{1/2}(q,q)_\infty\theta(y^{-1};q) \\ \text{with} \quad y = e^{2\pi iz}, \quad q = e^{2\pi i\tau}.\end{gathered} \tag{B.2}$$

Dedekind eta function is

$$\eta(q) = q^{1/24}\prod_{k=1}^{\infty}(1 - q^k). \tag{B.3}$$

C. Elliptic Gamma Function

The elliptic gamma function is a meromorphic function of three complex variables with double infinite product [128]

$$\Gamma(u;\tau,\sigma) = \prod_{i,j=0}^{\infty}\frac{1 - e^{2\pi i((1+j)\tau+(1+i)\sigma-u)}}{1 - e^{2\pi i(j\tau+i\sigma+u)}}, \tag{C.1}$$

here $u, \sigma, \tau \in \mathbb{C}$ and $\mathrm{Im}\,\tau, \mathrm{Im}\,\sigma > 0$. It is convenient to do the following reparameterization

$$p = e^{2\pi i\tau}, \quad q = e^{2\pi i\sigma}, \quad z = e^{2\pi iu}, \tag{C.2}$$

and get the following form

$$\Gamma(z;p,q) := \prod_{i,j=0}^{\infty}\frac{1 - z^{-1}p^{i+1}q^{j+1}}{1 - zp^iq^j}, \tag{C.3}$$

for $|p|, |q| < 1$ and $z \in \mathbb{C}^*$.

The elliptic gamma function satisfies many interesting properties such as symmetry under exchange of parameters p and q

$$\Gamma(z;p,q) = \Gamma(z;q,p), \tag{C.4}$$

the functional relations

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q), \tag{C.5}$$

$$\Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q), \tag{C.6}$$

and the reflection property

$$\Gamma(z;p,q)\Gamma\left(\frac{pq}{z};p,q\right) = 1. \tag{C.7}$$

The elliptic gamma function has zeros at

$$z \in (p^{i+1}q^{j+1}); \quad (i,j) \in \mathbb{Z}^{\geq 0}, \tag{C.8}$$

poles at

$$z \in (p^{-i}q^{-j}); \quad (i,j) \in \mathbb{Z}^{\geq 0}, \tag{C.9}$$

and the residue

$$\mathrm{Res}_{z=1}\Gamma(z;p,q) = -\frac{1}{(p,p)_\infty(q,q)_\infty}. \tag{C.10}$$

Th elliptic Gamma function is an automorphic form of degree 1 associated to a 2-cocycle and it has an $SL(3,Z)$ modular property [129] based on the following relations

$$\Gamma(u+\tau,\tau,\tau+\sigma)\Gamma(u,\tau+\sigma,\sigma) = \Gamma(u,\tau,\sigma), \tag{C.11}$$

$$\Gamma\left(\frac{z}{\sigma};\frac{\tau}{\sigma},\frac{1}{\sigma}\right) = e^{i\pi Q(z,\tau,\sigma)}\Gamma\left(\frac{z-\sigma}{\tau};\frac{1}{\tau},\frac{\sigma}{\tau}\right)\Gamma(z;\tau,\sigma). \tag{C.12}$$

Note that the elliptic gamma function is related to the Barnes multiple gamma function of order three [130]. Probably this relationship has connection to its modular property.

Lens elliptic gamma function is defined as

$$\begin{aligned}&\Gamma_e(z,m;\sigma,\tau) \\ &= e^{\phi_e(z,m;\sigma,\tau)}\prod_{i,j=0}^{\infty}\frac{1 - z^{-1}p^{-m}(pq)^{i+1}p^{r(j+1)}}{1 - zp^m(pq)^ip^{rj}} \\ &\times\frac{1 - z^{-1}q^{-r+m}(pq)^{i+1}q^{r(j+1)}}{1 - zq^{r-m}(pq)^ip^{rj}},\end{aligned} \tag{C.13}$$

$$\phi_e(z,m;\sigma,\tau) = 2\pi i\Big(R_2(z,0;\sigma,\tau) + R_2\Big(0,m;\frac{1}{2},-\frac{1}{2}\Big) - R_2(z,m;\sigma,\tau)\Big), \quad \text{(C.14)}$$

$$R_2(z,m;\sigma,\tau) = R(z+m\sigma;r\sigma,\sigma+\tau) + R(z+m\sigma;r\sigma,\sigma+\tau), \quad \text{(C.15)}$$

where $z \in \mathbb{C}$, $m \in \mathbb{Z}$ and $r \in \{0,1,2,\ldots\}$.

Here

$$R(z;\sigma,\tau) = \frac{B_{3,3}(z,\sigma,\tau,-1) + B_{3,3}(z-1,\sigma,\tau,-1)}{12}, \quad \text{(C.16)}$$

and the third order Bernoulli polynomial is

$$B_{3,3}(z,\omega_1,\omega_2,\omega_3) = \frac{z^3}{\omega_1\omega_2\omega_3} - \frac{3(\omega_1+\omega_2+\omega_3)z^2}{2\omega_1\omega_2\omega_3} + \frac{(\omega_1^2+\omega_2^2+\omega_3^2)+3(\omega_1\omega_2+\omega_1\omega_3+\omega_2\omega_3)z}{2\omega_1\omega_2\omega_3} - \frac{(\omega_1+\omega_2+\omega_3)(\omega_1\omega_2+\omega_1\omega_3+\omega_2\omega_3)}{4\omega_1\omega_2\omega_3}, \quad \text{(C.17)}$$

where $z \in \mathbb{C}$ and $\omega_1,\omega_2,\omega_3 \in \mathbb{C}\backslash\{0\}$.

D. Hyperbolic Gamma Function and Its Extensions

The hyperbolic gamma function is defined as

$$\gamma^{(2)}(u;\omega_1,\omega_2) = e^{-\pi i B_{2,2}(u;\omega)/2}\frac{(e^{2\pi i u/\omega_1}\tilde{q};\tilde{q})}{(e^{2\pi i u/\omega_1};q)} \quad \text{with} \quad q = e^{2\pi i\omega_1/\omega_2},\ \tilde{q} = e^{-2\pi i\omega_2/\omega_1}, \quad \text{(D.1)}$$

where $B_{2,2}(u;\omega)$ is the second order Bernoulli polynomial,

$$B_{2,2}(u;\omega) = \frac{u^2}{\omega_1\omega_2} - \frac{u}{\omega_1} - \frac{u}{\omega_2} + \frac{\omega_1}{6\omega_2} + \frac{\omega_2}{6\omega_1} + \frac{1}{2}. \quad \text{(D.2)}$$

There is a version of the hyperbolic gamma function, called the double sine function which is defined as follows

$$s_b(z) = e^{-\frac{i\pi z^2}{2}}\frac{\prod_{k=0}^{\infty}\left(1+e^{2\pi b z}e^{2\pi i b^2\left(k+\frac{1}{2}\right)}\right)}{\prod_{k=0}^{\infty}\left(1+e^{2\pi z/b}e^{-2\pi i/b^2\left(k+\frac{1}{2}\right)}\right)}. \quad \text{(D.3)}$$

The relation between hyperbolic gamma function and double sine functions reads as

$$\gamma^{(2)}(u;\omega_1,\omega_2) = e^{-\frac{\pi i}{2}(B_{2,2}(u;\omega_1,\omega_2)-u^2-\frac{1}{4}(\omega_1+\omega_2)^2)+u(\omega_1+\omega_2)))} \times s_b^{-1}\Big(iu - \frac{i}{2}(\omega_1+\omega_2)\Big), \quad \text{(D.4)}$$

for $\omega_1 = b$, $\omega_2 = b^{-1}$ and $u = \frac{1}{2}(b+b^{-1}) - iz$.

The reflection identity for a hyperbolic gamma-function is as follows

$$\gamma^{(2)}(z;\omega_1,\omega_2)\gamma^{(2)}(\omega_1+\omega_2-z;\omega_1,\omega_2) = 1, \quad \text{(D.5)}$$

and the asymptotic formulas are

$$\lim_{u\to\infty} e^{\frac{\pi i}{2}B_{2,2}(u;\omega_1,\omega_2)}\gamma^{(2)}(u;\omega_1,\omega_2) = 1, \quad \text{for} \quad \arg\omega_1 < \arg u < \arg\omega_2 + \pi, \quad \text{(D.6)}$$

$$\lim_{u\to\infty} e^{-\frac{\pi i}{2}B_{2,2}(u;\omega_1,\omega_2)}\gamma^{(2)}(u;\omega_1,\omega_2) = 1, \quad \text{for} \quad \arg\omega_1 - \pi < \arg u < \arg\omega_2. \quad \text{(D.7)}$$

It has the following useful properties

$$\gamma^{(2)}(z+\omega_2;\omega_1,\omega_2) = 2\sin\left(\frac{\pi z}{\omega_1}\right)\gamma^{(2)}(z;\omega_1,\omega_2), \quad \text{(D.8)}$$

$$\gamma^{(2)}(z+\omega_1;\omega_1,\omega_2) = 2\sin\left(\frac{\pi z}{\omega_2}\right)\gamma^{(2)}(z;\omega_1,\omega_2).$$

The hyperbolic gamma function has zeros at

$$z = \omega_1\mathbb{Z}^{\geq 1} + \omega_2\mathbb{Z}^{\geq 1}, \quad \text{(D.9)}$$

poles at

$$z = -\omega_1\mathbb{Z}^{\leq 0} - \omega_2\mathbb{Z}^{\leq 0}. \quad \text{(D.10)}$$

There are different notations and modifications of hyperbolic gamma function, relations between some of them can be found in [25].

ACKNOWLEDGMENTS

The authors would like to thank all participants of "Quantum integrability" seminars at the Physics Department of Mimar Sinan Fine Arts University, especially to Deniz Bozkurt, Doğu Dönmez and Sinan Sevim for discussions on the subject and on some parts of the paper. The authors are also thankful to Zainab Nazari for fruitful discussions. IG would like to express his sincere thanks to Vyacheslav Spiridonov, Andrew Kels, Hjalmar Rosengren, Masahito Yamazaki and Gonenc Mogol for valuable discussions on the subject. Sh.J is thankful to ICTP (Trieste, Italy) Postgraduate Diploma Program for supporting and the Max-Planck Institut für Gravitationsphysik (Albert-Einstein-Institut) for warm hospitality. IG would like to thank The Institut des HautesEtudes Scientifiques (IHES) and Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna where some parts of the work were done for the hospitality. This research is financially supported by BAP Project (no. 2018-18), funded by Mimar Sinan Fine Arts University, Istanbul, Turkey.

REFERENCES

1. J. B. McGuire, "Study of exactly soluble one-dimensional n-body problems," J. Math. Phys. **5**, 622–636 (1964).
2. C.-N. Yang, "Some exact results for the many body problems in one dimension with repulsive delta function interaction," Phys. Rev. Lett. **19**, 1312–1314 (1967).
3. R. J. Baxter, "Partition function of the eight vertex lattice model," Ann. Phys. **70**, 193–228 (1972); Ann. Phys. **281**, 187 (2000).
4. R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1982).
5. M. Jimbo, "Introduction to the Yang–Baxter equation," Int. J. Mod. Phys. A **4**, 3759–3777 (1989).
6. P. P. Kulish and E. K. Sklyanin, "On the solution of the Yang-Baxter equation," J. Sov. Math. **19**, 1596–1620 (1982).
7. L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan, "The quantum inverse problem method. 1," Theor. Math. Phys. **40**, 688–706 (1980).
8. L. A. Takhtadzhan and L. D. Faddeev, "The quantum method of the inverse problem and the Heisenberg xyz model," Russ. Math. Surv. **34** (5), 11 (1979). http://stacks.iop.org/0036-0279/34/i=5/a=R02.
9. L. D. Faddeev, "How algebraic bethe ansatz works for integrable model," in *Proceedigns of the School of Physics, Les Houches, France, 1995,* pp. 149–219; arXiv:hep-th/9605187 [hep-th].
10. E. Ising, "Contribution to the theory of ferromagnetism," Z. Phys. **31**, 253–258 (1925).
11. L. Onsager, "Crystal statistics. 1. A two-dimensional model with an order disorder transition," Phys. Rev. **65**, 117–149 (1944).
12. R. J. Baxter, "Star-triangle and star-star relations in statistical mechanics," Int. J. Mod. Phys. B **11**, 27–37 (1997).
13. H. Au-Yang, J. H. Perk, et al., "Onsager's star-triangle equation: master key to integrability," in *Integrable Systems in Quantum Field Theory and Statistical Mechanics* (Elsevier, Amsterdam, 1989), pp. 57–94.
14. P. P. Kulish, N. Yu. Reshetikhin, and E. K. Sklyanin, "Yang-Baxter equation and representation theory. 1," Lett. Math. Phys. **5**, 393–403 (1981).
15. V. A. Fateev and A. B. Zamolodchikov, "Selfdual solutions of the star triangle relations in Z(N) models," Phys. Lett. A **92**, 37–39 (1982).
16. M. Kashiwara and T. Miwa, "A class of elliptic solutions to the star triangle relation," Nucl. Phys. B **275**, 121 (1986).
17. K. Hasegawa and Y. Yamada, "Algebraic derivation of the broken Z(N) symmetric model," Phys. Lett. A **146**, 387–396 (1990).
18. M. Gaudin, "La relation etoile-triangle d'un modele elliptique Z(N)," J. Phys. I **1**, 351–361 (1991).
19. G. von Gehlen and V. Rittenberg, "Z(N) symmetric quantum chains with an infinite set of conserved charges and Z(N) zero modes," Nucl. Phys. B **257**, 351 (1985).
20. H. Au-Yang, B. M. McCoy, J. H. H. Perk, S. Tang, and M.-L. Yan, "Commuting transfer matrices in the chiral potts models: solutions of star triangle equations with genus > 1," Phys. Lett. A **123**, 219–223 (1987).
21. R. J. Baxter, J. H. H. Perk, and H. Au-Yang, "New solutions of the star triangle relations for the chiral potts model," Phys. Lett. A **128**, 138–142 (1988).
22. L. D. Faddeev, "Current - like variables in massive and massless integrable models," in *Proceedings of the Conference on Quantum Groups and Their Applications in Physics, 1994,* pp. 117–136; arXiv:hep-th/9408041 [hep-th].
23. A. Yu. Volkov, "Quantum Volterra model," Phys. Lett. A **167**, 345–355 (1992).
24. V. V. Bazhanov and S. M. Sergeev, "A master solution of the quantum Yang–Baxter equation and classical discrete integrable equations," Adv. Theor. Math. Phys. **16**, 65–95 (2012); arXiv:1006.0651 [math-ph].
25. V. P. Spiridonov, "Elliptic beta integrals and solvable models of statistical mechanics," Contemp. Math. **563**, 181–211 (2012); arXiv:1011.3798 [hep-th].
26. M. Yamazaki, "Quivers, YBE and 3-manifolds," J. High Energy Phys. **05**, 147 (2012); arXiv:1203.5784 [hep-th].
27. M. Yamazaki, "New integrable models from the gauge/YBE correspondence," J. Stat. Phys. **154**, 895 (2014); arXiv:1307.1128 [hep-th].
28. J. Yagi, "Quiver gauge theories and integrable lattice models," J. High Energy Phys. **10**, 065 (2015); arXiv:1504.04055 [hep-th].
29. M. Yamazaki and W. Yan, "Integrability from 2d N = (2. 2) dualities," J. Phys. A **48**, 394001 (2015); arXiv:1504.05540 [hep-th].
30. I. Gahramanov and V. P. Spiridonov, "The star-triangle relation and 3d superconformal indices," J. High Energy Phys. **08**, 040 (2015); arXiv:1505.00765 [hep-th].
31. A. P. Kels, "New solutions of the star - triangle relation with discrete and continuous spin variables," J. Phys. A **48**, 435201 (2015); arXiv:1504.07074 [math-ph].
32. K. Maruyoshi and J. Yagi, "Surface defects as transfer matrices," PTEP **2016**, 113B01 (2016); arXiv:1606.01041 [hep-th].
33. I. Gahramanov and A. P. Kels, "The star-triangle relation, lens partition function, and hypergeometric sum/integrals," J. High Energy Phys. **02**, 040 (2017); arXiv:1610.09229 [math-ph].
34. M. Yamazaki, "Cluster-enriched Yang-Baxter equation from SUSY gauge theories," arXiv:1611.07522 [hep-th].
35. J. Yagi, "Branes and integrable lattice models," Mod. Phys. Lett. A **32**, 1730003 (2016); arXiv:1610.05584 [hep-th].
36. J. Yagi, "Surface defects and elliptic quantum groups," J. High Energy Phys. **06**, 013 (2017); arXiv:1701.05562 [hep-th].
37. A. P. Kels and M. Yamazaki, "Elliptic hypergeometric sum/integral transformations and supersymmetric lens index," arXiv:1704.03159 [math-ph].
38. A. P. Kels, "Exactly solved models on planar graphs with vertices in Z3," J. Phys. A **50**, 495202 (2017); arXiv:1705.06528 [math-ph].
39. N. A. Nekrasov and S. L. Shatashvili, "Quantization of integrable systems and four dimensional gauge theo-

ries," in Proceedings of the 16th International Congress on Mathematical Physics ICMP09, 2009, pp. 265–289; arXiv:0908.4052 [hep-th].

40. N. A. Nekrasov and S. L. Shatashvili, "Supersymmetric vacua and Bethe ansatz," Nucl. Phys. Proc. Suppl. **192–193**, 91–112 (2009); arXiv:0901.4744 hep-th.
41. V. V. Bazhanov, A. P. Kels, and S. M. Sergeev, "Quasiclassical expansion of the star-triangle relation and integrable systems on quad-graphs," J. Phys. A **49** (2016, in press); arXiv:1602.07076.
42. J. H. Perk and H. Au-Yang, "Yang-Baxter equations," in *Encyclopedia of Mathematical Physics* (Elsevier Science, Oxford, 2006), Vol. 5, pp. 465–473; math-ph/0606053.
43. M. P. Bellon, J. M. Maillard, and C. Viallet, "On the symmetries of integrability," Int. J. Mod. Phys. B **6**, 1881–1904 (1992).
44. T. Deguchi, "Introduction to solvable lattice models in statistical and mathematical physics," in *Classical and Quantum Nonlinear Integrable Systems: Theory and Application* (CRC, Boca Raton, FL, 2003), p. 107.
45. H. Saleur and J. B. Zuber, "Integrable lattice models and quantum groups," in *Proceedings of the Spring School on String Theory and Quantum Gravity Trieste, Italy, Apr. 23–May 1, 1990,* p. 0001-54.
46. S. Jafarzade and Z. Nazari, "A new integrable ising-type model from 2d N=(2,2) dualities (2017)," arXiv:1709.00070 [hep-th].
47. S. E. Derkachov and A. N. Manashov, "General solution of the Yang-Baxter equation with symmetry group SL(N,C)," St. Petersburg Math. J. **21**, 513–577 (2010).
48. S. E. Derkachov and V. P. Spiridonov, "Yang–Baxter equation, parameter permutations, and the elliptic beta integral," Russ. Math. Surv. **68**, 1027–1072 (2013); arXiv:1205.3520 [math-ph].
49. D. Chicherin, S. E. Derkachov, and V. P. Spiridonov, "New elliptic solutions of the Yang-Baxter equation," arXiv:1412.3383 [math-ph].
50. R. J. Baxter, "Hard hexagons: exact solution," J. Phys. A: Math. Gen. **13**, L61 (1980).
51. R. J. Baxter, "Eight vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. 1. Some fundamental eigenvectors," Ann. Phys. **76**, 1–24 (1973).
52. R. J. Baxter, "Eight vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. 2. Equivalence to a generalized ice-type lattice model," Ann. Phys. **76**, 25–47 (1973).
53. R. J. Baxter, "Eight vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. 1. Eigenvectors of the transfer matrix and Hamiltonian," Ann. Phys. **76**, 48–71 (1973).
54. P. A. Pearce and K. A. Seaton, "A solvable hierarchy of cyclic solid-on-solid lattice models," Phys. Rev. Lett. **60**, 1347–1350 (1988).
55. P. A. Pearce and K. A. Seaton, "Exact solution of cyclic solid-on-solid lattice models," Ann. Phys. **193**, 326–366 (1989).
56. G. E. Andrews, R. J. Baxter, and P. J. Forrester, "Eight vertex SOS model and generalized Rogers-Ramanujan type identities," J. Stat. Phys. **35**, 193–266 (1984).
57. E. H. Lieb, "Residual entropy of square ice," Phys. Rev. **162**, 162–172 (1967).
58. B. Sutherland, "Exact solution of a two-dimensional model for hydrogen-bonded crystals," Phys. Rev. Lett. **19**, 103 (1967).
59. A. G. Izergin and V. E. Korepin, "The inverse scattering method approach to the quantum Shabat-Mikhailov model," Commun. Math. Phys. **79**, 303 (1981).
60. J. H. H. Perk and F. Y. Wu, "Graphical approach to the nonintersecting string model: star triangle equation, inversion relation and exact solution," Phys. A (Amsterdam, Neth.) **138**, 100–124 (1986).
61. R. J. Baxter, "The inversion relation method for some two-dimensional exactly solved models in lattice statistics," J. Stat. Phys. **28**, 1–41 (1982).
62. K. A. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories and electric-magnetic duality," Nucl. Phys. Proc. Suppl. **45BC**, 1–28 (1996); arXiv:hep-th/9509066 [hep-th].
63. M. J. Strassler, "Erice lectures on confinement and duality," Subnucl. Ser. **40**, 154–193 (2003); ICTP Lect. Notes Ser. **7**, 105 (2002).
64. M. J. Strassler, "The Duality cascade," in *Proceedings of the Summer School on Progress in string theory, TASI 2003, Boulder, USA, June 2–27, 2003* (2005), pp. 419–510; arXiv:hep-th/0505153 [hep-th].
65. J. Terning, *Modern Supersymmetry: Dynamics and Duality* (Oxford Univ. Press, Oxford, 2006).
66. N. Seiberg, "Electric - magnetic duality in supersymmetric non-Abelian gauge theories," Nucl. Phys. B **435**, 129–146 (1995); arXiv:hep-th/9411149 [hep-th].
67. D. Berenstein and M. R. Douglas, "Seiberg duality for quiver gauge theories," arXiv:hep-th/0207027 [hep-th].
68. M. Yamazaki, "Brane tilings and their applications," Fortsch. Phys. **56**, 555–686 (2008); arXiv:0803.4474 [hep-th].
69. A. Hanany and D. Vegh, "Quivers, tilings, branes and rhombi," J. High Energy Phys. **10**, 029 (2007); arXiv:hep-th/0511063 [hep-th].
70. I. Gahramanov, "Mathematical structures behind supersymmetric dualities," Archivum Math. **51**, 273–286 (2015); arXiv:1505.05656 [math-ph].
71. I. B. Gahramanov and G. S. Vartanov, "Superconformal indices and partition functions for supersymmetric field theories," in *Proceedings of the 17th International Congress on Mathematical Physics, 2013,* pp. 69–703; arXiv:1310.8507 [hep-th].
72. M. Yamazaki, "Four-dimensional superconformal index reloaded," Theor. Math. Phys. **174**, 154–166 (2013).
73. J. Teschner and G. Vartanov, "6j symbols for the modular double, quantum hyperbolic geometry, and supersymmetric gauge theories," Lett. Math. Phys. **104**, 527–551 (2014); arXiv:1202.4698 [hep-th].
74. F. A. H. Dolan, V. P. Spiridonov, and G. S. Vartanov, "From 4d superconformal indices to 3d partition functions," Phys. Lett. B **704**, 234–241 (2011); arXiv:1104.1787 [hep-th].

75. V. P. Spiridonov, "Elliptic hypergeometric terms," SMF Semin. Congr. **23**, 385–405 (2011); arXiv:1003.4491 [math.CA]. http://inspire-hep.net/record/849713/files/arXiv:1003.4491.pdf.
76. A. P. Kels, "A new solution of the star-triangle relation," J. Phys. A **47**, 055203 (2014); arXiv:1302.3025 [math-ph].
77. O. Aharony, S. S. Razamat, N. Seiberg, and B. Willett, "3d dualities from 4d dualities," J. High Energy Phys. **07**, 149 (2013); arXiv:1305.3924 [hep-th].
78. O. Aharony, S. S. Razamat, and B. Willett, "From 3d duality to 2d duality," J. High Energy Phys. **11**, 090 (2017); arXiv:1710.00926 [hep-th].
79. Yu. G. Stroganov, "A new calculation method for partition functions in some lattice models," Phys. Lett. A **74**, 116–118 (1979).
80. V. P. Spiridonov and G. S. Vartanov, "Vanishing superconformal indices and the chiral symmetry breaking," J. High Energy Phys. **06**, 062 (2014); arXiv:1402.2312 [hep-th].
81. S. Elitzur, A. Giveon, and D. Kutasov, "Branes and N=1 duality in string theory," Phys. Lett. B **400**, 269–274 (1997); arXiv:hep-th/9702014 [hep-th].
82. S. Elitzur, A. Giveon, D. Kutasov, E. Rabinovici, and A. Schwimmer, "Brane dynamics and N=1 supersymmetric gauge theory," Nucl. Phys. B **505**, 202–250 (1997); arXiv:9704104 [hep-th].
83. A. Giveon and D. Kutasov, "Brane dynamics and gauge theory," Rev. Mod. Phys. **71**, 983–1084 (1999); arXiv:hep-th/9802067 [hep-th].
84. V. V. Bazhanov, A. P. Kels, and S. M. Sergeev, "Comment on star-star relations in statistical mechanics and elliptic gamma-function identities," J. Phys. A **46**, 152001 (2013); arXiv:1301.5775 [math-ph].
85. V. V. Bazhanov and S. M. Sergeev, "Elliptic gamma-function and multi-spin solutions of the Yang-Baxter equation," Nucl. Phys. B **856**, 475–496 (2012); arXiv:1106.5874 [math-ph].
86. I. Gahramanov and S. Jafarzade, "Comments on the multi-spin solution to the Yang-Baxter equation and basic hypergeometric sum/integral identity," arXiv:1710.09106 [math-ph] (2017).
87. V. P. Spiridonov, "Theta hypergeometric integrals," St. Petersburg Math. J. **15**, 929–967 (2004).
88. F. Benini, T. Nishioka, and M. Yamazaki, "4d index to 3d index and 2d TQFT," Phys. Rev. D **86**, 065015 (2012); arXiv:1109.0283 [hep-th].
89. J. Kinney, J. M. Maldacena, S. Minwalla, and S. Raju, "An index for 4 dimensional super conformal theories," Commun. Math. Phys. **275**, 209–254 (2007); arXiv:hep-th/0510251 [hep-th].
90. C. Romelsberger, "Counting chiral primaries in N = 1, d=4 superconformal field theories," Nucl. Phys. B **747**, 329–353 (2006); arXiv:hep-th/0510060 [hep-th].
91. F. A. Dolan and H. Osborn, "Applications of the superconformal index for protected operators and q-hypergeometric identities to N=1 dual theories," Nucl. Phys. B **818**, 137–178 (2009); arXiv:0801.4947 [hep-th].
92. V. P. Spiridonov, "On the elliptic beta function," Russ. Math. Surv. **56**, 185 (2001). http://stacks.iop.org/0036-0279/56/i=1/a=L21.
93. V. P. Spiridonov, "A Bailey tree for integrals," Theor. Math. Phys. **139**, 536–541 (2004); math/0312502.
94. S. Kim, "The complete superconformal index for N=6 Chern–Simons theory," Nucl. Phys. B **821**, 241–284 (2009); arXiv:0903.4172 [hep-th]; Nucl. Phys. B **864**, 884(E) (2012).
95. J. Bhattacharya and S. Minwalla, "Superconformal indices for N = 6 Chern–Simons theories," J. High Energy Phys. **01**, 014 (2009); arXiv:0806.3251 [hep-th].
96. A. Kapustin, B. Willett, and I. Yaakov, "Exact results for Wilson loops in superconformal Chern–Simons theories with matter," J. High Energy Phys. **03**, 089 (2010); arXiv:0909.4559 [hep-th].
97. I. Gahramanov and G. Vartanov, "Extended global symmetries for 4D N = 1 SQCD theories," J. Phys. A **46**, 285403 (2013); arXiv:1303.1443 [hep-th].
98. I. Gahramanov and H. Rosengren, "A new pentagon identity for the tetrahedron index," J. High Energy Phys. **11**, 128 (2013); arXiv:1309.2195 [hep-th].
99. I. Gahramanov and H. Rosengren, "Integral pentagon relations for 3d superconformal indices," in *Proceedings of the String-Math 2014, Alberta, Canada, June 9–13, 2014,* arXiv:1412.2926 [hep-th].
100. I. Gahramanov and H. Rosengren, "Basic hypergeometry of supersymmetric dualities," Nucl. Phys. B **913**, 747–768 (2016); arXiv:1606.08185 [hep-th].
101. H. Rosengren, "Rahman's biorthogonal functions and superconformal indices," arXiv:1612.05051 [math.CA].
102. Y. Imamura and D. Yokoyama, "S3=Zn partition function and dualities," J. High Energy Phys. **11**, 122 (2012); arXiv:1208.1404 [hep-th].
103. F. Nieri and S. Pasquetti, "Factorisation and holomorphic blocks in 4d," J. High Energy Phys. **11**, 155 (2015); arXiv:1507.00261 [hep-th].
104. Y. Imamura and D. Yokoyama, "N = 2 supersymmetric theories on squashed three-sphere," Int. J. Mod. Phys. Conf. Ser. **21**, 171–172 (2013).
105. K. Hosomichi, "A review on SUSY gauge theories on S3," in *New Dualities of Supersymmetric Gauge Theories* (Springer Int., Switzerland, 2016), pp. 307–338; arXiv:1412.7128 [hep-th].
106. J. Nian, "Localization of supersymmetric Chern–Simons-matter theory on a squashed S3 with SU(1.2) U(1.1) isometry," J. High Energy Phys. **07**, 126 (2014); arXiv:1309.3266 [hep-th].
107. N. Hama, K. Hosomichi, and S. Lee, "SUSY gauge theories on squashed three-spheres," J. High Energy Phys. **05**, 014 (2011); arXiv:1102.4716 [hep-th].
108. Y. Imamura and D. Yokoyama, "N=2 supersymmetric theories on squashed three-sphere," Phys. Rev. D **85**, 025015 (2012); arXiv:1109.4734 [hep-th].
109. J. V. Stokman, "Hyperbolic beta integrals," Adv. Math. **190**, 119–160 (2005).
110. A. Y. Volkov and L. D. Faddeev, "Yang-baxterization of the quantum dilogarithm," J. Math. Sci. **88**, 202–207 (1998).

111. V. V. Bazhanov, V. V. Mangazeev, and S. M. Sergeev, "Faddeev–Volkov solution of the Yang–Baxter equation and discrete conformal symmetry," Nucl. Phys. B **784**, 234–258 (2007); arXiv:hep-th/0703041 [hep-th].

112. V. V. Bazhanov, V. V. Mangazeev, and S. M. Sergeev, "Exact solution of the Faddeev–Volkov model," Phys. Lett. A **372**, 1547–1550 (2008); arXiv:0706.3077 [cond-mat.stat-mech].

113. F. J. van de Bult, "Hyperbolic hypergeometric functions," PhD Thesis (Univ. Amsterdam, 2007).

114. V. P. Spiridonov and G. S. Vartanov, "Elliptic hypergeometry of supersymmetric dualities II. Orthogonal groups, knots, and vortices," Commun. Math. Phys. **325**, 421–486 (2014); arXiv:1107.5788 [hep-th].

115. I. Gahramanov, S. Jafarzde, and G. Mogol, (2018, to appear).

116. F. Benini and S. Cremonesi, "Partition functions of N = (2. 2) gauge theories on S2 and vortices," Commun. Math. Phys. **334**, 1483–1527 (2015); arXiv:1206.2356 [hep-th].

117. N. Doroud, J. Gomis, B. le Floch, and S. Lee, "Exact results in D = 2 supersymmetric gauge theories," J. High Energy Phys. **05**, 093 (2013); arXiv:1206.2606 [hep-th].

118. A. Gadde and S. Gukov, "2D Index and Surface operators," J. High Energy Phys. **03**, 080 (2014); arXiv:1305.0266 [hep-th].

119. F. Benini, R. Eager, K. Hori, and Y. Tachikawa, "Elliptic genera of 2d N = 2 gauge theories," Commun. Math. Phys. **333**, 1241–1286 (2015); arXiv:1308.4896 [hep-th].

120. F. Benini, R. Eager, K. Hori, and Y. Tachikawa, "Elliptic genera of two-dimensional N=2 gauge theories with rank-one gauge groups," Lett. Math. Phys. **104**, 465–493 (2014); arXiv:1305.0533 [hep-th].

121. A. Kuniba, M. Okado, and S. Sergeev, "Tetrahedron equation and generalized quantum groups," J. Phys. A **48**, 304001 (2015); arXiv:1503.08536 [math.QA].

122. V. V. Bazhanov and S. M. Sergeev, "Zamolodchikov's tetrahedron equation and hidden structure of quantum groups," J. Phys. A **39**, 3295–3310 (2006); arXiv:hep-th/0509181 [hep-th].

123. Yu. G. Stroganov, "Tetrahedron equation and spin integrable models on a cubic lattice," Theor. Math. Phys. **110**, 141–167 (1997).

124. V. V. Bazhanov and Yu. G. Stroganov, "On commutativity conditions for transfer matrices on multidimensional lattice," Theor. Math. Phys. **52**, 685–691 (1982).

125. I. Frenkel and G. Moore, "Simplex equations and their solutions," Commun. Math. Phys. **138**, 259–271 (1991).

126. A. Zamolodchikov, "Tetrahedra equations and integrable systems in three-dimensional space," Sov. Phys. JETP **52**, 325 (1980).

127. S. Fomin and A. Zelevinsky, "Cluster algebras I: foundations," J. Am. Math. Soc. **15**, 497–529 (2002); math/0104151.

128. S. N. M. Ruijsenaars, "First order analytic difference equations and integrable quantum systems," J. Math. Phys. **38**, 1069–1146 (1997).

129. G. Felder and A. Varchenko, "The elliptic gamma function and SL(3.Z) Z3," Adv. Math. **156**, 44–76 (2000); arXiv:math/9907061.

130. E. Friedman and S. Ruijsenaars, "Shintani-barnes zeta and gamma functions," Adv. Math. **187**, 362–395 (2004).