ANALYTIC SOLUTIONS OF SCALAR FIELD COSMOLOGY WITH MINIMAL AND NONMINIMAL COUPLING AND DEFORMED DISCRETE AND FINITE QUANTUM SYSTEMS

by

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ABSTRACT

ANALYTIC SOLUTIONS OF SCALAR FIELD COSMOLOGY WITH MINIMAL AND NONMINIMAL COUPLING AND DEFORMED DISCRETE AND FINITE QUANTUM SYSTEMS

In this thesis first, we study analytic solutions of cosmology. We investigate the most general cosmological model with real scalar field which is minimally coupled to gravity and Brans-Dicke cosmology. Field equations consist of three differential equations. We switch independent variable from time to scale factor by change of variable $\dot{a}/a = H(a)$. Thus a new set of differential equations are analytically solvable with known methods. $a(t)$ can be explicitly found as long as methods of integration techniques are available. We investigate the dynamics of the universe at early times as well as at late times in light of these formulas. We find mathematical machinery which turns on and turns off early accelerated expansion. On the other hand late time accelerated expansion is explained by cosmic domain walls. ϕ^4 potential is studied in Brans-Dicke Cosmology. In this thesis we also study discrete and finite quantum systems. We define a deformed kinetic energy operator for a discrete position space with a finite number of points. The structure may be either periodic or nonperiodic with well-defined end points. It is shown that for the nonperiodic case the translation operator becomes nonunitary due to the end points. This uniquely defines an algebra which has the desired unique representation. Energy eigenvalues and energy wave functions for both cases are found. In addition, we uncover the mathematical structure of the Schwinger algebra and introduce almost unitary Schwinger operators which are derived by considering translation operators on a finite lattice.

ÖZET

SKALAR ALANLA MİNİMAL VE MİNİMAL OLMAYAN BAĞLANTIDAKİ KOZMOLOJİNİN ANALİTİK ÇÖZÜMLERİ VE DEFORME AYRIK VE SONLU KUANTUM SİSTEMLERİ

Bu tezde öncelikle kozmolojinin analitik çözümlerini inceliyoruz. Gerçek skalar alanla yerçekimi arasında minimal bağlantının olduğu en genel kozmolojik modeli ve Brans-Dicke kozmolojisi araştırıyoruz. Alan denklemleri üç diferansiyel denklemden oluşur. $\dot{a}/a = H(a)$ dönüşümü ile bağımsız değişkeni zamandan ölçek faktörüne değiştiririz. Böylece yeni diferansiyel denklem seti, bilinen yöntemlerle analitik olarak çözülebilir. Entegrasyon teknikleri yöntemleri mevcut olduğu sürece, $a(t)$ açıkça bulunabilir. Evrenin dinamiklerini bu formüller ışığında hem erken hem de geç zamanlar için inceliyoruz. Erken ivmelenen genişlemeyi açan ve kapatan matematiksel mekanizma buluyoruz. Öte yandan, geç zamandaki ivmelenen genişlemeyi, kozmik alan duvarları ile açıkladık. ϕ^4 potansiyeli Brans-Dicke Kozmolojisinde incelendi. Bu tezde ayrıca ayrık ve sonlu kuantum sistemlerini de inceliyoruz. Sonlu sayıda noktaya sahip ayrık bir konum uzayı için deforme olmuş bir kinetik enerji operatörü tanımlıyoruz. Yapı, iyi tanımlanmış uç noktaları olan periyodik veya periyodik olmayan olabilir. Periyodik olmayan durum için, öteleme operatörünün bitiş noktalarından dolayı üniter olmayan hale geldiği gösterilmiştir. Bu, istenen tek temsile sahip bir cebiri tek şekilde tanımlar. Her iki durum için de enerji özdeğerleri ve enerji öz vektörleri bulunur. Ek olarak, Schwinger cebirinin matematiksel yapısını ortaya çıkardık ve sonlu bir kafes üzerindeki öteleme operatörlerini dikkate alarak türetilen neredeyse üniter Schwinger operatörlerini tanıttık.

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LIST OF SYMBOLS

$(a)_n$	Pochhammer symbol
$\mathscr{A}_d$	An algebra on d-dimensional vector space
dx^α	Coordinate basis one-form
e^i	Orthonormal basis one-form
e_{ij}	Basis of $M_N(C)$
$F(a,b;c;z)$	The hypergeometric function
${}_pF_q(a;b;z)$	The generalized hypergeometric function
$g_{\mu\nu}$	Metric tensor
$-g$	Determinant of the metric tensor $g_{\mu\nu}$
G_N	Newtonian gravitational constant
$G_{\mu\nu}$	Einstein tensor
$H(a)$	Hubble function
H_0	Present value of the Hubble function
$\mathcal{L}$	Lagrangian density
$\mathscr{L}_d$	A linear lattice with d elements
$M_N(C)$	Complex Matrix algebra in N-dimension
$R^\mu_{\nu\sigma\rho}$	Riemann curvature tensor
R^μ_ν	Ricci tensor
R	Ricci scalar
S	Action
$T_{\mu\nu}$	Energy momentum tensor
u^μ	4-velocity vector for fluid
Φ	Original Brans-Dicke scalar field
ϕ	Scalar field
Ω	Density parameter
Ω^μ_{nu}	Curvature two-form
$\Gamma^\sigma_{\mu\nu}$	Christoffel symbol

ω	Brans-Dicke coupling constant
ω^i_j	Connection one-form

LIST OF ACRONYMS/ABBREVIATIONS

BDJT	Brans Dicke Jordan Thiry
CMB	Cosmic microwave background
EFE	External field effect
FLWR	Friedmann Lemaitre Robertson Walker
MOND	Modified Newtonian Dynamics
Mpc	Mega parsec
RMOND	Relativistic Modified Newtonian Dynamics
ΛCDM	Lambda cold dark matter

1. INTRODUCTION

This thesis consists of two main parts: cosmology and deformed quantum mechanical finite systems. In the first part we aim to find exact solutions of field equations both in the most general cosmological model with real scalar field which is minimally coupled to gravity and in Brans-Dicke cosmology. Early epoch and late-time era of the universe are investigated. In the second part we focus on a discrete position space with a finite number of points. We study momentum and translation operators both on periodic and nonperiodic structures. We construct a relation between these translation operators and Schwinger operators.

1.1. Analytic Solutions of Cosmology

The scalar field plays an important role in many parts of modern physics. Its usage in cosmology was seen firstly in Nordstrom's studies after Newton's gravity which has a scalar potential field. Although he introduced scalar theory of gravity [1–4] in 1912-1913, none of them have been verified by observation [5]. Then in 1916 Einstein's theory of general gravity was established. This is a purely tensor theory. Seeds of some alternative theories which incorporates a scalar field were conceived by Jordan [6] and Dirac [7].

Since Einstein's General Theory of Relativity, many new modified theories have been introduced to extend it. Dirac proposed that Newton's constant may be varying with time [7], after influential ideas were developed by Weyl [8, 9] and Eddington [10]. Spacetime dependent gravitational constant can be described according to Jordan's study in [6]. Thus his work has been accepted as the origin of the scalar-tensor theory. Scalar fields appear through a nonminimal coupling term. This term arises with the idea of the space-time dependent gravitational constant which was conceived by Jordan [6] and Dirac [7].

A general form of the Lagrangian density for scalar-tensor theory [11–14] can be written as

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}[f(\Phi)R - g(\Phi)\nabla_\mu\Phi\nabla^\mu\Phi - 2\Lambda(\Phi)] + \mathcal{L}_m(\psi, h(\Phi)g_{\mu\nu}), \qquad (1.1)$$

where f, g, h and Λ are arbitrary functions of the scalar field Φ and $\mathcal{L}_m$ is the Lagrangian density of the matter fields ψ.

Two different revolutionary steps have been taken in cosmology since 1980. First one was the construction of inflationary cosmology by A.H. Guth and A.D.Linde. It had been proposed as a solution for problems of the standard model of cosmology which are the flatness problem, the horizon problem and the monopole problem [15, 16]. In these studies it has been shown that one or more scalar fields drive the early phase of accelerated expansion. Second one was the observational evidence of accelerated expansion of the present universe [17–19]. Standard cosmology explains this behaviour by contribution of dark energy ($\sim 68\%$), dark matter ($\sim 27\%$) and baryonic matter ($\sim 5\%$) to the total density parameter. A Simple explanation of dark energy just as a cosmological constant in standard model of cosmology is problematic [20]. Hence to explain dark energy many different studies have been developed by using scalar fields similar to early inflationary theories. All these models are widely explained in the review article [21]. The second main part of the universe consists of dark matter. This still keeps its secrets. It has not been explained properly yet. A scalar field is again a candidate to reveal its nature [22].

In 1961 Brans and Dicke [23] developed a new approach to general relativity based on Mach's principle. Their main idea was to replace the effective gravitational constant with the reciprocal of a scalar field. The so called Jordan-Brans-Dicke Lagrangian has given inspiration to many scientists. For example the same Lagrangian with spontaneous symmetry breaking potential has been studied by Zee [24] and Smolin [25]. Induced-gravity inflation models have been constructed by using these ideas [26–31]. Moreover in extended inflation, a simple potential causes accelerated expansion [32–34].

On the other hand, more complicated nonminimal coupling terms have been studied in [35–43]. Finally, one should also remember multifield inflationary scenarios which have been developed in the last decade [44–50]. These theories include multiple nonminimally coupled fields.

1.1.1. Problems of Cosmology

Inflationary cosmology was established to solve the problems of Hot Big Bang scenario. Two main problems can be explained as briefly;

1.1.1.1. Flatness Problem. Einstein's equations which will be introduced in Section 2 state that

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3} \sum_i \rho_i(t). \tag{1.2}$$

The critical energy density is defined for a flat universe as $\rho_c = \dfrac{3H^2}{8\pi G}$. Then one can arrange (1.2) as

$$\frac{\mid \rho_{total}(t) - \rho_c \mid}{\rho_c} = \mid \Omega_k \mid, \qquad \Omega_k = -\frac{k}{a^2 H^2}. \tag{1.3}$$

Present observations indicate that the ratio $\Omega = \dfrac{\rho}{\rho_c}$ is very close to one [51, 52]. One can show that at the Planck time $t \sim 5 \times 10^{-44} s$, $\mid \Omega_P - 1 \mid \lesssim 10^{-60}$. The coincidence of this fine tuning initial condition is known as the flatness problem.

1.1.1.2. Horizon Problem. Cosmic microwave background radiation implies homogeneity and isotropy of our universe while it has temperature variation $\dfrac{\delta T}{T} \simeq 10^{-5}$. However if one takes two opposite points which are separated by 180^0 on the last scattering surface a proper distance between them is $1.96 d_{hor}(t_0)$ [53]. Thus there is the problem how causally disconnected regions can be in thermal equilibrium.

1.1.1.3. Inflationary Cosmology. To solve these two basic problems it was shown that there should be inflationary era preceding the radiation dominated epoch. An acceleration of the scale factor $\ddot{a} > 0$ or $-\frac{\dot{H}}{H^2} < 1$ are konown as the condition for inflation. The usual assumption states that the Hubble function was constant during this period of time. Thus an exponential expansion $a \sim e^{H_i(t-t_i)}$ of the universe exists [54, 55]. Then at the end of the inflation

$$\mid \Omega_f - 1 \mid = e^{-2N} \mid \Omega_i - 1 \mid, \tag{1.4}$$

$$\mid \Omega_{kf} \mid = e^{-2N} \mid \Omega_{ki} \mid, \tag{1.5}$$

$$N = (t_f - t_i)H_i, \tag{1.6}$$

where i and f denote beginning and end of the inflation. Calculations in [54] indicate that $N > 62$. If at the beginning of inflation the universe was curved such that $\mid \Omega_\kappa \mid \sim 1$ exponential expansion prevents flatness problem.

When we trace back the exponential or accelerated expansion of the universe, today's casually disconnected patches of the cosmic sky have a chance to communicate with each other. Hence inflationary solution also explain the horizon problem.

Inflationary period is assumed to be driven by the scalar field called the *inflaton*. Thus cosmic fluid is dominated by the scalar field. Field equations are solved by two main assumptions,

$$\dot{\phi}^2 \ll V(\phi), \qquad \mid \ddot{\phi} \mid \ll H \mid \dot{\phi} \mid . \tag{1.7}$$

These are called the *slow-roll approximations* [54, 55].

1.1.1.4. Cosmological Constant Problem. Vacuum energy or cosmological constant is known as the term which causes accelerated expansion of the universe although it was

initially introduced to obtain static universe. Einstein field equations were written as

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{1.8}$$

after adding cosmological constant Λ where metric sign is $(-,+,+,+)$. Hence it contributes to the right side of (1.2) as $\rho_{vac} = \frac{\Lambda}{8\pi G}$. By using observational evidence $\Omega_{\Lambda 0} = 0.7 \pm 0.1$ where $\Omega_\Lambda = \frac{8\pi G}{3H^2}\rho_\Lambda$ and $H_0 = 70 kms^{-1}Mpc^{-1}$, we found $\rho_\Lambda \approx 10^{-29} g/cm^3 \approx 3 \times 10^{-47} GeV^4$ which is consistent with [56].

In quantum field theory the vacuum corresponds to the lowest energy state. Spontaneous virtual particle creation and annihilation occurs in this empty space. These quantum fluctuations contribute to the total energy density of the vacuum. Zero-point energies of the field with mass m is; $\frac{\hbar w}{2}$ and $w = \sqrt{m^2 + k^2}$ where k is the wave number. Summation of all modes with wave number cutoff gives $\rho_{vac} \sim 10^{72} GeV^4$ [20, 56]. There is 119 order of magnitude difference between theoretical prediction and observational results. If one is gudied by calculations in quantum chromodynamics which indicate that $\rho_{vac} \sim 10^{-6} GeV^4$ [20] the gap is at the order of magnitude 41.

1.1.2. Dark Matter and Mond Theory

According to Keplerian behavior, rotation speed of stars at a radius r in a galaxy is found as

$$v(r) = \sqrt{\frac{Gm(r)}{r}}, \tag{1.9}$$

where $m(r)$ is the total mass contained in a volume with radius r. Since luminous mass is concentrated at the center of the galaxy we expect to see decrease in velocity of the rotating object. However according to observational data velocities of the stars increase as their distance from the center increase. Thus it has been concluded that there must be nonluminous mass such that as r increases $m(r)$ increases [57]. This missing matter is known as the dark matter.

MOND(Modified Newtonian Dynamics) was established as an alternative to hidden mass [58]. In this theory instead of a Newtonian force acting on a mass m, $\vec{F} = m\vec{a}$ Milgrom introduced

$$\vec{F} = m\mu(\frac{a}{a_0})\vec{a}, \tag{1.10}$$

$$\mu(x \gg 1) \approx 1, \qquad \mu(x \ll 1) \approx x, \tag{1.11}$$

where $a_0 \approx 2 \times 10^{-8} cms^{-2}$. After performing some fundamental physics one shows that $v = (GMa_0)^{1/4}$ for distant stars of galaxies where accelerations of stars are very small. Although MOND was successful when fitting rotation curves of galaxies, none of the relativistic versions of MOND (RMOND) were compatible with data when CMB and MPS (matter power spectra) were computed. However in 2021 the first RMOND theory which achieved to yield CMB and MPS in agremment with data was established [59]. In addition in 2020 EFE (external field effect) in MOND which was proposed as an alternative to dark matter was investigated and observational evidence was found [60].

1.1.3. Cosmic Domain Walls

In a grand unified theory [61] strong, weak and electromagnetic forces are combined into a single force at extremely high temperatures. As the universe cools phase transitions occur. As a result symmetries between interactions are broken. During this phenomena topological defects may occur [62]. For example the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi), \tag{1.12}$$

$$V(\phi) = -\frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda\phi^4, \tag{1.13}$$

is invariant under reflection ($\phi \to -\phi$) symmetry. The potential has two stable minima at $\phi = \pm\sqrt{\frac{m^2}{\lambda}}$, since

$$V(\pm\sqrt{\frac{m^2}{\lambda}}) = -\frac{m^4}{4\lambda}, \qquad V''(\pm\sqrt{\frac{m^2}{\lambda}}) = 2m^2. \tag{1.14}$$

The case $\phi = 0$ corresponds to an unstable extremum since $V''(0) < 0$. One can add a constant term to the potential without affecting equations of motion. Thus we can write

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - \eta^2)^2, \qquad \eta^2 = \frac{m^2}{\lambda}. \tag{1.15}$$

Hence vacuum energy becomes zero. If we choose one of the vacuum state and set $\phi = \eta + \psi$ our Lagrangian is written as

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\psi)^2 - \frac{1}{2}m_\psi^2\psi^2 - \sqrt{\frac{\lambda}{2}}m_\psi\psi^3 - \frac{1}{4}\lambda\psi^4, \tag{1.16}$$

where $m_\psi = \sqrt{2}m$. Then the symmetry of the Lagrangian is spontaneously broken. Now space has two ground states. One can visualize a universe consisting of different parts; one has $\phi_{ground} = +\eta$, other part has $\phi_{ground} = -\eta$. At the smooth transition of ϕ from one part to other part there is a region where $\phi = 0$ and $V(\phi) \neq 0$. Domain walls are interpreted as this interpolation region. They are two-dimensional defetcs while cosmic-strings are one-dimensional defects and magnetic monopoles are zero-dimensional defects [63–65]. To include them in cosmology is very appealing because they appear in a field theory which has spontaneously broken discrete symmetries [65].

Domain walls differentiate among various candidates for dark energy. They supply the required accelerated expansion with negative pressure $p = -(2/3)\rho$. Altough cosmic fluids with a negative equation of state have an imaginary sound speed there have been several studies indicating that cosmic walls are not ruled out in cosmology [66–71].

1.1.4. Field Equations

In the last decades cosmologists established several modified theories of gravity to investigate all these concepts [55, 72–80]. All of these theories have different field equations which govern the dynamics of the universe. Thus all these theories have different physical conclusions.

In this thesis we study the most general cosmological model with scalar field which is minimally coupled to gravity and the Brans-Dicke cosmology where there exists non-minimal coupling to gravity.

The field equations which govern the universe are ordinary differential equations. To be able to solve them many different approaches have been developed. For the case of minimal coupling we can briefly sumarize these studies as follows. One of them is the dynamical systems methods in which stabiliy analysis of systems of nonlinear differential equations are investigated. Detailed studies have been performed by this method in [81–84] . Other methods are based on assumptions or approximations. The "slow-roll approximation" is the most common one which is applied in scenarios of the inflationary universe [85, 86]. Lastly, the generating function method is proposed as a method which gives exact solution of the field equations in [87]. Some of the searches for exact solutions of the field equations where the scalar field is minimally coouppled to gravity have been presented in [88–91].

In recent years searching for exact solutions of the field equations of scalar tensor theories also has taken more attention [89–94]. In addition in 1990 it has been shown that there are potentials $V(\phi)$ leading to desired behaviour for the scale factor without the use of 'slow-roll approximation' usually assumed in inflationary models [95].

Up to now, for Brans-Dicke cosmology some exact solutions for specific cases have been presented. While for a radiation filled universe exact solution was given in [96], parametric solution for the case $p = \epsilon\rho$ was given in [97]. For Bianchi type

universes with arbitrary barotropic perfect fluid exact solution were found [98]. Parametric exact solutions for potential free universe filled with ordinary matter were given in [99]. On the other hand Noether's theorem was used to determine conservation laws in Brans-Dicke cosmology which includes two scalar fields and exact solutions were found [100]. An exact solution of modified Brans-Dicke theory was studied in [101].

We have three main goals in our cosmological studies. The first is solving field equations analytically. The second is finding mathematical machinery which causes to turn on and turn of accelerated expansion in early universe. Last is explaining late time accelerated expansion without dark energy. We study the most general cosmological model with scalar field which is minimally coupled to gravity in the second chapter. Brans-Dicke cosmology is investigated in the chapter 3 and its complementary part is given in chapter 4.

1.2. Deformed Discrete and Finite Quantum Systems

The conventional formulation [102] of quantum mechanics starts with a position space which is the set of real numbers. This leads to physical states which are vectors of a separable Hilbert space. The observables are then described by Hermitian operators on this Hilbert space. Angular momentum and spin one half are the well known systems in which observables have finite and discrete values. Furthermore, quantum information theory, quantum optics and the lattice models are all realized in a finite dimensional Hilbert space. One can find a guide to the literature on the applications in Vourdas' work [103], where quantum systems with finite Hilbert space are considered and phase-space methods are discussed.

Our motivation is to construct new operators so that one can calculate the energy spectrum. In chapter 5 we define a deformed momentum operator and nonunitary translation operators. Eigenvalues and eigenfunctions of the Hamiltonian for periodic and nonperiodic cases are calculated.

On the other hand quantum mechanics on a finite periodic lattice is a well known subject which has been studied repeatedly since Schwinger's famous 1960 paper [104]. He developed the generators of a complete unitary operator basis. Applications of Schwinger approach have been used in quantum optics, quantum communications, quantum probability and Galois quantum systems [105–115]. In addition one can find the review of the literature on quantum systems with finite Hilbert space and the link between this theory and the other research fields in Vourdas [116].

In chapter 6 we recognize that when the lattice is not periodic the end points give rise to almost unitary Schwinger operator. We construct the projection operators in terms of the almost unitary translation operators and in terms of the unitary Schwinger operators. Since projection operators play the key role in relations between these two algebras we investigate their properties. Then we are able to write each algebra in terms of the other one. We also find two new representations where the standard basis of $M_N(C)$ is constructed in terms of the projection operators in each algebra. Finally, we formulate an algebra which is related to representing a multi-dimensional lattice in terms of one-dimensional lattices in each direction.

2. ANALYTIC SOLUTIONS OF SCALAR FIELD COSMOLOGY WITH MINIMAL COUPLING

In this chapter we have three main purposes. The first is solving field equations analytically. The second is finding a mathematical machinery which causes to turn on and to turn of accelerated expansion in early universe. Last is explaining late time accelerated expansion without dark energy. In Section 2.1 we use a mathematical tool which is a change of independent variable. Thus the field equations are converted to a new set of differential equations. In Section 2.2 we exactly solve this new set of equations and present solutions in four different forms. In Section 2.3 we investigate single-component universes. In Section 2.4 we examine two-component universes and we find a new exotic matter which causes mathematical mechanisms which turns on and turns off accelerated expansion in an early universe. In Section 2.5 we show that a universe which contains matter and cosmic walls results in accelerated expansion. Furthermore we compare our results with supernova Ia data. Results are quite satisfactory. Then we examine dark energy dominated universe with the same procedure. Our discussion is given in Section 2.6.

2.1. Field Equations

2.1.1. Original Form

Action of general relativity with scalar field and the cosmological constant is given by

$$S = \int d^4x\sqrt{-g}[\frac{1}{2\kappa}(R-2\Lambda) - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \mathcal{L}_M], \tag{2.1}$$

where R is the Ricci scalar and $\kappa = 8\pi G c^{-4}$. We will use FLWR metric with space dominant metric sign $(-,+,+,+)$ and units with $\hbar = 1$, $c = 1$

$$ds^2 = -dt^2 + a^2(t)[\frac{dr^2}{1 - k\frac{r^2}{L^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)], \tag{2.2}$$

where $k = -1, 0, 1$ and $a(t) = \dfrac{R(t)}{R(t_0)}$ is normalized scale factor, with the convention $a(t_0) = 1$, $L = R(t_0)$ and r has dimension of lenght.

Energy-momentum tensor for the field is defined as

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S_\phi}{\delta g^{\mu\nu}}, \tag{2.3}$$

where $T^{\mu\nu} = \{\rho, p, p, p\}$.

In standard cosmology field equations have been found as

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{L^2 a^2}, \tag{2.4}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}, \tag{2.5}$$

where $\rho = \rho_{ord} + \rho_\phi$ and $p = p_{ord} + p_\phi$ are known as Einstein equations. *ord* stands for ordinary and represents matter-energy distribution placed in Einstein equations by hand as a function the scale size of the universe.

In addition, variation of the Lagrangian density with respect to the field ϕ gives another equation

$$\ddot{\phi} + 3H\dot{\phi} + \frac{dV}{d\phi} = 0. \tag{2.6}$$

For scalar field dominated universe we have

$$\rho = \frac{\dot{\phi}^2}{2} + V(\phi), \tag{2.7}$$

$$p = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{2.8}$$

where we assume that ϕ is a function only of t. Then Equation (2.6) is equivalent to continuity equation which is given by

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0. \tag{2.9}$$

For the details of the continuity equation and evolution of cosmological fluid see the Appendix A.8.

We name our potential as an effective potential in the sense that it may reflect another more basic physical theory. We will call Equation (2.4) as the first Einstein equation, Equation (2.5) as the second Einstein equation and the Equation (2.6) as the ϕ equation.

2.1.2. New Form of Field Equations

If independent variable does not appear explicitly in the differential equation one can define a new variable in terms of dependent variables so that the order of the differential equation can be reduced by one [117]. In our field equations independent variable is "t". We define our new dependent variable as Hubble function

$$H(a) = \frac{\dot{a}}{a}. \tag{2.10}$$

Thus our new independent variable becomes the scale factor a. For this reason we write all other variables in terms of the new variable;

$$\phi = \phi(a) \qquad \text{and} \qquad V(\phi) = V(a). \tag{2.11}$$

Expressions for derivatives of a and ϕ with respect to time in terms of derivatives with respect to new independent variable a are given in Appendix A.3. In addition Λ can be included in $V(\phi)$ so we will not carry it anymore.

One can easily write the field equations, energy density of the scalar field and the pressure of the scalar field as

$$H^{2} = \frac{8\pi G}{3}\rho - \frac{k}{L^2 a^2}, \tag{2.12}$$

$$H^{'}Ha + H^2 = -\frac{4\pi G}{3}(\rho + 3p), \tag{2.13}$$

$$\phi^{''}a^2H^2 + 4\phi^{'}aH^2 + \phi^{'}a^2HH^{'} + V^{'}\frac{1}{\phi^{'}} = 0, \tag{2.14}$$

$$\rho(a) = \frac{1}{2}(\phi^{'}aH)^2 + V(a), \tag{2.15}$$

$$p(a) = \frac{1}{2}(\phi^{'}aH)^2 - V(a), \tag{2.16}$$

where prime denotes $\dfrac{d}{da}$.

When this set of differential equations is solved exactly we will obtain all unknown functions; $H(a)$, $\phi(a)$, $V(a)$, $\rho(a)$, $p(a)$ and the deceleration parameter $q(a)$ as a function of scale factor, a. Thus it will be possible to track the dynamical history of the universe backward and forward in time. Indeed in some cases it will be possible to formulate some of these functions as a function of time.

2.2. Solution for Field Equations

When solving this differential equation set one should be careful. By taking time derivative of first Einstein equation and using the continuity equation we reach the second Einstein equation. Thus by taking time derivative of first Einstein equation and using the second Einstein equation we reach the continuity equation. In addition by substitution $\rho(t)$ and and $p(t)$ in the continuity equation one can reach ϕ equation. One of the field equations can be derivable from other two of them. One can combine these 3 equations in 3 different pairs such that when their solutions are plugged in the remaining differential equation it will be satisfied automatically.

First combination is the easiest one. We take the first Einstein equation and the ϕ equation. Then we multiply the ϕ equation by $\phi^{'}$ and obtain

$$a^2\phi^{'}\phi^{''}H^2 + 4a\phi^{'2}H^2 + a^2\phi^{'2}HH^{'} + V^{'}(a) = 0. \tag{2.17}$$

We define

$$\gamma(a) = \frac{\phi^{'2}H^2a^2}{2}, \tag{2.18}$$

to be able to solve the last differential equation. Therefore this equation is converted to

$$\gamma^{'} + \frac{6}{a}\gamma = -V^{'}. \tag{2.19}$$

This is a first order linear differential equation and it's solution can be found easily as

$$\gamma(a) = \frac{1}{a^6}[\int_{a_{in}}^{a}(-a^{'6}V^{'}(a^{'}))da^{'} + a_{in}^6\gamma(a_{in})]. \tag{2.20}$$

Hence by rewriting Equation (2.18) we obtain

$$\frac{\phi'^2 H^2 a^2}{2} = \gamma(a),$$
$$\phi'^2 H^2 a^2 = \frac{2}{a^6}[\int_{a_{in}}^{a} (-a'^6 V'(a'))da' + a_{in}^6 \gamma(a_{in})]. \tag{2.21}$$

It is apparent that to be able to solve this field equation one needs the knowledge of one of the following functions; $V(a)$, $H(a)$, $\phi(a)$. There is one more function which can be used as a starting point of calculations. This is the energy density. The relation between $\rho(a)$ and Equation (2.21) will be studied in Section 2.2.4.

We have gone further by plugging energy density into the first Einstein equation

$$H^2 = \frac{8\pi G}{3}[\frac{1}{2}\phi'^2 H^2 a^2 + V(a)] - \frac{k}{L^2 a^2}, \tag{2.22}$$
$$H^2 = \frac{\frac{8\pi G}{3}V(a) - \frac{k}{L^2 a^2}}{1 - \frac{4\pi G}{3}\phi'^2 a^2}. \tag{2.23}$$

We will refer the last equation as our Friedmann equation.

2.2.1. Solution for Given $V(a)$

In this section we start our calculations by using our Friedmann equation. Substituting (2.23) in (2.21) we obtain

$$\phi'^2[\frac{\frac{8\pi G}{3}V(a) - \frac{k}{L^2 a^2}}{1 - \frac{4\pi G}{3}\phi'^2 a^2}]a^2 = 2\gamma(a). \tag{2.24}$$

Then one can reach the following results

$$\phi^{'2} = \frac{2\gamma(a)}{a^2[\frac{8\pi G}{3}(V(a)+\gamma(a)) - \frac{k}{L^2a^2}]}, \tag{2.25}$$

$$\phi(a) = \pm \int_{a_{in}}^{a} \sqrt{\frac{2\gamma(a^{'})}{a^{'2}[\frac{8\pi G}{3}(V(a^{'})+\gamma(a^{'})) - \frac{k}{L^2a^{'2}}}} da^{'} + \phi_{a_{in}}. \tag{2.26}$$

One should decide to pick one of the $\pm$ sign in front of the right side of $\phi(a)$ such that the value of the scalar field increase or decrease as the universe expands. $H(a)$ has been found by using the formula of the scalar field in our Friedmann equation as

$$H(a) = \sqrt{\frac{8\pi G}{3}(V(a)+\gamma(a)) - \frac{k}{L^2a^2}}, \tag{2.27}$$

where

$$\gamma(a) = \frac{1}{a^6}[\int_{a_{in}}^{a} (-a^{'6}V^{'}(a^{'}))da^{'} + a_{in}^6\gamma(a_{in})]. \tag{2.28}$$

It is apparent that knowledge of the potential energy function $V(a)$ is sufficient to formulate the scalar field $\phi(a)$ and the Hubble function $H(a)$ as an exact solution of the field equations.

We would like to mention that the results of this subsection are similar to results of [118]. They have reduced the differential equations to quadrature problems by writing $V(a)$ in a complicated way. Then exponential potentials and hyperbolic potentials were focused in their examples.

2.2.2. Solution for Given $\phi(a)$

In some cases one may need to solve the field equations for a specific scalar field. In this calculation $V(a)$ becomes the unknown dependent variable in (2.24). To be able

to go further first we write the solution of the ϕ equation as

$$\gamma(a) = -V(a) + \frac{1}{a^6}[6\int_{a_{in}}^{a} V(a^{'})a^{'5}da^{'} + a_{in}^6\tilde{\gamma}(a_{in})], \quad \tilde{\gamma}_(a_{in}) = \gamma(a_{in}) + V(a_{in}), \tag{2.29}$$

where we have applied integration by parts to Equation (2.20). Details of this calculation are given in the Appendix A.4.

2.2.2.1. Singular Case. Firstly we will investigate the special form of the scalar field which causes this singularity in the denominator of right side of the equation. From (2.23) we have

$$1 - \frac{4\pi G}{3}\phi^{'2}a^2 = 0. \tag{2.30}$$

Therefore

$$\int_{\phi_{in}}^{\phi} d\phi^{'} = \pm\sqrt{\frac{3}{4\pi G}}\int_{a_{in}}^{a} \frac{da'}{a^{'}}, \tag{2.31}$$

and

$$\phi = \begin{cases} \sqrt{\frac{3}{4\pi G}}ln(\frac{a}{a_{in}}) + \phi_{in}, \\ -\sqrt{\frac{3}{4\pi G}}ln(\frac{a}{a_{in}}) + \phi_{in}. \end{cases} \tag{2.32}$$

Since $a_{in} \leq a$, " + " sign indicates that the scalar field always increases. On the other hand minus sign implies that one will have positive and decreasing scalar field when ϕ_{in} big enough.

One easily obtains the potential by plugging the field into (2.22)

$$V(a) = \frac{3}{8\pi G}(\frac{k}{L^2a^2}). \tag{2.33}$$

Then the Hubble function is formulated just by substitution of $\phi(a)$ and $V(a)$ into the solution of the ϕ equation which is given by the (2.21)

$$H^2(a) = \frac{2}{a^6}\{\int_{a_{in}}^{a} \frac{k}{L^2} a^{'3} da^{'} + \frac{4\pi G}{3} a_{in}^6 \gamma(a_{in})\}, \tag{2.34}$$

$$H(a) = \sqrt{\frac{k}{2L^2a^2} + \frac{8\pi G}{3}\frac{a_{in}^6 \tilde{\gamma}(a_{in})}{a^6}}, \qquad \gamma(a_{in}) = \frac{3k}{16\pi G L^2 a_{in}^2} + \tilde{\gamma}(a_{in}). \tag{2.35}$$

At first sight one can say that the spatially flat universe is static by choosing $\tilde{\gamma}(a_{in}) = 0$. However this statement is incorrect because it is incomplete. Firstly we would like to remind that our choice at the change of variable $\frac{\dot{a}}{a} = H(a)$ works only for the dynamic universes where $H(a) \neq 0$. Secondly by using the first Einstein equation one can easily deduce that the static and spatially flat universe must be empty. Therefore complete and correct interpretation says that the spatially flat and dynamic universes have time varying energy density.

Considering the solution for $k = 0$, it is seen from the Equation (2.33), $V = 0$ for spatially flat universe. Thus we jump back to Equation (2.19) and it turns to

$$\begin{aligned} &\gamma^{'} + \frac{6}{a}\gamma = 0, \\ &\int_{\gamma_{in}}^{\gamma} \frac{d\gamma^{'}}{\gamma^{'}} = -6\int_{a_{in}}^{a} \frac{da^{'}}{a^{'}}, \end{aligned} \tag{2.36}$$

and

$$\begin{aligned} ln\gamma - ln\gamma_{in} &= -6(lna - lna_{in}), \\ \gamma(a) = \frac{\tilde{\gamma}(a_{in})}{a^6} \quad &\text{and} \quad \tilde{\gamma}(a_{in}) = \gamma(a_{in}) a_{in}^6. \end{aligned} \tag{2.37}$$

Hence by plugging γ and ϕ in (2.18) we have obtain the Hubble function as

$$H = \sqrt{\frac{8\pi G}{3}\frac{\tilde{\gamma}(a_{in})}{a^6}}. \tag{2.38}$$

2.2.2.2. Non-Singular Case. In this case we investigate general form of the scalar field where $\phi(a) \neq \sqrt{\frac{3}{4\pi G}} ln(a)$. We have start this case by using (2.29) in (2.25)

$$\phi^{'2} = \frac{2\{-V(a) + \frac{1}{a^6}[6\int_{a_{in}}^{a} V(a^{'})a^{'5}da^{'} + a_{in}^6\tilde{\gamma}(a_{in})]\}}{a^2\{\frac{8\pi G}{3a^6}[6\int_{a_{in}}^{a} V(a^{'})a^{'5}da^{'} + a_{in}^6\tilde{\gamma}(a_{in})] - \frac{k}{L^2a^2}\}}. \tag{2.39}$$

To be able to calculate the potential $V(a)$, one should define a new function

$$\alpha(a) = 6\int_{a_{in}}^{a} V(a^{'})a^{'5}da^{'} + a_{in}^6\tilde{\gamma}(a_{in}), \qquad V(a) = \frac{\alpha^{'}}{6a^5}, \tag{2.40}$$

where $\alpha(a_{in}) = a_{in}^6\tilde{\gamma}_{in}$. Then (2.39) turns into a first order linear differential equation which is obtained as

$$\alpha^{'} + (8\pi Ga\phi^{'2} - \frac{6}{a})\alpha = (\frac{3ka^5}{L^2})\phi^{'2}. \tag{2.41}$$

The solution is found as

$$\begin{aligned}\alpha(a) = exp[\int_{a_{in}}^{a}(\frac{6}{a^{'}} - 8\pi Ga^{'}\phi^{'2})da^{'}] \\ \times\Big\{\alpha(a_{in}) + \int_{a_{in}}^{a} exp[\int_{a_{in}}^{a^{'}}(-\frac{6}{a^{''}} + 8\pi Ga^{''}\phi^{'2})da^{''}](\frac{3ka^{'5}\phi^{'2}}{L^2})da^{'}\Big\}.\end{aligned} \tag{2.42}$$

Then according to relation (2.40) the potential $V(a)$ is found as

$$V(a) = \frac{\alpha^{'}}{6a^5},$$

$$V(a) = (1 - \frac{4\pi G}{3}a^2\phi^{'2})e^{\lambda(a)}\beta(a) + \frac{k\phi^{'2}}{2L^2}, \tag{2.43}$$

$$\lambda(a) = -\int_{a_{in}}^{a} 8\pi Ga^{'}\phi^{'2}da^{'}, \tag{2.44}$$

$$\beta(a) = \alpha(a_{in}) + \int_{a_{in}}^{a} e^{-\lambda(a^{'})}\frac{3ka_{in}^6\phi^{'2}}{L^2a^{'}}da^{'}]. \tag{2.45}$$

$H(a)$ has been found by substituting this potential and the specific scalar field into our Friedmann equation

$$H^2 = \frac{\frac{8\pi G}{3}V(a) - \frac{k}{L^2a^2}}{1 - \frac{4\pi G}{3}\phi'^2a^2},$$
$$H(a) = \sqrt{\frac{8\pi G}{3a_{in}^6}e^{\lambda(a)}\beta(a) - \frac{k}{L^2a^2}}, \tag{2.46}$$

where $\lambda(a)$ and $\beta(a)$ are given by (2.44) and (2.45). Hence for a specific scalar field exact solution of the field equations are given by the last two equations.

Both these two cases have a common physical result. If there is only scalar field without any kinds of matter except the dark energy this universe has dynamic behaviour as a result of it being curved by the scalar field.

For $k = 0$ the solution changes. Equation (2.41) is solved as

$$\alpha' = (\frac{6}{a} - 8\pi Ga\phi'^2)\alpha, \tag{2.47}$$
$$\int_{\alpha_{in}}^{\alpha}\frac{d\alpha'}{\alpha'} = \int_{a_{in}}^{a}(\frac{6}{a'} - 8\pi Ga'\phi'^2(a'))da',$$
$$ln\alpha - ln\alpha_{in} = (lna^6 - lna_{in}^6) - \int_{a_{in}}^{a}8\pi Ga'\phi'^2(a')da',$$
$$\frac{\alpha}{\alpha_{in}} = \frac{a^6}{a_{in}^6}exp[-\int_{a_{in}}^{a}8\pi Ga'\phi'^2(a')da'],$$
$$\alpha = \frac{\alpha_{in}a^6}{a_{in}^6}exp[-\int_{a_{in}}^{a}8\pi Ga'\phi'^2(a')da']. \tag{2.48}$$

Then we formulate the potential by using Equation (2.40)

$$V(a) = \frac{\alpha'}{6a^5},$$
$$V(a) = \frac{\alpha_{in}}{a_{in}^6}[1 - \frac{4\pi G}{3}a^2\phi'^2(a)]e^{\lambda(a)}, \tag{2.49}$$

where $\lambda(a)$ is given by Equation (2.44). The Hubble function is found just by substuting the potential and the scalar field into our Friedmann equation

$$H(a) = \sqrt{\frac{8\pi G}{3}\frac{\alpha_{in}}{a_{in}^6}e^{\lambda(a)}}, \tag{2.50}$$

where $\lambda(a)$ is given in (2.44).

2.2.3. Solution for Given $H(a)$

In this section we will start our calculations by rewriting our Friedmann equation in the form

$$H^2 = \frac{8\pi G}{3}(\frac{1}{2}\phi'^2 H^2 a^2 + V(a)) - \frac{k}{L^2 a^2}.$$

This equation is easily converted to

$$\frac{\phi'^2 a^2 H^2}{2} = \frac{3}{8\pi G}(H^2 + \frac{k}{L^2 a^2}) - V(a). \tag{2.51}$$

Therefore one can recognize the first term on the left side of Equation (2.51) as $\gamma(a)$ which is the variable found as a solution of the ϕ equation at the beginning of the Section 2.2. Hence (2.51) turns into the following form

$$\gamma(a) = \frac{3}{8\pi G}(H^2 + \frac{k}{L^2 a^2}) - V(a). \tag{2.52}$$

By using the last form of $\gamma(a)$ which is formulated in (2.29)

$$\gamma(a) = -V(a) + \frac{1}{a^6}[6\int_{a_{in}}^{a} V(a')a'^5 da' + a_{in}^6 \tilde{\gamma}(a_{in})].$$

So we have obtained

$$-V(a)+\frac{1}{a^6}[6\int_{a_{in}}^{a}V(a^{'})a^{'5}da^{'}+a_{in}^6\tilde{\gamma}(a_{in})]=[\frac{3}{8\pi G}(H^2+\frac{k}{L^2a^2})-V(a)]. \qquad (2.53)$$

As we have done in the previous section we now find the potential energy. The last equation can be easily solved so that $\alpha(a)$

$$\alpha(a)=6\int_{a_{in}}^{a}V(a^{'})a^{'5}da^{'}+a_{in}^6\tilde{\gamma}(a_{in}) \qquad \text{so} \qquad V(a)=\frac{\alpha^{'}}{6a^5}.$$

Therefore $\alpha(a)$ is found algebraically from Equation (2.53) as

$$\alpha(a)=\frac{3a^6}{8\pi G}(H^2+\frac{k}{L^2a^2}). \qquad (2.54)$$

As a result the potential energy is calculated as

$$V(a)=\frac{3}{8\pi G}[(H^2+\frac{k}{L^2a^2})+\frac{a}{3}(HH^{'}-\frac{k}{L^2a^3})]. \qquad (2.55)$$

The scalar field is found by substituting the potential into Equation (2.51) as

$$\phi(a)=\pm\int_{a_{in}}^{a}\sqrt{\frac{1}{4\pi GaH^2}(-HH'+\frac{k}{L^2a^{'3}})}da^{'}+\phi(a_{in}). \qquad (2.56)$$

Therefore last equations can be used to construct the scalar field and the potential for a given Hubble function.

2.2.4. Solution for Given $\rho(a)$

When one starts the calculations with one of the following functions $V(a)$, $\phi(a)$, $H(a)$ one can end up with some unusual forms of the energy density. To avoid this possibility one should start the calculations for desired energy density. It is written in

terms of our new independent variable "a" as

$$\rho(a) = \frac{1}{2}(\phi^{'} aH)^2 + V(a).$$

One can recognize the first term on the right side of this equation as $\gamma(a)$. Hence we obtain

$$V(a) = \rho(a) - \gamma(a) \quad \text{and} \quad \gamma(a) = \frac{(\phi^{'} aH)^2}{2}. \tag{2.57}$$

Then we substitute this into the ϕ equation

$$\gamma^{'} + \frac{6}{a}\gamma = -V^{'},$$

and we obtain

$$\gamma(a) = -\frac{a}{6}\rho'. \tag{2.58}$$

By using the definition of $\gamma(a)$ we also obtain

$$\phi^{'2} = \frac{2\gamma(a)}{a^2H^2}. \tag{2.59}$$

The Hubble function is known just by inserting the energy density into the original form of the first Einstein equation

$$H(a) = \sqrt{\frac{8\pi G}{3}\rho - \frac{k}{L^2a^2}}. \tag{2.60}$$

Therefore the scalar field is formulated as

$$\phi(a) = \pm \int_{a_{in}}^{a} \sqrt{\frac{-\frac{a^{'}}{3}\rho'}{a'^2[\frac{8\pi G}{3}\rho - \frac{k}{L^2a^{'}}]}} da^{'} + \phi(a_{in}). \tag{2.61}$$

The potential energy is written by substitution of (2.58) into (2.57) as

$$V(a) = \rho(a) + \frac{a}{6}\rho'. \tag{2.62}$$

Exact solution of the field equations for a desired energy density are given by the formulas in (2.60-2.62).

2.3. Single Component Universes

We present some general solutions for a universe which has a single component. This purpose is easily achieved for a given $\rho(a)$ in Section 2.3.1 and for a given $V(a)$ in Section 2.3.2. In addition we have performed calculations for both a curved universe and a spatially flat universe. Therefore one can see the effect of curvature term in dynamics of the universe.

2.3.1. General Solution for $\rho(a) = \frac{\rho_n}{a^n}$

To satisfy the weak energy condition $\rho \geq 0$ and $\rho + p \geq 0$ one should start calculations with a given energy density.

2.3.1.1. $k \neq 0$. We begin this subsection by taking the energy density as in the form of perfect fluid

$$\rho = \frac{\rho_n}{a^n}. \tag{2.63}$$

Then by applying the procedure which is explained in Section 2.2.4 we immediately obtain $H(a)$, $V(a)$, $\phi(a)$, $p(a)$, $q(a)$ as

$$H(a) = \sqrt{\frac{8\pi G\rho_n}{3a^n} - \frac{k}{a^2}}, \tag{2.64}$$

$$V(a) = \frac{(6-n)}{6}\frac{\rho_n}{a^n}, \tag{2.65}$$

$$\phi(a) = \pm\sqrt{\frac{n}{8\pi G}}\{ln(\frac{a}{a_{in}}) + \frac{1}{1-n/2}ln[\frac{1}{b}(1+\sqrt{1-\frac{3ka^{n-2}}{8\pi G\rho_n}})]\} + \phi(a_{in}), \tag{2.66}$$

$$b = (1+\sqrt{1-\frac{3ka_{in}^{n-2}}{8\pi G\rho_n}}),$$

$$p(a) = (\frac{n-3}{3})\frac{\rho_n}{a^n}, \tag{2.67}$$

$$q(a) = \frac{4(n-2)\pi G\rho_n}{8\pi G\rho_n - 3ka^{n-2}}. \tag{2.68}$$

First, we interpret these formulas generally. When we apply the boundaries on equation of state $\nu = \dfrac{p}{\rho}$ we obtain

$$-1 \leq \nu \leq 1, \quad -1 \leq \frac{n-3}{3} \leq 1, \quad 0 \leq n \leq 6. \tag{2.69}$$

Therefore all exotic fluids with energy density in the form of $\dfrac{\rho_n}{a^n}$ have $0 \leq n \leq 6$. This condition also makes the potential non-negative. Then special cases pops up immediately for $n = 0, 2, 6$. Furthermore we would like to add one more comment. The second term on the right side of (2.66) is always real. This is easily recognized when one writes the related components in terms of cosmological density parameters.

Case $n = 0$ corresponds to constant energy density $\rho = \rho_0$. One presents related functions for more comments,

$$H(a) = \sqrt{\frac{8\pi G\rho_0}{3} - \frac{k}{a^2}}, \tag{2.70}$$

$$V(a) = \rho_0, \tag{2.71}$$

$$\phi(a) = 0, \tag{2.72}$$

$$p(a) = -\rho_0, \tag{2.73}$$

$$q(a) = -1 + \frac{3k}{3k - 8\pi G\rho_0 a^2}. \tag{2.74}$$

Furthermore $a(t)$ can be formulated by the following steps

$$\int_0^t dt' = \int_{a_{in}}^{a} \frac{da'}{a' H(a')}, \tag{2.75}$$

$$a(t) = (\frac{a_{in}}{2} + \frac{\sqrt{\mu^2 a_{in}^2 - k}}{2\mu})e^{\mu t} + (\frac{a_{in}}{2} - \frac{\sqrt{\mu^2 a_{in}^2 - k}}{2\mu})e^{-\mu t}, \quad \mu = \sqrt{\frac{8\pi G\rho_0}{3}}. \tag{2.76}$$

Since the scalar field is zero in this case, our solutions reduce to solutions of standard cosmology with dark energy. According to results given in (2.70) and (2.76), to have a dynamic universe with real Hubble function and real scale factor cosmological constant must be big enough to overcome the smallness of the universe;

$$\frac{8\pi G\rho_0}{3} > \frac{k}{a_{in}^2}. \tag{2.77}$$

This is the mathematical reason which explains the big value of the cosmological constant in the early universe according to the standard model.

Case $n = 2$ creates a singularity in the scalar field as seen in (2.66). Thus we have calculated $\phi(a)$ separately and we have found it as

$$\phi(a) = \pm\sqrt{\frac{2\rho_2}{8\pi G\rho_2 - 3k}} ln(\frac{a}{a_{in}}) + \phi(a_{in}). \tag{2.78}$$

Furthermore $a(t)$ is

$$a(t) = \tau t + a_{in}, \qquad \tau = \sqrt{\frac{8\pi G\rho_2}{3} - k}, \tag{2.79}$$

which is consistent with (2.68) which tells us that for $n = 2$, the universe expands with constant speed.

Case $n = 6$ requires special attention. Potential becomes $V = 0$ and equation of state becomes $\nu = \frac{p}{\rho} = 1$. This case corresponds to a massless scalar field.

2.3.1.2. $k = 0$. When we study the spatially flat universe, nature of the scalar field changes. As a result of this change we can formulate the potential as a function of the scalar field. The field is written as

$$\phi(a) = \pm\sqrt{\frac{n}{8\pi G}} ln(\frac{a}{a_{in}}) + \phi(a_{in}). \tag{2.80}$$

Therefore one can formulate the scale factor and hence the potential as a function of the scalar field as

$$a = a_{in} exp[\pm\sqrt{\frac{8\pi G}{n}}(\phi - \phi(a_{in}))], \tag{2.81}$$

$$V(a) = (\frac{6-n}{6})\frac{\rho_n}{a_{in}^n} exp[\mp\sqrt{8\pi n G}(\phi - \phi(a_{in}))]. \tag{2.82}$$

Furthermore we can find $a(t)$ by

$$\int_0^t dt^{'} = \int_{a_{in}}^a \frac{da^{'}}{a^{'}H(a^{'})},$$

$$a(t) = (n\sqrt{\frac{2\pi G\rho_n}{3}}t + a_{in}^{n/2})^{2/n}, \tag{2.83}$$

where there is a singularity in the case $n = 0$. This case corresponds to the standard model with cosmological constant. Hence

$$\int_0^t dt^{'} = \int_{a_{in}}^a \frac{da^{'}}{a^{'}H(a^{'})},$$

$$a(t) = a_{in}e^{\mu t}, \qquad \mu = \sqrt{\frac{8\pi G\rho_0}{3}}. \tag{2.84}$$

As it is seen there is no constraint on this constant energy density. It can be big as well as it can be small.

2.3.2. General Solution for $V(a) = \dfrac{V_n}{a^n}$

To start calculations with given $V(a)$ is more fundamental. After getting intuition about the form of the potential which is required for the perfect fluid, we continue our work by choosing the potential. Results are important because they are surprisingly different than Section 2.3.1.

2.3.2.1. $k \neq 0$. We have plugged in $V(a) = \dfrac{V_n}{a^n}$ in the formulas given by (2.26-2.28) and we have obtained

$$\gamma(a) = \frac{1}{a^6}[\frac{nV_n}{6-n}a^{6-n} + a_{in}^6\tilde{\gamma}(a_{in})], \quad \tilde{\gamma}(a_{in}) = -\frac{nV_n}{6-n}a_{in}^{-n} + \gamma(a_{in}), \tag{2.85}$$

$$H(a) = \sqrt{\frac{16\pi G V_n}{(6-n)a^n} + \frac{8\pi G a_{in}^6\tilde{\gamma}(a_{in})}{3a^6} - \frac{k}{L^2a^2}}, \tag{2.86}$$

$$\phi(a) = \pm\sqrt{6}\int_{a_{in}}^{a}\sqrt{\frac{nV_n a^{'6} + (6-n)a_{in}^6\tilde{\gamma}(a_{in})a^{'n}}{48\pi G V_n a^{'8} + (n-6)(3ka^{'4} - 8\pi G a_{in}^6\tilde{\gamma}(a_{in}))a^{'n+2}}}da^{'} + \phi(a_{in}), \tag{2.87}$$

$$\rho(a) = \frac{6V_n}{(6-n)a^n} + \frac{a_{in}^6\tilde{\gamma}(a_{in})}{a^6}, \tag{2.88}$$

$$p(a) = \frac{2(n-3)V_n}{(6-n)a^n} + \frac{a_{in}^6\tilde{\gamma}(a_{in})}{a^6}, \tag{2.89}$$

$$q(a) = \frac{8\pi G[2(6-n)a_{in}^6\tilde{\gamma}(a_{in})a^n + 3(n-2)V_n a^6]}{48\pi G V_n a^6 + (n-6)(3ka^4 - 8\pi G a_{in}^6\tilde{\gamma}(a_{in}))a^n}. \tag{2.90}$$

For all n there is the term proportional to a^{-6} in energy density. Therefore not only for zero potential but also for each potential, universe contains the stiff fluid.

For case $n = 0$ one should perform the calculations by starting from Equation (2.19),

$$\int_{\gamma(a_{in})}^{\gamma}\frac{d\gamma}{\gamma} = -\int_{a_{in}}^{a}\frac{6}{a^{'}}da^{'}, \tag{2.91}$$

$$\gamma(a) = \frac{\gamma(a_{in})a_{in}^6}{a^6}. \tag{2.92}$$

Thus we have

$$H(a) = \sqrt{\frac{8\pi G}{3}(\frac{\gamma(a_{in})a_{in}^6}{a^6} + V_0) - \frac{k}{L^2a^2}}, \tag{2.93}$$

$$\phi(a) = \pm\sqrt{6}\int_{a_{in}}^{a} \sqrt{\frac{\gamma(a_{in})a_{in}^6}{8\pi G(\gamma(a_{in})a_{in}^6 + V_0a'^6)a'^2 - 3ka'^6}}da' + \phi(a_{in}), \tag{2.94}$$

$$\rho(a) = \frac{\gamma(a_{in})a_{in}^6}{a^6} + V_0, \tag{2.95}$$

$$p(a) = \frac{\gamma(a_{in})a_{in}^6}{a^6} - V_0, \tag{2.96}$$

$$q(a) = \frac{8\pi G[2\gamma(a_{in})a_{in}^6 - V_0a^6]}{8\pi G[\gamma(a_{in})a_{in}^6 + V_0a^6] - 3ka^4}. \tag{2.97}$$

In contrast to constant energy density case, constant potential differs from cosmological constant case.

As it is seen from formulas there is a singularity for $n = 6$. Thus we have investigated this case separately:

$$\gamma(a) = \frac{1}{a^6}[6V_6ln(\frac{a}{a_{in}}) + a_{in}^6\gamma(a_{in})], \tag{2.98}$$

$$H(a) = \sqrt{\frac{8\pi G}{3}[\frac{V_6 + a_{in}^6\gamma(a_{in})}{a^6} + \frac{6V_6ln(\frac{a}{a_{in}})}{a^6}] - \frac{k}{L^2a^2}}, \tag{2.99}$$

$$\phi(a) = \pm\sqrt{6}\int_{a_{in}}^{a} \sqrt{\frac{a_{in}^6\gamma(a_{in}) + 6V_6ln(\frac{a'}{a_{in}})}{[48\pi GV_6ln(\frac{a'}{a_{in}}) + 8\pi G(V_6 + a_{in}^6\gamma(a_{in})) - 3ka'^4]a'^2}}da' + \phi(a_{in}), \tag{2.100}$$

$$\rho(a) = \frac{a_{in}^6\gamma(a_{in}) + V_6}{a^6} + \frac{6V_6ln(\frac{a}{a_{in}})}{a^6}, \tag{2.101}$$

$$p(a) = \frac{a_{in}^6\gamma(a_{in}) - V_6}{a^6} + \frac{6V_6ln(\frac{a}{a_{in}})}{a^6}, \tag{2.102}$$

$$q(a) = \frac{8\pi G[2a_{in}^6\gamma(a_{in}) - V_6 + 12V_6ln(\frac{a}{a_{in}})]}{48\pi GV_6ln(\frac{a}{a_{in}}) + 8\pi G(a_{in}^6\gamma(a_{in}) + V_6) - 3ka^4}. \tag{2.103}$$

Energy density and pressure should be written in the following form

$$\rho(a) = \rho_1(a) + \rho_2(a), \tag{2.104}$$

$$\rho_1(a) = \frac{a_{in}^6 \gamma(a_{in})}{a^6}, \qquad \rho_2(a) = \frac{V_6}{a^6} ln[e(\frac{a}{a_{in}})^6], \tag{2.105}$$

$$p(a) = p_1(a) + p_2(a), \tag{2.106}$$

$$p_1(a) = \frac{a_{in}^6 \gamma(a_{in})}{a^6}, \qquad p_2(a) = \frac{V_6}{a^6} ln[\frac{1}{e}(\frac{a}{a_{in}})^6]. \tag{2.107}$$

Thus each component satisfies the continuity equation which is given by (2.9) according to perfect fluid theorem.

Furthermore investigation of equation of state for the second part of the fluid is important. First we write the pressure in the following form:

$$p_2(a) = \frac{V_6}{a^6} ln[\frac{1}{e}(\frac{a_{in}}{a})^6(\frac{a}{a_{in}})^{12}], \tag{2.108}$$

$$p_2(a) = \frac{V_6}{a^6} \{ln[\frac{1}{e}(\frac{a_{in}}{a})^6] + ln(\frac{a}{a_{in}})^{12}\}. \tag{2.109}$$

Then equation of state turns into

$$\nu_2 = \frac{p_2}{\rho_2}, \tag{2.110}$$

$$\nu_2 = \frac{ln[\frac{1}{e}(\frac{a_{in}}{a})^6] + ln(\frac{a}{a_{in}})^{12}}{ln[e(\frac{a}{a_{in}})^6]}, \tag{2.111}$$

$$\nu_2 = -1 + \frac{12 ln(\frac{a}{a_{in}})}{1 + 6 ln(\frac{a}{a_{in}})}. \tag{2.112}$$

In addition

$$\lim_{a \to a_{in}} \nu_2 = -1, \qquad \lim_{a \to \infty} \nu_2 = 1. \tag{2.113}$$

This phenomenon says that at the beginning of the universe there was a negative pressure. This pressure was huge because it is proportional to $\frac{1}{a^6}$. As the universe expands this pressure and the related energy density becomes negligible since both of them proportional to $\frac{1}{a^6}$.

2.3.2.2. k=0. First simplification occurs in the relation between cosmological time and the scale factor of the universe as

$$\begin{aligned} t &= \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}H(a^{'})}, \\ t &= \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}\sqrt{\frac{8\pi G}{3}(\frac{a_{in}^6\tilde{\gamma}(a_{in})}{a^{'6}} + \frac{6v_n}{(6-n)a^{'n}})}}, \end{aligned} \tag{2.114}$$

$$t = \sqrt{\frac{1}{24\pi G a_{in}^6\tilde{\gamma}(a_{in})}}\Big[a^{'3}{}_2F_1(\frac{1}{2}, \frac{3}{6-n}; \frac{3}{6-n}+1; -\frac{6v_n}{(6-n)a_{in}^6\tilde{\gamma}(a_{in})}a^{'(6-n)})\Big]_{a_{in}}^{a}. \tag{2.115}$$

${}_2F_1$ is the hypergeometric function

$${}_2F_1(1/2, b; b+1; u) = \sum_{n=0}^{\infty} \frac{(1/2)_n (b)_n}{(b+1)_n} \frac{u^n}{n!}, \tag{2.116}$$

where $(b)_n$ is the Pochhammer symbol which is defined as

$$(b)_n = \begin{cases} 1 \quad \text{if} \quad n = 0, \\ b(b+1)(b+2)...(b+n-1), \quad \text{if} \quad n = 1, 2 \ldots . \end{cases} \tag{2.117}$$

Details of this calculation is given in Appendix A.6 Our condition, $n < 6$ for positivity of energy density avoids the singularity of the hypergeometric function given in (2.115).

One can go further by choosing initial condition $a_{in}^6\tilde{\gamma}(a_{in}) = 0$. Results are very similar to what we have obtained in Section 2.3.1.2. The scale factor is found as

$$a(t) = (2n\sqrt{\frac{\pi G v_n}{(6-n)}}t + a_{in}^{n/2})^{2/n}. \tag{2.118}$$

Positivity condition $n < 6$ for energy density makes $a(t)$ real. The scalar field is found as

$$\phi(a) = \pm\sqrt{\frac{n}{8\pi G}}ln(\frac{a}{a_{in}}) + \phi(a_{in}). \tag{2.119}$$

Then

$$a = a_{in}exp(\pm\mu\psi), \quad \psi = \phi(a) - \phi(a_{in}) \quad \text{and} \quad \mu = 2\sqrt{\frac{2\pi G}{n}}. \tag{2.120}$$

Therefore

$$V(\psi) = V_n a_{in}^{-n} exp(\mp n\mu\psi) \quad \text{or } V(\phi) = V_n a_{in}^{-n} exp\{\mp n\mu[\phi(a) - \phi(a_{in})]\}. \tag{2.121}$$

In constant potential case where $n = 0$ one can obtain explicit form of the scalar field as

$$\phi(a) = \mp\frac{1}{2}\sqrt{\frac{1}{3\pi G}}ln[\frac{1}{ba^3}(1 + \sqrt{1 + \frac{V_0 a^6}{\gamma(a_{in})a_{in}^6}})] + \phi(a_{in}), \tag{2.122}$$
$$b = \frac{1}{a_{in}^3}(1 + \sqrt{1 + \frac{V_0}{\gamma(a_{in})}}).$$

Formulation of $a(t)$ is also possible

$$t = \int_{a_{in}}^{a} \frac{da'}{a'\sqrt{\frac{8\pi G}{3}(V_0 + \frac{\gamma(a_{in})a_{in}^6}{a'^6})}}, \tag{2.123}$$

$$a(t) = [\frac{b^2 V_0 e^{2\mu t} - \gamma(a_{in}) a_{in}^6 e^{-2\mu t}}{2bV_0}]^{1/3}, \tag{2.124}$$
$$b = a_{in}^3 + \sqrt{a_{in}^6 + \frac{\gamma(a_{in}) a_{in}^6}{V_0}} \quad \text{and} \quad \mu = \sqrt{6\pi G V_0}.$$

This case will be investigated in more detail in Section 2.4.3.

The case $n = 6$ has two simplification for a spatially flat universe. The scalar field has been found as

$$\phi(a) = \pm\frac{1}{12}\sqrt{\frac{3}{\pi G}}\frac{1}{V_6} I(a) + \phi(a_{in}), \tag{2.125}$$
$$I(a^{'}) = x(a^{'})\sqrt{1 - \frac{V_6}{x(a^{'})}} + \frac{V_6}{2} ln[-\frac{2x(a^{'})}{V_6} + 1 + \frac{2x(a^{'})}{V_6}\sqrt{1 - \frac{V_6}{x(a^{'})}}], \tag{2.126}$$
$$x(a^{'}) = a_{in}^6 \gamma(a_{in}) + V_6[1 + 6ln(\frac{a^{'}}{a_{in}})]. \tag{2.127}$$

Expression of $a(t)$ is

$$a(t) = a_{in} exp\Big\{\frac{1}{6}[2\Big(erfi^{-1}(\mu t + b)\Big)^2 - 1 - \frac{a_{in}^6 \gamma(a_{in})}{V_6}]\Big\}, \tag{2.128}$$
$$\mu = \sqrt{\frac{8\pi G}{3}}\frac{1}{a_{in}^3} exp[\frac{1}{2} + \frac{a_{in}^6 \gamma(a_{in})}{2V_6}], \qquad b = erfi[\frac{1}{2}\Big(1 + \frac{a_{in}^6 \gamma(a_{in})}{V_6}\Big)], \tag{2.129}$$
$$erfi(\theta) = \frac{2}{\sqrt{\pi}}\int_0^\theta exp(z^2) dz, \tag{2.130}$$

where $erfi(\theta)$ is the *imaginary error function*. Details of this calculation is given in Appendix A.7. When θ is real $erfi(\theta)$ is real [119]. Therefore it's inverse function $erfi^{-1}(\theta)$ becomes real for real θ.

2.4. Early Epoch of the Universe

There have been many studies which show that the early universe should expand with exponential expansion to be able to reach its size today. Thus our purpose in this section is to explore the mathematical turn on and turn of mechanism to start

and to end up exponential expansion. For this reason we have studied three different combinations for curved and spatially flat universes.

2.4.1. Dark Energy

We will search for the universe with energy density in the following form

$$\rho(a) = \frac{\rho_n}{a^n} + \rho_0. \tag{2.131}$$

2.4.1.1. $k \neq 0$. Firstly we take nonzero curvature. We have found $H(a)$, $V(a)$, $\phi(a)$, $p(a)$ and $q(a)$ as

$$H(a) = \sqrt{\frac{8\pi G}{3}(\frac{\rho_n}{a^n} + \rho_0) - \frac{k}{a^2}}, \tag{2.132}$$

$$V(a) = \frac{(6-n)}{6}\frac{\rho_n}{a^n} + \rho_0, \tag{2.133}$$

$$\phi(a) = \pm \int_{a_{in}}^{a} \sqrt{\frac{n\rho_n}{8\pi G(\rho_n a'^2 + \rho_0 a'^{2+n}) - 3ka'^n}} da' + \phi(a_{in}), \tag{2.134}$$

$$p(a) = \frac{(n-3)}{3}\frac{\rho_n}{a^n} - \rho_0, \tag{2.135}$$

$$q(a) = \frac{4\pi G[-2\rho_0 a^n + (n-2)\rho_n]a^2}{8\pi G[\rho_n a^2 + \rho_0 a^{2+n}] - 3ka^n}. \tag{2.136}$$

We continue to investigate the dynamics of the early universe for small a as

$$\lim_{a\to 0} q(a) = \frac{4\pi G[(n-2)\rho_n]a^2}{8\pi G\rho_n a^2 - 3ka^n}, \tag{2.137}$$

$$\lim_{a\to 0} q(a) = \begin{cases} -1 + \dfrac{n}{2} & \text{if} \quad 2 < n \leq 6, \\ 0 & \text{if} \quad 0 \leq n \leq 2. \end{cases} \tag{2.138}$$

One should check the roots of $q(a)$. Since the denominator of the $q(a) \sim H^2(a)$ and $H^2(a) > 0$ we are only interested in numerator of $q(a)$.

$$q(a) = 0 \qquad \Rightarrow \qquad a = [\frac{(n-2)\rho_n}{2\rho_0}]^{1/n}, \tag{2.139}$$

where there is no real and positive root for $n \leq 2$. Hence for $2 < n \leq 6$ the universe starts to expand with deceleration and then expansion turns to acceleration. For $0 < n \leq 2$ the universe starts with constant velocity and it immediately accelerates. In both cases although there are mathematical turn on mechanism to initiate acceleration there is no mathematical turn of mechanism to end acceleration in the this model.

2.4.1.2. $k = 0$. For $k = 0$, behaviour of the deceleration parameter changes as follows

$$\lim_{a\to 0} q(a) = -1 + \frac{n}{2} \qquad \text{if} \qquad 0 \leq n \leq 6. \tag{2.140}$$

Root of $q(a)$ is still given by (2.139). The difference between spatially flat and curved universe occurs just at the beginning of the universe for $n < 2$. The universe starts to expand with acceleration.

The scalar field can be simplified as

$$\phi(a) = \mp\frac{1}{\sqrt{2\pi G n}} ln[\frac{a^{-n/2} + \sqrt{a^{-n} + \frac{\rho_0}{\rho_n}}}{a_{in}^{-n/2} + \sqrt{a_{in}^{-n} + \frac{\rho_0}{\rho_n}}}] + \phi(a_{in}). \tag{2.141}$$

Then one can derive $V(\phi)$ as follows

$$\frac{a^{-n/2} + \sqrt{a^{-n} + \frac{\rho_0}{\rho_n}}}{a_{in}^{-n/2} + \sqrt{a_{in}^{-n} + \frac{\rho_0}{\rho_n}}} = exp(-\sqrt{n}\psi), \qquad \psi = \pm\sqrt{2\pi G}[\phi(a) - \phi(a_{in})]. \tag{2.142}$$

Thus for some specific values of n one obtains

$$V(\psi) = \begin{cases} \rho_0 \quad \text{if} \quad n = 6, \\ \frac{1}{3}\frac{\rho_4}{a^4(\psi)} + \rho_0 \quad \text{if} \quad n = 4, \\ \frac{1}{2}\frac{\rho_3}{a^3(\psi)} + \rho_0 \quad \text{if} \quad n = 3, \end{cases} \tag{2.143}$$

where

$$a(\psi) = \begin{cases} \sqrt{2}\{\frac{-a_{in}^2 e^{2\psi}[1 - \sqrt{1 + a_{in}^4\frac{\rho_0}{\rho_4}} + (1 + \sqrt{1 + a_{in}^4\frac{\rho_0}{\rho_4}})e^{4\psi}]}{a_{in}^4\frac{\rho_0}{\rho_4}(1 + e^{8\psi}) - 2(2 + a_{in}^4\frac{\rho_0}{\rho_4})e^{4\psi}}\}^{1/2} \quad \text{if} \quad n = 4, \\ 2^{2/3}\{\frac{a_{in}^{3/2} e^{\sqrt{3}\psi}[-1 + \sqrt{1 + a_{in}^3\frac{\rho_0}{\rho_3}} - (1 + \sqrt{1 + a_{in}^3\frac{\rho_0}{\rho_3}})e^{2\sqrt{3}\psi}]}{a_{in}^3\frac{\rho_0}{\rho_3}(1 + e^{4\sqrt{3}\psi}) - 2(2 + a_{in}^3\frac{\rho_0}{\rho_3})e^{2\sqrt{3}\psi}}\}^{2/3} \quad \text{if} \quad n = 3. \end{cases} \tag{2.144}$$

Formulation of $a(t)$ is possible for $k = 0$ as

$$t = \int_{a_{in}}^{a} \frac{da'}{a' H(a')}, \tag{2.145}$$

$$t = \sqrt{\frac{3}{8\pi G\rho_0}}\frac{2}{n} ln[\frac{a^{n/2} + \sqrt{a^n + \frac{\rho_n}{\rho_0}}}{a_{in}^{n/2} + \sqrt{a_{in}^n + \frac{\rho_n}{\rho_0}}}]. \tag{2.146}$$

For some specific n

$$a(t) = [\frac{(a_{in}^3 + \sqrt{a_{in}^6 + \frac{\rho_n}{\rho_0}})e^{3\mu t} + (a_{in}^3 - \sqrt{a_{in}^6 + \frac{\rho_n}{\rho_0}})e^{-3\mu t}}{2}]^{1/3}, \quad \text{if} \quad n = 6, \tag{2.147}$$

$$a(t) = [\frac{(a_{in}^2 + \sqrt{a_{in}^4 + \frac{\rho_n}{\rho_0}})e^{2\mu t} + (a_{in}^2 - \sqrt{a_{in}^4 + \frac{\rho_n}{\rho_0}})e^{-2\mu t}}{2}]^{1/2}, \quad \text{if} \quad n = 4, \quad (2.148)$$

$$a(t) = [\frac{(a_{in}^{3/2} + \sqrt{a_{in}^3 + \frac{\rho_n}{\rho_0}})e^{3\mu t/2} + (a_{in}^{3/2} - \sqrt{a_{in}^3 + \frac{\rho_n}{\rho_0}})e^{-3\mu t/2}}{2}]^{2/3}, \quad \text{if} \quad n = 3, \quad (2.149)$$

where $\mu = \sqrt{\frac{8\pi G\rho_0}{3}}$. If one chooses $a_{in} = 0$, for $t \ll 1$ $a(t)$ can be approximately written as

$$a(t) = \begin{cases} 3^{1/3}(\frac{\rho_6}{\rho_0})^{1/6}(\mu t)^{1/3} & \text{if} \quad n = 6, \\ 2^{1/2}(\frac{\rho_4}{\rho_0})^{1/4}(\mu t)^{1/2} & \text{if} \quad n = 4, \\ (\frac{3}{2})^{2/3}(\frac{\rho_3}{\rho_0})^{1/3}(\mu t)^{2/3} & \text{if} \quad n = 3. \end{cases} \quad (2.150)$$

2.4.2. Cosmic Domain Walls

Cosmic domain walls are known with their contribution to energy density with term $\rho \sim 1/a$. Their equation of state parameter is given by $\nu = -2/3$. Dynamics of the universe with two components where one of them is a domain wall are very similar to dynamics of universes with two components where one of them is dark energy. We have taken the energy density in the following form

$$\rho(a) = \frac{\rho_w}{a} + \frac{\rho_n}{a^n}. \quad (2.151)$$

2.4.2.1. $k \neq 0$. For a curved space results are found as

$$H(a) = \sqrt{\frac{8\pi G}{3}(\frac{\rho_w}{a} + \frac{\rho_n}{a^n}) - \frac{k}{L^2a^2}}, \quad (2.152)$$

$$V(a) = \frac{5\rho_w}{6a} + \frac{(6-n)}{6}\frac{\rho_n}{a^n}, \tag{2.153}$$

$$\phi(a) = \pm \int_{a_{in}}^{a} \sqrt{\frac{n\rho_n a^{'} + \rho_w a^{'n}}{8\pi G[\rho_w a^{'2+n} + \rho_n a^{'3}] - 3k a^{'1+n}}} da^{'} + \phi(a_{in}), \tag{2.154}$$

$$p(a) = \frac{-2\rho_w}{3a} + \frac{(n-3)}{3}\frac{\rho_n}{a^n}, \tag{2.155}$$

$$q(a) = \frac{4\pi G[-\rho_w a^n + (n-2)\rho_n a]a}{8\pi G(\rho_w a^n + \rho_n a)a - 3ka^n}. \tag{2.156}$$

Behaviour of the deceleration parameter at the beginning is obtained as

$$\lim_{a\to 0} q(a) = \begin{cases} -1 + \dfrac{n}{2} & \text{if} \quad 2 < n \leq 6, \\ 0 & \text{if} \quad 0 < n \leq 2. \end{cases} \tag{2.157}$$

Furthermore since denominator of $q(a) \sim H^2(a)$ as stated before, numerator of $q(a)$ determines dynamics of the universe. Roots of the deceleration parameter is found as

$$a = [(n-2)\frac{\rho_n}{\rho_w}]^{1/(n-1)}. \tag{2.158}$$

Therefore the universe start to expand with deceleration and then turns to accelerate for $2 < n \leq 6$. On the other hand for $0 \leq n \leq 2$ at the beginning of universe its velocity was constant and thus the universe starts to its expansion with acceleration.

2.4.2.2. $k = 0$. Behaviour of $q(a)$ changes as

$$\lim_{a\to 0} q(a) = \begin{cases} -1 + \dfrac{n}{2} & \text{if} \quad 1 < n \leq 6, \\ -\dfrac{1}{2} & \text{if} \quad 0 \leq n \leq 1. \end{cases} \tag{2.159}$$

Therefore the differences in dynamics of the universe when it is spatially flat is seen when $0 \leq n < 1$. In this case at the beginning its velocity is not constant and the universe starts its expansion with acceleration.

$\phi(a)$ has been simplified for three cases: for domain walls and stiff fluid as

$$\phi(a) = \pm\frac{1}{5\sqrt{2\pi G}}\{-\sqrt{6}ln[\frac{(\sqrt{6}\sqrt{\rho_s+\rho_w a^5}+\sqrt{6\rho_s+\rho_w a^5})a_{in}^{5/2}}{(\sqrt{6}\sqrt{\rho_s+\rho_w a_{in}^5}+\sqrt{6\rho_s+\rho_w a_{in}^5})a^{5/2}}]\}$$
$$+ ln[\frac{\sqrt{\rho_s+\rho_w a^5}+\sqrt{6\rho_s+\rho_w a^5}}{\sqrt{\rho_s+\rho_w a_{in}^5}+\sqrt{6\rho_s+\rho_w a_{in}^5}}] + \phi(a_{in}) \qquad \text{where} \qquad n=6, \tag{2.160}$$

for domain walls and radiation as

$$\phi(a) = \pm\frac{1}{3\sqrt{2\pi G}}\{-2ln[\frac{(2\sqrt{\rho_r+\rho_w a^3}+\sqrt{4\rho_r+\rho_w a^3})a_{in}^{3/2}}{(2\sqrt{\rho_r+\rho_w a_{in}^3}+\sqrt{4\rho_r+\rho_w a_{in}^3})a^{3/2}}]$$
$$+ ln[\frac{\sqrt{\rho_r+\rho_w a^3}+\sqrt{4\rho_r+\rho_w a^3}}{\sqrt{\rho_r+\rho_w a_{in}^3}+\sqrt{4\rho_r+\rho_w a_{in}^3}}]\} + \phi(a_{in}) \qquad \text{where} \qquad n=4, \tag{2.161}$$

for domain walls and matter as

$$\phi(a) = \pm\frac{1}{2\sqrt{2\pi G}}\{-\sqrt{3}ln[\frac{(\sqrt{3}\sqrt{\rho_m+\rho_w a^2}+\sqrt{3\rho_m+\rho_w a^2})a_{in}}{(\sqrt{3}\sqrt{\rho_m+\rho_w a_{in}^2}+\sqrt{3\rho_m+\rho_w a_{in}^2})a}]$$
$$+ ln[\frac{\sqrt{\rho_m+\rho_w a^2}+\sqrt{3\rho_m+\rho_w a^2}}{\sqrt{\rho_m+\rho_w a_{in}^2}+\sqrt{3\rho_m+\rho_w a_{in}^2}}]\} + \phi(a_{in}) \qquad \text{where} \qquad n=3. \tag{2.162}$$

It is possible to simply the relation between time and the scale factor as

$$t = \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}H(a^{'})},$$
$$t = \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}\sqrt{\frac{8\pi G}{3}(\frac{\rho_w}{a^{'}}+\frac{\rho_n}{a^{'n}})}}, \tag{2.163}$$
$$t = \sqrt{\frac{3}{2\pi G\rho_w}}\Big[\sqrt{a^{'}}\,{}_2F_1(\frac{1}{2},\frac{1}{2-2n};\frac{1}{2-2n}+1;-\frac{\rho_n}{\rho_w}a^{'(1-n)})\Big]_{a_{in}}^{a}, \tag{2.164}$$

where ${}_2F_1$ is the hypergeometric function which was introduced in Section 2.3.2.2. Details of this calculation is given in Appendix A.6. To avoid singularities in the hypergeometric function in (2.164) n should be chosen such that $3/2 < n$.

2.4.3. Dark Energy Revisited

We have already examined the case $V = V_0$ in Section 2.3.2. For spatially flat universe we have obtained

$$a(t) = [\frac{b^2 V_0 e^{2\mu t} - \gamma(a_{in}) a_{in}^6 e^{-2\mu t}}{2bV_0}]^{1/3},$$
$$b = a_{in}^3 + \sqrt{a_{in}^6 + \frac{\gamma(a_{in}) a_{in}^6}{V_0}} \qquad \text{and} \qquad \mu = \sqrt{6\pi G V_0}.$$

Then we formulate the Hubble function and the deceleration parameter as a function of t as

$$H(t) = \frac{2\mu}{3}(1 - \frac{2f}{f - b^2 V_0 e^{4\mu t}}), \qquad f = \gamma(a_{in}) a_{in}^6, \tag{2.165}$$
$$q(t) = -1 + \frac{12 b^2 f V_0 e^{4\mu t}}{(1 + b^2 V_0 e^{4\mu t})^2}. \tag{2.166}$$

First constraint on our parameters comes from positivity of the Hubble function

$$H(t) > 0 \qquad \Rightarrow \qquad f < b^2 V_0. \tag{2.167}$$

On the other hand $q(t)$ has only one real root

$$q(t) = 0 \qquad \Rightarrow \qquad t = \frac{1}{4\mu} ln[\frac{(5 + 2\sqrt{6})f}{b^2 V_0}]. \tag{2.168}$$

We choose

$$t > 0 \qquad \Rightarrow \qquad f > \frac{b^2 V_0}{\tau}, \qquad \tau = (5 + 2\sqrt{6}). \tag{2.169}$$

One can write

$$f = (\frac{1}{\tau} + \varepsilon) b^2 V_0, \quad 0 < \varepsilon < 1 - \frac{1}{\tau} \quad \Rightarrow \frac{b^2 V_0}{\tau} < f < b^2 V_0. \tag{2.170}$$

Now we will find the condition which results in acceleration at the beginning of the universe

$$\lim_{t->0} q(t) = -1 + \frac{12b^2 f V_0}{(f + b^2 V_0)^2} < 0. \tag{2.171}$$

Thus

$$5 - \frac{1}{\tau} - \sqrt{24} < \varepsilon < 1 - \frac{1}{\tau}, \tag{2.172}$$

$$4.3 \times 10^{-16} < \varepsilon < 0.90. \tag{2.173}$$

With these initial conditions universe starts to expand with acceleration and then turns into deceleration.

2.4.4. Combination Containing Exotic Matter

We have already explored the case of exotic matter in Section 2.3.2. Now we will study combination of this kind of matter and some ordinary matters in the early universe. We have taken the most general form of the potential as

$$V(a) = \frac{V_s}{a^6} + \frac{V_n}{a^n}. \tag{2.174}$$

Related cosmological functions have been found as

$$\gamma(a) = \frac{1}{a^6}[\frac{nV_n}{6-n}a^{6-n} + 6V_s ln(\frac{a}{a_{in}}) + a_{in}^6 \tilde{\gamma}(a_{in})], \quad \tilde{\gamma}(a_{in}) = \gamma(a_{in}) - \frac{nV_n}{(6-n)a_{in}^n}, \tag{2.175}$$

$$H(a) = \sqrt{\frac{8\pi G}{3}[\frac{6V_n}{(6-n)a^n} + \frac{a_{in}^6 \tilde{\gamma}(a_{in}) + V_s[1 + 6ln(\frac{a}{a_{in}})]}{a^6}] - \frac{k}{a^2}}, \tag{2.176}$$

$$\phi(a) = \pm\sqrt{6}\int_{a_{in}}^{a}\Big\{\Big[(6-n)[a_{in}^6\tilde{\gamma}(a_{in}) + 6V_s ln(\frac{a^{'}}{a_{in}})] + nV_n a^{'6}\Big]$$
$$/\Big[(6-n)\{8\pi G[a_{in}^6\tilde{\gamma}(a_{in}) + V_s(1+6ln(\frac{a^{'}}{a_{in}}))] - 3ka^{'4}\}a^{'2+n} + 48\pi G V_n a^{'8}\Big]\Big\}^{1/2} da^{'}$$
$$+\phi(a_{in}), \tag{2.177}$$

$$\rho(a) = \frac{6V_n}{(6-n)a^n} + \frac{V_s}{a^6}ln[e(\frac{a}{a_{in}})^6] + \frac{a_{in}^6\tilde{\gamma}(a_{in})}{a^6}, \tag{2.178}$$

$$p(a) = \frac{2(n-3)V_n}{(6-n)a^n} + \frac{V_s}{a^6}ln[\frac{1}{e}(\frac{a}{a_{in}})^6] + \frac{a_{in}^6\tilde{\gamma}(a_{in})}{a^6}, \tag{2.179}$$

$$q(a) = \frac{8\pi G\{(6-n)[2a_{in}^6\tilde{\gamma}(a_{in}) + V_s(-1+12ln(\frac{a}{a_{in}}))]a^n + 3(n-2)V_n a^6\}}{(6-n)\{8\pi G[a_{in}^6\tilde{\gamma}(a_{in}) + V_s(1+6ln(\frac{a}{a_{in}}))] - 3ka^4\}a^n + 48\pi G V_n a^6}. \tag{2.180}$$

2.4.4.1. Combination with Radiation. When the exotic matter is accompanied with radiation its dynamics in spatially flat universe is governed by the following deceleration parameter

$$q(a) = \frac{2a_{in}^6\tilde{\gamma}(a_{in}) + V_s[-1+12ln(\frac{a}{a_{in}})] + 3V_r a^2}{a_{in}^6\tilde{\gamma}(a_{in}) + V_s[1+6ln(\frac{a}{a_{in}})] + 3V_r a^2}, \quad \tilde{\gamma}(a_{in}) = \gamma(a_{in}) - \frac{2V_r}{a_{in}^4}. \tag{2.181}$$

One can trace its behaviour back into time as

$$q(a_{in}) = \frac{2a_{in}^6\gamma(a_{in}) - V_r a_{in}^2 - V_s}{a_{in}^6\gamma(a_{in}) + V_r a_{in}^2 + V_s}. \tag{2.182}$$

The following condition

$$V_s + V_r a_{in}^2 > 2a_{in}^6\gamma(a_{in}). \tag{2.183}$$

causes accelerated beginning for the universe. Moreover this choice also results in negative total pressure in the beginning as

$$p(a_{in}) = \frac{\gamma(a_{in})a_{in}^6 - V_r a_{in}^2 - V_s}{a_{in}^6} < 0, \tag{2.184}$$

while energy density remains positive. Thus negative pressure results in accelerated motion for a while. Then this behaviour changes as pressure becomes positive and the universe decelerates. Therefore this exotic matter and radiation with initial condition which satisfies (2.183) also has mathematical turn on and turn off mechanism for accelerated motion in the early universe.

2.4.4.2. Combination with Domain Walls. In spatially flat universe the deceleration parameter becomes

$$q(a) = \frac{10a_{in}^6\tilde{\gamma}(a_{in}) - 3v_w a^5 + 5v_s[-1 + 12ln(\frac{a}{a_{in}})]}{5a_{in}^6\tilde{\gamma}(a_{in}) + 6v_w a^5 + 5v_s[1 + 6ln(\frac{a}{a_{in}})]}. \tag{2.185}$$

Although this universe may start to expand with acceleration or deceleration, after a while it will accelerate because the leading term a^5 in the numerator has a negative coefficient.

2.4.4.3. Combination with Dark Energy. In spatially flat universe the deceleration parameter becomes

$$q(a) = \frac{2a_{in}^6\tilde{\gamma}(a_{in}) - v_0 a^6 + v_s[-1 + 12ln(\frac{a}{a_{in}})]}{a_{in}^6\tilde{\gamma}(a_{in}) + v_0 a^6 + v_s[1 + 6ln(\frac{a}{a_{in}})]}. \tag{2.186}$$

Although this universe may start to expand with acceleration or deceleration, after a while it will accelerate because the leading term a^6 in the numerator has a negative coefficient.

2.5. Late Time Expansion of the Universe

We try to understand the present era which is usually described by dark matter and dark energy. However we consider different forms of energy components. Thus in the first subsection we model a universe with domain walls and matter. After obtaining

formulas for functions we compare our results with supernova type Ia data just by curve fitting. In the second subsection we study dark energy dominated universe with the same steps.

2.5.1. Cosmic Walls

The hypothesis that the scalar field is the dark matter and the dark energy was investigated for flat universe in [120]. The results were compared with observations of type Ia supernovae which were available in 2000. In that study, matter part of the universe was neglected and it was found that $\rho_\phi \sim a^{-1.09}$ and $q_0 = -0.45$. Different from them we include matter component of the universe and we solve field equations analytically. Then we compare our results with the type Ia supernovae data released in 2018 [121]. Furthermore in this comparision we extract the value of Hubble constant H_0 and the value of absolute magnitude M with cosmological density parameters.

Today our universe is believed to be almost flat and contribution of radiation to the energy density is very tiny. For this reason we investigate the case in which

$$\rho(a) = \frac{\rho_m}{a^3} + \frac{\rho_w}{a}. \tag{2.187}$$

Domain wall dominated universes have been already studied in Section 2.4.2. Just plugging $n = 3$ in Section 2.4.2.2 we obtain the Hubble function, the scalar field and the potential as

$$H(a) = \sqrt{\frac{8\pi G}{3}(\frac{\rho_w}{a} + \frac{\rho_m}{a^3})}, \tag{2.188}$$

$$\begin{aligned}\phi(a) = \pm\frac{1}{2\sqrt{2\pi G}}\{-\sqrt{3}ln\Big[\frac{(\sqrt{3}\sqrt{\rho_m + \rho_w a^2} + \sqrt{3\rho_m + \rho_w a^2})a_{in}}{(\sqrt{3}\sqrt{\rho_m + \rho_w a_{in}^2} + \sqrt{3\rho_m + \rho_w a_{in}^2})a}\Big] \\ + ln\Big[\frac{\sqrt{\rho_m + \rho_w a^2} + \sqrt{3\rho_m + \rho_w a^2}}{\sqrt{\rho_m + \rho_w a_{in}^2} + \sqrt{3\rho_m + \rho_w a_{in}^2}}\Big]\} + \phi(a_{in}),\end{aligned} \tag{2.189}$$

$$V(a) = \frac{5\rho_w}{6a} + \frac{\rho_m}{2a^3}, \tag{2.190}$$

$$p(a) = -\frac{2\rho_w}{3a}, \tag{2.191}$$

$$q(a) = -\frac{1}{2} + \frac{\rho_m}{\rho_m + \rho_w a^2}, \tag{2.192}$$

$$t = \sqrt{\frac{3}{2\pi G\rho_w}}\Big[\sqrt{a'}\,{}_2F_1(\frac{1}{2}, -\frac{1}{4}; \frac{3}{4}; -\frac{\rho_m}{\rho_w a'^2})\Big]_{a_{in}}^{a}. \tag{2.193}$$

The last relation is obtained by substituting $n = 3$ in (2.164). Investigation of the deceleration parameter tells us

$$\lim_{a\to 0} q(a) = \frac{1}{2}, \qquad \lim_{a\to\infty} q(a) = -\frac{1}{2}, \tag{2.194}$$

$$q(a) = 0 \qquad \Rightarrow \qquad a = \sqrt{\frac{\rho_m}{\rho_w}}. \tag{2.195}$$

Hence if $\rho_m < \rho_w$ this model of the universe accelerates.

To test the reliability of the theoretical model we will use supernovae data. Luminosity distance-redshift (For definition see the Appendix A.9.) relation had been already constructed as [122]

$$d_L = (1+z)R_0 S(\chi(z)). \tag{2.196}$$

The comoving coordinate χ is given as

$$\chi(z) = \frac{c}{R_0}\int_0^z \frac{dz'}{H(z')}. \tag{2.197}$$

The function $r = S(\chi)$ is given by

$$S(\chi) = \begin{cases} Sin(\chi) & \text{if } k = 1, \\ \chi & \text{if } k = 0, \\ Sinh(\chi) & \text{if } k = -1. \end{cases}$$

Thus

$$R_0 S(\chi(z)) = \frac{c}{H_0} \begin{cases} |\Omega_{k,0}|^{-1/2} S(\sqrt{|\Omega_{k,0}|} E(z)) & \text{for } \Omega_k \neq 0, \\ E(z) & \text{for } \Omega_k = 0, \end{cases}$$

where $E(z) = \dfrac{R_0 H_0}{c} \chi(z)$.

For spatially flat universe with matter and domain walls it reduces to the following form

$$d_L = \frac{c(1+z)}{H_0} \int_0^z \frac{dz'}{[\Omega_m (1+z')^3 + \Omega_w (1+z')]^{1/2}} \tag{2.198}$$

where $\Omega_m = 1 - \Omega_w$. The relation between observational measurements and the theory are established as

$$m = 5 log_{10}(\frac{d_L}{1Mpc}) + 25 + M, \tag{2.199}$$

where m and M are the apparent and the absolute magnitudes respectively. Then the distance modulus is defined as $\mu = m - M$.

Before going further we would like to remind the Hubble tension and supernova absolute magnitude tension. Both of them are fundamental cosmological parameters. Their values must be presented precisely.

To determine the value of H_0 different methods have been applied. According to the Planck measurement of the cosmic microwave background (CMB) anisotropies, assuming the base-ΛCDM model [51] $H_0 = 67.36 \pm 0.54 kms^{-1}Mpc^{-1}$. On the other hand Hubble Space Telescope (HST) observations of Cepheids have been used to calibrate the measurements using type Ia supernovae in [123] and it has been declared $H_0 = 74.13 \pm 1.42 kms^{-1}Mpc^{-1}$.

In last years it has been pointed out that the absolute peak magnitude M_B of Type Ia supernovae is converted into a value of H_0 [124–127]. It's value has been stated as $M_B = -19.401 \pm 0.027$ mag [128] in 2020 and $M_B = -19.244 \pm 0.037$ mag [125] in 2021 by application of different methods.

The most recent data set for type Ia supernova observation wchich is called as Pantheon dataset was released in [121]. 1048 data points are presented as (m, z) pairs where $z < 2.3$. Since there is a debate on values of H_0 and M we include their values as parameters which are to be derived from curve fitting. Therefore we have to extract values of ρ_w, H_0 and M from data. To be able to see effects of these numbers on each other separately we applied the curve fitting method for three combinations of two parameter sets.

Then we applied the χ^2 test to measure the goodness of these fits. χ^2 per degrees of freedom, χ^2_ν is calculated according to following formula

$$\chi^2 = \sum_i^N \frac{(\mu_i^{data} - \mu_i^{model})^2}{\sigma_i^2}, \tag{2.200}$$

$$\chi^2_\nu = \frac{\chi^2}{\nu}, \qquad \nu = N - k, \tag{2.201}$$

where k is the number of parameters that will be extracted from the N number of data points.

First, we assign trial number for Ω_w and results are shown in Table 2.1. The best fit which is obtained for $\Omega_w = 0.90$ gives $H_0 = 72.1563 \pm 0.0001$, $M = -19.269 \pm 0.004$ and $\chi^2/\nu = 1.009$.

Table 2.1. Values of H_0 and M for Given Ω_w.

Ω_w	H_0	M	χ^2/ν
0.70	72.0756 ± 0.0001	-19.213 ± 0.004	1.124
0.75	72.0945 ± 0.0001	-19.227 ± 0.004	1.075
0.80	72.1142 ± 0.0001	-19.240 ± 0.004	1.038
0.85	72.1347 ± 0.0001	-19.255 ± 0.004	1.014
0.90	72.1563 ± 0.0001	-19.269 ± 0.004	1.009
0.95	72.1789 ± 0.0001	-19.285 ± 0.004	1.026
0.9999	72.2028 ± 0.0001	-19.301 ± 0.004	1.073

Table 2.2. Values of M and Ω_w for Given H_0.

H_0	M	Ω_w	χ^2/ν
74	-19.211 ± 0.006	0.888 ± 0.015	1.008
73	-19.240 ± 0.006	0.888 ± 0.015	1.008
72	-19.270 ± 0.006	0.888 ± 0.015	1.008
71	-19.301 ± 0.006	0.888 ± 0.015	1.008
70	-19.332 ± 0.006	0.888 ± 0.015	1.008
69	-19.363 ± 0.006	0.888 ± 0.015	1.008
68	-19.395 ± 0.006	0.888 ± 0.015	1.008
67	-19.427 ± 0.006	0.888 ± 0.015	1.008

After getting intuition about parameters we perform curve fitting for trial H_0 values. Results are given in Table 2.2. Numbers in Table 2.2 tell us that the hundredths digit of M is sensitively depended on the ones digit of H_0. In addition, more accurate result for Ω_w is obtained. All of the results have the same χ^2/ν value. Thus we choose

3rd line: $H = 72$, $M = -19.270 \pm 0.006$, $\Omega_w = 0.888 \pm 0.015$ and $\chi^2/\nu = 1.008$. These numbers are compatible with best fit of the table 2.1

Effect of trial values of M on H_0 and Ω_w are presented in Table 2.3. It is apparent that the value of the ones digit of H_0 is sensitively depended on the hundredths digit of M. Our choice is the 5th line: $M = 19.25$, $H_0 = 72.68 \pm 0.21$, $\Omega_w = 0.889 \pm 0.015$ and $\chi^2/\nu = 1.008$. These numbers are in agreement with best fit of the table 2.1.

Table 2.3. Values of H_0 and Ω_w for Given M.

M	H_0	Ω_w	χ^2/ν
-19.45	66.29 ± 0.19	0.889 ± 0.015	1.008
-19.40	67.83 ± 0.20	0.889 ± 0.015	1.008
-19.35	69.41 ± 0.20	0.889 ± 0.015	1.008
-19.30	71.03 ± 0.20	0.889 ± 0.015	1.008
-19.25	72.68 ± 0.21	0.889 ± 0.015	1.008
-19.20	74.38 ± 0.22	0.889 ± 0.015	1.008
-19.15	76.11 ± 0.22	0.889 ± 0.015	1.008

Assuming base ΛCDM cosmology, late universe parameters were found as $H_0 = 67.27 \pm 0.60$, $\Omega_m = 0.3166 \pm 0.0084$ and $\Omega_\Lambda = 0.6834 \pm 0.0084$ in [51]. It is known that $\Omega_m = \Omega_{bm} + \Omega_{dm}$ where $\Omega_{bm} \simeq 0.05$ and $\Omega_{dm} \simeq 0.27$. However our results indicate that late universe parameters as $H_0 = 72.68 \pm 0.21$, $\Omega_w = 0.889 \pm 0.015$ and $\Omega_m = 0.111 \pm 0.015$. Since $\Omega_{bm} \simeq 0.05$, $\Omega_{dm} \simeq 0.06$. Therefore domain wall dominated universe is a candidate to explain 94 percentage of the structures in the present universe while still 6 percentage of the universe remains as unknown.

To compare our results with Pantheon-data graphically we draw distance modulus μ vs redshift z plot. In Figure 2.1 we use results given in 5th line of Table 2.3 where $M = -19.25$, $H_0 = 72.68 \pm 0.21$, $\rho_w = 0.889 \pm 0.015$ and $\rho_m = 0.111 \pm 0.015$.

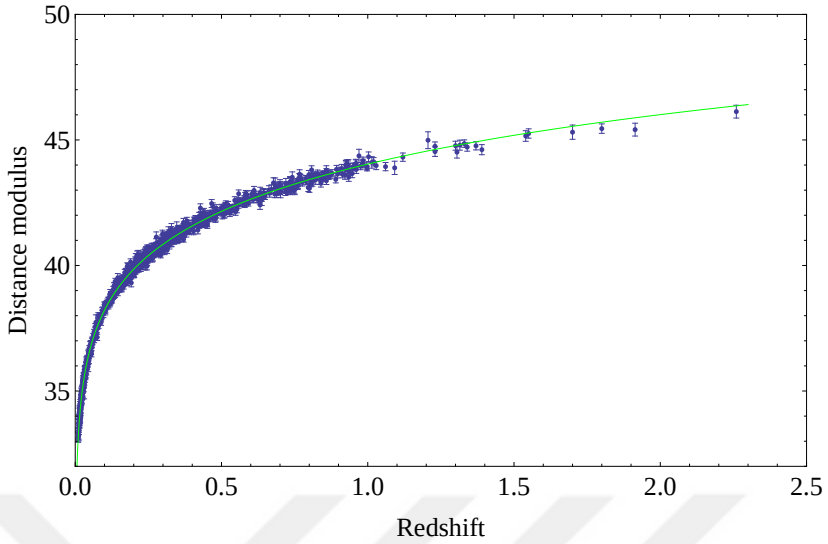

Figure 2.1. Distance Modulus vs Redshift Plot for Domain Wall Dominated Universe. Dots represent observation of Pantheon data while green line represents domain wall dominated universe.

We obtain $q_0 = -0.389$ by using values of cosmological density parameters in (2.192).

2.5.2. Dark Energy

Now we will present exact solutions for energy density given as

$$\rho = \rho_0 + \frac{\rho_m}{a^3}. \tag{2.202}$$

Actually this case corresponds to $n = 3$ in Section 2.4.1. For $k = 0$ we have already obtained the scalar factor as

$$a(t) = [\frac{(a_{in}^{3/2} + \sqrt{a_{in}^3 + \frac{\rho_m}{\rho_0}})e^{3\mu t/2} + (a_{in}^{3/2} - \sqrt{a_{in}^3 + \frac{\rho_m}{\rho_0}})e^{-3\mu t/2}}{2}]^{2/3}, \tag{2.203}$$

where $\mu = \sqrt{\frac{8\pi G\rho_0}{3}}$. In addition potential is formulated as

$$V(\psi) = \frac{1}{2}\frac{\rho_m}{a^3(\psi)} + \rho_0, \tag{2.204}$$

$$a(\psi) = 2^{2/3}\{\frac{a_{in}^{3/2}e^{\sqrt{3}\psi}[-1+\sqrt{1+a_{in}^3\frac{\rho_0}{\rho_m}}-(1+\sqrt{1+a_{in}^3\frac{\rho_0}{\rho_m}})e^{2\sqrt{3}\psi}]}{a_{in}^3\frac{\rho_0}{\rho_m}(1+e^{4\sqrt{3}\psi})-2(2+a_{in}^3\frac{\rho_0}{\rho_m})e^{2\sqrt{3}\psi}}\}^{2/3}, \tag{2.205}$$

$$\psi = \pm\sqrt{2\pi G}[\phi(a)-\phi(a_{in})]. \tag{2.206}$$

Behaviour of the deceleration parameter is shown by

$$q(a) = \frac{-2\rho_0 a^3+\rho_m}{2(\rho_0 a^3+\rho_m)}, \tag{2.207}$$

$$\lim_{a\to\infty} q(a) = -1. \tag{2.208}$$

To extract cosmological parameters from Pantheon data we apply the procedure as explained in the previous subsection with modification

$$d_L = \frac{c(1+z)}{H_0}\int_0^z \frac{dz'}{[\Omega_m(1+z')^3+\Omega_0]^{1/2}}, \tag{2.209}$$

where $\Omega_m = 1-\Omega_0$.

Table 2.4. Values of H_0 and M for Given Ω_0.

Ω_0	H_0	M	χ^2/ν
0.50	72.0674 ± 0.0001	-19.208 ± 0.004	1.199
0.60	72.1183 ± 0.0001	-19.243 ± 0.004	1.059
0.70	72.1749 ± 0.0001	-19.282 ± 0.004	0.992
0.80	72.2384 ± 0.0001	-19.324 ± 0.004	1.041
0.90	72.3116 ± 0.0001	-19.372 ± 0.005	1.296

First, we perform curve fitting for trial values of Ω_0. Results are shown in Table 2.4. The best fit occurs for $\Omega_0 = 0.70$. Thus $H_0 = 72.1749 \pm 0.0001$, $M = -19.282 \pm 0.004$ and $\chi^2/\nu = 0.992$.

Then we perform curve fitting for trial H_0 values. Results are given in Table 2.5. Results in the third line are compatible with the best fit of Table 2.4. $M = -19.294 \pm 0.007$, $\Omega_0 = 0.715 \pm 0.012$ and $\chi^2/\nu = 0.990$ are obtained for a given $H_0 = 72$.

Table 2.5. Values of M and Ω_0 for Given H_0.

H_0	M	Ω_0	χ^2/ν
74	-19.234 ± 0.007	0.715 ± 0.012	0.990
73	-19.264 ± 0.007	0.715 ± 0.012	0.990
72	-19.294 ± 0.007	0.715 ± 0.012	0.990
71	-19.324 ± 0.007	0.715 ± 0.012	0.990
70	-19.355 ± 0.007	0.715 ± 0.012	0.990
69	-19.386 ± 0.007	0.715 ± 0.012	0.990
68	-19.418 ± 0.007	0.715 ± 0.012	0.990
67	-19.450 ± 0.007	0.715 ± 0.012	0.990

Finally we test the effect of M on H_0 and Ω_0. Results are presented in Table 2.6. Numbers in the fourth line are compatible with the best fit of Table 2.4. $H_0 = 71.80 \pm 0.22$, $\Omega_0 = 0.715 \pm 0.012$ and $\chi^2/\nu = 0.990$ are found for a given $M = -19.30$.

All the tables in this section have a common interpretation: The number in ones digit of the Hubble constant H_0 is sensitively depended on the number in the hundredths digit of the absolute magnitude M in both models. In addition as the value of H_0 increases, the value of M decreases. We will stop to dig more about this argument here since it is beyond the scope of our aims. Cosmologists will continue to

reveal the relation between H_0 and M more clearly on further studies.

Now we would like to compare our results for a domain wall dominated universe and a dark energy dominated universe in the same plot. However this goal can not be achieved accurately, because best fit values of M are different for both models. For this reason we plot two figures.

Table 2.6. Values of H_0 and Ω_0 for Given M.

M	H_0	Ω_0	χ^2/ν
-19.45	67.00 ± 0.21	0.715 ± 0.012	0.990
-19.40	68.57 ± 0.21	0.715 ± 0.012	0.990
-19.35	70.16 ± 0.22	0.715 ± 0.012	0.990
-19.30	71.80 ± 0.22	0.715 ± 0.012	0.990
-19.25	73.47 ± 0.23	0.715 ± 0.012	0.990
-19.20	75.18 ± 0.23	0.715 ± 0.012	0.990
-19.15	76.93 ± 0.24	0.715 ± 0.012	0.990

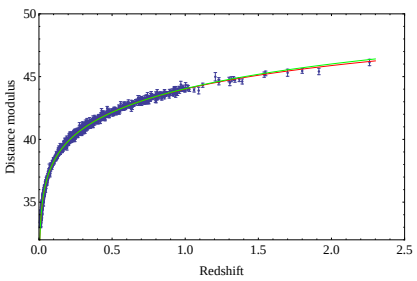

Figure 2.2. Distance Modulus vs Redshift Plot for $M = -19.25$.

Dots represent observations, green line represents domain wall dominated universe and red line represents dark energy dominated universe.

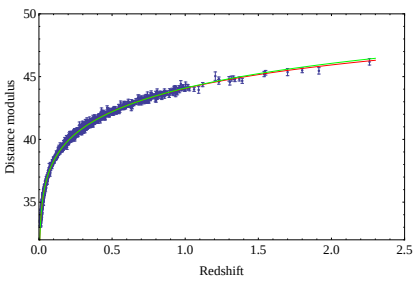

Figure 2.3. Distance Modulus vs Redshift Plot for $M = -19.30$.

Dots represent observations, green line represents domain wall dominated universe and red line represents dark energy dominated universe.

We draw our Figure 2.2 by taking $M = -19.25$ for which one of the best fits of the domain wall dominated universe is obtained with parameters $H_0 = 72.68 \pm 0.21$, $\Omega_w = 0.889 \pm 0.015$ and $\Omega_m = 0.111 \pm 0.015$. On the other hand we have obtained $H_0 = 73.47 \pm 0.23$, $\Omega_0 = 0.715 \pm 0.012$, $\Omega_m = 0.285 \pm 0.12$ and $\chi^2/\nu = 0.990$ for $M = -19.25$.

In Figure 2.3 we choose one of the best fits of the dark energy dominated universe for which $M = -19.30$, $H_0 = 71.80 \pm 0.22$, $\Omega_0 = 0.715 \pm 0.012$ and $\Omega_m = 0.285 \pm 0.012$. In the first model we have obtained $H_0 = 71.03 \pm 0.20$, $\Omega_w = 0.889 \pm 0.015$, $\Omega_m = 0.111 \pm 0.015$ and $\chi^2/\nu = 0.990 = 1.008$ for $M = -19.30$

Two figures are almost the same because χ^2/ν for both models are very close the 1. We need more data for bigger redshift values to decide whether one of the models is superior to the other one. We obtain $q_0 = -0.572$ by using these values of cosmological parameters in (2.207).

2.6. Discussion

We studied FLRW cosmology with real scalar field which is minimally coupled to gravity. Our main motivation in this chapter has been to assign the effective scalar field a source of all components of energy density. We applied a change of variable twice which is a more powerful method in the set of differential equations which represent dynamics of the universe. In the first one, we have replaced the independent variable "t" with "a". In the second one, we have changed the dependent variable of the ϕ equation so that it became a first order linear differential equation. We presented exact solutions in four different forms; solutions for given $V(a)$, solutions for given $\phi(a)^{'}$, solutions for given $H(a)$, and solutions for given $\rho(a)$.

Then we have examined single component universes and two component universes for a given energy density and for a given potential. In these solutions we have searched for mathematical mechanisms which create turn on and turn off for early

inflationary expansion. We have explored mathematical structure of a new exotic matter. Equation of state for this component changes with the scale factor or equivalently changes with time. A universe which consists of radiation and this exotic matter, has mathematical machinery to turn on and to turn off inflationary expansion in early epoch.

We have investigated the present era of the universe. Domain wall dominated universe and dark energy dominated universe have been studied. We have extracted numerical values of cosmological parameters from the most recent type Ia supernova data by taking care of the Hubble tension and the absolute magnitude tension. For domain wall dominated universe we have found that $\Omega_w = 0.889 \pm 0.015$, $\Omega_m = 0.111 \pm 0.015$ and $H_0 = 72.68 \pm 0.21$ for $M = -19.25$. This universe accelerates with $q_0 = -0.389$.

On the other hand for dark energy dominated universe cosmological parameters have been found as $\Omega_0 = 0.715 \pm 0.012$, $\Omega_m = 0.285 \pm 0.012$ and $H_0 = 71.80 \pm 0.22$ for $M = -19.30$. Deceleration parameter of this universe is $q = -0.572$. Detailed analysis for the relation between distance modulus and redshift have shown that the number in ones digit of the Hubble constant H_0 is sensitively depended on the number in the hundredths digit of the absolute magnitude M in both models.

These two analyses indicate that both models equivalently explains dynamics of late time accelerated expansion of the universe. The difference between these models most probably will be seen when bigger redshift data are available. One can also reach our calculations which are presented in this chapter in [129].

3. ANALYTIC SOLUTIONS OF BRANS-DICKE COSMOLOGY

The major purpose of this chapter is to solve field equations of Brans-Dicke cosmology analytically. First, we present field equations in Section 3.1. Then we introduce our mathematical techniques in Section 3.2 and we rewrite the field equations with the scale size "a" as the new independent variable. This set of equations can be reduced to a constraint equation and a first order differential equation. Exact solutions are found in Section 3.3 for given $\phi(a)$ and $\rho(a)$, in Section 3.4 for given $\phi(a)$ and $V(a)$, in Section 3.5 for given $\phi(a)$ and $H(a)$. A universe consisting of single component energy-matter density is investigated in Section 3.6. The early epoch of the universe is studied in Section 3.7. We present a solution for a universe which contains dark energy and matter in Section 3.8.1 and we present a solution for a universe which consists of cosmic domain walls and matter in Section 3.8.2. Comparison of both models with observation is illustrated by a plot. Discussion and summary are given in Section 3.9. In this study we choose Brans-Dicke parameter $\omega > 4 \times 10^4$ to be compatible with results of Einstein telescope [130] and time delay experiments [131].

3.1. Field Equations

In this study we will use FLWR metric given by,

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{3.1}$$

where k is the curvature parameter, $a(t) = \dfrac{R(t)}{R(t_0)}$ is the normalized scale factor of the universe. Here r has dimension of $lenght$, $a(t)$ is dimensionless and k has diemsion of

$lenght^{-2}$. In BDJT the Lagrange density can be written as [23, 132–135]

$$\mathcal{L} = \left[-\Phi R + \omega\frac{1}{\Phi}g^{\mu\nu}\partial_\mu\Phi\partial_\nu\Phi - V(\Phi) + L_M\right]\sqrt{-g}, \tag{3.2}$$

$$= \left[-\frac{1}{8\omega}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + L_M\right]\sqrt{-g}, \tag{3.3}$$

where we will call ϕ as the Jordan scalar field and Φ as the Brans-Dicke scalar field, the two being related by

$$\Phi = \frac{1}{8\omega}\phi^2, \tag{3.4}$$

where ω is the dimensionless Brans-Dicke parameter, R is the Ricci scalar and L_M represents the contribution due to matter fields. We use the metric signature $(+,-,-,-)$ and units with $\hbar = 1$, $c = 1$. We prefer to use the Jordan scalar for the scalar field since in flat spacetime the Lagrangian then becomes the standard Lagrangian for a scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - V(\phi). \tag{3.5}$$

The homogeneous and isotropic cosmological field equations obtained from this Lagrangian density have already been calculated for a potential $V(\phi) = \frac{1}{2}m^2\phi^2$ by Arik, Calik and Sheftel in [136–138]. Thus their general form for a potential $V(\phi)$ are given by,

$$\frac{3}{4\omega}\phi^2\left(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) - \frac{1}{2}\dot{\phi}^2 - V(\phi) + \frac{3}{2\omega}\frac{\dot{a}}{a}\dot{\phi}\phi = \rho_m, \tag{3.6}$$

$$\frac{-1}{4\omega}\phi^2\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) - \frac{1}{\omega}\frac{\dot{a}}{a}\dot{\phi}\phi - \frac{1}{2\omega}\ddot{\phi}\phi - \left(\frac{1}{2} + \frac{1}{2\omega}\right)\dot{\phi}^2 + V(\phi) = p_m, \tag{3.7}$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV(\phi)}{d\phi} - \frac{3}{2\omega}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)\phi = 0. \tag{3.8}$$

3.2. Mathematical Techniques

It is apparent that the independent variable time t is not seen explicitly in our set of differential equations given by (3.6-3.8). For this reason one can choose a new variable in terms of old dependent variables. Hence the order of differential equation can be reduced by one [117]. Our choice is $\dot{a} = aH(a)$. Then a becomes new independent variable and field equations transform into the following form

$$\frac{3}{4\omega}\phi^2\left(H^2+\frac{k}{a^2}\right)-\frac{1}{2}\left(\phi^{'}Ha\right)^2-V(a)+\frac{3}{2\omega}\phi\phi^{'}H^2a=\rho_m, \tag{3.9}$$

$$-\frac{1}{4\omega}\phi^2\left(3H^2+2H^{'}Ha+\frac{k}{a^2}\right)-\frac{3}{2\omega}\phi^{'}\phi H^2a-\frac{1}{2\omega}\left(\phi^{''}H^2a^2+\phi^{'}H^{'}Ha^2\right)\phi$$
$$-\left(\frac{1}{2}+\frac{1}{2\omega}\right)\left(\phi^{'}Ha\right)^2+V(a)=p_m, \tag{3.10}$$

$$\phi^{''}H^2a^2+\phi^{'}H^{'}Ha^2+4\phi^{'}H^2a+\frac{V^{'}}{\phi^{'}}-\frac{3}{2\omega}\left(2H^2+H^{'}Ha+\frac{k}{a^2}\right)\phi=0, \tag{3.11}$$

where the prime denotes $\dfrac{d}{da}$ (For details of the transformation see Appendix A.3.). We will name (3.9) as the energy density equation, (3.10) as the pressure equation and (3.11) as the ϕ equation.

These three equations are connected by the perfect fluid continuity equation,

$$\dot{\rho}(a)+3\frac{\dot{a}}{a}(\rho(a)+p(a))=0, \tag{3.12}$$

$$[\rho^{'}(a)a+3(\rho(a)+p(a))]H(a)=0. \tag{3.13}$$

In other words when we plug energy density and pressure which are given by (3.9-3.10) in (3.13) we end up with $-a\phi^{'}(a)H(a)\times$ (3.11). This shows that we have two independent equations and four unknown functions: $H(a)$, $V(a)$, $\phi(a)$ and $\rho(a)$ (one can choose $p(a)$ instead of $\rho(a)$).

In our calculations we will use the energy density equation and the ϕ equation. When we plug solutions in to the pressure equation, $\rho(a)$ and the resulting $p(a)$ will satisfy the perfect fluid equation of state.

It is seen that energy density equation is a first order variable coefficient and nonlinear differential equation for $\phi(a)$. In addition this is not a type of Bernoulli equation which is analytically solvable. In addition (3.11) is second order variable coefficient nonlinear differential equation for $\phi(a)$. Analytic solutions for both of them have not been established yet. To be able to solve them, $\phi(a)$ should be given. Then we will be left with $\rho(a)$, $V(a)$ and $H(a)$. Since (3.9) does not contain any derivative of last three functions it is just a constraint. On the other hand first derivatives of $H(a)$ and $V(a)$ are seen in (3.11). Thus our system consist of a constraint equation and a first order differential equation. One of the following functions; $H(a)$, $V(a)$ and $\phi(a)$ is still free. One of them should be accompanied with $\phi(a)$ as a given function. Three combinations exist; $\rho(a)$ and $\phi(a)$, $V(a)$ and $\phi(a)$, $H(a)$ and $\phi(a)$. We will solve our system for these three sets separately.

3.3. Solution for Given $\rho(a)$ and $\phi(a)$

One can solve the constraint Equation (3.9) and the first order differential Equation (3.11) by different methods:

- Method 1) First, find $V(a)$ from the constraint equation then substitute it in the differential equation and solve it for $H(a)$.
- Method 2) First, solve the differential equation for $V(a)$ then substitute it in the constraint equation and solve it for $H(a)$.
- Method 3) First, find $H(a)$ from the constraint equation then substitute it in the differential equation and solve it for $V(a)$.
- Mehod 4) First, solve the differential equation for $H(a)$ then substitute it in the constraint equation and solve it for $V(a)$.

Results are different representations of the same solution. However they may have different singular cases which require special attention. To reach complete set of solutions we perform all four methods. Moreover, most of the solutions include integrals of different functions and sometimes integration techniques can be inadequate . For this reason one of the form of solutions can be superior to others. We would like to remind that the ρ equation is equivalent to the constraint equation and the first order differential equation is equivalent to the ϕ equation. We will use both names.

3.3.1. Method 1

3.3.1.1. Non-Singular Case. Firstly we find $V(a)$ from the constraint equation

$$V(a) = \frac{3}{4\omega}\phi^2(a)(H^2(a) + \frac{k}{a^2}) - \frac{1}{2}(\phi^{'}(a)H(a)a)^2 + \frac{3}{2\omega}\phi\phi^{'}H^2a + \rho(a). \tag{3.14}$$

Then the ϕ equation becomes

$$H^{'}(a) + \frac{a[(1+2\omega)\phi^{'2}(a) + \phi(a)\phi^{''}(a)]}{\phi(a)(\phi(a) + a\phi^{'}(a))}H(a) = \frac{2\omega\rho^{'}a^3 + 3k\phi^2(a)}{3a^3\phi(a)(\phi(a) + a\phi^{'}(a))H(a)}, \tag{3.15}$$

which is the Bernoulli equation. Thus we apply change of variable $\gamma(a) = H^2(a)$ and we obtain

$$\gamma^{'}(a) + \frac{2a[(1+2\omega)\phi^{'2}(a) + \phi(a)\phi^{''}(a)]}{\phi(a)(\phi(a) + a\phi^{'}(a))}\gamma(a) = \frac{2(2\omega\rho^{'}a^3 + 3k\phi^2(a))}{3a^3\phi(a)(\phi(a) + a\phi^{'}(a))}. \tag{3.16}$$

Its solution is found as

$$\gamma(a) = \frac{exp\Big[-\int_{a_{in}}^{a} P(a^{'})da^{'}\Big]}{\Big[\phi(\phi + a\phi^{'})\Big]^2}\Big\{\int_{a_{in}}^{a} exp\Big[\int_{a_{in}}^{a^{'}} P(a^{''})da^{''}\Big]Q(a^{'})da^{'} + \tilde{\gamma}(a_{in})\Big\}, \tag{3.17}$$

$$P(a) = \frac{2[2\omega a\phi^{'2}(a) - 3\phi(a)\phi^{'}(a)]}{\phi(a)(\phi(a) + a\phi^{'}(a))}, \quad Q(a) = \frac{2(2\omega\rho^{'}a^3 + 3k\phi^2(a))\Big[\phi(a)(\phi(a) + a\phi^{'}(a))\Big]}{3a^3}, \tag{3.18}$$

$$\tilde{\gamma}(a_{in}) = \gamma(a_{in})\Big[\phi(a_{in})\Big(\phi(a_{in}) + a_{in}\phi^{'}(a_{in})\Big)\Big]^2. \tag{3.19}$$

Hence

$$H(a) = \sqrt{\gamma(a)}, \tag{3.20}$$

where $\gamma(a)$ is given by (3.17) (Details of this calculation is given in the Appendix B.3.). One should choose $\phi(a)$ such that $\gamma(a)$ is positive. Thus $H(a)$ will be a real function. However if one goes further by assigning specific form to the Hubble function and tries to solve (3.17) for $\phi(a)$, she will end up with a second order, variable coefficient, nonlinear differential equation for whose solution there is no known method. For this reason we will continue with trial $\phi(a)$ functions in our examples.

3.3.1.2. Singular Case. The denominator of the function $P(a)$ which is used to formulate the Hubble function becomes singular when $\phi(a) = F/a$. Hence this case needs special attention. $V(a)$ is obtained form the ρ equation,

$$V(a) = -\rho(a) + \frac{F^2\Big[3k - (3+2\omega)a^2H^2(a)\Big]}{4\omega a^4}. \tag{3.21}$$

Then the ϕ equation turns into the following form

$$\frac{3F^2\Big[k - (3+2\omega)a^2H^2(a)\Big] + 2\omega a^5\rho^{'}(a)}{2F\omega a^3} = 0. \tag{3.22}$$

The Hubble function is found as

$$H(a) = \sqrt{\frac{3F^2k + 2\omega a^5\rho^{'}(a)}{3(3+2\omega)F^2a^2}}. \tag{3.23}$$

Hence the final form of potential is given by

$$V(a) = -\rho(a) - \frac{a}{6}\rho^{'}(a) + \frac{kF^2}{2\omega a^4}. \tag{3.24}$$

If one inserts the energy density of an ordinary matters which is proportional to $1/a^n$ results are

$$H(a) = \sqrt{\frac{3F^2k - 2n\omega\rho_n a^{4-n}}{3(3+2\omega)F^2a^2}}, \tag{3.25}$$

$$V(a) = \frac{F^2k}{2\omega a^4} + \frac{(n-6)\rho_n}{a^n}, \tag{3.26}$$

$$V(\phi) = \frac{F^2k}{2\omega}\phi^4 + (n-6)\rho_n\phi^n. \tag{3.27}$$

Except the radiation dominated universe and dark energy dominated universe all these matters results in an imaginary form of the Hubble function and negative potential values as easily recognized from (3.25-3.26). In addition radiation dominated universe corresponds to $V = \lambda\phi^4/4$ which will be studied in chapter 3 of this thesis.

3.3.2. Method 2

3.3.2.1. Non-Singular Case. In the second method we start with the ϕ equation. By using it $V(a)$ is found as

$$V(a) = -\int_{a_{in}}^{a} \frac{\phi^{'}\Big\{-3\phi\Big[k + a^{'2}H(2H + a^{'}H^{'})\Big] + 2a^{'3}\omega H\Big[(4H + a^{'}H^{'})\phi^{'} + a^{'}H\phi^{''}\Big]\Big\}}{2a^{'2}\omega} da^{'}$$
$$+V(a_{in}). \tag{3.28}$$

Then we insert $V(a)$ in ρ equation and we take the derivative of both sides with respect to a. We obtain the following equation

$$H^{'}(a) + \frac{a[(1+2\omega)\phi^{'2}(a) + \phi(a)\phi^{''}(a)]}{\phi(a)(\phi(a) + a\phi^{'}(a))}H(a) = \frac{2\omega\rho^{'}(a)a^3 + 3k\phi^2(a)}{3a^3\phi(a)(\phi(a) + a\phi^{'}(a))H(a)}. \tag{3.29}$$

This equation is equivalent to (3.15). Its solution has been already presented in (3.17-3.20). Although forms of potentials which are given by (3.24) and (3.28) are different, one can show that they are equal to each other up to a constant.

We have just shown that results of method 1 and method 2 are the same, thus there is no need to repeat calculations for singular case. In this method we end up with two integration constants. One of them should be chosen such that when final forms of formulas are inserted in the constraint equation it must be satisfied.

3.3.3. Method 3

3.3.3.1. Non-Singular Case. First we find $H(a)$ from the ρ equation

$$H(a) = \sqrt{\frac{-3k\phi^2(a) + 4\omega a^2\Big[\rho(a) + V(a)\Big]}{a^2\Big[3\phi^2(a) + 6a\phi(a)\phi^{'}(a) - 2\omega a^2\phi^{'2}\Big]}}. \tag{3.30}$$

Then the ϕ equation turns into the following form

$$V^{'}(a) + P(a)V(a) = Q(a), \tag{3.31}$$

where

$$\begin{aligned} P(a) = &\Big\{2\phi^{'}(a)\Big[-6\phi^3(a) + (3+14\omega)a^2\phi(a)\phi^{'2}(a) - 2\omega(1+2\omega)a^3\phi^{'3}(a) \\ &+ a\phi^2(a)((-6+8\omega)\phi^{'}(a) + (3+2\omega)a\phi^{''}(a))\Big]\Big\} \\ &/\Big\{\phi(a)(\phi(a) + a\phi^{'}(a))\Big[3\phi^2(a) + 6a\phi(a)\phi^{'}(a) - 2\omega a^2\phi^{'2}(a)\Big]\Big\}, \end{aligned} \tag{3.32}$$

and

$$Q(a) = -\Big\{\phi^{'}(a)\Big[- 36\omega^2 a^4\phi(a)\rho^{'}(a)\phi^{'2}(a) + 8\omega^3 a^5\rho^{'}(a)\phi^{'3}(a)$$
$$- 12\omega a^2\phi^2(a)\phi^{'}(a)((-3+\omega)a\rho^{'}(a) + k(3+2\omega)\phi^{'2}(a)) + 9a\phi^3(a)(2\omega a\rho^{'}(a)$$
$$+ 3k(3+2\omega)\phi^{'2}(a)) + 9k(3+2\omega)\phi^4(a)(3\phi^{'}(a) + a\phi^{''}) + 12\omega a\rho(a)(6\phi^3(a)$$
$$- (3+14\omega)a^2\phi(a)\phi^{'2}(a) + 2\omega(1+2\omega)a^3\phi^{'3}(a) - a\phi^2(a)((-6+8\omega)\phi^{'}(a)$$
$$+ (3+2\omega)a\phi^{''}(a)))\Big]\Big\}$$
$$/\Big\{6\omega a\phi(a)(\phi(a) + a\phi^{'}(a))\Big[- 3\phi^2(a) - 6a\phi(a)\phi^{'}(a) + 2\omega a^2\phi^{'2}(a)\Big]\Big\}. \qquad (3.33)$$

Hence potential is

$$V(a) = exp\Big(- \int_{a_{in}}^{a} P(a^{'})da^{'}\Big)\Big\{ \int_{a_{in}}^{a} exp\Big(\int_{a_{in}}^{a^{'}} P(a^{''})da^{''}\Big)Q(a^{'})da^{'} + V(a_{in})\Big\}. \qquad (3.34)$$

3.3.3.2. Singular Cases. The solution for method 3 has three singular cases;

I) $\phi(a) = 0$,

II)$\phi(a) + a\phi^{'}(a) = 0$,

III)$-3\phi^2(a) - 6a\phi(a)\phi^{'}(a) + 2\omega a^2\phi^{'2}(a) = 0$.

For all of them both functions, $P(a)$ and $Q(a)$ which are used to formulate the potential become undefined.

First case is not acceptable because Brans-Dicke theory collapses for $\phi(a) = 0$. Second case has been already investigated in Method 1. We are left with the last one. We write $\phi(a) = Fexp(\int \alpha(a)da)$. Then singularity III becomes

$$2\omega a^2\alpha^2 - 6a\alpha - 3 = 0. \qquad (3.35)$$

Thus

$$\alpha(a) = \frac{3 \pm \sqrt{9+6\omega}}{2\omega a}, \qquad \phi(a) = F a^{(3\pm\sqrt{9+6\omega})/(2\omega)}. \tag{3.36}$$

Then the potential is found from the ρ equation

$$V(a) = \frac{3F^2 k}{4\omega} a^{(3-2\omega\pm\sqrt{9+6\omega})/\omega} - \rho(a). \tag{3.37}$$

After writing $H(a) = \sqrt{\gamma(a)}$, ϕ equation becomes

$$\gamma^{'}(a) + P(a)\gamma(a) = Q(a). \tag{3.38}$$

For $\alpha(a) = \dfrac{3+\sqrt{9+6\omega}}{2\omega a}$,

$$P(a) = \frac{2(3+3\omega+\sqrt{9+6\omega})}{\omega a}, \tag{3.39}$$

$$Q(a) = \frac{4\omega a^{-(3+3\omega+\sqrt{9+6\omega})/\omega}}{3F^2(3+2\omega+\sqrt{9+6\omega})}\Big[3F^2 k a^{(3+\sqrt{9+6\omega})/\omega} + 2\omega a^3 \rho^{'}(a)\Big]. \tag{3.40}$$

Then

$$\gamma(a) = \frac{1}{a^{2(3+3\omega+\sqrt{9+6\omega})/\omega}}\Big\{b\int_{a_{in}}^{a} a^{'(3+3\omega+\sqrt{9+6\omega})/\omega}\Big[3F^2 k a^{'(3+\sqrt{9+6\omega})/\omega} + 2\omega a^{'3}\rho^{'}\Big]da^{'} + \tilde{\gamma}(a_{in})\Big\}, \tag{3.41}$$

$$b = \frac{4\omega}{3F^2(3+2\omega+\sqrt{9+6\omega})}, \qquad \tilde{\gamma}(a_{in}) = a_{in}^{2(3+3\omega+\sqrt{9+6\omega})/\omega}\gamma(a_{in}). \tag{3.42}$$

For $\alpha(a) = \dfrac{3-\sqrt{9+6\omega}}{2\omega a}$,

$$P(a) = \frac{2(3+3\omega-\sqrt{9+6\omega})}{\omega a}, \tag{3.43}$$

$$Q(a) = \frac{4\omega a^{-3(1+\omega)/\omega}}{3F^2(3+2\omega-\sqrt{9+6\omega})}\Big[3F^2 k a^{3/\omega} + 2\omega a^{(3\omega+\sqrt{9+6\omega})/\omega}\rho^{'}(a)\Big]. \tag{3.44}$$

Then

$$\gamma(a) = \frac{1}{a^{2(3+3\omega-\sqrt{9+6\omega})/\omega}}\Big\{b\int_{a_{in}}^{a} a^{'(3+3\omega-2\sqrt{9+6\omega})/\omega}\Big[3F^2ka^{'3/\omega} + 2\omega a^{'(3\omega+\sqrt{9+6\omega})/\omega}\rho'\Big]da' + \tilde{\gamma}(a_{in})\Big\}, \tag{3.45}$$

$$b = \frac{4\omega}{3F^2(3+2\omega-\sqrt{9+6\omega})}, \qquad \tilde{\gamma}(a_{in}) = a_{in}^{2(3+3\omega-\sqrt{9+6\omega})/\omega}\gamma(a_{in}). \tag{3.46}$$

At the next step we examine nature of the Hubble function. First, we take

$$\phi(a) = Fa^{(3+\sqrt{9+6\omega})/(2\omega)}, \qquad \rho = \frac{\rho_n}{a^n}. \tag{3.47}$$

Hubble function and the potential are found as

$$H(a) = \frac{1}{a^{(3+3\omega+\sqrt{9+6\omega})/\omega}}\Big\{\frac{2\omega^2 a^{(3-(n-4)\omega+\sqrt{9+6\omega})/\omega}}{3F^2(3+2\omega+\sqrt{9+6\omega})^2[3+(6-n)\omega+\sqrt{9+6\omega}]} \times\Big[-4(3+2\omega+\sqrt{9+6\omega})n\omega\rho_n a^2 + 3F^2k\Big(3+(6-n)\omega+\sqrt{9+6\omega}\Big)a^{(3+n\omega+\sqrt{9+6\omega})/\omega}\Big] + \tilde{\gamma}(ain)\Big\}^{1/2}, \tag{3.48}$$

$$\tilde{\gamma}(a_{in}) = a_{in}^{2(3+3\omega+\sqrt{9+6\omega})/\omega}\gamma(a_{in}) - \Big\{\frac{2\omega^2 a_{in}^{(3-(n-4)\omega+\sqrt{9+6\omega})/\omega}}{3F^2(3+2\omega+\sqrt{9+6\omega})^2[3+(6-n)\omega+\sqrt{9+6\omega}]} \times\Big[-4(3+2\omega+\sqrt{9+6\omega})n\omega\rho_n a_{in}^2 + 3F^2k\Big(3+(6-n)\omega+\sqrt{9+6\omega}\Big)a_{in}^{(3+n\omega+\sqrt{9+6\omega})/\omega}\Big]\Big\}, \tag{3.49}$$

$$V(a) = -\frac{\rho_n}{a^n} + \frac{3F^2k}{4\omega a^{(2\omega-3-\sqrt{9+6\omega})/\omega}}. \tag{3.50}$$

The term containing energy density in $H(a)$ has a negative coefficient for $0 < n < 6$ and $\omega \gg 1$.

Then we take

$$\phi(a) = Fa^{(3-\sqrt{9+6\omega})/(2\omega)}, \qquad \rho = \frac{\rho_n}{a^n}. \tag{3.51}$$

Hubble function and the potential are found as

$$H(a) = \frac{1}{a^{(3+3\omega-\sqrt{9+6\omega})/\omega}}\Big\{\frac{2\omega^2 a^{(3-(n-4)\omega-2\sqrt{9+6\omega})/\omega}}{3F^2(-3-2\omega+\sqrt{9+6\omega})^2[-3+(n-6)\omega+\sqrt{9+6\omega}]}$$
$$\times\Big[4(3+2\omega-\sqrt{9+6\omega})n\omega\rho_n a^{(2\omega+\sqrt{9+6\omega})/\omega}$$
$$+3F^2k\Big(-3+(n-6)\omega+\sqrt{9+6\omega}\Big)a^{3/\omega+n}\Big]+\tilde{\gamma}(ain)\Big\}^{1/2}, \tag{3.52}$$

$$\tilde{\gamma}(a_{in}) = a_{in}^{2(3+3\omega-\sqrt{9+6\omega})/\omega}\gamma(a_{in})-$$
$$\Big\{\frac{2\omega^2 a_{in}^{(3-(n-4)\omega-2\sqrt{9+6\omega})/\omega}}{3F^2(-3-2\omega+\sqrt{9+6\omega})^2[-3+(n-6)\omega+\sqrt{9+6\omega}]}$$
$$\times\Big[4(3+2\omega-\sqrt{9+6\omega})n\omega\rho_n a_{in}^{(2\omega+\sqrt{9+6\omega})/\omega}$$
$$+3F^2k\Big(-3+(n-6)\omega+\sqrt{9+6\omega}\Big)a_{in}^{3/\omega+n}\Big]\Big\}, \tag{3.53}$$

$$V(a) = -\frac{\rho_n}{a^n}+\frac{3F^2k}{4\omega a^{(2\omega-3+\sqrt{9+6\omega})/\omega}}. \tag{3.54}$$

The term containing energy density in $H(a)$ has a negative coefficient for $0 < n < 6$ and $\omega \gg 1$.

3.3.4. Method 4

3.3.4.1. Non-Singular Case. Substitution of $H(a) = \sqrt{\gamma(a)}$ into ϕ equation results in the following form,

$$\gamma^{'}(a)+\tilde{P}(a)\gamma = \tilde{Q}(a), \tag{3.55}$$
$$\tilde{P}(a) = \frac{4\Big[-3\phi(a)+\omega a(4\phi^{'}(a)+a\phi^{''}(a))\Big]}{a(-3\phi(a)+2\omega a\phi^{'}(a))},$$
$$\tilde{Q}(a) = \frac{-4\omega a^2V^{'}(a)+6k\phi(a)\phi^{'}(a)}{a^3\phi^{'}(a)(-3\phi(a)+2\omega a\phi^{'}(a))}. \tag{3.56}$$

Hence

$$\gamma(a) = exp\Big(- \int_{a_{in}}^{a} \tilde{P}(a^{'})da^{'}\Big)\Big\{ \int_{a_{in}}^{a} exp\Big(\int_{a_{in}}^{a^{'}} \tilde{P}(a^{''})da^{''}\Big)\tilde{Q}(a^{'})da^{'} + \gamma(a_{in})\Big\}, \quad (3.57)$$

$$H(a) = \sqrt{\gamma(a)}. \quad (3.58)$$

After plugging $H(a)$ into ρ equation we multiply both sides by

$$\frac{exp\Big(\int_{a_{in}}^{a} \tilde{P}(a^{'})da^{'}\Big)}{3\phi^2(a) + 6a\phi(a)\phi^{'}(a) - 2\omega a^2\phi^{'2}}. \quad (3.59)$$

Then we take derivative of both sides with respect a. The resulting equation is equivalent to

$$V^{'}(a) + P(a)V(a) = Q(a), \quad (3.60)$$

where coefficients $P(a)$ and $Q(a)$ are given by (3.32) and (3.33) respectively. Thus solution is given by (3.34). In this method we end up with two integration constants. One of them should be chosen such that when final forms of formulas are inserted in the constraint equation it must be satisfied.

3.3.4.2. Singular Cases. This method has five singularities in its formulation. Three of them are the same singularities given in Method 3. Other two of them are seen in functions $\tilde{P}(a)$ and $\tilde{Q}(a)$

I)$\phi^{'}(a) = 0$,

II)$-3\phi(a) + 2\omega a\phi^{'}(a) = 0$.

First one makes Brans-Dicke cosmology equivalent to Einstein cosmology which have been studied in chapter 2. The second one implies

$$\phi(a) = Fa^{3/(2\omega)}. \quad (3.61)$$

Then ϕ equation becomes

$$\frac{a^{-3/\omega-2}}{12F\omega^2}\Big\{-9F^2a^{3/\omega}\Big[2k\omega-(3+2\omega)a^2H^2(a)\Big]+8\omega^3a^3V^{'}(a)\Big\}=0. \tag{3.62}$$

The Hubble function is found easily

$$H(a)=\frac{\sqrt{18F^2k\omega a^{3/\omega}-8\omega^3a^3V^{'}(a)}}{3\sqrt{(3+2\omega)F^2a^{3/\omega+2}}}. \tag{3.63}$$

Hence the ρ equation turns into the following form

$$V^{'}(a)+P(a)V(a)=Q(a), \tag{3.64}$$

$$P(a)=\frac{3}{\omega a},$$

$$Q(a)=\frac{3(3F^2ka^{3/\omega}-2\omega a^2\rho(a))}{2\omega^2a^3}. \tag{3.65}$$

Its solution is

$$V(a)=\frac{1}{a^{3/\omega}}\Big\{\int_{a_{in}}^{a}a^{'3/\omega}Q(a^{'})da^{'}+\tilde{V}(a_{in})\Big\}, \tag{3.66}$$

$$\tilde{V}(a_{in})=a_{in}^{3/\omega}V(a_{in}).$$

To investigate the nature of the Hubble function and the potential for usual forms of energy-matters we plug $\rho=\rho_n/a^n$. Thus we have

$$H(a)=\omega\Big\{\frac{2}{3(3+2\omega)}\Big[\frac{4a_{in}^{3/\omega}V(a_{in})}{F^2a^{6/\omega}}-\frac{3k}{(\omega-3)a^2}+\frac{4n\omega\rho_n}{(n\omega-3)F^2a^{3/\omega+n}}\Big]\Big\}^{1/2}, \tag{3.67}$$

$$V(a)=\frac{\tilde{V}(a_{in})}{a^{3/\omega}}+\frac{3\rho_n}{(n\omega-3)a^n}+\frac{9F^2k}{4\omega(3-\omega)a^{-3/\omega+2}}, \tag{3.68}$$

$$\tilde{V}(a_{in})=a_{in}^{3/\omega}V(a_{in})-\frac{3\rho_n}{(n\omega-3)a_{in}^n}-\frac{9F^2k}{4\omega(3-\omega)a_{in}^{-3/\omega+2}}. \tag{3.69}$$

Energy density related term in $H(a)$ has a positive coefficient therefore the Hubble function is always real.

3.4. Solution for Given $V(a)$ and $\phi(a)$

Special case of potential in the scalar-tensor theory is important. For this reason formulation of this case is significant. We establish the solution for our set of the constraint equation and the first order differential equation by two following different methods:

- Method 1) First, solve the ϕ equation for $H(a)$ then plug it into the constraint equation to obtain $\rho(a)$.
- Method 2) First, find $H(a)$ from the constraint equation then plug it in to the ϕ equation and solve it for $\rho(a)$.

3.4.1. Method1

3.4.1.1. Non-Singular Case. We start our calculations by changing the dependent variable as $H(a) = \sqrt{\gamma(a)}$ in ϕ equation. Hence it turns into the following form

$$\gamma^{'}(a) + P(a)\gamma(a) = Q(a), \tag{3.70}$$

$$P(a) = \frac{4\Big[-3\phi(a) + \omega a(4\phi^{'}(a) + a\phi^{''}(a))\Big]}{a(-3\phi(a) + 2\omega a\phi^{'}(a))}, \tag{3.71}$$

$$Q(a) = \frac{-4\omega a^2 V^{'}(a) + 6k\phi(a)\phi^{'}(a)}{a^3\phi^{'}(a)(-3\phi(a) + 2\omega a\phi^{'}(a))}. \tag{3.72}$$

Its solution is given by

$$\gamma(a) = exp\Big(-\int_{a_{in}}^{a} P(a^{'})da^{'}\Big)\Big\{\int_{a_{in}}^{a} exp\Big(\int_{a_{in}}^{a^{'}} P(a^{''})da^{''}\Big)Q(a^{'})da^{'} + \gamma(a_{in})\Big\}. \tag{3.73}$$

Energy density is obtained by inserting $H(a) = \sqrt{\gamma(a)}$ and $V(a)$ in the ρ equation.

3.4.1.2. Singular Case. When $-3\phi(a)+2\omega a\phi^{'}(a)=0$, previous solution becomes singular. This is the singularity of Section 3.3.4. Thus $\phi(a)=Fa^{3/(2\omega)}$ and ϕ equation is equal to (3.62) so $H(a)$ is found in (3.63). Then energy density is found as

$$\rho(a)=\frac{3F^2ka^{3/\omega-2}}{2\omega}-V(a)-\frac{\omega aV^{'}(a)}{3}. \tag{3.74}$$

When we inspect the results for $V=\frac{V_n}{a^n}$ we obtain

$$H(a)=\frac{1}{3\sqrt{3+2\omega}}\sqrt{\frac{18k\omega}{a^2}+\frac{8n\omega^3V_n}{F^2a^{3/\omega+n}}}, \tag{3.75}$$

$$\rho(a)=\frac{3F^2ka^{3/\omega-2}}{2\omega}+\frac{(n\omega-3)V_n}{3a^n}. \tag{3.76}$$

3.4.2. Method 2

3.4.2.1. Non-Singular Case. First we find $H(a)$ from the ρ equation

$$H(a)=\sqrt{\frac{-3k\phi^2(a)+4\omega a^2\Big[\rho(a)+V(a)\Big]}{a^2\Big[3\phi^2(a)+6a\phi(a)\phi'(a)-2\omega a^2\phi'^2\Big]}}. \tag{3.77}$$

Then ϕ equation turns into the following form

$$\rho^{'}(a)+P(a)\rho(a)=Q(a), \tag{3.78}$$

where

$$\begin{aligned}P(a)=&\Big\{6\Big[6\phi^3(a)-(3+14\omega)a^2\phi(a)\phi'^2(a)+2(1+2\omega)\omega a^3\phi'^3(a)\\&-a\phi^2(a)((-6+8\omega)\phi^{'}(a)+(3+2\omega)a\phi^{''}(a))\Big]\Big\}\\&/\Big\{a(-3\phi(a)+2\omega a\phi^{'}(a))\Big[-3\phi^2(a)-6a\phi(a)\phi^{'}(a)+2\omega a^2\phi'^2(a)\Big]\Big\},\end{aligned} \tag{3.79}$$

$$
\begin{aligned}
Q(a) = -\Big\{3\Big[&4\omega a^3\phi(a)(-(3+14\omega)V(a)+\omega aV^{'}(a))\phi^{'3}(a)+8\omega^2(1+2\omega)a^4V(a)\phi^{'4}(a)\\
&+3a\phi^3(a)\phi^{'}(a)(8\omega V(a)-6\omega aV^{'}(a)+3k(3+2\omega)\phi^{'2}(a))\\
&+4\omega a^2\phi^2(a)\phi^{'}(a)\Big(\phi^{'}(a)((6-8\omega)V(a)+(-3+\omega)aV^{'}(a)-k(3+2\omega)\phi^{'2}(a))\\
&-(3+2\omega)aV(a)\phi^{''}(a)\Big)\\
&+3\phi^4(a)\Big(-2\omega aV^{'}(a)+k(3+2\omega)\phi^{'}(a)(3\phi^{'}(a)+a\phi^{''}(a))\Big)\Big]\Big\}\\
&/\Big\{2\omega a^2\phi^{'}(a)(-3\phi(a)+2\omega a\phi^{'}(a))\Big[-3\phi^2(a)-6a\phi(a)\phi^{'}(a)+2\omega a^2\phi^{'2}(a)\Big]\Big\}.
\end{aligned}
\tag{3.80}
$$

Hence energy density is found as

$$
\rho(a) = exp\Big(-\int_{a_{in}}^{a} P(a^{'})da^{'}\Big)\Big\{\int_{a_{in}}^{a} exp\Big(\int_{a_{in}}^{a^{'}} P(a^{''})da^{''}\Big)Q(a^{'})da^{'}+\rho(a_{in})\Big\}. \tag{3.81}
$$

3.4.2.2. Singular Cases. This method has two singular cases for non-constant scalar field.

I)$-3\phi(a)+2\omega a\phi^{'}(a)=0$,

II)$-3\phi^2(a)-6a\phi(a)\phi^{'}(a)+2\omega a^2\phi^{'2}(a)=0$.

First case implies $\phi(a)=Fa^{3/(2\omega)}$. Thus the ρ equation becomes

$$
\frac{3F^2a^{3/\omega-2}\Big[2k\omega+(3+2\omega)a^2H^2(a)\Big]}{8\omega^2}-V(a)=\rho(a). \tag{3.82}
$$

The Hubble function is found as

$$
H(a)=\sqrt{\frac{\omega}{3+2\omega}}\sqrt{-\frac{2k}{a^2}+\frac{8\omega}{3F^2a^{3/\omega}}(\rho(a)+V(a))}. \tag{3.83}
$$

Then the ϕ equation becomes

$$
-\frac{3Fk}{\omega}a^{3/(2\omega)-2}+\frac{2a^{-3/(2\omega)}}{3F}\Big[3(\rho(a)+V(a))+\omega aV^{'}(a)\Big], \tag{3.84}
$$

and energy density is found as

$$\rho(a) = \frac{3F^2k}{2\omega}a^{3/\omega-2} - V(a) - \frac{\omega a V^{'}(a)}{3}. \tag{3.85}$$

Examination of this singularity with $V(a) = \frac{V_n}{a^n}$ gives us

$$H(a) = \sqrt{\frac{\omega}{3+2\omega}}\sqrt{\frac{2k}{a^2} + \frac{8n\omega^2 V_n}{9F^2a^{3/\omega+n}}}, \tag{3.86}$$

$$\rho(a) = \frac{3F^2k}{2\omega}a^{3/\omega-2} + \frac{(n\omega-3)V_n}{3a^n}. \tag{3.87}$$

Thus we have the energy density in usual form.

Second singularity has been already seen in Section 3.3.3.2. It implies $\phi(a) = Fa^{(3\pm\sqrt{9+6\omega})/(2\omega)}$.

Then the ρ equation has the following form

$$\frac{3F^2k}{4\omega}a^{(3-2\omega\pm\sqrt{9+6\omega})/\omega} - V(a) = \rho(a). \tag{3.88}$$

After writing $H(a) = \sqrt{\gamma(a)}$, ϕ equation becomes

$$\gamma^{'}(a) + P(a)\gamma(a) = Q(a). \tag{3.89}$$

For $\phi(a) = a^{(3+\sqrt{9+6\omega})/(2\omega)}$,

$$P(a) = \frac{2(3+3\omega+\sqrt{9+6\omega})}{\omega a}, \tag{3.90}$$

$$Q(a) = \frac{a^{-(3+3\omega+\sqrt{9+6\omega})/\omega}}{3\sqrt{3+2\omega}F^2}\Big[6\sqrt{3}F^2ka^{(3+\sqrt{9+6\omega})/\omega} + 4\omega(\sqrt{3}-\sqrt{3+2\omega})a^3V^{'}(a)\Big]. \tag{3.91}$$

Then

$$\gamma(a)=\frac{1}{a^{2(3+3\omega+\sqrt{9+6\omega})/\omega}}\Big\{b\int_{a_{in}}^{a}a^{'(3+3\omega+\sqrt{9+6\omega})/\omega}\Big[6\sqrt{3}F^2ka^{'(3+\sqrt{9+6\omega})/\omega}$$
$$+4\omega(\sqrt{3}-\sqrt{3+2\omega})a^{'3}V^{'}(a^{'})\Big]da^{'}+\tilde{\gamma}(a_{in})\Big\}, \tag{3.92}$$
$$b=\frac{1}{3\sqrt{3+2\omega}F^2},\qquad \tilde{\gamma}(a_{in})=a_{in}^{2(3+3\omega+\sqrt{9+6\omega})/\omega}\gamma(a_{in}). \tag{3.93}$$

For $\phi(a)=a^{(3-\sqrt{9+6\omega})/(2\omega)}$,

$$P(a)=\frac{2(3+3\omega-\sqrt{9+6\omega})}{\omega a}, \tag{3.94}$$
$$Q(a)=\frac{a^{-3(1+\omega)/\omega}}{3F^2(3+2\omega-\sqrt{9+6\omega})}\Big[6F^2k(3-\sqrt{9+6\omega})a^{3/\omega}-8\omega^2a^{(3\omega+\sqrt{9+6\omega})/\omega}V^{'}(a)\Big]. \tag{3.95}$$

Then

$$\gamma(a)=\frac{1}{a^{2(3+3\omega-\sqrt{9+6\omega})/\omega}}\Big\{b\int_{a_{in}}^{a}a^{'(3+3\omega-2\sqrt{9+6\omega})/\omega}\Big[6F^2k(3-\sqrt{9+6\omega})a^{'3/\omega}$$
$$+8\omega^2a^{'(3\omega+\sqrt{9+6\omega})/\omega}V^{'}(a^{'})\Big]da^{'}+\tilde{\gamma}(a_{in})\Big\}, \tag{3.96}$$
$$b=\frac{1}{3(3+2\omega-\sqrt{9+6\omega})F^2},\qquad \tilde{\gamma}(a_{in})=a_{in}^{2(3+3\omega-\sqrt{9+6\omega})/\omega}\gamma(a_{in}). \tag{3.97}$$

At the next step we examine nature of the Hubble function. First, we take

$$\phi(a)=Fa^{(3+\sqrt{9+6\omega})/(2\omega)},\qquad V(a)=\frac{V_n}{a^n}. \tag{3.98}$$

The Hubble function and the energy density are found as

$$H(a)=\Big\{\frac{\sqrt{3}k(\sqrt{3+2\omega}-\sqrt{3})}{2(3+2\omega)a^2}+\frac{8n\omega^3V_na^{-(3+n\omega+\sqrt{9+6\omega})/\omega}}{3(3+2\omega+\sqrt{9+6\omega})\Big[3+(6-n)\omega+\sqrt{9+6\omega}\Big]F^2}$$
$$+\tilde{\gamma}(a_{in})a^{-2(3+3\omega+\sqrt{9+6\omega})/\omega}\Big\}^{1/2}, \tag{3.99}$$

$$\tilde{\gamma}(a_{in}) = \Big\{\gamma(a_{in}) - \frac{\sqrt{3}k(\sqrt{3+2\omega}-\sqrt{3})}{2(3+2\omega)a_{in}^2} - \frac{8n\omega^3 V_n a_{in}^{-(3+n\omega+\sqrt{9+6\omega})/\omega}}{3(3+2\omega+\sqrt{9+6\omega})\Big[3+(6-n)\omega+\sqrt{9+6\omega}\Big]F^2}\Big\}a_{in}^{2(3+3\omega+\sqrt{9+6\omega})/\omega}, \tag{3.100}$$

$$\rho(a) = \frac{3F^2k}{4\omega a^{(2\omega-3-\sqrt{9+6\omega})/\omega}} - \frac{V_n}{a^n}. \tag{3.101}$$

Although the Hubble function is real for $0 < n < 6$ and $\omega \gg 1$, the energy density always has a negative component.

Then we take

$$\phi(a) = Fa^{(3-\sqrt{9+6\omega})/(2\omega)}, \qquad V(a) = \frac{V_n}{a^n}. \tag{3.102}$$

The Hubble function and the energy density are found as

$$H(a) = \Big\{ -\frac{\sqrt{3}k(\sqrt{3+2\omega}+\sqrt{3})}{2(3+2\omega)a^2} + \frac{8n\omega^3 V_n a^{-(3+n\omega-\sqrt{9+6\omega})/\omega}}{3(3+2\omega-\sqrt{9+6\omega})\Big[3+(6-n)\omega-\sqrt{9+6\omega}\Big]F^2} + \tilde{\gamma}(a_{in})a^{-2(3+3\omega-\sqrt{9+6\omega})/\omega}\Big\}^{1/2}, \tag{3.103}$$

$$\tilde{\gamma}(a_{in}) = \Big\{\gamma(a_{in}) + \frac{\sqrt{3}k(\sqrt{3+2\omega}+\sqrt{3})}{2(3+2\omega)a_{in}^2} - \frac{8n\omega^3 V_n a_{in}^{-(3+n\omega-\sqrt{9+6\omega})/\omega}}{3(3+2\omega-\sqrt{9+6\omega})\Big[3+(6-n)\omega-\sqrt{9+6\omega}\Big]F^2}\Big\}a_{in}^{2(3+3\omega-\sqrt{9+6\omega})/\omega}, \tag{3.104}$$

$$\rho(a) = \frac{3F^2k}{4\omega a^{(2\omega-3+\sqrt{9+6\omega})/\omega}} - \frac{V_n}{a^n}. \tag{3.105}$$

Although the Hubble function is real for $0 < n < 6$ and $\omega \gg 1$, the energy density always has a negative component.

3.5. Solution for Given $H(a)$ and $\phi(a)$

The solution for specific form of the Hubble function can be crucial. To construct the solution one can follow two different paths:

- Method 1) First, find $V(a)$ from the constraint equation then insert it into the ϕ equation and solve it for $\rho(a)$.
- Method 2) First, solve the ϕ equation for the $V(a)$ then insert it into the constraint equation and obtain $\rho(a)$.

3.5.1. Method1

We find $V(a)$ from the ρ equation

$$V(a) = -\rho(a) + \frac{3k\phi^2(a)}{4\omega a^2} + \Big[3\phi^2(a) + 6a\phi(a)\phi^{'}(a) - 2\omega a^2\phi^{'2}(a)\Big]\frac{H^2(a)}{4\omega}. \tag{3.106}$$

Then the ϕ equation turns into the following form

$$\rho^{'}(a) = Q(a), \tag{3.107}$$

$$Q(a) = \frac{-3\phi^2(k - a^3HH^{'}) + a^3[3(1+2\omega)aH^2\phi^{'2}] + 3a^4H\phi(H^{'}\phi^{'} + H\phi^{''})}{2\omega a^3}. \tag{3.108}$$

Thus

$$\rho(a) = \int_{a_{in}}^{a} Q(a^{'})da^{'} + \rho(a_{in}), \tag{3.109}$$

$$V(a) = -\Big[\int_{a_{in}}^{a} Q(a^{'})da^{'} + \rho(a_{in})\Big] + \frac{3k\phi^2(a)}{4\omega a^2} + \Big[3\phi^2(a) + 6a\phi(a)\phi^{'}(a) - 2\omega a^2\phi^{'2}(a)\Big]\frac{H^2(a)}{4\omega}. \tag{3.110}$$

3.5.2. Method2

We solve the ϕ equation for the potential,

$$V(a) = -\int_{a_{in}}^{a} \phi^{'}\Big\{ -3\phi\Big[k + a^{'2}H(2H + a^{'}H^{'})\Big] + 2\omega a^{'3}H\Big[(4H + a^{'}H^{'})\phi^{'} + a^{'}H\phi^{''}\Big]\Big\}da^{'} + V(a_{in}). \tag{3.111}$$

We obtain energy density by substituting $V(a)$ into the ρ equation

$$\rho(a) = \int_{a_{in}}^{a} \phi^{'}\Big\{ -3\phi\Big[k + a^{'2}H(2H + a^{'}H^{'})\Big] + 2\omega a^{'3}H\Big[(4H + a^{'}H^{'})\phi^{'} + a^{'}H\phi^{''}\Big]\Big\}da^{'} - V(a_{in}) + \frac{3k\phi^2(a)}{4\omega a^2} + \Big[3\phi^2(a) + 6a\phi(a)\phi^{'}(a) - 2\omega a^2\phi^{'2}(a)\Big]\frac{H^2(a)}{4\omega}. \tag{3.112}$$

3.6. Single Component Universe

We investigate the universe with given energy density and the scalar field which are in the following form

$$\rho(a) = \frac{\rho_n}{a^n}, \qquad \phi(a) = \frac{F}{a^d}. \tag{3.113}$$

We have obtained the Hubble function and the potential function by applying the procedure explained in Section 3.3.1, we obtain the following formulas

$$H(a) = \sqrt{\frac{k}{\mathcal{B}a^2} - \frac{4n\rho_n\omega a^{2d-n}}{3\mathcal{C}F^2} + \tilde{\gamma}(a_{in})a^{\mu}}, \tag{3.114}$$

$$V(a) = \frac{d^2(3+2\omega)F^2k}{2\mathcal{B}\omega a^{2+2d}} + \frac{\mathcal{A}\rho_n a^{-n}}{3\mathcal{C}} + \frac{\mathcal{F}\tilde{\gamma}(a_{in})F^2a^{-2d+\mu}}{4\omega}, \tag{3.115}$$

$$\tilde{\gamma}(a_{in}) = a_{in}^{-\mu}H^2(a_{in}) - \frac{ka_{in}^{-2-\mu}}{\mathcal{B}} + \frac{4n\rho_n\omega a_{in}^{2d-n-\mu}}{3\mathcal{C}F^2}, \tag{3.116}$$

$$\mu = \frac{2d[1+2d(1+\omega)]}{-1+d}, \tag{3.117}$$

$$\mathcal{A} = d\{3(-4+n)+2d[-3+(-6+n)\omega]\}, \tag{3.118}$$

$$\mathcal{B} = [-1+2d(1+d+d\omega)], \tag{3.119}$$

$$\mathcal{C} = \{-n+d[4+n+d(2+4\omega)]\}, \tag{3.120}$$

$$\mathcal{F} = [-3+2d(3+d\omega)], \tag{3.121}$$

$$p(a) = \frac{(n-3)\rho_n}{3a^n}. \tag{3.122}$$

We should choose d such that $\mathcal{B} < 0$ and $\mathcal{C} < 0$. Then we have real Hubble function. In addition if we also satisfy $\mu = 0$ at the same time this will have important physical implications which will be discussed soon. Thus

$$\mu = 0 \qquad \Rightarrow \qquad d = -\frac{1}{2(1+\omega)}, \tag{3.123}$$

$$\mathcal{B} = -1 - \frac{1}{2(1+\omega)}, \qquad \mathcal{C} = -\frac{(3+2\omega)(1+n+n\omega)}{2(1+\omega)^2}. \tag{3.124}$$

Then the Hubble function and potential are simplified to the following forms

$$H(a) = \sqrt{-\frac{2(1+\omega)k}{(3+2\omega)a^2} + \frac{8n(1+\omega)^2\omega\rho_n a^{-1/(1+\omega)-n}}{3(3+2\omega)[1+n(1+\omega)]F^2} + \tilde{\gamma}(ain)}, \tag{3.125}$$

$$V(a) = \frac{(-3+n)\rho_n a^{-n}}{3(1+n(1+\omega))} - \frac{kF^2 a^{1/(1+\omega)-2}}{4\omega(1+\omega)} + \frac{(3+2\omega)(4+3\omega)F^2}{8\omega(1+\omega)^2}\tilde{\gamma}(a_{in})a^{1/(1+\omega)}, \tag{3.126}$$

$$\tilde{\gamma}(a_{in}) = H^2(a_{in}) + \frac{2(1+\omega)k}{(3+2\omega)a_{in}^2} - \frac{8n(1+\omega)^2\omega\rho_n a_{in}^{-1/(1+\omega)-n}}{3(3+2\omega)[1+n(1+\omega)]F^2}. \tag{3.127}$$

The deceleration parameter is found by applying the chain rule

$$q(t) = \frac{d}{dt}(\frac{1}{H(t)}) - 1, \tag{3.128}$$

$$q(a) = \frac{d}{da}(\frac{1}{H(a)})aH(a) - 1, \tag{3.129}$$

$$q(a) = -1 + \{2(1+\omega)[1+n(1+\omega)][-3F^2ka^{1/(1+\omega)+n} + 2n\omega\rho_n a^2]\}$$
$$/\{8n\omega(1+\omega)^2\rho_n a^2 + 3[1+n(1+\omega)]F^2[-2k(1+\omega) + (3+2\omega)\tilde{\gamma}(a_{in})a^2]a^{1/(1+\omega)+n}\}. \tag{3.130}$$

When finding $q(a_{in})$ one should plug $\tilde{\gamma}(a_{in})$ which is given by (3.127)

$$q(a_{in}) = -1 + \frac{2(1+\omega)[-3F^2ka_{in}^{1/(1+\omega)+n} + 2n\omega\rho_n a_{in}^2]}{3(3+2\omega)F^2H^2(a_{in})a_{in}^2}. \tag{3.131}$$

Since $a_{in} \ll 1$, $q(a_{in})$ is simplified further

$$\lim_{a\to a_{in}} q(a) = \begin{cases} -1 + \dfrac{4(1+\omega)n\omega\rho_n}{3(3+2\omega)F^2H^2(a_{in})}, & if \quad n + \dfrac{1}{1+\omega} > 2, \\ -1 + \dfrac{2(1+\omega)[-3F^2k + 2n\omega\rho_n]}{3(3+2\omega)F^2H^2(a_{in})} & if \quad n + \dfrac{1}{1+\omega} = 2, \\ -1 - \dfrac{2(1+\omega)k}{(3+2\omega)H^2(a_{in})a_{in}^{-1/(1+\omega)-n+2}}, & if \quad 0 < n + \dfrac{1}{1+\omega} < 2. \end{cases}$$

When we set today values of the scale factor $a = 1$, ρ_n and F become today's values of the energy density and the scalar field respectively. Thus dynamics of the early universe depends on not only initial value of the Hubble function and the scale factor but also today's values of the energy density and scalar field. In addition behaviour of the universe also depends on the curvature parameter k for $0 < n + \dfrac{1}{1+\omega} \leq 2$.

The fate of the universe can be described by the following number

$$\lim_{a\to\infty} q(a) = -1. \tag{3.132}$$

Now, we formulate $a(t)$ for a spatially flat universe as

$$t = \int_{a_{in}}^{a} \frac{da'}{a'H(a')}, \tag{3.133}$$

$$H(a) = \sqrt{\delta a^{-1/(1+\omega)-n} + \alpha}, \qquad \delta = \frac{8n(1+\omega)^2\omega\rho_n}{3(3+2\omega)[1+n(1+\omega)]F^2}, \qquad \alpha = \tilde{\gamma}(a_{in}). \tag{3.134}$$

Thus

$$t = \frac{1}{\kappa\sqrt{\alpha}} ln\Big[\frac{a^\kappa + \sqrt{a^{2\kappa} + \frac{\delta}{\alpha}}}{a_{in}^\kappa + \sqrt{a_{in}^{2\kappa} + \frac{\delta}{\alpha}}}\Big], \qquad \kappa = \frac{1}{2}\Big[\frac{1 + n(1+\omega)}{1+\omega}\Big]. \tag{3.135}$$

Hence

$$a(t) = \Big[\frac{(a_{in}^\kappa + \sqrt{a_{in}^{2\kappa} + \frac{\delta}{\alpha}})e^{\kappa\sqrt{\alpha}t} + (a_{in}^\kappa - \sqrt{a_{in}^{2\kappa} + \frac{\delta}{\alpha}})e^{-\kappa\sqrt{\alpha}t}}{2}\Big]^{1/\kappa}, \tag{3.136}$$

where δ, α and κ are given in (3.134-3.135). This formula indicates a very significant interpretation. When there is no constant energy density, the universe still expands exponentially because of nonzero initial conditions. We will investigate this phenomena in depth by adding constant term to the energy density in Sections 3.7 and 3.8

As we said at the beginning of Section 3.1 we have obtained perfect fluid equation of state

$$\nu = \frac{p(a)}{\rho(a)} = \frac{(n-3)}{3}. \tag{3.137}$$

Potential formula as a function of the scalar field are written as

$$\begin{aligned} V(\phi) = {} & \frac{(-3+n)\rho_n}{3(1+n(1+\omega))}(\frac{\phi}{F})^{-2n(1+\omega)} - \frac{kF^2}{4\omega(1+\omega)}(\frac{\phi}{F})^{-2-4\omega} \\ & + \frac{(3+2\omega)(4+3\omega)}{8\omega(1+\omega)^2}\tilde{\gamma}(a_{in})\phi^2. \end{aligned} \tag{3.138}$$

Most probably ϕ^2 term in the potential is responsible for late time accelerated expansion. This formula is simpler than the corresponding formula $V(\phi)$ in our previous study [129].

3.7. Early Universe, Dark Energy and Radiation

In this section the scalar field and the energy density is taken as

$$\phi(a) = \frac{F}{a^d}, \qquad d = -\frac{1}{2(1+\omega)}, \qquad \rho(a) = \frac{\rho_r}{a^4} + \Lambda. \tag{3.139}$$

To see whether Λ contributes to $H(a)$ we have performed calculations in all four different methods. Results are found in the following form

$$H(a) = \sqrt{-\frac{2(1+\omega)k}{(3+2\omega)a^2} + \frac{32(1+\omega)^2\omega\rho_r a^{-1/(1+\omega)-4}}{3(3+2\omega)(5+4\omega)F^2} + \alpha}, \tag{3.140}$$

$$V(a) = -\frac{F^2 k a^{1/(1+\omega)-2}}{4\omega(1+\omega)} + \beta a^{1/(1+\omega)} + \frac{\rho_r}{3(5+4\omega)a^4} - \Lambda. \tag{3.141}$$

In method-1 we find

$$\alpha_1 = H^2(a_{in}) + \frac{2(1+\omega)k}{(3+2\omega)a_{in}^2} - \frac{32(1+\omega)^2\omega\rho_r a_{in}^{-1/(1+\omega)-4}}{3(3+2\omega)(5+4\omega)F^2}, \tag{3.142}$$

$$\beta_1 = \frac{(3+2\omega)(4+3\omega)F^2}{8\omega(1+\omega)^2}\alpha_1. \tag{3.143}$$

In method-2 we find

$$\alpha_2 = \alpha_1, \qquad \beta_2 = \beta_1. \tag{3.144}$$

In this method to be able to satisfy the constraint equation with final formulas of $V(a)$ and $H(a)$ initial conditions must satisfy the following equation

$$V(a_{in}) = -\frac{F^2 k a_{in}^{1/(1+\omega)-2}}{4\omega(1+\omega)} + \beta_2 a_{in}^{1/(1+\omega)} + \frac{\rho_r}{3(5+4\omega)a_{in}^4} - \Lambda, \tag{3.145}$$

which is consistent with the formula of $V(a)$.

In method-3 we find

$$\alpha_3 = \frac{8\omega(1+\omega)^2\tilde{V}(a_{in})}{(3+2\omega)(4+3\omega)F^2}, \tag{3.146}$$

$$\tilde{V}(a_{in}) = V(a_{in})a_{in}^{-1/(1+\omega)} - \frac{\rho_r a_{in}^{-1/(1+\omega)-4}}{3(5+4\omega)} + \frac{F^2k}{4\omega(1+\omega)a_{in}^2} + \Lambda a_{in}^{-1/(1+\omega)}, \tag{3.147}$$

$$\beta_3 = \frac{(3+2\omega)(4+3\omega)F^2}{8\omega(1+\omega)^2}\alpha_3. \tag{3.148}$$

Although at first sight it seems that $\alpha_1 \neq \alpha_3$, one can show that they are equal to each other by substituting a_{in} in $H(a)$ and solving the equation for $V(a_{in})$.

In method-4 we find

$$\alpha_4 = \alpha_3, \qquad \beta_4 = \beta_3. \tag{3.149}$$

By the same reasoning in method-2, constraint equation implies the following condition on initial values

$$a_{in}^{1/(1+\omega)+3}H^2(a_{in}) - \frac{32\omega(1+\omega)^2\rho_r}{3(3+2\omega)(5+4\omega)F^2a_{in}} - \frac{8\omega(1+\omega)^2\tilde{V}(a_{in})a_{in}^{1/(1+\omega)+3}}{(3+2\omega)(4+3\omega)F^2} + \frac{2(1+\omega)ka_{in}^{1/(1+\omega)+1}}{3+2\omega} = 0. \tag{3.150}$$

$V(a_{in})$ which is obtained by solving this equation is consistent with the formula of $V(a)$.

The deceleration parameter is found as

$$q(a) = -1 + \frac{2(1+\omega)(5+4\omega)[-3F^2ka^{1/(1+\omega)+2} + 8\omega\rho_r]}{32\omega(1+\omega)^2\rho_r + 3(5+4\omega)F^2[-2k(1+\omega) + (3+2\omega)\alpha_1a^2]a^{1/(1+\omega)+2}}, \tag{3.151}$$

$$\alpha_1 = H^2(a_{in}) + \frac{2(1+\omega)k}{(3+2\omega)a_{in}^2} - \frac{32(1+\omega)^2\omega\rho_r a_{in}^{-1/(1+\omega)-4}}{3(3+2\omega)(5+4\omega)F^2}. \tag{3.152}$$

Thus initial value of the deceleration parameter is found

$$q(a_{in}) = -1 + \frac{2(1+\omega)[-3F^2 k a_{in}^{1/(1+\omega)+2} + 8\omega\rho_r]}{3(3+2\omega)F^2 H^2(a_{in})}. \tag{3.153}$$

The deceleration parameter which is given in (3.151) is same as the deceleration parameter found in single component universe without constant term, since α_1 given in (3.152) is equal to $\tilde{\gamma}(a_{in})$ given in (3.127) with $n = 4$.

The scale factor as a function of time is written as

$$a(t) = \Big[\frac{(a_{in}^{\kappa} + \sqrt{a_{in}^{2\kappa} + \frac{\delta}{\alpha}})e^{\kappa\sqrt{\alpha}t} + (a_{in}^{\kappa} - \sqrt{a_{in}^{2\kappa} + \frac{\delta}{\alpha}})e^{-\kappa\sqrt{\alpha}t}}{2}\Big]^{1/\kappa}, \tag{3.154}$$

$$\kappa = \frac{1}{2}\Big(\frac{5+4\omega}{1+\omega}\Big), \qquad \delta = \frac{32(1+\omega)^2\omega\rho_r}{3(3+2\omega)(5+4\omega)F^2}, \tag{3.155}$$

where $\alpha = \alpha_1$ and α_1 is given by (3.142).

Formula of the potential as function of the scalar field is found as

$$V(\phi) = -\frac{F^2 k}{4\omega(1+\omega)}(\frac{\phi}{F})^{-2-4\omega} + \beta(\frac{\phi}{F})^2 + \frac{\rho_r}{3(5+4\omega)}(\frac{\phi}{F})^{-8(1+4\omega)} - \Lambda, \tag{3.156}$$

where $\beta = \beta_1$ which is given by (3.143). Pressure is found as in expected form

$$p(a) = \frac{\rho_r}{3a^4} - \Lambda. \tag{3.157}$$

3.8. Late Time Expansion of the Universe

3.8.1. Dark Energy Dominated Universe

We study in spatially flat universe where

$$\phi(a) = \frac{F}{a^d}, \qquad d = -\frac{1}{2(1+\omega)}, \qquad \rho(a) = \frac{\rho_m}{a^3} + \Lambda. \tag{3.158}$$

Then related function is formulated as

$$H(a) = \sqrt{\frac{8\omega(1+\omega)^2\rho_m}{(3+2\omega)(4+3\omega)F^2a^{1/(1+\omega)+3}} + \tilde{\gamma}(a_{in})}, \tag{3.159}$$

$$V(a) = \frac{(3+2\omega)(4+3\omega)}{8\omega(1+\omega)^2}F^2\tilde{\gamma}(a_{in})a^{1/(1+\omega)} - \Lambda, \tag{3.160}$$

$$q(a) = -1 + \frac{4\omega(1+\omega)(4+3\omega)\rho_m}{8\omega(1+\omega)^2\rho_m + (3+2\omega)(4+3\omega)F^2\tilde{\gamma}(a_{in})a^{1/(1+\omega)+3}}, \tag{3.161}$$

$$\tilde{\gamma}(a_{in}) = H^2(a_{in}) - \frac{8(1+\omega)^2\omega\rho_m a_{in}^{-1/(1+\omega)-3}}{(3+2\omega)(4+3\omega)F^2}, \tag{3.162}$$

$$p(a) = -\Lambda. \tag{3.163}$$

The scale factor as function of time is found as

$$a(t) = \Big[\frac{(a_{in}^{\kappa} + \sqrt{a_{in}^{2\kappa} + \frac{\delta}{\alpha}})e^{\kappa\sqrt{\alpha}t} + (a_{in}^{\kappa} - \sqrt{a_{in}^{2\kappa} + \frac{\delta}{\alpha}})e^{-\kappa\sqrt{\alpha}t}}{2}\Big]^{1/\kappa}, \tag{3.164}$$

$$\kappa = \frac{1}{2}\Big(\frac{4+3\omega}{1+\omega}\Big), \qquad \delta = \frac{8(1+\omega)^2\omega\rho_m}{(3+2\omega)(4+3\omega)F^2}. \tag{3.165}$$

where $\alpha = \tilde{\gamma}(a_{in})$ and $\tilde{\gamma}(a_{in})$ is given by (3.162).

In Brans-Dicke theory effective gravitational constant is defined as $G_{eff} = \frac{\omega}{2\pi\phi^2}$. Thus it's present value becomes $G_0 = \frac{\omega}{2\pi F^2}$ where $F = \phi(t_0)$. In addition when we take

observational value of Brans-Dicke parameter $\omega \gg 1$, the Hubble function becomes

$$H(a) = \sqrt{\frac{8\pi G_0 \rho_m}{3a^3} + \tilde{\gamma}(a_{in})}. \tag{3.166}$$

By using following definitions of cosmological parameters

$$\Omega_{m,0} = \frac{8\pi G}{3H_0^2}\rho_m, \qquad \Omega_{\tilde{\gamma},0} = \frac{8\pi G}{3H_0^2}\rho_{\tilde{\gamma}}, \qquad \rho_{\tilde{\gamma}} = \frac{3\tilde{\gamma}(a_{in})}{8\pi G}, \tag{3.167}$$

the Hubble function is written as

$$H(a) = H_0\sqrt{\frac{\Omega_{m,0}}{a^3} + \Omega_{\tilde{\gamma},0}}. \tag{3.168}$$

When one replace $\Omega_{\tilde{\gamma},0}$ with $\Omega_{\Lambda,0}$ in (3.166) we will have the Hubble function for the late epoch of the universe in standard cosmology where dark energy dominates.

We have already compared this model with observation of type Ia supernovae data in chapter 2. In our previous study we have obtained $H_0 = 71.80 \pm 0.22$, $\Omega_{\Lambda,0} = 0.715 \pm 0.012$, $\Omega_m = 0.285 \pm 0.012$ and $\chi^2/\nu = 0.990$ for absolute magnitude $M = -19.30$. Present value of deceleration parameter has been found as $q_0 = -0.572$ with this cosmological parameters.

On the other hand potential can be rewritten as

$$V(\phi) = \frac{(3+2\omega)(4+3\omega)}{8\omega(1+\omega)^2}\tilde{\gamma}(a_{in})\phi^2 - \Lambda. \tag{3.169}$$

Most probably ϕ^2 potential is responsible from the accelerated expansion.

3.8.2. Domain Wall Dominated Universe

In this section we study in spatially flat universe with scalar field and the energy density is given as

$$\phi(a) = \frac{F}{a^d}, \qquad d = -\frac{1}{2(1+\omega)}, \qquad \rho(a) = \frac{\rho_m}{a^3} + \frac{\rho_w}{a}. \tag{3.170}$$

The Hubble function, potential, deceleration parameter and the pressure is formulated as

$$H(a) = \sqrt{\frac{8\omega(1+\omega)^2}{3(3+2\omega)F^2}\Big[\frac{3\rho_m}{(4+3\omega)a^3} + \frac{\rho_w}{(2+\omega)a}\Big]a^{-1/(1+\omega)} + \tilde{\gamma}(a_{in})}, \tag{3.171}$$

$$V(a) = \frac{(3+2\omega)(4+3\omega)}{8\omega(1+\omega)^2}\tilde{\gamma}(a_{in})F^2a^{1/(1+\omega)} - \frac{2\rho_w}{3(2+\omega)a}, \tag{3.172}$$

$$\begin{aligned} q(a) = &-1 + \{4\omega(1+\omega)(2+\omega)(4+3\omega)(3\rho_m + \rho_w a^2)\} \\ &/\{3(2+\omega)(3+2\omega)(4+3\omega)\tilde{\gamma}(a_{in})F^2a^{1/(1+\omega)+3} + 8\omega(1+\omega)^2[3(2+\omega)\rho_m \\ &+ (4+3\omega)\rho_w a^2]\}, \end{aligned} \tag{3.173}$$

$$\tilde{\gamma}(a_{in}) = H^2(a_{in}) - \frac{8\omega(1+\omega)^2}{3(3+2\omega)F^2}\Big[\frac{3\rho_m}{(4+3\omega)a_{in}^3} + \frac{\rho_w}{(2+\omega)a_{in}}\Big]a_{in}^{-1/(1+\omega)}, \tag{3.174}$$

$$p(a) = -\frac{2\rho_w}{3a}. \tag{3.175}$$

Now we take $\tilde{\gamma}(a_{in}) = 0$, to investigate whether cosmic domain walls cause accelerated expansion of the universe or not. We again apply the fundamental idea of the Brans-Dicke theory as we did in Section 3.7. Thus we take $G_0 = \dfrac{\omega}{2\pi F^2}$ and we take $\omega \gg 1$. Then the Hubble function is simplified to

$$H(a) = \sqrt{\frac{8\pi G_0}{3}(\frac{\rho_m}{a^3} + \frac{\rho_w}{a})}. \tag{3.176}$$

The relation between the scale factor and time is found as

$$t = \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}H(a^{'})},$$
$$t = \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}\sqrt{\frac{8\pi G}{3}(\frac{\rho_m}{a^{'3}} + \frac{\rho_w}{a^{'}})}}, \tag{3.177}$$
$$t = \sqrt{\frac{3}{2\pi G\rho_w}}\Big[\sqrt{a^{'}}\,{}_2F_1(\frac{1}{2}, -\frac{1}{4}; \frac{3}{4}; -\frac{\rho_m}{\rho_w a^{'2}})\Big]_{a_{in}}^{a}. \tag{3.178}$$

${}_2F_1$ is the hypergeometric function

$${}_2F_1(1/2, b; b+1; u) = \sum_{n=0}^{\infty} \frac{(1/2)_n (b)_n}{(b+1)_n} \frac{u^n}{n!}, \tag{3.179}$$

where $(b)_n$ is the Pochhammer symbol which is defined as

$$(b)_n = \begin{cases} 1 \quad \text{if} \quad n = 0, \\ b(b+1)(b+2)...(b+n-1), \quad \text{if} \quad n = 1, 2 \ldots. \end{cases} \tag{3.180}$$

The result presented in (3.178) corresponds to the result given in (2.164) with $n = 3$.

Type Ia supernovaes are known as standard candles since their measurements makes comparison of theory and observations is possible. This is done by the following relation

$$m = 5log_{10}(\frac{d_L}{1Mpc}) + 25 + M, \tag{3.181}$$

where m and M are the apparent and the absolute magnitudes respectively. Then the distance modulus is defined as $\mu = m - M$.

Now we borrow one of our plots from chapter 2. In our previous study we have already performed curve fitting with type Ia supernovae data. We have found

$H_0 = 71.03 \pm 0.20$, $\Omega_{w,0} = 0.889 \pm 0.015$, $\Omega_{m,0} = 0.111 \pm 0.015$ and $\chi^2/\nu = 1.008$ for $M = -19.30$ in domain wall dominated universe. We obtain $q_0 = -0.389$ with these values of cosmological density parameters. On the other hand for dark energy dominated universe we have found $H_0 = 71.80 \pm 0.22$, $\Omega_{\Lambda,0} = 0.715 \pm 0.012$, $\Omega_{m,0} = 0.285 \pm 0.012$ and $\chi^2/\nu = 0.990$ for $M = -19.30$. We obtain $q_0 = -0.572$. All these results are shown in Figure 3.1.

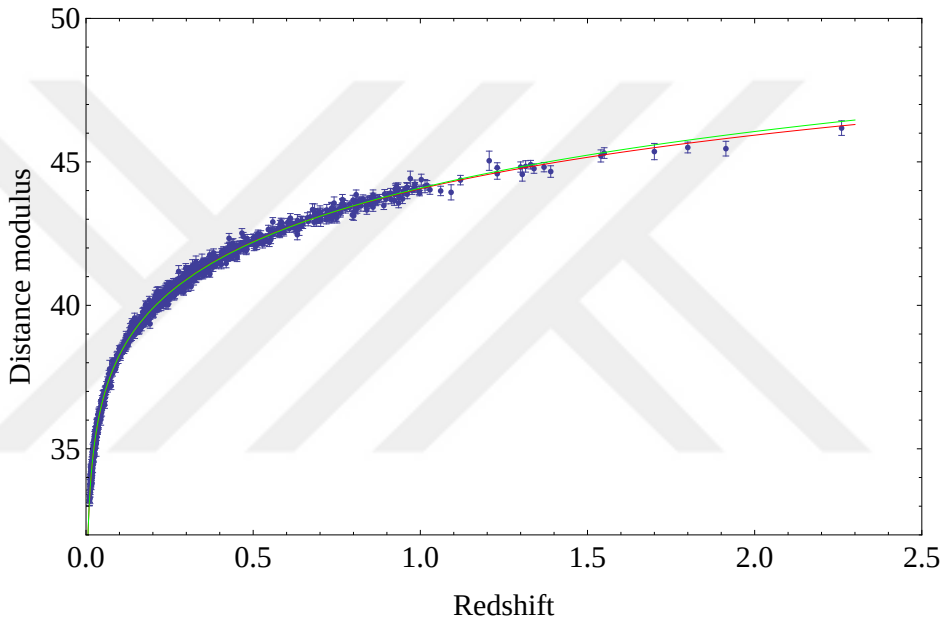

Figure 3.1. Distance Modulus vs Redshift Plot for $M = -19.30$ Reproduced

Dots represent observations of Pantheon data while green line represents domain wall dominated universe and red line represents dark energy dominated universe.

In addition

$$\lim_{a\to\infty} q(a) = -0.5. \tag{3.182}$$

$V(\phi)$ is formulated as

$$V(\phi) = -\frac{2\rho_w}{3(2+\omega)}(\frac{\phi}{F})^{-2(1+\omega)}. \tag{3.183}$$

Although power of ϕ in the potential is different from 2 it causes accelerated expansion of the universe.

3.9. Discussion

We have revealed three major points of cosmology in this study. These are the answers for the following questions:

(i) Is there any corresponding energy for the cosmological constant?
(ii) Is it possible to have accelerated universe without cosmological constant?
(iii) Is ratio of dark matter to baryonic mater smaller than $\frac{0.27}{0.05}$?

It is known that constant energy density contributes to the Hubble function. We have examined this phenomena in Section 3.7 and in Section 3.8.1. Firstly we have solved our system of equations for early universe by taking

$$\phi(a) = \frac{F}{a^d}, \qquad d = -\frac{1}{2(1+\omega)}, \qquad \rho(a) = \frac{\rho_r}{a^4} + \Lambda. \tag{3.184}$$

Then $H(a)$ is found in the form of (3.140) where there is a constant term α. At first sight there is no contribution of Λ in α_1 given by (3.142). However one can also claim that Λ contributes in α_3 by substituting (3.147) in (3.146). This dilemma can be

explained as follows. One can write

$$V(a_{in}) = \frac{3F^2 k a_{in}^{1/(1+\omega)-2}}{4\omega} + \frac{(3+2\omega)(4+3\omega)}{8\omega(1+\omega)^2} F^2 H^2(a_{in}) a^{1/(1+\omega)}(a_{in}) - \frac{\rho_r}{a_{in}^4} - \Lambda. \tag{3.185}$$

Then one can rewrite $H(a_{in})$ in terms of $V(a_{in})$ and Λ.

We have also examined the spatially flat late universe in Section 3.8.1 by supplying the following functions

$$\phi(a) = \frac{F}{a^d}, \qquad d = -\frac{1}{2(1+\omega)}, \qquad \rho(a) = \frac{\rho_m}{a^3} + \Lambda. \tag{3.186}$$

The Hubble function has been found in the form given in (3.159) which has a constant term. At first sight this constant term which is given by (3.162) does not contain Λ. One can apply the same trick and one can write

$$V(a_{in}) = \frac{(3+2\omega)(4+3\omega)}{8\omega(1+\omega)^2} F^2 H^2(a_{in}) a_{in}^{1/(1+\omega)} - \frac{\rho_m}{a_{in}^3} - \Lambda. \tag{3.187}$$

Again, one can rewrite $H(a_{in})$ in terms of $V(a_{in})$ and Λ and one can claim that $H(a)$ contains Λ.

There are two possible interpretations:

(i) Constant energy density does not contribute to the Hubble function, it only modifies value of potential at the beginning of the universe.

(ii) Initial value of the Hubble function $H(a_{in})$, can be rewritten in terms of $V(a_{in})$ and Λ so constant energy density contributes to the Hubble function.

Both of them can be reasonable according to one's perspective. Although if one chooses second explanation which is in agreement with common trend, one can still ask the question in another way: *Is it possible to have a constant term in the Hubble function*

when there is no constant energy density and no constant potential in the universe? To answer this question we have investigated single component universe in Section 3.6 by giving following functions

$$\phi(a) = \frac{F}{a^d}, \qquad d = -\frac{1}{2(1+\omega)}, \qquad \rho(a) = \frac{\rho_n}{a^n}. \tag{3.188}$$

Then $H(a)$ is found in the form given by (3.125). The Hubble function has a constant term although there is no constant energy density. This constant is given in (3.127). Now $V(a_{in})$ does not contain Λ. In addition, expected exponential expansion of $a(t)$ is given in (3.136). Thus there is one possible explanation. When there is no constant energy density, the Hubble function still has a constant term which causes exponential expansion.

We have shown that universe has an accelerated expansion when one introduces cosmic domain walls with matter in the energy density in Section 3.8.2. We had studied this case with Friedmann cosmology in chapter 2, and we have compared our model with the latest supernovae data. It is seen that both dark energy dominated and domain wall dominated universe have perfect fit with data. The differences between these two models most probably will be seen when bigger redshift data are available.

While comparison of dark energy dominated universe with observation has resulted in $\Omega_\Lambda = 0.715 \pm 0.012$ and $\Omega_m = 0.285 \pm 0.012$, comparison of domain wall dominated universe with observation has resulted in $\Omega_w = 0.889 \pm 0.015$ and $\Omega_m = 0.111 \pm 0.015$. Hence a new question appears; is the ratio of dark matter to baryonic matter less than $\dfrac{0.27}{0.05}$?

In 1980s Modified theories of Newtonian Dynamics or MOND, were proposed [58, 139]. It mainly claims that observational aspects of galaxies can be understood without dark matter. Recent observational evidence for the external field effect in MOND which was proposed as an alternative to dark matter is presented in [60]. A new relativistic MOND theory [59] successfully reproduces cosmic microwave back-

ground power spectra. These theories may explain non-baryonic part of domain wall dominated universe which covers six percent of the energy-matter content of the universe.

In this chapter we have applied change of variable $a = \dot{a}H(a)$ to field equations of Brans-Dicke theory and have written all equations in terms of independent variable "a". Then we have ended up with a constraint equation and a Bernoulli type differential equation which can be linearized. We have presented analytic solutions for supplied pairs of functions; ($\phi(a)$ and $\rho(a)$) , ($\phi(a)$ and $V(a)$), or ($\phi(a)$ and $H(a)$).

Investigation of single component universe has shown that one will have a constant term in the Hubble function although there is no constant energy density. Early epoch of the universe with dark energy and radiation have been studied, and exponential expansion is seen in $a(t)$. Late-time acceleration is obtained for both dark energy dominated universe and domain wall dominated universe. When Brans-Dicke parameter $\omega \gg 1$, the Hubble function reduces to $H(a)$ of the Einstein cosmology. In all cases potential is found as combination of power law potentials.

We would like to emphasize that so-called dark energy term may be just a number without corresponding constant energy-matter density. Details of this subject has been presented in the discussion. Hopefully astronomers and cosmologists will pay more attention for searches of cosmic domain walls. One can also reach our calculations which are presented in this chapter in [140].

4. INFLATION AND LINEAR EXPANSION IN THE RADIATION DOMINATED ERA IN BRANS-DICKE COSMOLOGY

In this chapter we study ϕ^4 potential in Brans-Dicke cosmology. This potential has been already recognized in Section 3.3.1.2. It results in real Hubble function and positive energy density. Our systematic studies have shown that choice of other power law potentials can not satisfy positivity of energy density together with real Hubble function. Here instead of presenting details of these parts, we just focus on ϕ^4 potential. Now we will pick time as a independent variable different than our choices in chapter 2 and chapter 3.

In this chapter we will use FLWR metric given by,

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{4.1}$$

where k is the curvature parameter with $k = -1, 0, 1$ corresponding to open, flat, closed universes respectively, $a(t)$ is the scale factor of the universe. Here r is dimensionless and $a(t)$ has the dimension of lenght. We use the Lagrangian which has been introduced in chapter 3. We choose

$$V(\phi) = \frac{1}{4}\lambda\phi^4, \tag{4.2}$$

which in flat spacetime leads to a renormalizable quantum field theory. Due to the choice of scale-invariant potential, the coupling constant λ is dimensionless, so that there are no dimensional parameters in the BDJT part of the Lagrangian density in (3.2) and (3.3).

We would like to remind the reader the field equations which have been derived in previous chapter,

$$\frac{3}{4\omega}\phi^2\left(\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)-\frac{1}{2}\dot{\phi}^2-V(\phi)+\frac{3}{2\omega}\frac{\dot{a}}{a}\dot{\phi}\phi=\rho_m, \tag{4.3}$$

$$\frac{-1}{4\omega}\phi^2\left(2\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)-\frac{1}{\omega}\frac{\dot{a}}{a}\dot{\phi}\phi-\frac{1}{2\omega}\ddot{\phi}\phi-\left(\frac{1}{2}+\frac{1}{2\omega}\right)\dot{\phi}^2+V(\phi)=p_m, \tag{4.4}$$

$$\ddot{\phi}+3\frac{\dot{a}}{a}\dot{\phi}+\frac{dV(\phi)}{\phi}-\frac{3}{2\omega}\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{k}{a^2}\right)\phi=0. \tag{4.5}$$

We will name (4.3) as the energy density equation, (4.4) as the pressure equation and (4.5) as the ϕ equation.

Sen and Seshadri [141] have investigated the nature of a potential relevant to the power law expansion in BD cosmology. In [136–138] a perturbation technique was applied to the above equations with $V(\phi)=\frac{1}{2}m^2\phi^2$. In addition, exact solutions of modified BD cosmological equations have been found by using symmetry analysis [142]. On the other hand the quartic potential, $V(\phi)=\frac{1}{4}\lambda\phi^4$ has been studied in [143] where solutions are comparable with the observed cosmological data only for small negative values of ω for spatially flat FRW geometry. Chubaryan *et al.* have studied the quartic potential with barotropic equation of state in [144]. This potential has also been studied in [145] in the presence of a generalized Brans-Dicke parameter $\omega_{GBD}(\phi)$. Santos and Gregory have found linearly and exponentially expanding solutions for vacuum cosmologies [146].

Although to investigate role of stiff matter in cosmology was not the main purpose of this work, it appears in our results. An exocit fluid with an equation of state $\nu=p/\rho=1$ was first introduced by Zeldovich [147]. This fluid is also called as the stiff fluid or Zeldovich fluid and gives energy density proportional to $1/a^6$. Many scientists have produced cosmological models with stiff matter [148–156]. In addition there exist studies on stiff matter in Brans-Dicke Theory [157–159]. In an another work a complex scalar field description of Bose-Einstein condensate dark matter was studied [160]. It has been found that the early universe evolves from stiff ($p=\rho$) to radiationlike

$(p = \rho/3)$. We have obtained a similar result in our calculations.

In this chapter we present exact solutions of (4.3-4.5) with $V(\phi) = \frac{1}{4}\lambda\phi^4$ in a $k = 1$ closed universe. We choose $\omega > 4 \times 10^4$ to be comfortable compatible with results of Einstein telescope [130] and time delay experiments [131]. Our starting point is to introduce a scale invariant solution in Section 4.1. Once we find the Jordan scalar field as a function of time, we calculate the scale-factor of the universe for different eras by tracking the behaviour of the field forward and backward in time. Therefore we will have three different $\phi(t)$, we will solve the ϕ equation for each case and we will obtain the scale factor as a function of time for each era. These solutions result in inflation in the early radiation dominated era, linear expansion in the late radiation dominated era and scale invariant solution with stiff fluid between these two eras.

Here we should take attention of the reader to the important remark. In classical cosmology the basic information obtained from observation is the value of the Hubble function. It is related with the theory by one of the Friedmann-Lemaitre equation

$$\dot{a}^2(t) = \frac{8\pi G}{3}\rho a^2(t) + \frac{1}{3}\Lambda c^2 a^2(t) - c^2 k. \tag{4.6}$$

In addition the followings are definitions of the Hubble function, total energy density, density parameters and the curvature density parameter

$$H \equiv \frac{\dot{a}}{a}, \qquad \rho(t) \equiv \rho_{m,0}[\frac{a_0}{a(t)}]^3 + \rho_{r,0}[\frac{a_0}{a(t)}]^4 + \rho_{\Lambda,0}, \tag{4.7}$$

$$\Omega_i(t) \equiv \frac{8\pi G}{3H^2(t)}\rho_i(t), \qquad \Omega_k(t) = -\frac{c^2 k}{H^2(t)a^2(t)}, \tag{4.8}$$

where the label i includes matter, radiation, and the vacuum. Comparison with observation is usually made by writing last equation in the form

$$H^2 = H_0^2\{\Omega_{m,0}[\frac{a_0}{a(t)}]^3 + \Omega_{r,0}[\frac{a_0}{a(t)}]^4 + \Omega_{\Lambda,0} + \Omega_{k,0}[\frac{a_0}{a(t)}]^2\}, \tag{4.9}$$

where 0 denotes present values of the related function. However in our model right hand side of the equation is different,

$$H^2 = \frac{\rho + \frac{1}{2}\dot{\phi}^2 + V(\phi) - \frac{3}{2\omega}\frac{\dot{a}}{a}\dot{\phi}\phi}{\frac{3}{4\omega}\phi^2} - \frac{k}{a^2}. \tag{4.10}$$

Thus our model should be compared with (4.9) after expressing ϕ and $\dot{\phi}$ as functions of a. The only mathematical constraint on the above equation is positivity of right side. For this reason one can obtain interesting negative energy density contributions by providing right hand side is positive as a whole. At first sight, negative energy density seems unreasonable. However in the literature there are many physical cases which contain negative energy. It was firsly discovered by Casimir in quantum field theory [161]. In 1975, Hawking found the negative energy density across the event horizon [162]. In the following years the negtaive energy fluxes in radiation from moving mirrors were reported [163, 164]. In different fields there are many studies which give rise to comments on negative energy [165–170].

We also show that introducing matter in the linearly expanding radiation era can give a decelerating universe with positive energy density perturbation or an accelerating universe with negative energy density perturbation depending on the choice of one of the constants in our model in Section 4.1. Then early inflation is investigated in Section 4.2. We find the Hubble function and the deceleration parameter for each era. Then we calculate temperature-time relations for each era we investigate in Section 4.3. We study the passage from big bang to the scale invariant solution and the passage from the scale invariant solution to the linearly expanding solution in Section 4.4. Finally we extrapolate our results to the present day and show that this model gives acceleration for the present universe, albeit with a deceleration parameter which is not as big as observed in Section 4.5.

4.1. Radiation Dominated Era

4.1.1. The Scale Invariant Solution

One important property of potential in (4.2) is that it does not introduce any dimensional parameters into the Lagrangian density so that the action and the resulting equations are scale invariant. In this part we have assumed that the relation between $\phi(t)$ and $a(t)$ preserves the scale invariance and is given by

$$\phi(t) = \frac{A}{a(t)}, \tag{4.11}$$

where A is a positive dimensionless constant. With this constraint, the ϕ equation becomes a second order nonlinear differential equation.

$$\frac{A[-3 + 2A^2\omega\lambda - (3+2\omega)\dot{a}^2 - (3+2\omega)a\ddot{a}]}{2\omega a^3} = 0, \tag{4.12}$$

where dot denotes derivative with respect to time. To be able to solve this equation we introduce new variable $\theta(t) = a^2(t)$. Then the differential equation reduces to

$$\ddot{\theta} = \frac{4\omega\lambda A^2 - 6}{3 + 2\omega}. \tag{4.13}$$

One can easily find θ and with appropriate choice of integration constants b_1 and b_2 the solution for the scale factor can be written as

$$a(t) = \sqrt{\left(\frac{2A^2\omega\lambda - 3}{3+2\omega}\right) t^2 + b_1 t + b_2}. \tag{4.14}$$

Behaviour of energy density and pressure are found to be

$$\rho = A^2 \left[\frac{6 - 3A^2\omega\lambda}{4a^4\omega} + \frac{-(3+2\omega)b_1^2 + 4(-3+2A^2\omega\lambda)b_2}{16a^6\omega} \right], \tag{4.15}$$

$$p = A^2 \left[\frac{2 - A^2\omega\lambda}{4a^4\omega} + \frac{-(3+2\omega)b_1^2 + 4(-3+2A^2\omega\lambda)b_2}{16a^6\omega} \right]. \tag{4.16}$$

The term proportional to $1/a^6$ is the stiff matter term [147]. By using the usual continuity equation which is satisfied by (4.3-4.5) the terms proportional to a^{-4} are readily recognized as radiation whereas the a^{-6} stiff fluid are related to maximal pressure $p = \rho$ without violation of positivity of energy. Positivity of both terms and real scale factor requires $3/2 < A^2\omega\lambda < 2$ and $b_2 > (3+2\omega)b_1^2/4(2A^2\omega\lambda - 3)$. We should note that constants b_1 and b_2 are important and they must not be chosen zero. Solutions before and after this era will be matched by adjusting b_1 and b_2. We will call this phase of the universe as the scale invariant phase.

As the universe expands the second term becomes negligible Then the equation of state becomes,

$$\nu = \frac{p}{\rho} = \frac{1}{3}, \tag{4.17}$$

as it should be in the radiation dominated era.

There are two more important cosmological parameters we should calculate are the Hubble function and the deceleration parameter;

$$H(t) = \frac{\dot{a}}{a}, \qquad q(t) = \frac{d}{dt}(\frac{1}{H}) - 1 = -\frac{a\ddot{a}}{\dot{a}^2}. \tag{4.18}$$

For this era these parameters are found as

$$H(t) = \frac{2(\frac{2A^2\omega\lambda - 3}{3+2\omega})t + b_1}{2[(\frac{2A^2\omega\lambda - 3}{3+2\omega})t^2 + b_1 t + b_2]}, \tag{4.19}$$

$$q(t) = \frac{(3+2\omega)[(3+2\omega)b_1^2 + 4(3 - 2A^2\omega\lambda)b_2]}{[2 + (2A^2\omega\lambda - 3) + (3+2\omega)b_1]^2}. \tag{4.20}$$

This era takes place after early inflation. Thus time is always greater than the value of t_1 which is found in Section 4.4 by (4.78). Therefore one can easily conclude that the Hubble parameter is always positive. Our condition on b_1 and b_2 which makes energy density positive, makes deceleration parameter negative. Thus we have an accelerated era.

4.1.2. Linearly Expanding Radiation Dominated Universe

As time increases its effect in (4.14) becomes larger so we can make the assumption

$$\phi(t) = \frac{B}{t}. \tag{4.21}$$

Then ϕ equation becomes

$$\frac{4\omega B a^2 - 6\omega B t a\dot{a} + 2\omega\lambda B^3 a^2 - 3Bt^2(a\ddot{a} + \dot{a}^2 + 1)}{2\omega t^3 a^2} = 0. \tag{4.22}$$

Firstly we set $a^2 = \theta$ in the above equation and we obtain

$$t^2\ddot{\theta} + 2\omega t\dot{\theta} - \frac{4\omega}{3}(2 + \lambda B^2)\theta = -2t^2. \tag{4.23}$$

Last equation is easily recognized as non-homogeneous Cauchy-Euler equation and its solution is found as

$$\theta(t) = c_1 t^{m_+} + c_2 t^{m_-} + \frac{3t^2}{2\omega(\lambda B^2 - 1) - 3}, \tag{4.24}$$

$$m_\pm = \frac{1}{2} - \omega \pm \sqrt{\omega^2 - \omega + \frac{1}{4} + \frac{4\omega}{3}(2 + \lambda B^2)}, \tag{4.25}$$

where c_1 and c_2 are integration constants (Details of this calculation is given in Appendix C). Unless one choose $c_1 = c_2 = 0$, it is impossible to obtain pressure and the density in the form c/a^n with constant c and rational n. Therefore we obtain the scale factor and the Jordan field as

$$a(t) = \sqrt{\frac{3}{2B^2\omega\lambda - (2\omega + 3)}}t, \quad \phi(t) = \sqrt{\frac{3}{2B^2\omega\lambda - (2\omega + 3)}}\frac{B}{a(t)}. \tag{4.26}$$

This choice satisfies the ϕ equation which is always equal to zero. By using the gravitational field equations one can easily calculate energy, pressure and equation of state as

$$\rho = \frac{9B^2[B^2\omega\lambda - 2(2\omega + 3)]}{[2B^2\omega\lambda - (2\omega + 3)]^2}\frac{1}{4\omega a^4}, \tag{4.27}$$

$$p = \frac{9B^2[B^2\omega\lambda - 2(2\omega + 3)]}{[2B^2\omega\lambda - (2\omega + 3)]^2}\frac{1}{12\omega a^4}, \tag{4.28}$$

$$\nu = \frac{p}{\rho} = \frac{1}{3}. \tag{4.29}$$

Note that in this era although we have not imposed the ansatz $\phi(t) = A'/a(t)$, we have ended up with it.

After this point we will continue with results of Section 4.4 where we match solutions for each era. Continuity of $\phi(t)$ and $a(t)$ at the passage from scale invariant

phase to linearly expanding era gives

$$a(t) = \sqrt{\frac{2A^2\omega\lambda - 3}{2\omega + 3}}t, \qquad B = A\sqrt{\frac{2\omega + 3}{2A^2\omega\lambda - 3}}. \tag{4.30}$$

Now let us calculate the Hubble function and the deceleration parameter for this era,

$$H(t) = \frac{1}{t}, \qquad q(t) = 0. \tag{4.31}$$

Thus universe expands with constant velocity.

4.1.3. Creation of Matter in the Late Radiation Dominated Era

When the temperature is much smaller than $m_e = 0.51\text{MeV}$ electrons and baryons can be considered non-relativistic. Pressure as compared to matter energy density can be neglected for a gas of non-relativistic particles. For this reason it can be accepted that at this stage of the universe non-relativistic matter starts to be created and eventually dominates radiation. We assume that creation of matter in radiation dominated era causes small changes in the Jordan field and in the scale factor. Thus we start the ansatz,

$$\tilde{\phi}(t) = \frac{A}{\mathcal{B}t} + \psi(t) \quad \text{with} \quad \psi(t) = ut^m, \tag{4.32}$$

$$\tilde{a}(t) = \mathcal{B}t + \alpha(t) \quad \text{with} \quad \alpha(t) = vt^n, \tag{4.33}$$

$$\mathcal{B} = \sqrt{\frac{2A^2\omega\lambda - 3}{2\omega + 3}}, \tag{4.34}$$

and ψ and α are small. First we write all three equations in terms of new functions. In addition we neglect second and higher order terms $(\alpha^2, \psi^2, \alpha\psi, ...)$ in the corrections in the perturbations of α and ψ. We easily obtain the constant $v \sim t^{2+m-n}$ from the ϕ equation. It follows that we must choose $m = n - 2$ to keep v constant. Then energy

density becomes

$$\rho = \frac{C_1}{t^4} + \frac{C_2}{t^{5-n}}, \tag{4.35}$$

with C_1 and C_2 are constants. Thus we choose $n = 2$ to have two component energy density which is composed of a radiation part and a matter part

$$\rho = \frac{C_r}{a^4} + \frac{C_m}{a^3}, \tag{4.36}$$

where

$$C_r = -\frac{3A^2(-2 + A^2\omega\lambda)}{4\omega}, \tag{4.37}$$

$$C_m = -\frac{Au[27 + 2\omega(6 + A^2\lambda(-12 - 5\omega + 2A^2\omega\lambda(2 + \omega)))]}{2\omega[-9 - 5\omega + 2A^2\omega\lambda(2 + \omega)]}. \tag{4.38}$$

Then pressure is found as,

$$p = \frac{C_r/3}{a^4} + \frac{P_m}{a^3}, \tag{4.39}$$

$$P_m = \frac{Au(-2 + A^2w\lambda)[-w + 2A^2w\lambda(1 + w)]}{w[-9 - 5\omega + 2A^2\omega\lambda(2 + \omega)]}. \tag{4.40}$$

The constant v which is part of the scale factor, is found as

$$v = \frac{u(-3 + 2A^2\omega\lambda)[-\omega + 2A^2\omega\lambda(1 + \omega)]}{A(3 + 2\omega)[-9 - 5\omega + 2A^2\omega\lambda(2 + \omega)]}. \tag{4.41}$$

We have already found $3/2 < A^2\omega\lambda < 2$ at the end of the discussion of Section 4.1.1. In addition A, λ, ω are all positive parameters. In this scope one can easily find the following results by inspection,

$$sign(C_m) \equiv sign(P_m) \equiv sign(u) \quad and \quad sign(v) \equiv -sign(u). \tag{4.42}$$

In addition it is also obvious that

$$\lim_{A^2\omega\lambda\to 2^-} \frac{p_{perturbation}}{\rho_{perturbation}} = \lim_{A^2\omega\lambda\to 2^-} \frac{P_m}{C_m} \to 0^+ \ll 1, \tag{4.43}$$

as it should be in the matter dominated era. Thus we can neglect the perturbation in pressure. In addition this result shows that we should take the value of $A^2\omega\lambda$ as 2^- that is a little less than 2. Other effects of this result on our model will be explained in the following sections.

The Hubble function and the deceleration parameter for this era are found as

$$H(t) = \frac{\sqrt{\frac{2A^2\omega\lambda-3}{3+2\omega}} + 2vt}{\sqrt{\frac{2A^2\omega\lambda-3}{3+2\omega}}t + vt^2}, \quad q(t) = -\frac{2tv(tv + \sqrt{\frac{2A^2\omega\lambda-3}{3+2\omega}})}{(\sqrt{\frac{2A^2\omega\lambda-3}{3+2\omega}} + 2tv)^2}. \tag{4.44}$$

By using $v \ll 1$ which is the condition to write the perturbation in the field equations, we obtain

$$H(t) \simeq \frac{1}{t} + \frac{v}{\sqrt{\frac{2A^2\omega\lambda-3}{3+2\omega}}}, \quad q(t) \simeq \frac{-2vt}{\sqrt{\frac{2A^2\omega\lambda-3}{3+2\omega}}}. \tag{4.45}$$

Now one should develop a solid argument to determine the sign of the constant v. Firstly let us start with positivity of energy density. In this scope the parameter u must be chosen as a positive number to ensure positivity of perturbation term in energy density. Thus v is found to be negative. This tell us that the Hubble function can be negative and the universe can decelerate. This means creation of matter can cause a big crunch depending on the values of the constants in our model.

But since the universe has expanded during this era it has the positive Hubble function. This makes the constant v positive and thus the constant u negative. At this moment one can immediately say that matter energy density is negative. However baryonic matter has positive energy density. Therefore one should write the matter perturbation to radiation energy density with two components where one of them is

positive and the other one is negative for example,

$$C_m = -\frac{Au[2\omega A^2\lambda(-12-5\omega)]}{2\omega[-9-5\omega+2A^2\omega\lambda(2+\omega)]} - \frac{Au[27+2\omega(6+A^2\lambda(2A^2\omega\lambda(2+\omega)))]}{2\omega[-9-5\omega+2A^2\omega\lambda(2+\omega)]}. \tag{4.46}$$

Thus we conclude that universe can accelerate in this era by using positivity of v. This result can be interpreted as baryonic matter being created together with matter of negative energy density. This causes acceleration which in the standard model would be described as dark matter. At this point we should once again mention that because of the dependence of Jordan scalar field $\phi(t)$ on the scale size $a(t)$ the contribution of this negative energy density term to observation will be different from Einstein's gravity as indicated in (4.9) and (4.10).

4.2. Early Inflation in the Radiation Dominated Era

Here we relax our constraint $\phi(t) = \dfrac{A}{a(t)}$. However we will keep matching our solution for the Jordan field which was found to be

$$\phi(t) = \frac{A}{\sqrt{\left(\frac{2A^2\omega\lambda-3}{3+2\omega}\right)t^2 + b_1 t + b_2}}. \tag{4.47}$$

We see that as t goes to zero, ϕ becomes constant and to investigate this behaviour we look for a solution

$$\phi(t) = F. \tag{4.48}$$

ϕ equation becomes

$$F^3\lambda - \frac{3F}{2wa^2}(1+\dot{a}^2+a\ddot{a}) = 0. \tag{4.49}$$

We have again used the change of variable method to solve the differential equation by introducing $\theta(t) = a^2(t)$. Then the equation immediately reduces to

$$\ddot{\theta} - \frac{4\omega\lambda F^2\theta}{3} = -2, \tag{4.50}$$

and the positive solution for the scale factor of the universe is

$$a(t) = \sqrt{c_1 e^{\alpha t} + c_2 e^{-\alpha t} + \frac{3}{2\omega\lambda F^2}}, \qquad \alpha = 2F\sqrt{\frac{\omega\lambda}{3}}. \tag{4.51}$$

For this solution $\phi(t)$ can be interpreted as an effective cosmological constant. Energy density and pressure are easily calculated by substitution of $a(t)$ and $\phi(t)$ in Equations (4.3) and (4.4) as

$$p = \frac{9 - 16c_1c_2F^4\omega^2\lambda^2}{48\omega^2\lambda a^4}, \tag{4.52}$$

$$\rho = \frac{9 - 16c_1c_2F^4\omega^2\lambda^2}{16\omega^2\lambda a^4}, \tag{4.53}$$

$$\nu = \frac{p}{\rho} = \frac{1}{3}. \tag{4.54}$$

To have $\rho > 0$ we must satisfy the condition $9 - 16c_1c_2F^4\omega^2\lambda^2 > 0$.

To simplify our result let us choose $c_1 = d_1/\alpha^2$ and $c_2 = d_2/\alpha^2$ where $d_1 > 0$. Then we have

$$a(t) = \frac{1}{\alpha}\sqrt{d_1 e^{\alpha t} + d_2 e^{-\alpha t} + 2}. \tag{4.55}$$

Choosing the integration constant d_1 and d_2 such that $a(t)$ is minimum at $t = 0$ gives us $d_1 = d_2$ and $a(t)$ must approach a positive non-zero value at $t = 0$. Therefore our

results become

$$a(t) = \frac{\sqrt{2}}{\alpha}\sqrt{d_1 \cosh(\alpha t) + 1}, \tag{4.56}$$

$$\rho(t) = \frac{9(1-d_1^2)}{16\omega^2\lambda a^4}, \quad p = \frac{\rho}{3}, \tag{4.57}$$

where $0 < d_1 < 1$.

For this solution the Hubble function and the deceleration parameter are found as

$$H(t) = \frac{d_1 \alpha \sinh(\alpha t)}{2(d_1 \cosh(\alpha t) + 1)}, \tag{4.58}$$

$$q(t) = 1 - 2\coth^2(\alpha t) - \frac{2}{d_1}\coth(\alpha t)\mathrm{csch}(\alpha t). \tag{4.59}$$

It is obvious that the Hubble function takes positive values. Let us make a change of variable $x = \cosh(\alpha t)$ to see the behaviour of the deceleration parameter

$$q(x) = 1 - 2(\frac{x^2 + x/d_1}{x^2 - 1}), \qquad x \geq 1. \tag{4.60}$$

Since $x \geq 1$, the value of $q(x)$ is always smaller than zero. Hence we have an accelerated expansion. In addition $t = 0$ corresponds $x = 1$ which creates singularity in the function $q(x)$. This negative infinite value of the deceleration parameter can explain the big bang.

We can also choose $a(t) = 0$ when $t = 0$ with $d_2 = -2 - d_1$. Then we obtain

$$a(t) = \frac{\sqrt{2}}{\alpha}\sqrt{d_1 \sinh(\alpha t) + 1 - e^{-\alpha t}}, \tag{4.61}$$

$$\rho(t) = \frac{9(1+d_1)^2}{16\omega^2\lambda a^4}, \quad p = \frac{\rho}{3}. \tag{4.62}$$

with $0 < d_1$.

For this solution the Hubble function and the deceleration parameter are found as

$$H(t) = \frac{\alpha[1 + d_1 e^{\alpha t}\cosh(\alpha t)]}{(e^{\alpha t} - 1)(2 + d_1 + d_1 e^{\alpha t})}, \tag{4.63}$$

$$q(t) = -1 + \frac{8e^{\alpha t}[1 + d_1 e^{\alpha t}(2 + d_1 - \sinh(\alpha t))]}{(2 + d_1 + d_1 e^{2\alpha t})^2}. \tag{4.64}$$

Again let us make a change of variable $x = e^{\alpha t}$ to understand the behaviour of the last functions. Thus we have

$$H(x) = \frac{\alpha[1 + d_1(x^2 + 1)/2]}{(x - 1)(2 + d_1 + d_1 x)}, \tag{4.65}$$

$$q(x) = -1 + \frac{8x[1 + d_1 x(2 + d_1 - (x - x^{-1})/2)]}{(2 + d_1 + d_1 x^2)^2}, \qquad x \geq 1. \tag{4.66}$$

Because $x \geq 1$ the Hubble function is always positive. In this kind of big bang the Hubble function becomes infinite when $x = 1$ ($t = 0$). Besides, at this moment $q(1) = 1$ and as x increases (as time increases) $q(x)$ becomes negative. Therefore expansion always starts with deceleration then acceleration occurs.

These results may indicate that early inflation took place in the radiation dominated era under the effect of the Brans-Dicke-Jordan field.

4.3. Temperature Calculations

The relation between energy density, pressure and temperature had already been derived as

$$\frac{dp(T)}{dT} = \frac{1}{T}[\rho(T) + p(T)]. \tag{4.67}$$

In our calculations pressure and energy density is a function of time. Thus we apply chain rule and obtain the temperature as

$$T = T_0 exp\left[\int_{t_0}^{t} \frac{\frac{dp(t^{'})}{dt^{'}}}{\rho(t^{'}) + p(t^{'})} dt^{'}\right], \tag{4.68}$$

where T_0 is the reference temperature of the universe when it evolves with the scale factor $a(t_0)$. For the early universe, both choices of the scale factor gives the same temperature-time relation which is found as

$$T = T_0 \frac{a(t_0)}{a(t)}. \tag{4.69}$$

For the scale-invariant phase we have

$$T = T_0 \frac{\tilde{a}^2(t) a^3(t_0)}{\tilde{a}^2(t_0) a^3(t)}, \tag{4.70}$$

$$\tilde{a}(t) = \sqrt{\left(\frac{2A^2\omega\lambda - 3}{3 + 2\omega}\right) t^2 + b_1 t + \frac{(3 + 2\omega) b_1^2 - 4b_2}{8(2 - A^2\omega\lambda)}}. \tag{4.71}$$

In the Section 4.1.3 we found that $A^2\omega\lambda$ is a little less than 2. Therefore temperature becomes $T_0 \frac{a^3(t_0)}{a^3(t)}$.

For the linearly expanding radiation dominated era we have obtained

$$T = T_0 \frac{a(t_0)}{a(t)} \quad \text{or} \quad T = \frac{\tilde{T}_0}{t}, \tag{4.72}$$

which is the standard time-temperature relation in the radiation dominated era.

For the era where we introduce matter into radiation temperature is more complicated than the other cases and we find the temperature to be given by

$$T = \frac{(c_1 + uc_2 t)^r}{t} \quad \text{with} \quad \frac{1}{4} < r < 1, \tag{4.73}$$

where c_1 and c_2 are constant. Note that the constant perturbation term $u \ll 1$. Hence by using a series expansion the temperature can be written as

$$T = T_\infty \left(1 + \frac{t_1}{t}\right), \tag{4.74}$$

where T_∞ and t_1 are constant. Physically T_∞ will denote the approximate temperature at the end of the radiation dominated era provided that t_1 can be chosen small compared to the time elapsed from big bang to the end of the radiation dominated era.

4.4. Matching the Solutions

In this section we will try to match solutions of the scale factor, the Jordan field and the energy density for different eras which follow each other in time. We will not be interested in pressure and temperature because microscopic events can affect them. Let us call the scale factor of the universe $a_1(t)$ for the early inflation era, $a_2(t)$ for the scale invariant era, and $a_3(t)$ for the late radiation dominated era. Similarly we name the Jordan field solutions as $\phi_1(t)$, $\phi_2(t)$ and $\phi_3(t)$ and the energy densities as $\rho_1(t)$, $\rho_2(t)$, $\rho_3(t)$ respectively. Initially we have tried to match $\phi_1(t)$ with $\phi_2(t)$ smoothly at a certain time. Then we try to match $a_1(t)$ with $a_2(t)$ smoothly at the same certain time. Thus we obtain four equations for continuity of $\phi(t)$, $\dot{\phi}(t)$, $a(t)$ and $\dot{a}(t)$. However it is only possible to have three equations satisfied at the same boundary. We eliminate

continuity of $\dot{a}(t)$. Consequently we end up with the following equations

$$\phi_1(t) = \phi_2(t) \quad \text{at } t = t_1, \tag{4.75}$$

$$\dot{\phi}_1(t) = \dot{\phi}_2(t) \quad \text{at } t = t_1, \tag{4.76}$$

$$a_1(t) = a_2(t) \quad \text{at } t = t_1. \tag{4.77}$$

The passage from early inflation to scale invariant phase occurs at time t_1 where

$$t_1 = -\frac{3+2\omega}{2(2A^2\omega\lambda - 3)} b_1, \tag{4.78}$$

thus $b_1 < 0$. When we use $a_1(t) = \frac{\sqrt{2}}{\alpha}\sqrt{d_1 \cosh(\alpha t) + 1}$ we obtain

$$d_1 = \frac{2A^2\omega\lambda - 3}{3\cosh(\alpha t_1)}, \tag{4.79}$$

so that the condition $0 < d_1 < 1$ satisfied. When we have $a_1(t) = \frac{\sqrt{2}}{\alpha}\sqrt{d_1 \sinh(\alpha t) + 1 - e^{-\alpha t}}$

$$d_1 = \frac{2A^2\omega\lambda - 3}{3\sinh(\alpha t_1)} + \frac{e^{-\alpha t_1}}{\sinh(\alpha t_1)}, \tag{4.80}$$

so that the condition $0 < d_1$ satisfied.

By using the information obtained above and $3/2 < A^2\omega\lambda < 2$ one can easily compare energy densities for the early universe and the scale-invariant phase. It is seen that there is a loss in energy density at the passage. In Section 4.1.3 we have concluded that the value of $3/2 < A^2\omega\lambda < 2$ is a little less than 2. This makes scale invariant era stiff fluid dominated and in this limit there is still a loss in energy density at the passage.

Now we will study matching the scale-invariant phase with the linearly expanding radiation dominated era. We can satisfy only continuity of $a(t)$ and $\phi(t)$. This gives

the following results

$$t_2 = \frac{-b_2}{b_1}, \qquad B = A\sqrt{\frac{2\omega+3}{2A^2\omega\lambda-3}}. \tag{4.81}$$

Therefore when we combine the outcomes of the two matching procedure, scale factors for second and third eras can be written as

$$a_2(t) = \sqrt{(\frac{2A^2\omega\lambda-3}{2\omega+3})[t^2+2t_1(t_2-t)]}, \tag{4.82}$$

$$a_3(t) = \sqrt{\frac{2A^2\omega\lambda-3}{2\omega+3}}t. \tag{4.83}$$

Again it is found that there is a loss in the energy density at the passage.

4.5. Discussion

There are three gravitational field equations given by (4.3-4.5). A standard mathematical procedure to solve such a set of differential equations which has nonlinear terms with variable coefficients has not been invented yet. Usually one chooses an appropriate energy density to find the solution for the desired era. However our approach to the problem is different. We have used the scale invariant ansatz $\phi(t) = A/a(t)$ and obtained an exact solution for the Jordan field and the scale factor which evolves from radiation and stiff fluid combined phase to a radiation phase. Similar results have been found in the standard model [160]. We have named this combined phase as the scale invariant phase and by investigating the behaviour of the Jordan field backward in time, we have found a universe which starts to expand exponentially at big bang with pure radiation. Similarly, investigating the behaviour of the Jordan field by extrapolating forward in time we again obtain a pure radiation dominated phase which expands linearly. We have found that introducing matter in this linearly expanding late radiation era may cause deceleration or acceleration. Furthermore we have presented the time-temperature relations for each era. As a result we have not only found the scale factor and the Jordan field for each era but also we have found the order of the

relevant eras in time. In the last part of the calculations we have matched solutions for $\phi(t)$ and $a(t)$ at the boundaries of the eras.

Importantly, we have formulated two different scale factors which cause inflation at the beginning of time. One of them always has an accelerated expansion and the other one starts with decelerated expansion and then turns into accelerated expansion. Acceleration of the universe continues in stiff fluid dominated phase. This acceleration decreases and the universe expands linearly. Introducing matter in this era may cause big crunch or may cause acceleration of the universe depending on our choice of one of the parameters in the perturbation. Thus the three important features of closed space-like section, radiation domination and primordial inflation can be explained by the JBDT model.

In our model we have not formulated a solution for behaviour of the present universe. However by using field equations one can approximately find the Jordan scalar field $\phi(t)$ near the present era. For this purpose effective gravitational constant is defined as $G_{eff}^{-1} = \frac{2\pi}{\omega}\phi^2$. We take the Friedmann equation for our model given by (4.10) and the definitions given by (4.7) and (4.8) which are used in standard cosmology. Then we set $G_{present} = \frac{\omega}{2\pi\phi^2(t_0)}$ and put time $t = t_0$ to both sides of the equation where 0 denotes today. Here we should remind that the cosmological constant Λ in our model is zero. We obtain the following equation

$$\Omega_{m,0} + \Omega_{r,0} + \Omega_{k,0} + \frac{4\omega}{3H_0^2}[\frac{\dot{\phi}(t_0)}{\phi(t_0)}]^2 + \frac{\lambda\omega}{3H_0^2}\phi^2(t_0) - \frac{2}{H_0}[\frac{\dot{\phi}(t_0)}{\phi(t_0)}] = 1. \tag{4.84}$$

We borrow the following equation

$$\Omega_{m,0} + \Omega_{r,0} + \Omega_{k,0} + \Omega_{\Lambda,0} = 1, \tag{4.85}$$

from the standard model of the cosmology. In addition we make a change of variable $\theta = \frac{\dot{\phi}(t_0)}{\phi(t_0)}$ and rearrange the terms for simplifications,

$$\theta^2 - \frac{3H_0}{2\omega}\theta + \frac{\lambda\omega}{8\pi G_{present}} - \frac{3H_0^2}{4\omega}\Omega_{\Lambda,0} = 0. \tag{4.86}$$

One can produce another useful formula by using today's value of pressure which is zero in the field equations. First we find $(-1/3)$(LHS of (4.3))+(LHS of (4.4))$= (-1/3)\rho$. Then we use the ϕ equation to get

$$(\frac{1}{3} + \frac{1}{2\omega})[\ddot{\phi}(t)\phi(t) + \dot{\phi}^2(t)] + (1 + \frac{3}{2\omega})\frac{\dot{a}(t)}{a(t)}\dot{\phi}(t)\phi(t) = -\frac{\rho}{3}. \tag{4.87}$$

We use the same tricks which have been used to find (4.86) with change of variable $\beta = \phi^2$. The result is

$$\ddot{\beta}(t_0) = \frac{-H_0^2}{4\pi G_{present}}\frac{(\Omega_{m,0} + \Omega_{r,0})}{(\frac{1}{3} + \frac{1}{2\omega})} - 3H_0\dot{\beta}(t_0). \tag{4.88}$$

First one should solve (4.86) and obtain θ. Then $\dot{\phi}(t_0) = \phi(t_0)\theta$ with $\phi(t_0) = \sqrt{\frac{\omega}{2\pi G_{present}}}$. Secondly one should use this information to find $\dot{\beta}(t_0) = 2\dot{\phi}(t_0)\phi(t_0)$. Then one should solve (4.88) and obtain $\ddot{\beta}$. Finally one should find $\ddot{\phi}(t_0) = \frac{\ddot{\beta} - 2\dot{\phi}^2(t_0)}{2\phi(t_0)}$. In conclusion one can approximately formulate the Jordan scalar field as

$$\phi(t) = \phi(t_0) + \dot{\phi}(t_0)(t - t_0) + \frac{1}{2}\ddot{\phi}(t_0)(t - t_0)^2, \tag{4.89}$$

by using the Taylor expansion up to second order.

For clarification one can substitute numerical values. We take $H_0 = 73.52(km/s)/Mpc$, $\Omega_{\Lambda,0} = 0.68$, $\Omega_k \simeq 0$, $\Omega_{m,0} + \Omega_{r,0} = 0.32$, $G = 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}$ and $\omega = 4 \times 10^4$. After writing all components in terms of eV we have found that λ must be at most

1.3199×10^{-130} to have real solutions. At this value of λ,

$$\dot{\phi}(t_0) = 2.86 \times 10^{-8} eV^2 \qquad \text{and} \qquad \ddot{\phi}(t_0) = -1.48 \times 10^{-40} eV^3. \tag{4.90}$$

At the value of $\lambda = 10^{-131}$ (4.86) gives two roots and our results are

$$\dot{\phi}(t_0) = -5.20 \times 10^{-6} eV^2 \qquad \text{and} \qquad \ddot{\phi}(t_0) = 2.44 \times 10^{-38} eV^3, \tag{4.91}$$

or

$$\dot{\phi}(t_0) = 5.26 \times 10^{-6} eV^2 \qquad \text{and} \qquad \ddot{\phi}(t_0) = -2.48 \times 10^{-38} eV^3. \tag{4.92}$$

One can calculate the value of G_{eff} 1.37×10^{10} years ago. When we use the highest allowed value of λ we find that G_{eff} was then $6.7243 \times 10^{-57} eV^{-2}$ and now it is $6.7236 \times 10^{-57} eV^{-2}$. We thus see that the expansion (4.89) can be extended to almost the end of the radiation dominated era.

Moreover one can calculate the deceleration parameter $q(t_0) = -\frac{a(t_0)\ddot{a}(t_0)}{\dot{a}^2(t_0)}$ after finding $\dot{\phi}(t_0)$ and $\ddot{\phi}(t_0)$. First we find 2(LHS of (4.3))+(LHS of (4.5))= 2ρ. Secondly we write $\frac{\ddot{a}(t)}{a(t)} = -q(t)H^2(t)$. Then we set $G_{present} = \frac{\omega}{2\pi\phi^2(t_0)}$ and put time $t = t_0$ to both sides of the equation. As a result we obtain

$$q(t_0) = \Omega_{m,0} + \Omega_{r,0} - \frac{2\omega}{3H_0^2}[\frac{\ddot{\phi}(t_0)}{\phi(t_0)} - (\frac{\dot{\phi}(t_0)}{\phi(t_0)})^2 + 3(1 + \frac{1}{\omega})H_0\frac{\dot{\phi}(t_0)}{\phi(t_0)} + \lambda\frac{\phi^2(t_0)}{2}]. \tag{4.93}$$

Numeric results are as follows. When λ has the highest allowed value $q(t_0) = -0.2$ which indicates present-universe is in a phase of accelerated expansion. As the value of λ decreases, the value of $q(t_0)$ approaches zero. On the other hand observational results were found as $q(t_0) = -1.0 \pm 0.4$ [17]. One can also reach our calculations which are presented in this chapter in [171, 172].

5. QUANTUM MECHANICS IN A SPACE WITH A FINITE NUMBER OF POINTS

In this chapter we would like to investigate what happens when the position space or the momentum space consists of a finite number of points so that the Hilbert space associated with the quantum mechanics is finite dimensional. Schwinger [104] has given a most instructive example where the position space and the momentum space each consist of d points and are both periodic. The starting point for this consideration is the well-known simple quantum mechanics on the circle S^1. In this case the position space is continuous whereas the momenta are given by $p_n = \frac{\hbar}{r}n, (n = 0, \pm 1, \pm 2, ...)$ where r is the radius of the circle. Now restricting the position space S^1 to integer multiples of angle $2\pi/d$, the position eigenvectors can be denoted by $|n\rangle, n = 0, 1, 2, ..., d-1$ such that

$$X|n\rangle = \frac{2\pi r}{d}n|n\rangle . \tag{5.1}$$

Note that this equation is not well-defined because the operator X is defined modulo $2\pi r$. A well defined operator is obtained by putting $V = e^{iX/r}$, which satisfies

$$V|n\rangle = e^{\frac{i2\pi}{d}n}|n\rangle = q^n|n\rangle , \qquad \text{where} \qquad q = e^{\frac{2\pi i}{d}}. \tag{5.2}$$

By the standard interpretation of quantum mechanics, V can be regarded as the unitary translation operator in momentum space. On the other hand, the translation operator in position space should be defined by

$$U|n\rangle = |n+1\rangle , \; n = 0, 1, 2, ..., d-2, \tag{5.3}$$

$$U|d-1\rangle = |0\rangle . \tag{5.4}$$

It follows that U and V satisfy [104]

$$U^d = V^d = 1, \tag{5.5}$$

$$VU = qUV, \qquad \text{where} \qquad q = e^{\frac{2\pi i}{d}}, \tag{5.6}$$

$$U^\dagger = U^{-1},\ V^\dagger = V^{-1}, \tag{5.7}$$

which can be taken as the defining relations of quantum mechanics with d points in periodic position space and periodic momentum space. Equation (5.6) is usually taken as the starting point of quantum mechanics in a space with a finite number of points. It can be shown that by taking the limit where the number of points is infinite, the Heisenberg commutation relation for P and X is obtained [173]. Thus one can intuitively think that the correct choice of the Hamiltonian at the corresponding continuum limit will result in our calculations in this chapter. However, until now, the formulation of the Hamiltonian has not been considered.

Taking Equation (5.6) as the starting point necessarily leads to periodicity in position space and momentum space. In this chapter we will show that it is also possible to have quantum mechanics in a space with a finite number of points where the space is not periodic.

In standard quantum mechanics both the translation operator in position space denoted by U and the translation operator in momentum space denoted by V are unitary. In terms of the momentum operator P and the position operator X,

$$U(a) = e^{\frac{-iaP}{\hbar}}, \tag{5.8}$$

$$V(b) = e^{\frac{ibX}{\hbar}}, \tag{5.9}$$

$$V(b)U(a) = e^{\frac{iab}{\hbar}}U(a)V(b), \tag{5.10}$$

where P and X are well-defined Hermitian operators. However for the discrete finite case only U and V are well-defined.

Bonatsos et al. [174] have considered the position and momentum operators for the q-deformed oscillator with q being a root of unity. They have shown that the phase space of this oscillator has a lattice structure, which is a non-uniformly distributed grid. In contrast, in this chapter we assume that the grid is uniform.

We define a deformed momentum operator

$$\tilde{P} = \frac{2\pi\hbar}{ad}\frac{U^{-1/2} - U^{1/2}}{q^{1/2} - q^{-1/2}}, \qquad \text{where} \qquad q = e^{\frac{2\pi i}{d}}, \qquad U = e^{\frac{-iaP}{\hbar}}, \tag{5.11}$$

where in the last equation P is not well-defined since momentum is periodic. As $a \to 0$ and $d \to \infty$ so that $ad = finite$

$$\tilde{P} = P + \mathcal{O}(P^3). \tag{5.12}$$

Although the half integer power in the exponent may look problematic, for the choice of the free particle Hamiltonian the half integer power is irrelevant. Thus our choice of the Hamiltonian is given by

$$H = \frac{1}{2M}\tilde{P}^2 = \frac{\pi^2\hbar^2}{(ad)^2 M}\frac{[2 - (U + U^\dagger)]}{2sin^2(\pi/d)}, \tag{5.13}$$

where M is the mass of the particle.

For a Hamiltonian to be acceptable it should reduce to $\frac{p^2}{2m}$ in the continuum limit. Our choice of the Hamiltonian is not unique. However, it is the simplest Hamiltonian which gives $\frac{p^2}{2m}$ in the continuum limit because it contains the first power of U and $U^\dagger$.

5.1. Nonunitary Translation Operators

We would also like to address the quantum mechanics when the position space is not periodic. We consider a position space of d points where a particle located at

position $X = na$ is described by the ket vector $|n\rangle, n = 0, 1, ..., d-1$ where

$$X|n\rangle = na|n\rangle. \tag{5.14}$$

We regard the points $x = 0$ and $x = (d-1)a$ as the end points of this discrete position space. We have to decide how the translation operator acts at the end points. For the right translation we define an operator u_+ whose action on the position eigenstates $|n\rangle$ is given by

$$u_+|n\rangle = |n+1\rangle, \qquad n = 0, 1, ..., d-2, \tag{5.15}$$

$$u_+|d-1\rangle = 0. \tag{5.16}$$

Similarly, the left translation operator will be denoted u_- which satisfies

$$u_-|n\rangle = |n-1\rangle, \qquad n = 1, ..., d-1, \tag{5.17}$$

$$u_-|0\rangle = 0. \tag{5.18}$$

Note that this is the simplest kind of generalized oscillator which has a finite number of states. The most well-known of these generalized oscillators is the Biedenharn-Macfarlane oscillator [175, 176] with the spectrum

$$a^\dagger a = \frac{q^N - q^{-N}}{q - q^{-1}}, \tag{5.19}$$

with real q. The spectrum also becomes well defined when q is a root of unity. The representations of the q-deformed oscillator algebra [177] with q a root of unity are discussed in [178].

Taking the position eigenstates as orthonormal, the translation operators u_+ and u_-, instead of being unitary, satisfy the algebra

$$u_+^d = u_-^d = 0, \tag{5.20}$$

$$u_+^\dagger = u_-, \tag{5.21}$$

$$u_- u_+ = 1 - u_+^{d-1} u_-^{d-1}, \tag{5.22}$$

$$u_+ u_- = 1 - u_-^{d-1} u_+^{d-1}. \tag{5.23}$$

Because our space has a finite number of points with $0 \leq n \leq d-1$,

$$u_+^d \left|n\right\rangle = u_-^d \left|n\right\rangle = 0. \tag{5.24}$$

Thus Equation (5.20) is satisfied. One can easily define u_+ and u_- in terms of position eigenstates and can arrive at Equation (5.21) because these states are orthonormal.

Then we check the unitarity of the operators

$$u_+ u_+^\dagger \left|n\right\rangle = u_+ u_- \left|n\right\rangle \neq \left|n\right\rangle, \qquad \text{only when} \qquad \left|n\right\rangle = \left|0\right\rangle, \tag{5.25}$$

$$u_- u_-^\dagger \left|n\right\rangle = u_- u_+ \left|n\right\rangle \neq \left|n\right\rangle, \qquad \text{only when} \qquad \left|n\right\rangle = \left|d-1\right\rangle, \tag{5.26}$$

otherwise

$$u_+ u_+^\dagger \left|n\right\rangle = u_+ u_- \left|n\right\rangle = \left|n\right\rangle, \qquad n = 1, 2, ...d-1, \tag{5.27}$$

$$u_- u_-^\dagger \left|n\right\rangle = u_- u_+ \left|n\right\rangle = \left|n\right\rangle, \qquad n = 0, 1, ...d-2. \tag{5.28}$$

This means that we should have an additional term that breaks unitarity. This is the second term on the right hand side of Equations (5.22) and (5.23). One can easily show that Equations (5.22) and (5.23) are not only satisfied at the end states but also satisfied at intermediate states. The minimal set of relations which define the algebra

generated by u and $u^\dagger$ is given by

$$uu^\dagger = 1 - u^{\dagger d-1}u^{d-1}, \qquad \text{and} \qquad u^d = 0,$$

where $u_+ = u^\dagger$ and $u_- = u$.

Now we will show that these relations imply a unique representation with the desired properties and lead to a set of relations satisfied by u and $u^\dagger$. We define two useful operators

$$P_0 = 1, \tag{5.29}$$

$$P_n = u^{\dagger n}u^n. \tag{5.30}$$

If we can show that

$$P_nP_m = P_m, \qquad m \geq n, \tag{5.31}$$

we will have the inclusion relation

$$P_n^2 = P_n. \tag{5.32}$$

Thus we can consider them as the projection operators. We will prove Equations (5.31) by induction. For $n = 1$, we write

$$\begin{aligned} P_1P_m &= u^\dagger uu^{\dagger m}u^m, \\ &= u^\dagger(uu^\dagger)u^{\dagger m-1}u^m, \end{aligned} \tag{5.33}$$

using (5.22)

$$P_1P_m = u^\dagger(1 - P_{d-1})u^{\dagger m-1}u^m, \tag{5.34}$$

and using (5.20)

$$P_1 P_m = u^{\dagger m} u^m = P_m. \tag{5.35}$$

For $n = l + 1$, we have

$$\begin{aligned} P_{l+1} P_m &= u^\dagger P_l (u u^\dagger) P_{m-1} u, \\ &= u^\dagger P_l (1 - P_{d-1}) P_{m-1} u, \\ &= u^\dagger P_l P_{m-1} u - u^\dagger P_l P_{d-1} P_{m-1} u, \qquad \text{where} \qquad l + 1 \leq m. \end{aligned} \tag{5.36}$$

Assuming (5.31) holds for $n = l$ and remembering $l + 1 \leq m$ implies $l \leq m - 1$

$$P_{l+1} P_m = u^\dagger P_{m-1} u - u^\dagger P_{d-1} P_{m-1} u, \tag{5.37}$$

using $u^\dagger P_{d-1} = 0$,

$$P_{l+1} P_m = u^\dagger P_{m-1} u = P_m. \tag{5.38}$$

Thus the proof is complete.

Before going further, we will supply two more useful equations

$$\begin{aligned} P_m u^\dagger &= u^{\dagger m} u^{m-1} (u u^\dagger), \\ &= u^\dagger P_{m-1} (1 - P_{d-1}), \\ &= u^\dagger P_{m-1} - u^\dagger P_{d-1}, \\ &= u^\dagger P_{m-1}, \end{aligned} \tag{5.39}$$

and

$$\begin{aligned} uP_m &= (uu^\dagger)P_{m-1}u, \\ &= (1 - P_{d-1})P_{m-1}u, \\ &= P_{m-1}u - P_{d-1}u, \\ &= P_{m-1}u. \end{aligned} \tag{5.40}$$

By replacing m with $m + 1$, we obtain

$$P_m u = uP_{m+1}. \tag{5.41}$$

Now we will prove

$$u^n u^{\dagger n} = 1 - P_{d-n}, \tag{5.42}$$

which implies the last part of our algebra (5.23) when $n = d - 1$

$$u^{d-1}u^{\dagger d-1} = 1 - u^\dagger u.$$

Our method is proof by induction. For $n = 1$

$$uu^\dagger = 1 - P_{d-1}, \tag{5.43}$$

which is (5.22).

For $n = k + 1$, we have

$$u^{k+1}u^{\dagger k+1} = u(u^k u^{\dagger k})u^\dagger. \tag{5.44}$$

For $n = k$ we assume $u^k u^{\dagger k} = 1 - P_{d-k}$. By substituting this into the last equation, we obtain

$$\begin{aligned} u^{k+1}u^{\dagger k+1} &= u(1 - P_{d-k})u^\dagger, \\ &= uu^\dagger - uP_{d-k}u^\dagger. \end{aligned} \tag{5.45}$$

Using (5.22) for the first term on the RHS and (5.41) for the second term on the RHS, we get

$$u^{k+1}u^{\dagger k+1} = 1 - P_{d-1} - P_{d-k-1}uu^\dagger, \tag{5.46}$$

using (5.22) once more for the term $uu^\dagger$,

$$u^{k+1}u^{\dagger k+1} = 1 - P_{d-1} - P_{d-k-1}(1 - P_{d-1}), \tag{5.47}$$

remembering $P_n P_m = P_m$ for $m \geq n$, which was our first proved equation

$$\begin{aligned} u^{k+1}u^{\dagger k+1} &= 1 - P_{d-1} - P_{d-k-1} + P_{d-1}, \\ u^{k+1}u^{\dagger k+1} &= 1 - P_{d-(k+1)} \quad \text{Q.E.D.}. \end{aligned} \tag{5.48}$$

5.2. Eigenvalues and Eigenfunctions of the Hamiltonian for the Nonperiodic Space

In a manner similar to (5.13), we choose the Hamiltonian for the non-periodic case

$$H = \frac{\pi^2\hbar^2}{(ad)^2 M}\frac{[2 - (u + u^\dagger)]}{2sin^2(\pi/d)}. \tag{5.49}$$

We now calculate the eigenvalues and eigenvectors of $(u + u^\dagger)$

$$|\lambda\rangle = \sum_{n=0}^{d-1} \lambda_n |n\rangle , \tag{5.50}$$

$$(u + u^\dagger)|\lambda\rangle = \sum_{n=0}^{d-1} (\lambda_n |n-1\rangle + \lambda_n |n+1\rangle) = \sum_{n=0}^{d-1} \lambda\lambda_n |n\rangle , \tag{5.51}$$

so

$$\lambda_{n+1} + \lambda_{n-1} = \lambda\lambda_n, \quad \text{with} \quad \lambda_{-1} = 0, \quad \text{and} \quad \lambda_d = 0. \tag{5.52}$$

We obtain the general solution $\lambda_n = Ae^{in\theta} + Be^{-in\theta}$, where $cos\theta = \lambda/2$. Using the boundary conditions $\lambda_{-1} = 0$ and $\lambda_d = 0$, we get $B = -Ae^{-2i\theta}$ and $\theta = \dfrac{\pi\, m}{d+1}$ where $m = 1, 2, ..., d$. $m = 0$ is excluded because it does not yield a non-zero eigenvector. Thus for each value of n, we obtain m different eigenvalues, given by

$$\lambda_n^{(m)} = A_m(e^{i\alpha nm} - e^{-2i\alpha m}e^{-i\alpha nm}) \quad \text{where} \quad \alpha = \pi/(d+1), \tag{5.53}$$

$$= A_m 2ie^{\frac{-im\pi}{1+d}} sin[\frac{m(1+n)\pi}{1+d}]. \tag{5.54}$$

If we normalize $\left|\lambda^{(m)}\right\rangle$, we obtain

$$A_m = \sqrt{\frac{1}{2(d+1)}}. \tag{5.55}$$

As a result the eigenvectors are written as

$$\left|\lambda^{(m)}\right\rangle = \frac{2ie^{\frac{-im\pi}{1+d}}}{\sqrt{2(d+1)}} \sum_{n=0}^{d-1} sin[\frac{m(1+n)\pi}{1+d}] |n\rangle , \tag{5.56}$$

and the corresponding eigenvalues are given by

$$\lambda^{(m)} = 2cos\theta = 2cos(\frac{\pi m}{d+1}) \quad \text{with} \quad m = 1, 2, ...d. \tag{5.57}$$

Now, it is time to compare what we have found with the usual quantum particle in an infinite one-dimensional square well. We calculate the eigenvalues of the Hamiltonian, by applying (5.49) to eigenvectors given by (5.56). We find

$$E_m = \frac{2\pi^2\hbar^2}{(ad)^2 M} \frac{sin^2[\frac{\pi m}{2(d+1)}]}{sin^2(\frac{\pi}{d})} \quad \text{where} \quad m = 1, 2, ..., d. \tag{5.58}$$

In the continuum limit $a \to 0$, $d \to \infty$ and $a(d-1) = L$, these energy eigenvalues become

$$E_m = \frac{\pi^2\hbar^2 m^2}{2ML^2}, \tag{5.59}$$

which are the same energy eigenvalues in the case of a particle confined in an infinite one dimensional square well of width L. The wave functions are given by

$$\psi_m(n) = C_n^{(m)} \left\langle n \right| \left. \lambda^{(m)} \right\rangle = C_n^{(m)} \sqrt{\frac{2}{d+1}} sin[\frac{m(1+n)\pi}{d+1}]. \tag{5.60}$$

We can easily find $C_n^{(m)}$ by normalization of the wave function,

$$\sum_{n=0}^{d-1} |\psi_m(n)|^2 a = 1, \qquad C_n^{(m)} = \frac{1}{\sqrt{a}}. \tag{5.61}$$

Thus the wave functions can be written as

$$\psi_m(n) = \sqrt{\frac{2}{(d+1)a}} sin[\frac{m(1+n)\pi}{d+1}] \quad \text{where} \quad m = 1, 2, ..., d. \tag{5.62}$$

In the continuum limit the wave function becomes

$$\psi_m(x) = \sqrt{\frac{2}{L}} sin[\frac{m\pi x}{L}], \tag{5.63}$$

which agrees with the wave functions of a square well in the interval $0 \leq x \leq L$.

5.3. Eigenvalues and Eigenfunctions of the Hamiltonian for the Periodic Space

For the periodic case, we need to find eigenvalues of $(U + U^\dagger)$ using the position basis. Our choice is the same as (5.50). If we apply $(U + U^\dagger)$ to this eigenvector, we will have

$$\lambda_{n+1} + \lambda_{n-1} = \lambda\lambda_n \quad \text{with} \quad \lambda_0 = \lambda_d. \tag{5.64}$$

Since the equation is linear, we can superpose linearly independent solutions to find the general solution. Since we desire the linearly independent solutions to be real in order to be able to plot the wave function, we choose λ_n as

$$\lambda_n = Ccos(n\theta) + Dsin(n\theta), \tag{5.65}$$

which satisfies (5.64) with $\lambda = 2cos\theta$. If we impose the boundary condition $\lambda_n = \lambda_{d+n}$, we obtain $\theta = \dfrac{2\pi m}{d}$ with $m = 0, 1, ..., d-1$ and

$$\left|\lambda^{(m)}\right\rangle = \sum_{n=0}^{d-1} [Ccos(\frac{2\pi nm}{d}) + Dsin(\frac{2\pi nm}{d})] \left|n\right\rangle . \tag{5.66}$$

At this point there are options regarding the choice of C and D. By introducing a phase angle we can write this equation as

$$\left|\lambda^{(m)}\right\rangle = \sum_{n=0}^{d-1} C_m cos(\frac{2\pi nm}{d} - \gamma_m) \left|n\right\rangle \qquad \text{where} \qquad \gamma_m = \frac{2\pi n_0 m}{d}. \tag{5.67}$$

By selecting n_0 and $n = -\frac{d-1}{2}, -\frac{d-1}{2}+1, ..., \frac{d-1}{2}$, the solutions become parity eigenstates. Thus parity corresponds to $n \to -n$. As a result, $n_0 = \frac{d-1}{2}$ and for normalized

eigenvectors we get

$$C_m = \begin{cases} \sqrt{\frac{1}{d}} & \text{if} \quad m = 0, \\ \sqrt{\frac{2}{d}} & \text{if} \quad m \neq 0. \end{cases} \tag{5.68}$$

The final form of eigenvectors is given by

$$\left|\lambda^{(m)}\right\rangle = C_m \sum_{n=0}^{d-1} cos[\frac{2\pi nm}{d} - \frac{2\pi(d-1)m}{2d}] \left|n\right\rangle, \tag{5.69}$$

and the eigenvalues are given by

$$\begin{aligned} \lambda^{(m)} &= 2cos\theta, \\ &= 2cos(\frac{2\pi m}{d}) \quad \text{where} \quad m = \begin{cases} 0, 1, ..., int(\frac{d-1}{2}) & \text{for positive parity states,} \\ 1, 2, ..., int(\frac{d}{2}) & \text{for negative parity states.} \end{cases} \end{aligned} \tag{5.70}$$

Hence we find the energy eigenvalues as

$$E_m = \frac{2\pi^2\hbar^2}{(ad)^2 M} \frac{sin^2[\frac{m\pi}{d}]}{sin^2(\frac{\pi}{d})}. \tag{5.71}$$

In the continuum limit, we set $ad = L$ and find

$$E_m = \frac{2\pi^2\hbar^2 m^2}{ML^2}, \tag{5.72}$$

which are the same as energy eigenvalues in the case of particle in a box with periodic boundary conditions.

We continue by obtaining the wave functions. Applying the same method that we used for the nonperiodic case, we find that the ground state is unique and given by

$$\psi_0(n) = \sqrt{\frac{1}{da}}. \tag{5.73}$$

For odd d the excited states are doubly degenerate, whereas for even d the excited states are doubly degenerate except the highest energy state. The eigenvalues of the position operator are given by

$$X\left|n\right\rangle = an\left|n\right\rangle, \qquad n = -\frac{d-1}{2}, -\frac{d-1}{2}+1, ..., \frac{d-1}{2}. \tag{5.74}$$

Then the positive parity excited states are given by the wave functions

$$\psi_m^+(n) = \sqrt{\frac{2}{da}} cos(\frac{2\pi mn}{d}), \qquad m = 1, 2, ..., int(\frac{d-1}{2}), \tag{5.75}$$

and the negative parity excited states are given by the wave functions

$$\psi_m^-(n) = \sqrt{\frac{2}{da}} sin(\frac{2\pi mn}{d}), \qquad m = 1, 2, ..., int(\frac{d}{2}). \tag{5.76}$$

If one plots the wave functions, it is observed that the wave functions of the omitted m values are the same as the wave functions of the indicated m values. The total number of states, including the ground state, is d.

At the continuum limit, the ground state wave function, the even parity wave functions and the odd parity wave functions become, respectively,

$$\psi_0^+(x) = \sqrt{\frac{1}{L}}, \tag{5.77}$$

$$\psi_m^+(x) = \sqrt{\frac{2}{L}} cos[\frac{2m\pi x}{L}], \tag{5.78}$$

$$\psi_m^-(x) = \sqrt{\frac{2}{L}} sin[\frac{2m\pi x}{L}], \tag{5.79}$$

which are the even parity solutions and the odd parity solutions for a particle in a box in an interval $-L/2 \leq x \leq L/2$ with periodic boundary conditions.

5.4. Discussion

The $d = 2$ case gives the fermionic oscillator. For this case $u^\dagger$ and u correspond to the creation and annihilation operators for a single fermionic degree of freedoom. For the periodic case Schwinger [179] has shown that for $d = 2$, U and V in fact generate the Pauli algebra. Our work shows that even for the non-periodic case, $d = 2$ leads to Pauli-matrices through the relations,

$$\sigma_1 = u^\dagger + u, \tag{5.80}$$

$$\sigma_2 = i(u^\dagger - u), \tag{5.81}$$

$$\sigma_3 = uu^\dagger - u^\dagger u, \tag{5.82}$$

(For details of these equations see Appendix D) .

We should also note that a universe with a finite number of points may be a physical reality. The size of the visible universe is 10^{27} m, whereas the smallest classical length is the Planck length, 10^{-35} m. This means that the physical space continuum can be regarded as consisting of $d_{universe}/l_{planck} = 10^{62}$ points lying along one dimension. Thus, it may be that what we call the space continuum can be described by quantum mechanics with $d = 10^{62}$ points along one dimension. Another possibility is that in a Klauza-Klein like theory [180] the internal space can consist of a finite number of points. One can also reach our calculations which are presented in this chapter in [181].

6. QUANTUM MECHANICS ON PERIODIC AND NON-PERIODIC LATTICES AND ALMOST UNITARY SCHWINGER OPERATORS

Schwinger considered a periodic lattice on which the translation operator U is unitary due to the periodicity of the lattice. On such a lattice the position can be again expressed by a unitary operator V such that

$$\begin{aligned} VU = qUV \quad \text{where} \quad q = e^{\frac{2\pi i}{d}}, \\ V^d = U^d = 1 \quad \text{and} \quad VV^\dagger = UU^\dagger = 1, \end{aligned} \tag{6.1}$$

where d is the number of points on the periodic lattice. Schwinger chose the integer d to be prime and in this case the relation $q = e^{\frac{2\pi i}{d}}$ can be omitted since it is already implied by the other equations.

In the previous chapter and in [181] it has been shown that a finite lattice has an almost unitary quasi-translation operator a which satisfies

$$aa^\dagger = 1 - a^{\dagger d-1}a^{d-1}, \quad \text{and} \quad a^{\dagger d} = 0, \tag{6.2}$$

$$a^\dagger a = 1 - a^{d-1}a^{\dagger d-1}, \quad \text{and} \quad a^d = 0. \tag{6.3}$$

The operators $a^\dagger$ and a in the above relations can be respectively regarded as the right quasi-translation operator and the left quasi-translation operator since an end point can be translated only in one direction. A point which lies at the right end of the finite lattice can only be translated left end vice versa. Equation (6.2) gives the minimal set of relations that define the algebra generated by a and $a^\dagger$. The second set of the relations written in equation (6.3) can be derived using (6.2). Although the algebras defined by Equation (6.1) and by equation (6.2) look very different, physically they accomplish basically the same concept. Therefore the exact mathematical relation be-

tween them should be unveiled.

It is obvious that the translation operators U and $U^\dagger$ of the Schwinger algebra commute. On the other hand quasi-translation operators a and $a^\dagger$ do not commute as can be seen from equations (6.2) and (6.3). In this sense discrete non-periodic space is a deformation of discrete cyclic space. Deformation emerges as a result of having the end points.

6.1. Mathematical Structure of the Almost Unitary Translation Operators

In the previous chapter [181] projection operators P_n were defined as,

$$P_n = a^{\dagger n} a^n, \quad \text{where} \quad P_0 = 1, \tag{6.4}$$

and it was shown that

$$a^n a^{\dagger n} = 1 - P_{d-n}. \tag{6.5}$$

Thus we can also define another projection operator such as,

$$R_n = a^n a^{\dagger n}, \quad \text{where} \quad R_0 = 1. \tag{6.6}$$

Therefore one can easily see the following relations between the projection operators P_n and R_n

$$P_n = 1 - R_{d-n} \quad \text{and} \quad R_n = 1 - P_{d-n}. \tag{6.7}$$

Their properties which are calculated in the Appendix E are summarized as,

$$P_n = a^{\dagger n} a^n, \quad P_0 = 1, \quad P_m a^\dagger = a^\dagger P_{m-1}, \quad a P_m = P_{m-1} a, \tag{6.8}$$
$$P_n P_m = P_j \quad \text{where} \quad j = \max(n, m),$$
$$P_m a^n = a^{\dagger n} P_m = 0 \quad \text{for} \quad n + m \geq d,$$

and

$$R_n = a^n a^{\dagger n}, \quad R_0 = 1, \quad R_m a^\dagger = a^\dagger R_{m+1}, \quad a R_m = R_{m+1} a, \tag{6.9}$$
$$R_n R_m = R_j \quad \text{where} \quad j = \max(n, m),$$
$$a^n R_m = R_m a^{\dagger n} = 0 \quad \text{for} \quad n + m \geq d.$$

In the previous chapter and also in [181] we have already considered a position space of d points where a particle located at position $X = \beta n$ is described by the ket vector $|n\rangle, n = 0, 1, ..., d-1$ where

$$X |n\rangle = \beta n |n\rangle, \tag{6.10}$$

with β as the grid spacing. We can define the position operator as

$$X = \beta \sum_{m=1}^{d-1} P_m, \tag{6.11}$$
$$X = \beta \{a^\dagger a + ... + a^{\dagger n} a^n + ... + a^{\dagger d-1} a^{d-1}\}.$$

Applying this to $|n\rangle$ one gets the desired result.

6.2. The Schwinger Algebra in Terms of the Almost Unitary Translation Operators

We define unitary operators U and V that cyclically permute the vectors of a given system in terms of almost unitary operators a and $a^\dagger$

$$U = a^\dagger + a^{d-1}, \tag{6.12}$$

$$V = \sum_{n=0}^{d-1} q^n (P_n - P_{n+1}). \tag{6.13}$$

Then we will show that these definitions satisfy the Schwinger algebra given by Equation (6.1). The first relation we will prove is

$$VV^k = \sum_{n=0}^{d-1} q^{(k+1)n} (P_n - P_{n+1}), \tag{6.14}$$

which has the inclusion relation

$$\begin{aligned} V^d = VV^{d-1} &= \sum_{n=0}^{d-1} q^{dn} (P_n - P_{n+1}) \\ &= \sum_{n=0}^{d-1} (P_n - P_{n+1}) \\ &= (1 + a^\dagger a + a^{\dagger 2} a^2 + \cdots + a^{\dagger d-1} a^{d-1}) - (a^\dagger a + a^{\dagger 2} a^2 + \cdots + a^{\dagger d-1} a^{d-1} + a^{\dagger d} a^d) \\ &= 1, \end{aligned} \tag{6.15}$$

where we used the algebra relation $a^{\dagger d} = 0$.

We will prove the equation (6.14) using the method of proof by induction.

For $k = 1$ we have

$$\begin{aligned} VV &= \sum_{n,m=0}^{d-1} q^{n+m}(P_n - P_{n+1})(P_m - P_{m+1}), \\ &= \sum_{n,m=0}^{d-1} q^{n+m}(P_nP_m - P_nP_{m+1} - P_{n+1}P_m + P_{n+1}P_{m+1}), \\ &= \sum_{\substack{n,m=0 \\ n\geq m+1}}^{d-1} q^{n+m}(P_nP_m - P_nP_{m+1} - P_{n+1}P_m + P_{n+1}P_{m+1}), \\ &+ \sum_{\substack{n,m=0 \\ n=m}}^{d-1} q^{n+m}(P_nP_m - P_nP_{m+1} - P_{n+1}P_m + P_{n+1}P_{m+1}), \\ &+ \sum_{\substack{n,m=0 \\ n<m}}^{d-1} q^{n+m}(P_nP_m - P_nP_{m+1} - P_{n+1}P_m + P_{n+1}P_{m+1}). \end{aligned} \tag{6.16}$$

Using the property $P_nP_m = P_j$ where $j = \max(n, m)$, we obtain

$$\begin{aligned} VV &= 0 + \sum_{\substack{n,m=0 \\ n=m}}^{d-1} q^{n+m}(P_n - P_{m+1}) + 0, \\ &= \sum_{n=0}^{d-1} q^{2n}(P_n - P_{n+1}). \end{aligned} \tag{6.17}$$

We assume that for $k = l$

$$VV^l = \sum_{n=0}^{d-1} q^{(l+1)n}(P_n - P_{n+1}). \tag{6.18}$$

For $k = l + 1$,

$$\begin{aligned} VV^{l+1} &= (VV^l)V, \\ &= \sum_{n=0}^{d-1} q^{(l+1)n}(P_n - P_{n+1}) \sum_{m=0}^{d-1} q^m(P_m - P_{m+1}), \end{aligned} \tag{6.19}$$

$$
\begin{aligned}
&= \sum_{n,m=0}^{d-1} q^{(l+1)n+m}(P_nP_m - P_nP_{m+1} - P_{n+1}P_m + P_{n+1}P_{m+1}),\\
&= \sum_{\substack{n,m=0\\ n\geq m+1}}^{d-1} q^{(l+1)n+m}(P_nP_m - P_nP_{m+1} - P_{n+1}P_m + P_{n+1}P_{m+1}),\\
&+ \sum_{\substack{n,m=0\\ n=m}}^{d-1} q^{(l+1)n+m}(P_nP_m - P_nP_{m+1} - P_{n+1}P_m + P_{n+1}P_{m+1}),\\
&+ \sum_{\substack{n,m=0\\ n<m}}^{d-1} q^{(l+1)n+m}(P_nP_m - P_nP_{m+1} - P_{n+1}P_m + P_{n+1}P_{m+1}).
\end{aligned}
$$

Using the same property $P_nP_m = P_j$ where $j =\max(n, m)$, we obtain

$$
\begin{aligned}
VV^{l+1} &= 0 + \sum_{\substack{n,m=0\\ n=m}}^{d-1} q^{(l+1)n+m}(P_n - P_{m+1}) + 0, \qquad (6.20)\\
&= \sum_{n=0}^{d-1} q^{(l+2)n}(P_n - P_{n+1}) \quad \text{Q.E.D..}
\end{aligned}
$$

Then we will show that $VV^\dagger = 1$ by using definition of V.

Since $P_n = P_n^\dagger$ and $q^{\dagger n} = q^{d-n}$, we have

$$
\begin{aligned}
VV^\dagger &= \sum_{m=0}^{d-1}\sum_{n=0}^{d-1} q^m q^{d-n}(P_m - P_{m+1})(P_n - P_{n+1}), \qquad (6.21)\\
&= \sum_{m,n=0}^{d-1} q^m q^{d-n}(P_mP_n - P_mP_{n+1} - P_{m+1}P_n + P_{m+1}P_{n+1}),\\
&= \sum_{\substack{m,n=0\\ m\geq n+1}}^{d-1} q^{m+d-n}(P_mP_n - P_mP_{n+1} - P_{m+1}P_n + P_{m+1}P_{n+1}),\\
&+ \sum_{\substack{m,n=0\\ m=n}}^{d-1} q^{m+d-n}(P_mP_n - P_mP_{n+1} - P_{m+1}P_n + P_{m+1}P_{n+1}),
\end{aligned}
$$

$$+ \sum_{\substack{m,n=0 \\ m<n}}^{d-1} q^{m+d-n}(P_m P_n - P_m P_{n+1} - P_{m+1} P_n + P_{m+1} P_{n+1}),$$
$$= 0 + \sum_{m=0}^{d-1} q^d (P_m - P_{m+1}) + 0,$$
$$= (1 + a^\dagger a + a^{\dagger 2} a^2 + \cdots + a^{\dagger d-1} a^{d-1}) - (a^\dagger a + a^{\dagger 2} a^2 + \cdots + a^{\dagger d-1} a^{d-1} + a^{\dagger d} a^d),$$
$$= 1 \quad \text{Q.E.D..}$$

Next, we will show that

$$UU^n = a^{\dagger n+1} + a^{d-(n+1)} \quad \text{with} \quad n = 0, 1, \cdots, d-1, \tag{6.22}$$

which implies

$$\begin{aligned} U^d &= UU^{d-1}, \\ &= a^{\dagger d} + a^{d-d}, \\ &= 1. \end{aligned} \tag{6.23}$$

Our method is proof by induction. For $n = 1$, we have

$$\begin{aligned} UU &= (a^\dagger + a^{d-1})(a^\dagger + a^{d-1}), \\ &= a^\dagger a^\dagger + a^\dagger a^{d-1} + a^{d-1} a^\dagger + a^{d-1} a^{d-1}, \\ &= a^{\dagger 2} + (a^\dagger a) a^{d-2} + a^{d-2}(a a^\dagger), \\ &= a^{\dagger 2} + P_1 a^{d-2} + a^{d-2} R_1, \\ &= a^{\dagger 2} + a^{d-2} P_{1+d-2} + a^{d-2}(1 - P_{d-1}), \\ &= a^{\dagger 2} + a^{d-2} P_{d-1} + a^{d-2} - a^{d-2} P_{d-1}, \qquad = a^{\dagger 2} + a^{d-2}. \end{aligned} \tag{6.24}$$

For $n = l$, we assume that

$$UU^l = a^{\dagger l+1} + a^{d-(l+1)}. \tag{6.25}$$

For $n = l + 1$, we obtain

$$\begin{aligned} UU^{l+1} &= (UU^l)U, \\ &= (a^{\dagger l+1} + a^{d-(l+1)})(a^\dagger + a^{d-1}), \\ &= a^{\dagger l+2} + a^{\dagger l+1}a^{d-1} + a^{d-(l+1)}a^\dagger + a^{2d-(l+2)}, \\ &= a^{\dagger l+2} + (a^{\dagger l+1}a^{l+1})a^{d-1-(l+1)} + a^{d-(l+2)}(aa^\dagger), \\ &= a^{\dagger l+2} + P_{l+1}a^{d-1-(l+1)} + a^{d-(l+2)}R_1 \quad \text{use (8)}, \\ &= a^{\dagger l+2} + a^{d-(l+2)}P_{l+1+d-l-2} + a^{d-(l+2)}(1 - P_{d-1}), \\ &= a^{\dagger l+2} + a^{d-(l+2)}P_{d-1} + a^{d-(l+2)} - a^{d-(l+2)}P_{d-1}, \\ &= a^{\dagger l+2} + a^{d-(l+2)} \quad \text{Q.E.D.}. \end{aligned} \tag{6.26}$$

The last term in the third line is zero because at most $l = d - 2$ according to equation (6.22).

We will obtain $UU^\dagger = 1$ just by substitution

$$\begin{aligned} UU^\dagger &= (a^\dagger + a^{d-1})(a + a^{\dagger d-1}), \\ &= a^\dagger a + a^{\dagger d} + a^d + a^{d-1}a^{\dagger d-1}, \\ &= P_1 + R_{d-1}, \\ &= 1 \quad \text{Q.E.D.}. \end{aligned} \tag{6.27}$$

where we have used (6.2), (6.3) and (6.7).

We have the formula for U and V, so we will show that $VU = qUV$ just by substitution. Thus left hand side of the formula is equal to

$$\begin{aligned} VU &= \sum_{n=0}^{d-1} q^n (P_n - P_{n+1})(a^\dagger + a^{d-1}), \\ &= \sum_{n=0}^{d-1} q^n (P_n a^\dagger + P_n a^{d-1} - P_{n+1}a^\dagger - P_{n+1}a^{d-1}). \end{aligned} \tag{6.28}$$

Since $P_n = a^\dagger a$ and $P_0 = 1$, $P_n a^{d-1} = 0$ except for $n = 0$ and $P_{n+1}a^{d-1} = 0$ for all n. Therefore we obtain

$$\begin{aligned} VU &= \sum_{n=0}^{d-1} q^n (P_n a^\dagger - P_{n+1} a^\dagger) + q^0 a^{d-1}, \\ &= P_0 a^\dagger - P_1 a^\dagger + \sum_{n=1}^{d-1} q^n (a^\dagger P_{n-1} - a^\dagger P_n) + a^{d-1}, \\ &= a^\dagger - a^\dagger P_0 + \sum_{n=1}^{d-1} q^n (a^\dagger P_{n-1} - a^\dagger P_n) + a^{d-1}, \\ &= \sum_{n=1}^{d-1} q^n a^\dagger (P_{n-1} - P_n) + a^{d-1}, \end{aligned} \tag{6.29}$$

where we have used (6.8). To obtain right hand side of the formula $VU = qUV$ we calculate,

$$\begin{aligned} UV &= (a^\dagger + a^{d-1}) \sum_{n=0}^{d-1} q^n (P_n - P_{n+1}), \\ &= \sum_{n=0}^{d-1} q^n (a^\dagger (P_n - P_{n+1}) + a^{d-1}(P_n - P_{n+1})), \\ &= \sum_{n=0}^{d-1} q^n a^\dagger (P_n - P_{n+1}) + \sum_{n=0}^{d-1} q^n a^{d-1} (P_n - P_{n+1}), \\ &= \sum_{n=0}^{d-1} q^n a^\dagger (P_n - P_{n+1}) + \sum_{n=0}^{d-1} q^n a^{d-1} (1 - R_{d-n} - (1 - R_{d-n-1})), \\ &= \sum_{n=0}^{d-1} q^n a^\dagger (P_n - P_{n+1}) + \sum_{n=0}^{d-1} q^n a^{d-1} (R_{d-n-1} - R_{d-n}). \end{aligned} \tag{6.30}$$

Since $R_m = a^m a^{\dagger m}$, $a^{d-1}R_m = 0$ except $m = 0$. The element $R_{d-n-1} = R_0 = 1$ for $n = d-1$ so we have,

$$\begin{aligned} UV &= \sum_{n=0}^{d-1} q^n a^\dagger (P_n - P_{n+1}) + q^{d-1} a^{d-1}, \\ qUV &= \sum_{n=0}^{d-1} q^{n+1} a^\dagger (P_n - P_{n+1}) + q^d a^{d-1}, \\ &= \sum_{n=1}^{d} q^n a^\dagger (P_{n-1} - P_n) + a^{d-1}, \\ &= \sum_{n=1}^{d-1} q^n a^\dagger (P_{n-1} - P_n) + a^{d-1}, \end{aligned} \tag{6.31}$$

where we have used the facts that $q^d = 1$, $P_d = 0$ and $a^\dagger P_{d-1} = 0$. which is the same result given by equation (6.29).

6.3. Almost Unitary Operators in Terms of the Schwinger Algebra

It is also possible to write the almost unitary operators a and $a^\dagger$ in terms of U and V

$$a^\dagger = U - (\frac{1 + V + V^2 + \cdots + V^{d-1}}{d})U. \tag{6.32}$$

The expression in the parentheses is called $\mathscr{P}_0$. It is shown that $\mathscr{P}_0$ is a projection operator (see Appendix E). We have found that more general projection operators $\mathscr{P}_n$ are written as

$$\mathscr{P}_n = \frac{(1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)})}{d}. \tag{6.33}$$

From the definition, it is easily seen that

$$\mathscr{P}_{d-l} = \mathscr{P}_{-l} \quad \text{and} \quad \mathscr{P}_{d+l} = \mathscr{P}_l \quad \text{and} \quad \mathscr{P}_d = \mathscr{P}_0. \tag{6.34}$$

The relationships between unitary operators and projection operators are found as

$$\begin{aligned} \mathscr{P}_n U^m &= U^m \mathscr{P}_{n+m}, \\ U^{\dagger m} \mathscr{P}_n &= \mathscr{P}_{n+m} U^{\dagger m}, \end{aligned} \tag{6.35}$$

in the Appendix E. In addition, multiplication of $\mathscr{P}_n$ and $\mathscr{P}_m$ is found in the Appendix E as

$$\begin{aligned} \mathscr{P}_n \mathscr{P}_m &= 0 \quad \text{for } m \neq n, \\ \mathscr{P}_n \mathscr{P}_n &= \mathscr{P}_n. \end{aligned} \tag{6.36}$$

We will show that

$$a^\dagger a^{\dagger l} = U^{l+1}(1 - \mathscr{P}_1 - \mathscr{P}_2 - \cdots - \mathscr{P}_{l+1}), \tag{6.37}$$

which implies the following relation

$$\begin{aligned} a^{\dagger d} &= a^\dagger a^{\dagger d-1}, \\ &= U^d[1 - (\mathscr{P}_1 + \mathscr{P}_2 + \cdots + \mathscr{P}_d)], \\ &= U^d[1 - (\mathscr{P}_0 + \mathscr{P}_1 + \cdots + \mathscr{P}_{d-1})], \\ &= \mathbb{1}(1 - 1), \\ &= 0, \end{aligned} \tag{6.38}$$

where we have used (6.34) and (E.23). The equation (6.38) is the second equation of the unitary algebra defined in (6.2). We will prove the equation (6.37) by the method

of proof by induction. For $n = 1$ we have

$$\begin{aligned} a^\dagger a^\dagger &= (U - \mathscr{P}_0 U)(U - \mathscr{P}_0 U), \\ &= U(1 - \mathscr{P}_1)U(1 - \mathscr{P}_1), \\ &= U^2(1 - \mathscr{P}_2)(1 - \mathscr{P}_1), \\ &= U^2(1 - \mathscr{P}_1 - \mathscr{P}_2), \end{aligned} \tag{6.39}$$

where we have used (6.35) and (6.36). For $n = l$ we assume that

$$a^\dagger a^{\dagger l} = U^{l+1}(1 - \mathscr{P}_1 - \mathscr{P}_2 - \cdots - \mathscr{P}_{l+1}). \tag{6.40}$$

Therefore for $n = l + 1$ we obtain

$$\begin{aligned} a^\dagger a^{\dagger l+1} &= (a^\dagger a^{\dagger l})a^\dagger, \\ &= U^{l+1}(1 - \mathscr{P}_1 - \mathscr{P}_2 - \cdots - \mathscr{P}_{l+1})U(1 - \mathscr{P}_1), \\ &= U^{l+2}(1 - \mathscr{P}_2 - \mathscr{P}_3 - \cdots - \mathscr{P}_{l+2})(1 - \mathscr{P}_1), \\ &= U^{l+2}(1 - \mathscr{P}_1 - \mathscr{P}_2 + \mathscr{P}_2\mathscr{P}_1 - \cdots - \mathscr{P}_{l+2} + \mathscr{P}_{l+2}\mathscr{P}_1), \\ &= U^{l+2}(1 - \mathscr{P}_1 - \mathscr{P}_2 - \cdots - \mathscr{P}_{l+2}).\square \end{aligned} \tag{6.41}$$

Now we will show that $aa^\dagger = 1 - a^{\dagger d-1}a^{d-1}$ in terms of U and V, at the left hand side we have,

$$\begin{aligned} aa^\dagger &= U^\dagger(1 - \mathscr{P}_0)(1 - \mathscr{P}_0)U, \\ &= U^\dagger(1 - \mathscr{P}_0 - \mathscr{P}_0 + \mathscr{P}_0\mathscr{P}_0)U, \\ &= U^\dagger(1 - \mathscr{P}_0)U, \\ &= U^\dagger U(1 - \mathscr{P}_1), \\ &= 1 - \mathscr{P}_1, \end{aligned} \tag{6.42}$$

where we have used (6.35) and (1). By using (6.37) we easily obtain

$$\begin{aligned} a^{\dagger d-1} &= U^{d-1}(1 - \mathscr{P}_1 - \cdots - \mathscr{P}_{d-1}), \\ &= U^{d-1}(1 - (\mathscr{P}_1 + \mathscr{P}_2 + \cdots + \mathscr{P}_{d-1})), \\ &= U^{d-1}(1 - (1 - \mathscr{P}_0)), \\ &= U^{d-1}\mathscr{P}_0, \end{aligned} \tag{6.43}$$

where we have used (E.23). Taking Hermitian conjugate of this equation and using hermicity property of the projection operators which is shown by (E.21) we get

$$a^{d-1} = \mathscr{P}_0 U^{\dagger d-1}. \tag{6.44}$$

Then for the right hand side of $aa^\dagger = 1 - a^{\dagger d-1}a^{d-1}$ we have

$$\begin{aligned} 1 - a^{\dagger d-1}a^{d-1} &= 1 - U^{d-1}\mathscr{P}_0\mathscr{P}_0 U^{\dagger d-1}, \\ &= 1 - U^{d-1}\mathscr{P}_0 U^{\dagger d-1}, \\ &= 1 - U^{d-1}U^{\dagger d-1}\mathscr{P}_1, \\ &= 1 - \mathscr{P}_1, \\ &= aa^\dagger, \end{aligned} \tag{6.45}$$

at the second line we used the relations given by (6.36) and at the last line we have used (6.42). This is the first equation defining the almost unitary algebra given by (6.2).

6.4. New Representations for Basis of $M_N(C)$

The e_{ij} satisfying (6.46) form the standard basis of $M_N(C)$

$$e_{ij}e_{kl} = \delta_{jk}e_{il} \quad e_{ij}^\dagger = e_{ji}. \tag{6.46}$$

In this section, we give two new representations of the basis matrices. One of them is written in terms of the almost unitary algebra as

$$e_{mn} = a^{\dagger m} R_{d-1} a^n \quad \text{where} \quad R_n = a^n a^{\dagger n} \quad \text{and} \quad m, n = 0, 1, \cdots, d-1. \tag{6.47}$$

The other one is written in terms of Schwinger U and V operators

$$e_{mn} = \begin{cases} U^{m-n} \mathscr{P}_{d-n} & \text{for} \quad m > n, \\ \mathscr{P}_{d-n} & \text{for} \quad m = n, \\ U^{\dagger n-m} \mathscr{P}_{d-n} & \text{for} \quad m < n, \end{cases} \tag{6.48}$$

with

$$\mathscr{P}_n = \frac{(1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)})}{d}.$$

We prove these representations satisfy (6.46) in the last part of the Appendix E.

6.5. Multi-Dimensional Lattice in Terms of Lower Dimensional Lattices

Denoting a linear lattice with d elements by $\mathscr{L}_d$, we can show the Cartesian product $\mathscr{L}_{d_1} \bigtimes \mathscr{L}_{d_2}$ by the dots in the following figure. Corresponding to this Cartesian product of the lattices, we have the tensor product of the algebra $\mathscr{A}_{d_2} \otimes \mathscr{A}_{d_1}$. On the Cartesian product shown in the figure, the right translation operator corresponds to $a^\dagger \otimes \mathbb{1}$ and the up translation operator corresponds to $\mathbb{1} \otimes a^\dagger$. We denote the (right) translation operator on $\mathscr{A}_{d_1}$, $\mathscr{A}_{d_2}$, $\mathscr{A}_d$ respectively by $a^\dagger_{d_1}$, $a^\dagger_{d_2}$, $a^\dagger_d$ where $d = d_1 \times d_2$ and consider $\mathscr{L}_{d_1}$, $\mathscr{L}_{d_2}$ as a one one-dimensional lattice as shown by the arrows in the figure. This satisfies an isomorphism

$$\mathscr{A}_d \xrightarrow{\triangle} \mathscr{A}_{d_2} \otimes \mathscr{A}_{d_1}, \tag{6.49}$$

and one can write

$$\triangle(a_d^\dagger) = \mathbb{1}_{d_2} \otimes a_{d_1}^\dagger + a_{d_2}^\dagger \otimes a_{d_1}^{d_1-1}. \tag{6.50}$$

One immediately can check that the action of $a_d^\dagger$ is given by the arrows in the figure and satisfies the correct algebraic relations.

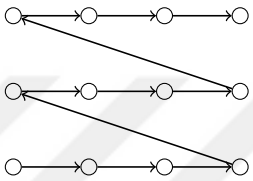

Figure 6.1. $d_1 \bigtimes d_2$ Lattice for $d_1 = 4$ and $d_2 = 3$

Similarly, we can express the translation operator for a $d_1 \times d_2$ dimensional periodic lattice by

$$\triangle(U_d) = \mathbb{1}_{d_2} \otimes a_{d_1}^\dagger + a_{d_2}^\dagger \otimes a_{d_1}^{d_1-1} + a_{d_2}^{d_2-1} \otimes a_{d_1}^{d_1-1}. \tag{6.51}$$

6.6. Discussion

We have shown that the Schwinger algebra can also be given by almost unitary operators which are physically related to the shift operators on a finite lattice. We have named these operators as almost unitary operators because relations

$$UU^\dagger = 1, \quad VV^\dagger = 1 \quad \text{and} \quad VU = qUV \quad \text{where} \quad q = e^{\frac{2\pi i}{d}}, \tag{6.52}$$

are replaced by

$$aa^\dagger = 1 - a^{\dagger d-1}a^{d-1} \qquad \text{and} \qquad a^\dagger a = 1 - a^{d-1}a^{\dagger d-1}, \tag{6.53}$$

and the terms $a^{\dagger d-1}a^{d-1}$, $a^{d-1}a^{\dagger d-1}$ reflect violation of unitarity for a and $a^\dagger$. For U we have the relation $U^\dagger = U^{-1}$ due to the periodic nature of the lattice. Similarly a and $a^\dagger$ can be considered as inverse of each other except at the end points. Note that a and $a^\dagger$ play the role of U and $U^\dagger$ where as V can be defined in terms of a and $a^\dagger$. It takes quite an effort to construct V which is given by (6.13) in terms of a and $a^\dagger$

In usual quantum mechanics where the position and the angular momentum operators have continuous eigenvalues the following equations are equivalent to each other

$$V(p_0)U(x_0) = e^{\frac{ix_0p_0}{\hbar}}U(x_0)V(p_0), \tag{6.54}$$

$$[X, P] = i\hbar, \tag{6.55}$$

$$[X, U(x_0)] = x_0U(x_0), \tag{6.56}$$

$$\text{here} \quad U(x_0) = exp(\frac{-ix_0P}{\hbar}), \quad V(p_0) = exp(\frac{ip_0X}{\hbar}). \tag{6.57}$$

For the discrete periodic case X is defined only modulo $2\pi r$. However U and V are well defined. Thus we have only one corresponding equation

$$VU = qUV \quad \text{where} \quad q = e^{\frac{2\pi i}{d}}. \tag{6.58}$$

On the other hand for the discrete non-periodic case, the position operator X, the right quasi-translation operator $a^\dagger$ and the left quasi-translation operator a are well defined. Therefore we have only one equation corresponding to (6.56)

$$[X, a^\dagger] = \beta a^\dagger, \tag{6.59}$$

where β is the grid spacing.

We have shown how to construct almost unitary translation operators a, $a^\dagger$ in terms of U, V and vice versa. In addition we have found the relation between basis matrices of $M_N(C)$ and the almost unitary operators and the relation between basis matrices of $M_N(C)$ and the Schwinger algebra. Furthermore we established an isomorphism between a multi-dimensional and periodic or non periodic linear lattices. One can also reach our calculations which are presented in this chapter in [182].

7. CONCLUSION

Studies in chapter 2 and chapter 3 differ from what has been done up to now by its mathematical techniques and their wide content. Field equations which govern the dynamics of the universe form a set of differential equations. In the original form they are nonlinear with independent variable time. We switch independent variable from time to scale factor "a". Then in the second chapter field equations of scalar field cosmology with minimal coupling can be reduced to linear first order differential equations. We have succeed to solve all of them simultaneously. We have constructed the most general solution in four different forms: solution for a given $V(a)$, solution for a given $\phi(a)$, solution for a given $H(a)$ and solution for a given $\rho(a)$. We would like to emphasize that our results are the most general. The scalar field has been taken as an effective field which averagely represents the underlying, more basic theories. Both single component universes and double component universes have been studied. We have explored mathematical structure for exotic matter which has time varying equation of state. Combination of this exotic matter and radiation in the early universe has a mathematical turn on and turn of structure for accelerated expansion. More importantly we have explained the late time accelerated expansion of the universe by cosmic domain walls. Results have been compared with observations of type Ia supernovae by taking care of the Hubble tension and the absolute magnitude tension. We have found $\Omega_\omega = 0.889$ and $\Omega_m = 0.111$ where ω denotes cosmic domain walls and m denotes matter.

On the other hand in the third chapter a change of independent variable turns the field equations of Brans-Dicke cosmology into a constraint equation and a Bernoulli type differential equation which can be linearized. We have constructed the most general solutions for supplied pairs of functions; ($\phi(a)$ and $\rho(a)$), ($\phi(a)$ and $V(a)$) and ($\phi(a)$ and $H(a)$). Early epoch of the universe and late-time era of the universe have been investigated. Both dark energy and domain wall dominated universe have been studied. It has been seen that the Hubble function shrinks to $H(a)$ of the Einstein cos-

mology when $\omega \gg 1$. The most significant result of this part comes from the question: *Does the cosmological constant need to correspond to a constant energy density or a constant potential?* We have asked the question in another way *Is it possible to have a constant term in the Hubble function when there is no constant energy density and no constant potential in the universe?* We have found that *it is possible to formulate the constant term in the Hubble function as a combination of initial values of the universe in Brans-Dicke cosmology which DOES NOT contain a constant energy density and a constant potential term.* Furthermore we have found $a(t)$ in an exponential form.

In the fourth chapter we have studied ϕ^4 potential in Brans-Dicke cosmology. We have taken time as an independent variable and we have proceeded by using the ansatz $\phi(t) = \dfrac{A}{a(t)}$. The exact solution results in radiation and stiff fluid mixed universe. Then we have studied early epoch of the universe by extrapolating our result in time. Two different exponential solutions have been found. We have also studied creation of matter in the late radiation dominated era. We have derived the today's value of the deceleration parameter by a new numerical method which has been introduced in Section 4.5.

In chapter 5 we have defined a deformed kinetic energy operator for a discrete position space with a finite number of points. Eigenvalues and eigenfunctions of the Hamiltonian for the periodic space and nonperiodic space have been calculated. As expected, in the continuum limit the solution for the nonperiodic case becomes the same as the solution of an infinite one dimensional square well and the periodic case solution becomes the same as the solution of a particle in a box with periodic boundary conditions.

In chapter 6 we have presented properties of the almost unitary Schwinger operators and properties of the unitary Schwinger operators. We have found relations between them. The famous Schwinger algebra has many applications in quantum optics, quantum communications, quantum probability and Galois quantum systems. We have not discussed these in detail, however we believe that our study will contribute to

these branches of physics. Our results are interesting and can be useful as they allow potential transformations between different existing systems. We hope that our work will motivate scientists to apply the differences and the relations between these two sets of operators to their research fields.

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APPENDIX A: APPENDIX FOR CHAPTER 2

A.1. Einstein Tensor

Since derivation of the field equations is independent of the sign of the metric we continue with the metric sign $\{+,-,-,-\}$. Thus

$$ds^2 = dt^2 - a^2(t)[\frac{dr^2}{1-k\frac{r^2}{L^2}} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)]. \tag{A.1}$$

First, we would like remind the relation between coordinate basis vectors $\frac{\partial}{\partial x^\alpha}$ and coordinate basis one-forms dx^α

$$< \frac{\partial}{\partial x^\alpha}, dx^\beta >= \delta^\beta_\alpha. \tag{A.2}$$

On the other hand orthonormal basis 1-forms are defined as

$$e^a = e^a_\mu dx^\mu, \tag{A.3}$$

obey the similar rule

$$< \vec{e}_a, e^b >= \delta^b_a, \tag{A.4}$$

where $\vec{e}_\alpha = E^\nu_\alpha \frac{\partial}{\partial x^\nu}$. Thus we obtain

$$e^0 = dt, \qquad e^1 = \frac{iadr}{\sqrt{1-\frac{kr^2}{L^2}}}, \tag{A.5}$$

$$e^2 = iard\theta, \qquad e^3 = iarSin\theta d\varphi. \tag{A.6}$$

Then we apply Cartan's first structure equation with no torsion;

$$de^a + \omega^a_b \wedge e^b = 0. \tag{A.7}$$

The connection 1-form has following properties

$$\omega^i_j = \omega_{ij} = \omega^{ij}, \qquad \omega_{ij} = -\omega_{ji}. \tag{A.8}$$

By using these features non-zero connection 1-forms are found as

$$\begin{aligned} \omega_{01} &= -\frac{\dot{a}}{a}e^1, \qquad \omega_{12} = \frac{i\sqrt{1-kr^2/L^2}}{ar}e^2, \\ \omega_{02} &= -\frac{\dot{a}}{a}e^2, \qquad \omega_{13} = \frac{i\sqrt{1-kr^2/L^2}}{ar}e^3, \\ \omega_{03} &= -\frac{\dot{a}}{a}e^3, \qquad \omega_{23} = \frac{icot\theta}{ar}e^3. \end{aligned} \tag{A.9}$$

Now we can calculate curvature two-forms by applying Cartan's second structure equation;

$$\Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b. \tag{A.10}$$

Hence we obtain

$$\Omega^0_1 = -\frac{\ddot{a}}{a}e^0 \wedge e^1, \qquad \Omega^1_2 = -(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2})e^1 \wedge e^2, \tag{A.11}$$

$$\begin{aligned} \Omega^0_2 &= -\frac{\ddot{a}}{a}e^0 \wedge e^2, \qquad \Omega^1_3 = -(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2})e^1 \wedge e^3, \\ \Omega^0_3 &= -\frac{\ddot{a}}{a}e^0 \wedge e^3, \qquad \Omega^2_3 = -(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2})e^2 \wedge e^3. \end{aligned} \tag{A.12}$$

The relation between the Riemann curvature tensor and curvature two-forms is given by

$$\Omega^a_b = \frac{1}{2}R^a_{bcd}e^c \wedge e^d = \sum_{c<d} R^a_{bcd}e^c \wedge e^d. \tag{A.13}$$

Hence related terms are

$$R^0_{101} = R^0_{202} = R^0_{303} = -\frac{\ddot{a}}{a}, \tag{A.14}$$
$$R^1_{212} = R^1_{313} = R^2_{323} = -(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2}).$$

Symmetry properties of the Riemann tensor according to it's indices are

$$R^a_{bcd} = -R^a_{bdc} = -R^b_{acd}. \tag{A.15}$$

Thus

$$R^1_{010} = R^2_{020} = R^3_{030} = -\frac{\ddot{a}}{a}, \tag{A.16}$$
$$R^2_{121} = R^3_{131} = R^3_{232} = -(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2}).$$

The Ricci tensor is defined as

$$R_{ab} = R^c_{acb}. \tag{A.17}$$

Then for orthonormal basis we find

$$R_{00} = R^0_{000} + R^1_{010} + R^2_{020} + R^3_{030} = -3\frac{\ddot{a}}{a}, \tag{A.18}$$
$$R_{11} = R^0_{101} + R^1_{111} + R^2_{121} + R^3_{131} = -\frac{\ddot{a}}{a} - 2(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2}),$$
$$R_{22} = R^0_{202} + R^1_{212} + R^2_{222} + R^3_{232} = -\frac{\ddot{a}}{a} - 2(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2}),$$
$$R_{33} = R^0_{303} + R^1_{313} + R^2_{323} + R^3_{333} = -\frac{\ddot{a}}{a} - 2(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2}), \tag{A.19}$$

where $R^0_{000} = R^1_{111} = R^2_{222} = R^3_{333} = 0$.

To find the Ricci tensor in coordinate basis we use the following relation

$$R_{ab}e^a \otimes e^b = R_{\mu\nu}e^\mu \otimes e^\nu, \tag{A.20}$$

where e^a are presented in (A.5-A.6). Therefore Ricci tensor in coordinate basis are

$$R_{00} = -3\frac{\ddot{a}}{a}, \qquad R_{ii} = -\Big[\frac{\ddot{a}}{a} + 2(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2})\Big]g_{ii}. \tag{A.21}$$

Then the Ricci scalar is found as

$$R = R^\mu_\mu = g^{\mu\nu}R_{\mu\nu} = -6(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a}). \tag{A.22}$$

Einstein tensor is defined as

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R. \tag{A.23}$$

Thus components of the Einstein tensor is found as

$$G_{00} = 3(\frac{\dot{a}^2}{a^2} + \frac{k}{L^2a^2}), \tag{A.24}$$

$$G_{ii} = (\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2} + 2\frac{\ddot{a}}{a})g_{ii}. \tag{A.25}$$

A.2. Action Variation for Non-minimal Coupling

A.2.1. Action Variation with Respect to Metric

Since we flip the sign of the metric to $\{+,-,-,-\}$ we flip the sign in front of the related terms in the action. According to (A.34) and (A.35) the Riemann tensor does not change sign when we change sign of the metric. Thus the Ricci tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ remains the same. However the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ changes sign. Thus for the

case of minimal coupling action is given by

$$S = \int d^4x\sqrt{-g}[-\frac{1}{2\kappa}(R + 2\Lambda) + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + \mathcal{L}_M], \tag{A.26}$$

where R is the Ricci scalar and $\kappa = 8\pi G$.

We would like to present some useful formulas before going further. If one starts from

$$g^{\mu\lambda}g_{\lambda\nu} = \delta^\mu_\nu, \tag{A.27}$$

one can obtain

$$\delta g_{\mu\nu} = -g_{\mu\alpha}g_{\lambda\nu}(\delta g^{\alpha\lambda}). \tag{A.28}$$

By using

$$ln(detM) = Tr(lnM), \tag{A.29}$$

$$\frac{1}{detM}\delta(detM) = Tr(M^{-1}\delta M), \tag{A.30}$$

with $M = g_{\mu\nu}$ one can show that

$$\delta g = -gg_{\mu\nu}\delta g^{\mu\nu}. \tag{A.31}$$

Here $g = detg_{\mu\nu} < 0$ so $\sqrt{-g} = \sqrt{|g|}$. Thus

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}. \tag{A.32}$$

On the other hand we also have

$$\delta R = \delta(g^{\mu\nu}R_{\mu\nu}) = \delta(g^{\mu\nu})R_{\mu\nu} + g^{\mu\nu}\delta(R_{\mu\nu}). \tag{A.33}$$

To calculate $\delta R_{\mu\nu}$ we follow the Palatini approach. The Riemann tensor has been defined as

$$R^{\mu}_{\nu\sigma\rho} = \partial_\sigma \Gamma^{\mu}_{\nu\rho} - \partial_\rho \Gamma^{\mu}_{\sigma\nu} + \Gamma^{\mu}_{\sigma\lambda}\Gamma^{\lambda}_{\rho\nu} - \Gamma^{\mu}_{\lambda\rho}\Gamma^{\lambda}_{\sigma\nu}, \tag{A.34}$$

where connection coefficients are given as

$$\Gamma^{\mu}_{\nu\sigma} = \frac{1}{2}g^{\mu\lambda}(\partial_\nu g_{\lambda\sigma} + \partial_\sigma g_{\nu\lambda} - \partial_\lambda g_{\sigma\nu}). \tag{A.35}$$

Thus we have

$$\delta R_{\mu\nu} = \delta R^{\lambda}_{\mu\lambda\nu} = \nabla_\lambda \delta\Gamma^{\lambda}_{\mu\nu} - \nabla_\nu \delta\Gamma^{\lambda}_{\mu\lambda}. \tag{A.36}$$

In this formalism connection coefficients and metric are accepted as two independent fields [183, 184]. Therefore we should vary the action with respect to inverse of the metric and connection coefficients. It is obvious that variation of $R_{\mu\nu}$ with respect to $g^{\mu\nu}$ is zero while variation of $R_{\mu\nu}$ with respect to connection coefficients imply (A.35).

One can write the action sum of the following parts

$$S_{EH\Lambda} = \int d^4x\sqrt{-g}[-\frac{1}{2\kappa}(R + 2\Lambda)], \tag{A.37}$$

$$S_\phi = \int d^4x\sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)], \tag{A.38}$$

$$S_M = \int d^4x\sqrt{-g}\mathcal{L}_M. \tag{A.39}$$

Then

$$\delta S_{EH\Lambda} = \int d^4x \Big\{ \delta(\sqrt{-g})[-\frac{1}{2\kappa}(R+2\Lambda)] - \sqrt{-g}(\frac{1}{\kappa}\delta R) \Big\}. \tag{A.40}$$

By using (A.32-A.36) we obtain

$$\delta S_{EH\Lambda} = \int d^4x \Big\{ \frac{1}{2}(\sqrt{-g}) g_{\mu\nu}\delta g^{\mu\nu}[\frac{1}{2\kappa}(R+2\Lambda)] - \sqrt{-g}[\frac{1}{\kappa}\delta(g^{\mu\nu})R_{\mu\nu}] \Big\}. \tag{A.41}$$

Then variation with respect to $g^{\mu\nu}$ leads to

$$\frac{\delta S_{EH\Lambda}}{\sqrt{-g}\delta g^{\mu\nu}} = \frac{1}{2\kappa}\Big[- R_{\mu\nu} + \frac{1}{2} g_{\mu\nu}(R+2\Lambda) \Big] + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}} = 0. \tag{A.42}$$

The last term is zero and this will be shown later. We use the energy momentum tensor which is defined as

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{A.43}$$

where

$$T_{\mu\nu} = g_{\mu\lambda} T^{\lambda}\,\nu, \qquad T^{\mu}_{\nu} = diag\{\rho, -p, -p-p\}. \tag{A.44}$$

Hence we obtain

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}, \tag{A.45}$$

where we have used $\kappa = 8\pi G_N$. By using definition of the Einstein tensor which is given by (A.23) the last equation becomes

$$G_{\mu\nu} - \Lambda g_{\mu\nu} = 8\pi G_N T_{\mu\nu}. \tag{A.46}$$

By using (A.24), (A.25) and (A.44) we end up with Einstein field equations

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{\Lambda}{3} - \frac{k}{L^2 a^2}, \tag{A.47}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3}. \tag{A.48}$$

Now we take care of the remaining term according to the action principle

$$\delta S_\phi = \int d^4x \Big\{ \delta(\sqrt{-g})[\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)] + \sqrt{-g}[\frac{1}{2}(\delta g^{\mu\nu})\partial_\mu\phi\partial_\nu\phi] \Big\}, \tag{A.49}$$

$$\delta S_\phi = \int d^4x \Big\{ -\frac{1}{2}\sqrt{-g} g_{\mu\nu}(\delta g^{\mu\nu})[\frac{1}{2} g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi - V(\phi)] + \sqrt{-g}[\frac{1}{2}(\delta g^{\mu\nu})\partial_\mu\phi\partial_\nu\phi] \Big\}, \tag{A.50}$$

By comparing the last equation with (A.57) we have

$$\begin{aligned} T^\phi_{\mu\nu} &= 2\frac{\delta S_\phi}{\sqrt{-g}\delta g^{\mu\nu}}, \\ &= \partial_\mu\phi\partial_\nu\phi - g_{\mu\nu}[\frac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi)]. \end{aligned} \tag{A.51}$$

By using perfect fluid energy momentum tensor for a scalar field

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - p g_{\mu\nu}, \tag{A.52}$$

we obtain

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi) + \frac{1}{2}(\nabla\phi)^2, \tag{A.53}$$

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi) + \frac{1}{2}(\nabla\phi)^2. \tag{A.54}$$

A.2.1.1. Action Principle. According to action principle S_M is invariant under coordinate transformations [54]. By adding infinitesimal vector field $\varepsilon^\mu(x)$ to our coordinates

we have

$$x^{'\mu} = x^{\mu} + \varepsilon^{\mu}(x). \tag{A.55}$$

One can show that [122]

$$\delta g_{\mu\nu} = -(\nabla_{\mu}\varepsilon_{\nu} + \nabla_{\nu}\varepsilon_{\mu}). \tag{A.56}$$

On the other hand by using definition given (A.43) one can write

$$\delta S_M \equiv \int_{\mathcal{R}} \frac{\delta \mathcal{L}_M}{\delta g^{\mu\nu}} \delta g^{\mu\nu} d^4x = \frac{1}{2}\int_{\mathcal{R}} T_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g}d^4x \quad = -\frac{1}{2}\int_{\mathcal{R}} T^{\mu\nu}\delta g_{\mu\nu}\sqrt{-g}d^4x, \tag{A.57}$$

where we have used $\delta g^{\mu\lambda} g_{\lambda\nu} = -g^{\mu\lambda}\delta g_{\lambda\nu}$. By using (A.56) and remembering $T^{\mu\nu}$ is symmetric we obtain

$$\delta S_M = \int_{\mathcal{R}} T^{\mu\nu}(\nabla_{\mu}\varepsilon_{\nu})\sqrt{-g}d^4x = 0. \tag{A.58}$$

By applying integration by parts, one can write

$$\delta S_M = \int_{\mathcal{R}} \nabla_{\mu}(T^{\mu\nu}\varepsilon_{\nu})\sqrt{-g}d^4x - \int_{\mathcal{R}} \nabla_{\mu}(T^{\mu\nu})\varepsilon_{\nu}\sqrt{-g}d^4x = 0. \tag{A.59}$$

By using divergence theorem

$$\delta S_M = \int_{\partial\mathcal{R}} n_{\mu}T^{\mu\nu}\varepsilon_{\nu}\sqrt{|\gamma|}d^3x - \int_{\mathcal{R}} \nabla_{\mu}(T^{\mu\nu})\varepsilon_{\nu}\sqrt{-g}d^4x = 0, \tag{A.60}$$

where γ is determinant of the induced metric and n_{μ} is the unit normal to the boundary. Since surface integral vanishes we end up with

$$\nabla_{\mu}(T^{\mu\nu}) = 0. \tag{A.61}$$

A.2.2. Action Variation with Respect to the Scalar Field

In general the action is written as

$$S = \int \mathcal{L}(\phi^a, \nabla_\mu \phi^a)\sqrt{-g}d^4x. \tag{A.62}$$

Under variation of the scalar field

$$\nabla_\mu \phi^a \to \nabla_\mu \phi^a + \nabla_\mu(\delta\phi^a), \tag{A.63}$$

$$\nabla_\mu(\delta\phi^a) = \delta(\nabla_\mu \phi^a). \tag{A.64}$$

Then

$$\delta S = \int \delta\mathcal{L}\sqrt{-g}d^4x, \tag{A.65}$$

$$= \int \Big[\frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi^a)}\delta(\nabla_\mu\phi^a)\Big]\sqrt{-g}d^4x. \tag{A.66}$$

By applying (A.64) and by applying integration by parts respectively, the second term on the right side becomes

$$\int \frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi^a)}\nabla_\mu(\delta\phi^a)\sqrt{-g}d^4x = \int \nabla_\mu[\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi^a)}\delta\phi^a]\sqrt{-g}d^4x - \int \nabla_\mu[\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi^a)}]\delta\phi^a\sqrt{-g}d^4x, \tag{A.67}$$

where the first term vanishes at the boundaries. Thus we are left with

$$\delta S = \int \Big\{\frac{\partial\mathcal{L}}{\partial\phi^a} - \nabla_\mu[\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi^a)]}\Big\}\delta\phi^a\sqrt{-g}d^4x, \tag{A.68}$$

$$= \int \frac{\delta\mathcal{L}}{\delta\phi^a}\delta\phi^a\sqrt{-g}d^4x. \tag{A.69}$$

where

$$\frac{\delta\mathcal{L}}{\delta\phi^a} = \frac{\partial\mathcal{L}}{\partial\phi^a} - \nabla_\mu[\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi^a)}] = 0. \tag{A.70}$$

Thus we have shown that variation of the action with respect to the scalar field results in Euler-Lagrange equations.

Now we have

$$S_\phi = \int d^4x\sqrt{-g}[\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]. \tag{A.71}$$

Since for scalars, the covariant derivative equals to a partial derivative we can write

$$S_\phi = \int d^4x\sqrt{-g}[\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)]. \tag{A.72}$$

Then

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\frac{dV(\phi)}{d\phi}, \tag{A.73}$$

$$\frac{\partial\mathcal{L}}{\partial(\nabla_\mu\phi)} = \frac{\partial}{\partial(\nabla_\mu\phi)}[\frac{1}{2}g^{\rho\sigma}\nabla_\rho\phi\nabla_\sigma\phi] = g^{\mu\nu}\nabla_\nu\phi. \tag{A.74}$$

Hence the Euler-Lagrange equation becomes

$$\frac{dV(\phi)}{d\phi} + g^{\mu\nu}\nabla_\nu\phi = 0. \tag{A.75}$$

Since $\nabla_u g^{\mu\nu} = 0$, we can write

$$\Box\phi + \frac{dV(\phi)}{d\phi} = 0, \tag{A.76}$$

where $\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$. Then by using the following property

$$\begin{aligned}\nabla_i\nabla_j\phi &= \nabla_i\partial_j\phi, \\ &= \partial_i\partial_j\phi - \partial_k\phi\Gamma^k_{ij},\end{aligned} \tag{A.77}$$

one can show that

$$\Box\phi = \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi}. \tag{A.78}$$

Finally we obtain our last equation as

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV(\phi)}{d\phi} = 0. \tag{A.79}$$

A.3. Change of Independent Variable

$$H(a) = \frac{\dot{a}}{a}, \tag{A.80}$$

therefore our new independent variable becomes a scale factor $'a'$. For this reason we write all other variables in terms of the new variable as

$$\phi = \phi(a), \qquad V(\phi(a)) = V(a), \qquad \dot{a} = aH(a). \tag{A.81}$$

As a result we obtain by change of variable

$$\frac{d\phi}{dt} = \frac{d\phi(a)}{dt}\frac{da}{dt} = \phi^{'}\dot{a} = \phi^{'}aH, \tag{A.82}$$

$$\dot{\phi} = \phi^{'}aH, \tag{A.83}$$

$$\frac{d^2\phi}{dt^2} = \frac{d}{dt}(\phi^{'}aH) = \frac{d}{da}(\phi^{'}aH)\frac{da}{dt}, \tag{A.84}$$

$$\ddot{\phi} = \phi^{''}a^2H^2 + \phi^{'}aH^2 + \phi^{'}a^2HH^{'} \tag{A.85}$$

By the help of the chain rule

$$\frac{dV}{d\phi} = \frac{dV}{da}\frac{da}{d\phi} = \frac{dV}{da}\frac{1}{\frac{d\phi}{da}}, \tag{A.86}$$

$$\frac{dV}{d\phi} = V^{'}\frac{1}{\phi^{'}}. \tag{A.87}$$

On the other hand by starting from our definition we get followings

$$\frac{\dot{a}}{a} = H(a), \tag{A.88}$$

$$\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} = \frac{dH}{da}\frac{da}{dt}, \tag{A.89}$$

$$\frac{\ddot{a}}{a} = H^{'}aH + H^2. \tag{A.90}$$

A.4. Integration by Parts

By using functions u and v a theorem integration by parts is written as

$$\int_{x_0}^{x} u(x)dv(x) = u(x)v(x)\Big|_{x_0}^{x} - \int_{x_0}^{x} v(x)du(x.) \tag{A.91}$$

When calculating the function $\gamma(a)$ we choose

$$u = -a^{'6}, \qquad dv = V^{'}da^{'}, \tag{A.92}$$

thus

$$du = -6a^{'5}da^{'}, \qquad v = V(a^{'}), \tag{A.93}$$

results in

$$\begin{aligned}\int_{a_{in}}^{a}(-a^{'6}V^{'})da^{'} &= -a^{'6}V(a^{'})\Big|_{a_{in}}^{a} + 6\int_{a_{in}}^{a}V(a^{'})a^{'5}da^{'},\\ \int_{a_{in}}^{a}(-a^{'6}V^{'})da^{'} &= -a^{6}V(a) + 6\int_{a_{in}}^{a}V(a^{'})a^{'5}da^{'} + a_{in}^{6}V(a_{in}).\end{aligned} \tag{A.94}$$

A.5. The Linear First Order Differential Equation

The initial value problem

$$\frac{dy}{dx} + P(x)y = Q(x), \qquad y(x_0) = y_0, \tag{A.95}$$

has the following solution [185]

$$\begin{aligned}y(x) &= \frac{1}{\rho(x)}\Big[\int_{x_0}^{x}\rho(x^{'})Q(x^{'})dx^{'} + y_0\Big],\\ \rho(x) &= exp\Big(\int_{x_0}^{x}P(x^{'})dx^{'}\Big).\end{aligned} \tag{A.96}$$

A.6. Hypergeometric Functions

The function $(1-z)^s$ can be represented by Maclaurin series as

$$\begin{aligned}(1-z)^s &= \sum_{k=0}^{\infty}\frac{[f^{(k)(z)}]_{z=0}}{k!}z^k,\\ &= \sum_{k=0}^{\infty}\frac{(-1)^k s(s-1)\dots(s-k+1)}{k!}z^k.\end{aligned} \tag{A.97}$$

By using Pochhammer symbol $(b)_n$ which is defined as

$$(b)_n = \begin{cases} 1 \quad \text{if} \quad n = 0, \\ b(b+1)(b+2)...(b+n-1), \quad \text{if} \quad n = 1, 2, \ldots. \end{cases} \tag{A.98}$$

one can write

$$(1-z)^s = \sum_{k=0}^{\infty} \frac{(-s)_k}{k!} z^k, \qquad (-s)_k = (-1)^k (s-k+1)_k. \tag{A.99}$$

The hypergeometric function $F(a, b; c; z)$ is defined as [186]

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n. \tag{A.100}$$

On the other hand the generalized Hypergeometric Function is defined as

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \ldots (a_p)_n}{n!(b_1)_n (b_2)_n \ldots (b_q)_n} z^n. \tag{A.101}$$

This function is also denoted as ${}_pF_q(a; b; z)$. By comparing (A.100) and (A.101) one can write

$${}_2F_1(a, b; c; z) = F(a, b; c; z). \tag{A.102}$$

Thus we can express the function $(1-z)^s$ as

$$(1-z)^s = F(-s, b; b; z) = {}_1F_0(-s; -; z) = \sum_{k=0}^{\infty} \frac{(-s)_k}{k!} z^k. \tag{A.103}$$

Furthermore, by using definition (A.101) one can show that

$$\frac{1}{a}(z\frac{d}{dz}+a)_2F_1(a,b;c;z) = \frac{1}{a}\sum_{n=0}^{\infty}\frac{(a+n)(a)_n(b)_n}{(c)_n n!}z^n, \tag{A.104}$$
$$= {}_2F_1(a+1,b;c;z).$$

where we have used $(a+n) = a(a+1)_n$. When we choose $c = a+1$ the last equation reduce to

$$\frac{1}{a}(z\frac{d}{dz}+a)_2F_1(a,b;c;z) = {}_1F_0(b;-;z), \tag{A.105}$$
$$= (1-z)^{-b}.$$

We use the following trick

$$\frac{1}{a}(z\frac{d}{dz}+a)f(z) = \frac{1}{a}z^{1-a}\frac{d}{dz}[z^a f(z)], \tag{A.106}$$

which is valid for all functions. Thus

$$\frac{1}{a}z^{1-a}\frac{d}{dz}[z^a{}_2F_1(a,b;a+1;z)] = {}_1F_0(b;-;z), \tag{A.107}$$

where we have used (A.105). Then we obtain

$$\frac{d}{dz}[z^a{}_2F_1(a,b;a+1;z)] = az^{a-1}{}_1F_0(b;-;z), \tag{A.108}$$
$$= az^{a-1}(1-z)^{-b}.$$

where we have used second line of (A.105).

Second trick is to replace $a = k + 1$ and $b = -m$. According to (A.100) we can swap the places of a and b, thus we obtain

$$\frac{d}{du}[u^{k+1}{}_2F_1(-m, k+1; k+2; u)] = (k+1)u^k(1-u)^m, \tag{A.109}$$

where we have changed our argument of the function form z to u since it will be useful later. Then by taking integral of both sides we find

$$\int (1-u)^m u^k du = \frac{u^{k+1}}{k+1}{}_2F_1(-m, k+1; k+2; u). \tag{A.110}$$

A.6.1. Some Integrals Expressed as Hypergeometric Function

We have used following technique to find relation between time and the scale factor at the integral given in Section 2.3.2.2 by the (2.114)

$$\begin{aligned} t &= \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}H(a^{'})}, \\ t &= \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}\sqrt{\frac{8\pi G}{3}(\frac{f}{a^{'6}} + \frac{6v_n}{(6-n)a^{'n}})}}, \end{aligned} \tag{A.111}$$

where $f = a_{in}^6\tilde{\gamma}(a_{in})$. We continue our calculations as

$$t = \sqrt{\frac{3}{8\pi G f}} \int_{a_{in}}^{a} \frac{a^{'2}}{\sqrt{1 + \frac{6v_n a^{'(6-n)}}{(6-n)f}}} da^{'}, \tag{A.112}$$

$$t = \sqrt{\frac{3}{8\pi G f}} \int_{a_{in}}^{a} a^{'2}[1 + \frac{6v_n a^{'(6-n)}}{(6-n)f}]^{-1/2} da^{'}. \tag{A.113}$$

Now we apply the following change of variable

$$u = -\frac{6v_n a^{'(6-n)}}{(6-n)f}. \tag{A.114}$$

Thus

$$a = [-\frac{(6-n)f}{6v_n}u]^{\frac{1}{6-n}}, \qquad da = \frac{1}{6-n}[-\frac{(6-n)f}{6v_n}]^{\frac{1}{6-n}}u^{\frac{1}{6-n}-1}. \tag{A.115}$$

Then we have

$$t = \sqrt{\frac{3}{8\pi Gf}}\frac{1}{(6-n)}[-\frac{(6-n)f}{6v_n}]^{\frac{3}{6-n}}\int_{a_{in}}^{a} u^{\frac{3}{6-n}-1}(1-u)^{-1/2}du. \tag{A.116}$$

By comparing the integral at the right side with the integral given in (A.110) we obtain

$$m = -\frac{1}{2}, \qquad k = \frac{3}{6-n} - 1. \tag{A.117}$$

Hence

$$t = \sqrt{\frac{3}{8\pi Gf}}\frac{1}{(6-n)}[-\frac{(6-n)f}{6v_n}]^{\frac{3}{6-n}}\Big[\frac{u^{3/(6-n)}}{\frac{3}{6-n}}{}_2F_1(\frac{1}{2},\frac{3}{6-n};\frac{3}{6-n}+1;u)\Big]_{u(a_{in})}^{u(a)}. \tag{A.118}$$

By back substitution $u = -\dfrac{6v_n a'^{(6-n)}}{(6-n)f}$ we obtain

$$t = \sqrt{\frac{1}{24\pi Ga_{in}^6\tilde{\gamma}(a_{in})}}\Big[a'^3{}_2F_1(\frac{1}{2},\frac{3}{6-n};\frac{3}{6-n}+1;-\frac{6v_n}{(6-n)a_{in}^6\tilde{\gamma}(a_{in})}a'^{(6-n)})\Big]_{a_{in}}^{a}. \tag{A.119}$$

We have used the same technique to find relation between time and the scale factor at the integral given in Section 2.4.2.2 by the (2.163)

$$
\begin{aligned}
t &= \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}H(a^{'})}, \\
t &= \int_{a_{in}}^{a} \frac{da^{'}}{a^{'}\sqrt{\frac{8\pi G}{3}(\frac{\rho_w}{a^{'}} + \frac{\rho_n}{a^{'n}})}}, \\
t &= \sqrt{\frac{3}{8\pi G\rho_w}} \int_{a_{in}}^{a} a^{'-1/2}(1 + \frac{\rho_n a^{'(1-n)}}{\rho_w})^{-1/2} da^{'}.
\end{aligned} \tag{A.120}
$$

We choose the following change of variable

$$
u = -\frac{\rho_n}{\rho_w} a^{'(1-n)}. \tag{A.121}
$$

Thus

$$
a = (-\frac{\rho_w}{\rho_n} u)^{\frac{1}{1-n}}, \qquad da = \frac{1}{1-n}(-\frac{\rho_w}{\rho_n})^{\frac{1}{1-n}} u^{\frac{1}{1-n}-1} du. \tag{A.122}
$$

Then we have

$$
t = (\frac{1}{1-n})\sqrt{\frac{3}{8\pi G\rho_w}}(-\frac{\rho_w}{\rho_n})^{\frac{1/2}{1-n}} \int_{a_{in}}^{a} u^{\frac{1}{2-2n}-1}(1-u)^{-1/2} du. \tag{A.123}
$$

By comparing the last integral with the integral given in (A.110) yields

$$
m = -\frac{1}{2}, \qquad k = \frac{1}{2-2n} - 1. \tag{A.124}
$$

Thus

$$
t = (\frac{1}{1-n})\sqrt{\frac{3}{8\pi G\rho_w}}(-\frac{\rho_w}{\rho_n})^{\frac{1/2}{1-n}} \Big[\frac{u^{\frac{1}{2-2n}}}{\frac{1}{2-2n}} {}_2F_1(\frac{1}{2}, \frac{1}{2-2n}; \frac{1}{2-2n}+1; u) \Big]_{u(a_{in})}^{u(a)}. \tag{A.125}
$$

By back substitution $u = -\frac{\rho_n}{\rho_w} a^{'(1-n)}$ we obtain

$$t = \sqrt{\frac{3}{2\pi G\rho_w}} \Big[\sqrt{a^{'}}\, {}_2F_1(\frac{1}{2}, \frac{1}{2-2n}; \frac{1}{2-2n} + 1; -\frac{\rho_n}{\rho_w} a^{'(1-n)}) \Big]_{a_{in}}^{a}. \tag{A.126}$$

A.7. Integral Expressed as Erfi Function

We calculate the scale factor as a function of time in Section 2.3.2.2 for case $n = 6$ by following steps. First, we use the $H(a)$ given in (2.99) with k=0, so we have

$$t = \int_{a_{in}}^{a} \frac{da^{'}}{a^{'} H(a^{'})}, \tag{A.127}$$

$$t = \int_{a_{in}}^{a} \frac{da^{'}}{a^{'} \sqrt{\frac{8\pi G}{3} (\frac{f}{a^{'6}} + \frac{gln(\frac{a^{'}}{c})}{a^{'6}})}}, \tag{A.128}$$

where $f = V_6 + a_{in}^6 \gamma(a_{in})$, $g = 6V_6$ and $c = a_{in}$. Then we have

$$t = \sqrt{\frac{3}{8\pi G}} \int_{a_{in}}^{a} \frac{a^{'2} da^{'}}{\sqrt{f + gln(\frac{a^{'}}{c})}}. \tag{A.129}$$

We apply the change of variable as $u = \frac{a^{'}}{c}$ and $du = \frac{da^{'}}{c}$ so

$$\begin{aligned} I &= \int \frac{a^{'2} da^{'}}{\sqrt{f + gln(\frac{a^{'}}{c})}}, \\ &= c^3 \int \frac{u^2}{\sqrt{f + gln(u)}} du. \end{aligned} \tag{A.130}$$

We apply the second change of variable as $v = f + glnu$ and $dv = \frac{g}{u} du$ so

$$I = \frac{c^3 e^{-3f/g}}{g} \int \frac{e^{3v/g}}{\sqrt{v}} dv. \tag{A.131}$$

Now we choose $w = \sqrt{\frac{3v}{g}}$ so $dw = \frac{1}{2}\sqrt{\frac{3}{g}}\frac{1}{\sqrt{v}}dv$. Thus the integral I becomes

$$I = \sqrt{\frac{g}{3}}\frac{c^3 e^{-3f/g}}{g}\sqrt{\pi}\int \frac{2e^{w^2}}{\sqrt{\pi}}dw, \tag{A.132}$$

$$= \frac{c^3 e^{-3f/g}}{\sqrt{3g}}\sqrt{\pi}erfi(w), \tag{A.133}$$

where we have used the following definition

$$\int exp[(\alpha z + \beta)^2]dz = \frac{\sqrt{\pi}}{2\alpha}erfi(\alpha z + \beta), \tag{A.134}$$

with $\alpha = 1$ and $\beta = 0$ [119]. Thus when one has a definite integral this definition becomes

$$erfi(\theta) = \frac{2}{\sqrt{\pi}}\int_0^\theta exp(z^2)dz. \tag{A.135}$$

According to this definition $erfi(0) = 0$. Then by applying back substitutions $w = \sqrt{\frac{3v}{g}}$, $v = f + glnu$ and $u = \frac{a^{'}}{c}$ respectively we obtain

$$I = \sqrt{\frac{\pi}{3g}}c^3 e^{-3f/g}erfi[\sqrt{\frac{3(f + gln(\frac{a^{'}}{c}))}{g}}]. \tag{A.136}$$

By writing our abbreviations in their original form $f = V_6 + a_{in}^6\gamma(a_{in})$, $g = 6V_6$ and $c = a_{in}$ we have

$$t = \sqrt{\frac{3}{8\pi G}}a_{in}^3 exp[-\frac{1}{2} - \frac{a_{in}^6\gamma(a_{in})}{2V_6}]\Big\{erfi[\sqrt{\frac{1}{2}(1 + 6ln(\frac{a}{a_{in}}) + \frac{a_{in}^6\gamma(a_{in})}{2V_6})}] \tag{A.137}$$

$$- erfi[\sqrt{\frac{1}{2}(1 + \frac{a_{in}^6\gamma(a_{in})}{2V_6})}]\Big\}.$$

As a result we have

$$a(t) = a_{in} exp\Big\{\frac{1}{6}[2\Big(erfi^{-1}(\mu t + b)\Big)^2 - 1 - \frac{a_{in}^6 \gamma(a_{in})}{V_6}]\Big\}, \tag{A.138}$$

$$\mu = \sqrt{\frac{8\pi G}{3}} \frac{1}{a_{in}^3} exp[\frac{1}{2} + \frac{a_{in}^6 \gamma(a_{in})}{2V_6}], \qquad b = erfi[\frac{1}{2}\Big(1 + \frac{a_{in}^6 \gamma(a_{in})}{V_6}\Big)]. \tag{A.139}$$

A.8. Cosmological Fluid

The energy-momentum conservation states that

$$\nabla_\mu T^{\mu\nu} = 0, \tag{A.140}$$

where

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}. \tag{A.141}$$

Then (A.140) becomes [122]

$$\dot{\rho} + 3(\frac{\dot{a}}{a})(\rho + p) = 0. \tag{A.142}$$

The last equation is also known as the continuity equation in cosmology. One can write it in the following form

$$\frac{d(\rho a^3)}{da} = -3pa^2. \tag{A.143}$$

In cosmology it is usually assumed that each component of the fluid obeys the equation of state which is given by

$$p = \nu\rho. \tag{A.144}$$

Then (A.143) turns into the following form

$$\frac{d(\rho a^3)}{da} = -3\nu\rho a^2, \tag{A.145}$$

which has a solution

$$\rho \sim a^{-3(1+\nu)}. \tag{A.146}$$

It is known that $\nu = -1$ corresponds to cosmological constant while $\nu = -2/3$ stands for cosmic domain walls. Pressureless dust which is nonrelativistic matter has $\nu = 0$ and radiation which corresponds to relativistic particles and photons has $\nu = 1/3$. Stiff fluid can be defined with $\nu = 1$.

A.9. Redshift in Cosmology

Suppose we measure a wavelength of a observed distant galaxy as λ_{ob} where it emits the wavelength λ_{em}. Then redshift of this galaxy is calculated according to the following formula

$$z = \frac{\lambda_{ob} - \lambda_{em}}{\lambda_{em}}. \tag{A.147}$$

In [53] it is shown that the relation between the redshift and the scale factor of the universe is

$$1 + z = \frac{a(t_0)}{a(t_{em})} = \frac{1}{a(t_{em})}, \tag{A.148}$$

where we have used the usual agreement such that $a(t_0) = 1$.

APPENDIX B: APPENDIX FOR CHAPTER 3

B.1. Action Variation for Brans-Dicke Cosmology

We use the Brans-Dicke action in the following form

$$S = \int d^4x\sqrt{-g}\left[-\frac{1}{8\omega}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + L_M\right]. \tag{B.1}$$

B.1.1. Action Variation with Respect to Metric

Let us start variation of the action with respect to metric by using $R = R_{\mu\nu}g^{\mu\nu}$

$$\begin{aligned}\delta S = \int d^4x\Big\{(\delta\sqrt{-g})\left[-\frac{1}{8\omega}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + L_M\right] \\ + \left[-\frac{1}{8\omega}\phi^2(\delta R_{\mu\nu})g^{\mu\nu} - \frac{1}{8\omega}\phi^2 R_{\mu\nu}(\delta g^{\mu\nu}) + \frac{1}{2}(\delta g^{\mu\nu})\partial_\mu\phi\partial_\nu\phi\right]\Big\} \\ + \int d^4x\sqrt{-g}\frac{\delta g^{\mu\nu}T_{\mu\nu}}{2},\end{aligned} \tag{B.2}$$

where we have used

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S_M}{\delta g^{\mu\nu}}, \qquad S_M = \int d^4x\sqrt{-g}L_M. \tag{B.3}$$

Then we use (A.28) and (A.23) we obtain

$$\delta S = \int d^4x\sqrt{-g}\delta(g^{\mu\nu})\Big\{-\frac{1}{8\omega}\phi^2 G_{\mu\nu} - \frac{1}{4}g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi + g_{\mu\nu}\frac{V(\phi)}{2} \tag{B.4}$$

$$+ \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{T_{\mu\nu}}{2}\Big\} - \frac{1}{8\omega}\int d^4x\sqrt{-g}\phi^2(\delta R_{\mu\nu})g^{\mu\nu}. \tag{B.5}$$

In the original Brans-Dicke and Jordan variations, S was taken as a functional of $g^{\mu\nu}$ and ϕ. We would like to notice that the palatini approach gives different field

equations [187–189]. Thus one has [56]

$$\int d^4x\sqrt{-g}\phi^2(\delta R_{\mu\nu})g^{\mu\nu} = \int d^4x\sqrt{-g}\delta g^{\mu\nu}(g_{\mu\nu}\Box\phi^2 - \nabla_\mu\nabla_\nu\phi^2), \tag{B.6}$$

where one uses (A.36), $\nabla_\lambda g^{\mu\nu} = 0$ and integration by parts. Thus we obtain

$$\frac{\delta S}{\delta g^{\mu\nu}} = -\frac{1}{8\omega}\Big[G_{\mu\nu}\phi^2 + g_{\mu\nu}\Box\phi^2 - \nabla_\mu\nabla_\nu\phi^2\Big] - \frac{1}{4}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi \tag{B.7}$$

$$+ g_{\mu\nu}\frac{V(\phi)}{2} + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{T_{\mu\nu}}{2} = 0. \tag{B.8}$$

We choose $\mu = \nu = 0$ and by using (A.24), (A.44) and (B.17) we obtain

$$\frac{3}{4\omega}(\frac{\dot{a}^2}{a^2} + \frac{k}{a^2})\phi^2 + \frac{3}{2\omega}\frac{\dot{a}}{a}\phi\dot{\phi} - \frac{\dot{\phi}^2}{2} - V(\phi) = \rho_m. \tag{B.9}$$

Then we choose $\mu = \nu = i$ and by using (A.25), (A.44) and (B.18) we obtain

$$-\frac{1}{4\omega}\phi^2\left(2\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right) - \frac{1}{\omega}\frac{\dot{a}}{a}\dot{\phi}\phi - \frac{1}{2\omega}\ddot{\phi}\phi - \left(\frac{1}{2} + \frac{1}{2\omega}\right)\dot{\phi}^2 + V(\phi) = p_m. \tag{B.10}$$

B.1.1.1. Some Necessary Calculations. To find $(g_{\mu\nu}\Box\phi^2 - \nabla_\mu\nabla_\nu\phi^2)$ we start with

$$\Box\phi^2 = g^{\mu\nu}\nabla_\mu\nabla_\nu\phi^2, \tag{B.11}$$

$$= g^{00}\nabla_0\nabla_0\phi^2 + g^{ii}\nabla_i\nabla_i\phi^2. \tag{B.12}$$

Before going further we write

$$g_{00} = 1, \quad g_{11} = -\frac{a^2}{1-kr^2}, \quad g_{22} = -a^2r^2, \quad g_{33} = -a^2r^2Sin^2\theta, \tag{B.13}$$

$$g^{00} = 1, \quad g^{11} = -\frac{(1-kr^2)}{a^2}, \quad g^{22} = -\frac{1}{a^2r^2}, \quad g^{33} = -\frac{1}{a^2r^2Sin^2\theta}. \tag{B.14}$$

We choose $\mu = \nu = 0$

$$
\begin{aligned}
(g_{00}\Box - \nabla_0\nabla_0)\phi^2 &= g_{00}(g^{00}\nabla_0\nabla_0\phi^2 + g^{ii}\nabla_i\nabla_i\phi^2) - \nabla_0\nabla_0\phi^2, \\
&= g_{00}g^{ii}\nabla_i\nabla_i\phi^2, \\
&= g^{ii}\nabla_i\nabla_i\phi^2.
\end{aligned} \tag{B.15}
$$

Then we use the definition given in (A.77) and we obtain

$$
\begin{aligned}
g^{ii}\nabla_i\nabla_i\phi^2 &= g^{ii}(\partial_i\partial_i\phi^2 - \partial_k\phi^2\Gamma^k_{ii}), \\
&= -g^{ii}\partial_0\phi^2\Gamma^0_{ii}, \\
&= -2\phi\dot{\phi}[g^{11}\Gamma^0_{11} + g^{22}\Gamma^0_{22} + g^{33}\Gamma^0_{33}], \\
&= -2\phi\dot{\phi}\{-\frac{(1-kr^2)}{a^2}\frac{a\dot{a}}{(1-kr^2)} - \frac{1}{a^2r^2}a\dot{a}r^2 - \frac{1}{a^2r^2Sin^2\theta}a\dot{a}r^2Sin^2\theta\}, \\
&= 6\frac{\dot{a}}{a}\phi\dot{\phi},
\end{aligned} \tag{B.16}
$$

where we have assumed that scalar field depends only on the time and we have used (B.13-B.14). Hence

$$
(g_{00}\Box - \nabla_0\nabla_0)\phi^2 = 6\frac{\dot{a}}{a}\phi\dot{\phi}. \tag{B.17}
$$

Now we choose $\mu = \nu = i$. By following the similar steps we obtain

$$
(g_{ii}\Box - \nabla_i\nabla_i)\phi^2 = g_{ii}(2\dot{\phi}^2 + 2\phi\ddot{\phi} + 4\phi\dot{\phi}\frac{\dot{a}}{a}). \tag{B.18}
$$

B.1.2. Action Variation with Respect to the Scalar Field

As we explained before action variation with respect to the scalar field results in Euler-Lagrange equations. In Brans-Dicke cosmology we have

$$S_\phi = \int d^4x\sqrt{-g}[-\frac{1}{8\omega}\phi^2 R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)]. \tag{B.19}$$

The differences between this action and the action given in (A.38) is the term $-\frac{1}{8\omega}\phi^2 R$. It only modifies the following term as

$$\frac{\partial\mathcal{L}}{\partial\phi} = -\frac{1}{4\omega}\phi R - \frac{dV(\phi)}{\phi}. \tag{B.20}$$

Therefore Euler-Lagrange equation given in (A.79) becomes

$$\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} + \frac{dV(\phi)}{d\phi} - \frac{3}{2\omega}\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right)\phi = 0, \tag{B.21}$$

where we have used $R = -6(\frac{k}{L^2a^2} + \frac{\dot{a}^2}{a^2} + \frac{\ddot{a}}{a})$ which is derived in Appendix A.

B.2. The Bernoulli Equation

A first order differential equation which is in the following form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \tag{B.22}$$

is known as a Bernoulli equation. It is linearized by a transformation

$$v = y^{1-n}. \tag{B.23}$$

Thus the original nonlinear differential equation turns into the linear first order differential equation [185]

$$\frac{dv}{dx} + (1-n)P(x)v = (1-n)Q(x). \tag{B.24}$$

B.3. Some Necessary Calculations

In Section 3.3.1.1 Equation (3.16) is given as

$$\gamma^{'}(a) + \frac{2a[(1+2\omega)\phi^{'2}(a) + \phi(a)\phi^{''}(a)]}{\phi(a)(\phi(a) + a\phi^{'}(a))}\gamma(a) = \frac{2(2\omega\rho^{'}a^3 + 3k\phi^2(a))}{3a^3\phi(a)(\phi(a) + a\phi^{'}(a))}. \tag{B.25}$$

For simplification we choose

$$\alpha = \phi^2 + a\phi\phi^{'}. \tag{B.26}$$

Then we have

$$\phi\phi^{''} = \frac{\alpha^{'} - 3\phi\phi^{'} - a\phi^{'2}}{a}. \tag{B.27}$$

Therefore we can write

$$\begin{aligned} \tilde{P}(a) &= \frac{2a[(1+2\omega)\phi^{'2}(a) + \phi(a)\phi^{''}(a)]}{\phi(a)(\phi(a) + a\phi^{'}(a))}, \\ &= 2\frac{\alpha^{'}}{\alpha} + \frac{2[2\omega a\phi^{'2} - 3\phi\phi^{'}]}{\phi(\phi + a\phi^{'})}, \\ &= 2\frac{\alpha^{'}}{\alpha} + P(a), \end{aligned} \tag{B.28}$$

where

$$P(a) = \frac{2[2\omega a\phi^{'2} - 3\phi\phi^{'}]}{\phi(\phi + a\phi^{'})}. \tag{B.29}$$

Thus we find

$$\gamma(a) = exp\Big(- \int_{a_{in}}^{a} \tilde{P}(a^{'})da^{'}\Big)\Big\{ \int_{a_{in}}^{a} exp\Big(\int_{a_{in}}^{a^{'}} \tilde{P}(a^{''})da^{''}\Big)[\frac{2(2\omega\rho^{'}a^3 + 3k\phi^2(a))}{3a^3\phi(a)(\phi(a) + a\phi^{'}(a))}]da^{'} \tag{B.30}$$

$$+\gamma(a_{in})\Big\}.$$

By using (B.28) we obtain

$$exp\Big(- \int_{a_{in}}^{a} \tilde{P}(a^{'})da^{'}\Big) = exp\Big(- \int_{a_{in}}^{a} [\frac{2\alpha^{'}}{\alpha} + P(a^{'})]da^{'}\Big), \tag{B.31}$$

$$= \frac{[\phi(a_{in})\Big(\phi(a_{in}) + a_{in}\phi^{'}(a_{in})\Big)]^2}{[\phi(a)\Big(\phi(a) + a\phi^{'}(a)\Big)]^2} exp\Big(- \int_{a_{in}}^{a} P(a^{'})da^{'}\Big).$$

As a result we have

$$\gamma(a) = \frac{exp\Big[- \int_{a_{in}}^{a} P(a^{'})da^{'}\Big]}{\Big[\phi(\phi + a\phi^{'})\Big]^2}\Big\{ \int_{a_{in}}^{a} exp\Big[\int_{a_{in}}^{a^{'}} P(a^{''})da^{''}\Big]Q(a^{'})da^{'} + \tilde{\gamma}(a_{in})\Big\}, \tag{B.32}$$

$$P(a) = \frac{2[2\omega a\phi^{'2}(a) - 3\phi(a)\phi^{'}(a)]}{\phi(a)(\phi(a) + a\phi^{'}(a))},$$

$$Q(a) = \frac{2(2\omega\rho^{'}a^3 + 3k\phi^2(a))\Big[\phi(a)(\phi(a) + a\phi^{'}(a))\Big]}{3a^3}, \tag{B.33}$$

$$\tilde{\gamma}(a_{in}) = \gamma(a_{in})\Big[\phi(a_{in})\Big(\phi(a_{in}) + a_{in}\phi^{'}(a_{in})\Big)\Big]^2. \tag{B.34}$$

APPENDIX C: APPENDIX FOR CHAPTER 4

Now we will solve the differential equation which is given in (4.23)

$$t^2\ddot{\theta} + 2\omega t\dot{\theta} - \frac{4\omega}{3}(2 + \lambda B^2)\theta = -2t^2. \tag{C.1}$$

It is recognized as non-homogeneous Cauchy-Euler equation. Thus we take

$$t = e^x \Rightarrow tD = D_x, \quad \text{and} \quad t^2D^2 = D_x(D_x - 1). \tag{C.2}$$

First, we take right side zero so (C.1) becomes

$$[D_x^2 + (2\omega - 1)D_x - \frac{4\omega}{3}(2 + \lambda B^2)]\theta = 0. \tag{C.3}$$

Then substitution $\theta = e^{mx}$ gives us the following auxiliary equation

$$m^2 + (2\omega - 1)m - \frac{4\omega}{3}(2 + \lambda B^2) = 0. \tag{C.4}$$

It's solution yields

$$m_\pm = \frac{1}{2} - \omega \pm \sqrt{\omega^2 - \omega + \frac{1}{4} + \frac{4\omega}{3}(2 + \lambda B^2)}. \tag{C.5}$$

Hence we obtain the complementary solution as

$$\theta_c(t) = c_1 t^{m_+} + c_2 t^{m_-}. \tag{C.6}$$

where c_1 and c_2 are integration constants.

By inspection the particular solution is found as

$$\theta_p(t) = -\frac{3t^2}{2\omega(1-\lambda B^2)+3}. \tag{C.7}$$

Therefore

$$\begin{aligned}\theta(t) &= \theta_c + \theta_p(t), \\ &= c_1 t^{m_+} + c_2 t^{m_-} + \frac{3t^2}{2\omega(\lambda B^2 - 1) - 3}.\end{aligned} \tag{C.8}$$

APPENDIX D: APPENDIX FOR CHAPTER 5

In chapter 5 we have introduced the right translation operator as u_+ which satisfies

$$u_+ |n\rangle = |n+1\rangle , \qquad n = 0, 1, ..., d-2, \tag{D.1}$$

$$u_+ |d-1\rangle = 0. \tag{D.2}$$

Similarly, the left translation operator have been denoted u_- which satisfies

$$u_- |n\rangle = |n-1\rangle , \qquad n = 1, ..., d-1, \tag{D.3}$$

$$u_- |0\rangle = 0. \tag{D.4}$$

Then we have switched our notation as $u_+ = u^\dagger$ and $u = u$. By regrading rules given in (D.1-D.4) one can easily construct the matrix form of the translation operators for $d = 2$ as

$$u^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{D.5}$$

Then we obtain

$$u^\dagger + u = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1, \tag{D.6}$$

$$i(u^\dagger - u) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2, \tag{D.7}$$

$$uu^\dagger - u^\dagger u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma_3. \tag{D.8}$$

APPENDIX E: APPENDIX FOR CHAPTER 6

E.1. Projection Operators in Terms of a and $a^\dagger$

We have found two projection operators P_n and R_n;

$$P_n = a^{\dagger n} a^n \quad \text{and} \quad R_n = a^n a^{\dagger n}. \tag{E.1}$$

The properties

$$\begin{aligned} P_m a^n = a^{\dagger n} P_m = 0 \quad \text{for} \quad n + m \geq d, \\ a^n R_m = R_m a^{\dagger n} = 0 \quad \text{for} \quad n + m \geq d, \end{aligned} \tag{E.2}$$

are immediate results of definition of the projection operators and the algebra property $a^d = a^{\dagger d} = 0$.

The following properties have already been proved in chapter 5 and in [181]

$$P_n P_m = P_m \quad \text{where} \quad m \geq n, \tag{E.3}$$

$$P_m a^\dagger = a^\dagger P_{m-1} \quad \text{and} \quad P_m a = a P_{m+1}. \tag{E.4}$$

Now, we will prove $R_n R_m = R_m$ where $m \geq n$. Our method is proof by induction.

For $n = 1$,

$$\begin{aligned} R_1 R_m &= (aa^\dagger)(a^m a^{\dagger m}), \\ &= a(a^\dagger a) a^{m-1} a^{\dagger m}, \end{aligned} \tag{E.5}$$

$$
\begin{aligned}
&= a(1 - a^{d-1}a^{\dagger d-1})a^{m-1}a^{\dagger m}, \\
&= aa^{m-1}a^{\dagger m}, \\
&= a^m a^{\dagger m}, \\
R_1 R_m &= R_m,
\end{aligned}
$$

where we have used $a^d = 0$. For $n = l$, we assume

$$R_l R_m = R_m \quad \text{for} \quad m \geq l. \tag{E.6}$$

For $n = l + 1$, we have

$$
\begin{aligned}
R_{l+1}R_m &= (a^{l+1}a^{\dagger l+1})(a^m a^{\dagger m}), \\
&= a(a^l a^{\dagger l})a^\dagger a(a^{m-1}a^{\dagger m-1})a^\dagger, \\
&= aR_l(a^\dagger a)R_{m-1}a^\dagger, \\
&= aR_l(1 - a^{d-1}a^{\dagger d-1})R_{m-1}a^\dagger, \\
&= aR_l R_{m-1}a^\dagger - aR_l R_{d-1}R_{m-1}a^\dagger, \\
&= aR_l R_{m-1}a^\dagger - aR_{d-1}R_{m-1}a^\dagger,
\end{aligned} \tag{E.7}
$$

where we have used (6.2) and (E.6). By using $aR_{d-1} = a^d a^{\dagger d-1} = 0$ and the assumption for $n = l$ with the fact that $l + 1 \leq m$ implies $l \leq m - 1$, so we have $R_l R_{m-1} = R_{m-1}$ and $aR_{m-1}a^\dagger = R_m$. Therefore

$$R_{l+1}R_m = R_m \quad \text{Q.E.D..} \tag{E.8}$$

Then we will show that $R_m R_n = R_m$ for $m \geq n$ by using the relation $R_n = 1 - P_{d-n}$ which is Equation (6.7). Thus

$$
\begin{aligned}
R_m R_n &= (1 - P_{d-m})(1 - P_{d-n}) \\
&= 1 - P_{d-n} - P_{d-m} + P_{d-m}P_{d-n}
\end{aligned} \tag{E.9}
$$

using $P_nP_m = P_m$ where $m \geq n$ with the fact that $m \geq n$ implies $d - n \geq d - m$

$$\begin{aligned} R_mR_n &= 1 - P_{d-n} - P_{d-m} + P_{d-n} \\ &= 1 - P_{d-m} \\ &= R_m \quad \text{Q.E.D..} \end{aligned} \tag{E.10}$$

By using the last two proofs, we conclude that

$$R_nR_m = R_j \quad \text{where} \quad j = \max(n, m). \tag{E.11}$$

Similarly we will show that $P_mP_n = P_m$ where $m \geq n$ by using $P_n = 1 - R_{d-n}$ which is Equation (6.7). Thus

$$\begin{aligned} P_mP_n &= (1 - R_{d-m})(1 - R_{d-n}), \\ &= 1 - R_{d-m} - R_{d-n} + R_{d-m}R_{d-n}, \end{aligned} \tag{E.12}$$

using $R_nR_m = R_m$ where $m \geq n$ with the fact that $m \geq n$ implies $d - n \geq d - m$

$$\begin{aligned} P_mP_n &= 1 - R_{d-m} - R_{d-n} + R_{d-n}, \\ &= 1 - R_{d-m}, \\ &= P_m \quad \text{Q.E.D..} \end{aligned} \tag{E.13}$$

This result and equation given by (E.3) can be expressed in one equation as

$$P_nP_m = P_j \quad \text{where} \quad j = \max(n, m). \tag{E.14}$$

Now, we will calculate relations between the projection operator R_m and shift operators a and $a^\dagger$.

$$\begin{aligned} R_m a &= a^m a^{\dagger m} a, \\ &= a^m a^{\dagger m-1}(a^\dagger a), \\ &= a(a^{m-1}a^{\dagger m-1})P_1, \\ &= aR_{m-1}(1 - R_{d-1}), \\ &= aR_{m-1} - aR_{m-1}R_{d-1}, \\ &= aR_{m-1} - aR_{d-1}, \\ &= aR_{m-1}, \end{aligned} \tag{E.15}$$

where we have used (6.7), (E.11) and the fact that $aR_{d-1} = a^d a^{\dagger d-1} = 0$. Then

$$\begin{aligned} a^\dagger R_m &= a^\dagger a^m a^{\dagger m}, \\ &= (a^\dagger a)(a^{m-1}a^{\dagger m-1})a^\dagger, \\ &= P_1 R_{m-1} a^\dagger, \\ &= (1 - R_{d-1})R_{m-1}a^\dagger, \\ &= R_{m-1}a^\dagger - R_{d-1}R_{m-1}a^\dagger, \\ &= R_{m-1}a^\dagger - R_{d-1}a^\dagger, \\ &= R_{m-1}a^\dagger, \end{aligned} \tag{E.16}$$

where we have used $R_{d-1}a^\dagger = a^{d-1}a^{\dagger d} = 0$. Let's summarize what we have derived about the projection operators P_n and R_n up to now;

$$\begin{aligned} P_n = a^{\dagger n}a^n, \quad P_0 = 1, \quad P_m a^\dagger = a^\dagger P_{m-1}, \quad aP_m = P_{m-1}a, \\ P_n P_m = P_j \quad \text{where} \quad j = \max(n, m), \\ P_m a^n = a^{\dagger n} P_m = 0 \quad \text{for} \quad n + m \geq d, \end{aligned}$$

and

$$R_n = a^n a^{\dagger n}, \quad R_0 = 1, \quad R_m a^\dagger = a^\dagger R_{m+1}, \quad aR_m = R_{m+1}a,$$
$$R_n R_m = R_j \quad \text{where} \quad j = \max(n, m),$$
$$a^n R_m = R_m a^{\dagger n} = 0 \quad \text{for} \quad n + m \geq d,$$

where

$$P_n = 1 - R_{d-n} \quad \text{and} \quad R_n = 1 - P_{d-n}.$$

E.2. Projection Operators in Terms of U and V

We have found two projection operators $\mathscr{P}_n$ and $\mathscr{R}_n$;

$$\mathscr{P}_n = \frac{(1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)})}{d}, \tag{E.17}$$
$$\mathscr{R}_n = 1 - \mathscr{P}_n. \tag{E.18}$$

Then

$$\begin{aligned}(\mathscr{P}_n)^2 &= \frac{(1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)})}{d} \\ &+ \frac{(q^n V + q^{2n} V^2 + q^{3n} V^3 + \cdots + q^{dn} V^d)}{d} \\ &+ \cdots, \\ &+ \frac{q^{(d-1)n} V^{(d-1)} + q^{dn} V^d + q^{(d+1)n} V^{(d+1)} + \cdots + q^{2(d-n)} V^{2(d-1)}}{d}, \\ &= d\frac{(1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)})}{d}, \\ &= \mathscr{P}_n,\end{aligned} \tag{E.19}$$

where we have used $V^d = 1$ and $q^d = 1$.

By using definition of $\mathscr{P}_n$ and Schwinger equation $VU = qUV$, one can easily obtain

$$\mathscr{P}_n U^m = U^m \mathscr{P}_{n+m}. \tag{E.20}$$

Hermitian conjugate of $\mathscr{P}_n$ is easily calculated as,

$$\begin{aligned}(\mathscr{P}_n)^\dagger &= \frac{(1 + (q^n)^\dagger V^\dagger + (q^{2n})^\dagger (V^2)^\dagger + \cdots + (q^{(d-1)n})^\dagger (V^{(d-1)})^\dagger)}{d}, \\ &= \frac{(1 + q^{d-n} V^{d-1} + q^{(d-2)n} V^{d-2} + \cdots + q^n V)}{d}, \\ &= \mathscr{P}_n,\end{aligned} \tag{E.21}$$

where we have used the unitary property of V and properties of complex number $q = exp(\frac{2\pi i}{d})$. By using hermicity property of $\mathscr{P}_n$ we will calculate,

$$\begin{aligned}&(\mathscr{P}_n U^m = U^m \mathscr{P}_{n+m})^\dagger, \\ &U^{\dagger m} \mathscr{P}_n = \mathscr{P}_{n+m} U^{\dagger m}.\end{aligned} \tag{E.22}$$

The other property of $\mathscr{P}_n$ is found by the following steps,

$$\begin{aligned}\mathscr{P}_0 + \mathscr{P}_1 + \mathscr{P}_2 + \cdots + \mathscr{P}_{(d-1)} = &\frac{(1 + V + V^2 + \cdots + V^{(d-1)})}{d} \\ &+ \frac{(1 + qV + q^2 V^2 + \cdots + q^{(d-1)} V^{(d-1)})}{d} \\ &+ \frac{(1 + q^2 V + q^4 V^2 + \cdots + q^{2(d-1)} V^{(d-1)})}{d} \\ &\cdots \\ &+ \frac{(1 + q^{d-1} V + q^{2(d-1)} V^2 + \cdots + q^{(d-1)(d-1)} V^{(d-1)})}{d},\end{aligned} \tag{E.23}$$

$$
\begin{aligned}
&= d/d + (1 + q + q^2 + \cdots + q^{(d-1)})V/d \\
&+ (1 + q^2 + q^4 + \cdots + q^{2(d-1)})V^2/d \\
&\cdots \\
&+ (1 + q^{(d-1)} + q^{2(d-1)} + \cdots + q^{(d-1)(d-1)})V^{(d-1)}/d. \\
&= 1.
\end{aligned}
$$

In the last equation we used the fact that the sum of roots of unity gives zero,

$$(1 + q + q^2) + \cdots + q^{d-1} = 0 \quad \text{with} \quad q = exp(\frac{2\pi i}{d}). \tag{E.24}$$

Furthermore, when we take unity as $1 = exp(2\pi in)$ where n is a integer, its roots will be $1, q^n, q^{2n}, \cdots, q^{n(d-1)}$. Thus the sum of the terms appear in parentheses in the last lines are equal to 0.

Then sum of $\mathscr{R}_n$ given by Equation (E.18) is found easily,

$$
\begin{aligned}
\mathscr{R}_0 + \mathscr{R}_1 + \cdots + \mathscr{R}_{d-1} &= (1 - \mathscr{P}_0) + (1 - \mathscr{P}_1) + \cdots + (1 - \mathscr{P}_{d-1}), \\
&= d - 1.
\end{aligned} \tag{E.25}
$$

Now we will show that $\mathscr{P}_n\mathscr{P}_m = 0$ unless $n \neq m$ by direct substitution of definition of projection operators

$$
\begin{aligned}
\mathscr{P}_n\mathscr{P}_m &= (1 + q^nV + q^{2n}V^2 + \cdots + q^{(d-1)n}V^{d-1})(1 + q^mV + q^{2m}V^2 + \cdots \\
&+ q^{(d-1)m}V^{d-1})/d^2 \\
&= \{(1 + q^mV + q^{2m}V^2 + \cdots + q^{(d-1)m}V^{d-1}) \\
&+ (q^nV + q^{n+m}V^2 + q^{n+2m}V^3 \cdots + q^{(d-1)m+n}V^d) \\
&+ (q^{2n}V^2 + q^{2n+m}V^3 + q^{2n+2m}V^4 \cdots + q^{(d-1)m+2n}V^{d+1})
\end{aligned} \tag{E.26}
$$

$$
\begin{aligned}
&+\cdots \\
&+ (q^{(d-1)n}V^{d-1} + q^{(d-1)n+m}V^{d} + q^{(d-1)n+2m}V^{(d+1)} \cdots + q^{(d-1)(m+n)}V^{2d-2})\}/d^2, \\
&= \{1 + (q^{(d-1)m+n} + q^{(d-2)m+2n} + \cdots + q^{(d-1)n+m})V^{d} \\
&+ (q^{m} + q^{n} + q^{(d-1)m+2n} + \cdots + q^{(d-1)n+2m})V \\
&+ (q^{2m} + q^{n+m} + q^{2n} + \cdots + q^{(d-1)n+3m})V^{2} \\
&+\cdots \\
&+ (q^{(d-1)m} + q^{(d-2)m+n} + \cdots + q^{(d-1)n})V^{d-1}\}/d^2,
\end{aligned}
$$

with the help of $V^{d+n} = V^n$. Now replace $n - m$ by k and use $q^d = 1$

$$
\begin{aligned}
\mathscr{P}_n\mathscr{P}_m = \{&(1 + q^k + q^{2k} + \cdots + q^{(d-1)k}) \\
&+ q^m(1 + q^k + q^{2k} + \cdots + q^{(d-1)k})V \\
&+ q^{2m}(1 + q^k + q^{2k} + \cdots + q^{(d-1)k})V^2 \\
&+ \cdots \\
&+ q^{(d-1)m}(1 + q^k + q^{2k} + \cdots + q^{(d-1)k})V^{d-1}\}/d^2.
\end{aligned} \tag{E.27}
$$

Furthermore, taking unity as 1, $1 + q^k + \cdots + q^{(d-1)k}$ will be equal to Equation (E.24) in different order. Hence $(1 + q^k + \cdots + q^{(d-1)k}) = 0$. As a result each parentheses in the Equation (E.27) are equal to zero except $k = 0$ (corresponds $n = m$). At the exception each paratheses sum to d. We can summarize our results as

$$
\begin{aligned}
\mathscr{P}_n\mathscr{P}_m &= 0 \quad \text{for } m \neq n, \\
\mathscr{P}_n\mathscr{P}_n &= d(1 + q^nV + \cdots + q^{(d-1)n}V^{d-1})/d^2, \\
&= \mathscr{P}_n.
\end{aligned} \tag{E.28}
$$

E.3. $M_N(C)$ in Terms of a and $a^\dagger$

The standard basis of $M_N(C)$ is given by the operator e_{ij} which satisfies

$$e_{ij}e_{kl} = \delta_{jk}e_{il}, \quad e_{ij}^\dagger = e_{ji}, \tag{E.29}$$

e_{mn} can be expressed in terms of a and $a^\dagger$ as,

$$e_{mn} = a^{\dagger m}R_{d-1}a^n \quad m, n = 0, 1, \cdots, d-1. \tag{E.30}$$

Then,

$$\begin{aligned}
e_{ij}e_{kl} &= (a^{\dagger i}R_{d-1}a^j)(a^{\dagger k}R_{d-1}a^l), \\
&= (a^{\dagger i}a^j R_{d-1-j})(R_{d-1-k}a^{\dagger k}a^l), \\
&= a^{\dagger i}a^j(R_{d-1-j}R_{d-1-k})a^{\dagger k}a^l, \\
\text{if} \quad j = k \quad &= a^{\dagger i}a^j R_{d-1-j}a^{\dagger j}a^l, \\
&= a^{\dagger i}R_{d-1}a^l, \\
&= e_{il}, \\
\text{if} \quad j > k \quad &= a^{\dagger i}a^j R_{d-1-k}a^{\dagger k}a^l, \\
&= a^{\dagger i}a^j a^{\dagger k}R_{d-1}a^l, \\
&= a^{\dagger i}a^j a^{\dagger k}R_k R_{d-1}a^l, \\
&= a^{\dagger i}a^{j-k}R_{d-1}a^l, \\
&= 0, \\
\text{if} \quad j < k \quad &= a^{\dagger i}a^j R_{d-1-j}a^{\dagger k}a^l,
\end{aligned} \tag{E.31}$$

$$
\begin{aligned}
&= a^{\dagger i} R_{d-1} a^j a^{\dagger k} a^l, \\
&= a^{\dagger i} R_{d-1} R_j a^{k-j} a^l, \\
&= a^{\dagger i} R_{d-1} a^{k-j} a^l, \\
&= a^{\dagger i} R_{d-1} R_j a^{k-j} a^l, \\
&= 0,
\end{aligned}
$$

where we have used (6.9). Thus $e_{ij}e_{kl} = \delta_{jk}e_{il}$. It is the time to check second part of Equation (E.29)

$$
\begin{aligned}
(e_{ij} &= a^{\dagger i} R_{d-1} a^j)^*, \\
e^*_{ij} &= a^{\dagger j} R^*_{d-1} a^i, \\
e^*_{ij} &= a^{\dagger j} R_{d-1} a^i, \\
e^*_{ij} &= e_{ji} \quad \text{Q.E.D..}
\end{aligned}
\tag{E.32}
$$

E.4. $M_N(C)$ in Terms of U and V

e_{mn} can be written in terms of operators U and V as

$$
e_{mn} = \begin{cases} U^{m-n} \mathscr{P}_{d-n} & \text{for} \quad m > n, \\ \mathscr{P}_{d-n} & \text{for} \quad m = n, \\ U^{\dagger n-m} \mathscr{P}_{d-n} & \text{for} \quad m < n, \end{cases}
\tag{E.33}
$$

with

$$
\mathscr{P}_n = \frac{(1 + q^n V + q^{2n} V^2 + \cdots + q^{(d-1)n} V^{(d-1)})}{d}.
$$

For $m > n$

$$
\begin{aligned}
e_{ij}e_{kl} &= U^{i-j}\mathscr{P}_{d-j}U^{k-l}\mathscr{P}_{d-l}, \\
&= U^{i-j}U^{k-l}\mathscr{P}_{d-j+k-l}\mathscr{P}_{d-l}, \\
\text{if} \quad j = k \qquad &= U^{i-j}U^{j-l}\mathscr{P}_{d-l}\mathscr{P}_{d-l}, \\
&= U^{i-l}\mathscr{P}_{d-l}, \\
&= e_{il}, \\
\text{if} \quad j \neq k \qquad &= U^{i-j}U^{k-l}\mathscr{P}_{d-j+k-l}\mathscr{P}_{d-l}, \\
&= 0.
\end{aligned} \tag{E.34}
$$

Thus $e_{ij}e_{kl} = \delta_{jk}e_{il}$.

Now let us check second part of the Equation (E.29) for $m > n$

$$
\begin{aligned}
(e_{mn} &= U^{m-n}\mathscr{P}_{d-n})^{\dagger}, \\
e^{\dagger}_{mn} &= \mathscr{P}^{\dagger}_{d-n}U^{\dagger m-n}, \\
e^{\dagger}_{mn} &= \mathscr{P}_{d-n}U^{\dagger m-n}, \\
e^{\dagger}_{mn} &= U^{\dagger m-n}\mathscr{P}_{d-n-(m-n)}, \\
e^{\dagger}_{mn} &= U^{\dagger m-n}\mathscr{P}_{d-m}, \\
e^{\dagger}_{mn} &= U^{n-m}\mathscr{P}_{d-m}, \\
e^{\dagger}_{mn} &= e_{nm},
\end{aligned} \tag{E.35}
$$

where we have used (E.21), (E.22) and (6.1). Next for $m = n$ we have

$$
\begin{aligned}
e_{ii}e_{kk} &= \mathscr{P}_{d-i}\mathscr{P}_{d-k}, \\
\text{if} \quad i = k, \qquad &= 1, \\
\text{if} \quad i \neq k, \qquad &= 0,
\end{aligned} \tag{E.36}
$$

where we have used (E.28). Thus $e_{ij}e_{kl} = e_{il}$. Now, hermicity of property of the projection operators given by Equation (E.21) imply the second part of the matrix algebra given by the Equation (E.29). Finally for $m < n$ we have

$$
\begin{aligned}
e_{ij}e_{kl} &= U^{\dagger j-i}\mathscr{P}_{d-j}U^{\dagger l-k}\mathscr{P}_{d-l}, \qquad \text{(E.37)}\\
&= U^{\dagger j-i}U^{\dagger l-k}\mathscr{P}_{d-j-l+k}\mathscr{P}_{d-l},\\
\text{if} \quad j = k \qquad &= U^{\dagger j-i}U^{\dagger l-k}\mathscr{P}_{d-l}\mathscr{P}_{d-l},\\
&= U^{\dagger l-i}\mathscr{P}_{d-l},\\
&= e_{il},\\
\text{if} \quad j \neq k \qquad &= U^{\dagger j-i}U^{\dagger l-k}\mathscr{P}_{d-j-l+k}\mathscr{P}_{d-l},\\
&= 0.
\end{aligned}
$$

Thus we can conclude that $e_{ij}e_{kl} = \delta_{jk}e_{il}$.

Now let us check second part of the Equation (E.29) for $m < n$ we have

$$
\begin{aligned}
(e_{mn} &= U^{\dagger n-m}\mathscr{P}_{d-n})^{\dagger}, \qquad \text{(E.38)}\\
&= \mathscr{P}^{\dagger}_{d-n}U^{n-m},\\
&= U^{n-m}\mathscr{P}_{d-n+n-m},\\
&= U^{n-m}\mathscr{P}_{d-m},\\
&= e_{nm} \quad \text{Q.E.D..}
\end{aligned}
$$