THE ROLE OF QUADRATIC LYAPUNOV FUNCTIONS IN AVERAGING BASED CONSENSUS ALGORITHMS

by

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ABSTRACT

THE ROLE OF QUADRATIC LYAPUNOV FUNCTIONS IN AVERAGING BASED CONSENSUS ALGORITHMS

Distributed consensus, that is described as the notion of achieving a common value by local information exchange, has been a popular research subject in recent years. Averaging based distributed consensus algorithms are applied in many areas such as distributed estimation, coverage control, distributed task assignment and clock synchronization. The focus of this thesis is to study the existence and the role of common quadratic Lyapunov functions in convergence analysis of averaging based distributed consensus algorithms. We first consider a discrete-time switched system model of the consensus algorithm where the network graph has a spanning tree. Although the algorithm is known to converge in this case, we show that there exists no common quadratic Lyapunov function for networks with five or more nodes. Subsequently, an approach is presented to determine whether a quadratic function is a Lyapunov function for a linear system with a stochastic system matrix. We also provide a sufficient condition that relates the coefficient of ergodicity to the existence of a special type of quadratic Lyapunov function for such systems. Based on these results, we generate a common quadratic Lyapunov function for systems defined by products of averaging algorithm matrices, which provides an alternative way of proving convergence of averaging based distributed consensus algorithms in networks having a spanning tree.

ÖZET

KARESEL LYAPUNOV FONKSİYONLARININ ORTALAMA HESAPLAMA TABANLI ONAYLAŞIM ALGORİTMALARINDAKİ ROLÜ

Ortak bir değere ulaşma olarak tanımlanan dağıtık onaylaşım, son yıllarda popüler bir araştırma konusudur. Ortalama hesaplama tabanlı dağıtık onaylaşım algoritmaları genellikle dağıtık kestirim, kaplam kontrolü, dağıtık görev tahsisi, saat eşzamanlama gibi birçok alanda uygulanır. Bu tezde, ortalama hesaplama tabanlı dağıtık onaylaşım algoritmalarının analizinde karesel Lyapunov fonksiyonlarının oluşumu ve rolü incelenmiştir. Öncelikle, bir kapsama ağacı içeren ağlarda onaylaşım algoritmasının ayrık zamanlı anahtar sistem modeli oluşturulmuş ve bu durumda algoritmanın yakınsayacağının bilinmesine rağmen, ortak karesel Lyapunov fonksiyonunun beş ya da daha fazla düğüm içeren ağlarda olmadığı gösterilmiştir. Daha sonra bir fonksiyonun stokastik sistem matrisli bir doğrusal sistem için Lyapunov fonksiyonu olup olmadığını belirlemek üzere yeni bir yaklaşım sunulmuştur ve söz konusu sistemler için karesel Lyapunov fonksi- yonlarından özel bir türün karesel Lyapunov fonksiyon olmasını döngelik katsayısıyla ilişkilendiren yeter koşul belirlenmiştir. Bu sonuçlar kullanılarak, ortalama hesaplama tabanlı algoritmaların matrislerinin çarpımlarıyla tanımlanan sistemler için ortak bir karesel Lyapunov fonksiyonu türetilmiştir. Önerilen yöntem, kapsama ağacı içeren ağlarda ortalama hesaplama tabanlı onaylaşım algoritmalarının yakınsamasını ispatlamak için alternatif bir yol sağlamaktadır.

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1. INTRODUCTION

With the development of engineering technology, systems are becoming too complex to analyse and design with classical methods. Controlling systems that consist of numerous sensors, drivers and actuators working together is a challenging problem in several engineering disciplines. The study of averaging algorithms has an important role in understanding the operation and the control of multi-agent systems [4–7, 21].

Consider a group of n agents with values $x_1(t)$, $x_2(t)$, ..., $x_n(t)$. Each agent tries to achieve consensus on a common value, x_{eq}, in a distributed way by changing information via communication links between them. Determining necessary and sufficient conditions for consensus, preferably fast convergence, is a critical problem for the multi-agent systems in which agents act based on the shared information in the presence of unreliable and limited information exchange and dynamically changing interaction structures.

The focus of this thesis is the convergence analysis of averaging based consensus algorithms. More specifically, we study the existence and non-existence of quadratic Lyapunov functions for averaging algorithms which help us analyse their convergence properties and the rate of convergence.

In this introductory chapter, we first present some motivating scenarios, i.e., application areas of consensus algorithms, more specifically averaging algorithms. Secondly, we take a glance at the related literature. Then, we briefly state the main contributions of this thesis. Lastly, we summarize the contents of the thesis.

1.1. Motivating Examples

Averaging based consensus algorithms are applicable for systems with multiple agents from many different disciplines. Several common application areas of the averaging algorithms are listed below:

- Distributed estimation: A set of sensors strives to estimate an unknown parameter from local observations at each sensor [4].
- Coverage control: A group of robots wish to position themselves so as to optimally monitor an environment of interest [5].
- Distributed task assignment: Allocate a collection of tasks among agents with individual preferences in a distributed way [6].
- Clock synchronization: A collection of clocks are constantly drifting apart, and would like to maintain a common time as much as this is possible. Various pairs of clocks can measure noisy versions of the time offsets between them [7, 8].

Now, let us briefly explain one of the scenarios that is listed above.

Consider a set of n sensors that are to be used to estimate an unknown parameter $\theta \in \mathbb{R}^k$ [25]. Some of these sensors are assumed to measure a noisy version of θ. More precisely, we have

$$x_i = \theta + w_i \tag{1.1}$$

where x_i is the measurement of the $i-th$ sensor; w_i is the noise term that is modelled as jointly Gaussian and independent at different sensors with statistics $w_i \sim N(0, \sigma_i^2)$. In this case, the maximum likelihood estimate is just the average of measurements, i.e.,

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i/\sigma_i^2}{\sum_{i=1}^{n} 1/\sigma_i^2} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{1.2}$$

The above estimate can be computed in a distributed way, i.e., there is not a 'computation center' that all sensors transmit measurements; instead, each sensor has to compute its estimate by receiving data from its neighbours.

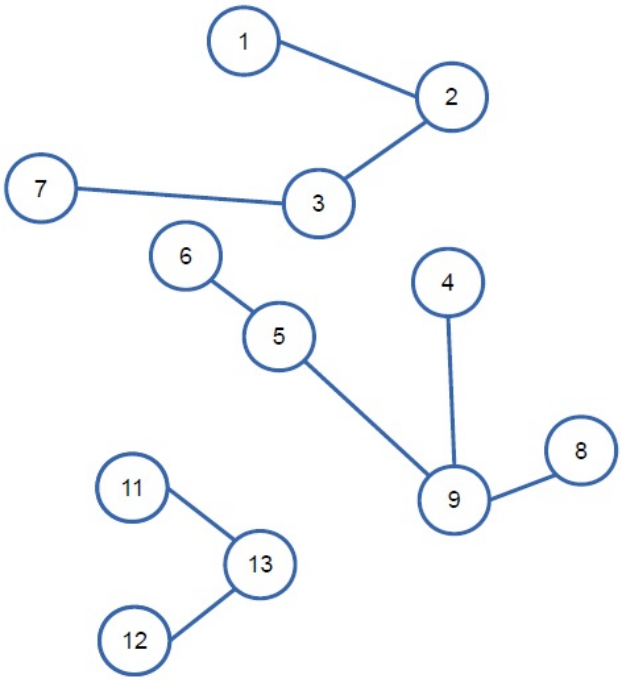

Figure 1.1. Set of online links between sensors at a given time.

Another problem that can arise in sensor networks is unreliable links between sensors. One link can be online for some time then may fail forever. In such a case, at any given time the network can split into disconnected clusters (see Figure 1.1).

More precisely, we assume a discrete sequence of times $t = 1, 2, 3, ...$, during which the sensors compute their estimates. At time t, sensor i can receive information from its neighbors. This can be modelled as a digraph where sensor i receives information from sensor j if and only if there is a link from node j to node i in the graph representation of the sensor network. The problem is to design distributed fast converging algorithms for estimating $\hat{\theta}$ and to determine the minimal connectivity requirements on $G(t)$.

There are several methodologies to solve the sensor network problem. One of the techniques is "flooding", where each sensor has an identifier and sends its identifier to its neighbors. Eventually every sensor has the knowledge of every single measurement that has been made. The advantage of flooding is its minimal requirements on connectivity. On the other hand, it has the disadvantage of hard storage requirements since each sensor needs to store n pieces of information [25].

Another method with better storage requirement is "leader election based protocols" [26]. In this scheme, a group of sensors elect one member as the leader. There are several protocols for doing this, e.g., choosing the sensor with the largest identifier number. Then, they build a spanning tree where the leader is the root. Finally, the sensors transmit all their information to the root, which can compute the estimate and send it back. Note that leader based schemes have less stringent storage requirements whereas they require connectivity.

On the other hand, a distributed averaging algorithm imposes minimal restrictions on connectivity and at the same time has better storage requirements. It only needs nodes to store a vector of constant real numbers and the connectivity condition that all unions of graphs in a finite time interval for all times should be connected is generally satisfied.

1.2. Related Literature

One of the first studies on distributed averaging algorithms was published by DeGroot who considered a simple "attaining algorithm" and gave a condition for convergence [17]. Chatterjee and Seneta extended this result to the case of varying weighting coefficients [18]. The problem of determining conditions on weighting coefficients and communication topologies necessary for consensus was studied in many works in the context of parallel computation of processors for a while. The set-up in these works is as follows: There are n processors with initial numbers x_i, $i = 1, ..., n$ stored in memory. The problem is to agree on a single value between the values $\min_i x_i$ and $\max_i x_i$. One can glance at Tsitsiklis' thesis [9] for a good summary.

Several years later, Vicsek *et al.* proposed a discrete time distributed algorithm for updating headings of n autonomous particles driven with a constant speed but with different headings in [3]; they simulated the algorithm and observed consensus. Then, Jadbabaei *et al.* show in [1] that consensus is guaranteed for the equal neighbor model described in [3] if the union of interaction graphs for the agents is connected frequently enough as the system evolves. As opposed to [3], authors in [19] considered

directed graphs and proved average consensus for strongly connected and balanced graphs. Moreau applied a set-valued Lyapunov approach to directed graphs with time-dependent communication links in [12]. In [13], Ren and Beard also study directed graphs with time-varying topologies and they show that consensus is achieved under dynamically changing interaction topologies if the union of the collection of the graphs has a spanning tree frequently enough.

The effects of delay on averaging based consensus algorithms have been investigated in [19, 32–35]. In [34], the authors provide a convergence analysis of linear and nonlinear protocols for undirected networks in presence of communication time-delays. In [19], the authors consider the directed networks and show that the maximum time-delay that can be tolerated by a network of integrators applying averaging based consensus algorithm is inversely proportional to the second largest eigenvalue of the averaging matrix. Liu *et al.* consider 1−D discrete-time asynchronous swarm model with a time-invariant network topology, and determine the conditions for asymptotic convergence in [32]. In [33], the results are extended to the multidimensional case. Gazi states the condition for consensus with time-dependent communication links in presence of time-delays as the old information being uniformly purged from the system in [35].

On the other hand, the agreement problem over random communication networks have been investigated in [36, 38]. In [36], the authors address the asymptotic convergence of the random graph model proposed by Erdős–Rényi in [37] via notions from stochastic stability. In [38], Akar and Shorten describe a probabilistic synchronization algorithm and analyse its convergence properties using tools of row-stochastic matrices.

One of the important ways of demonstrating the convergence of switched systems is by using the Lyapunov functions. In [22], Narendra and Balakrishnan show that a common quadratic Lyapunov function exists for two asymptotically stable matrices that commute and the system is stable for arbitrary switching between these two matrices. They have also extended their result for the set which contains finite number of matrices. In [23], Mori *et al.* show that triangular system matrices have common

quadratic Lyapunov functions. Moreover, Mori *et al.* relaxed these conditions further in [24]. More specifically, they prove that bounding a system matrix parameters' perturbations with the maximum allowable perturbations of the matrix parameters from a triangularizable set of matrices guarantees the existence of a common quadratic Lyapunov function for the system matrix. Necessary and sufficient conditions for the existence of common quadratic Lyapunov functions for continuous time and discrete time systems are derived in [29, 30]. Liberzon *et al.* present a Lie algebraic condition in [31]. Ögren *et al.* apply quadratic Lyapunov-function approach to analyse stability of formation control schemes [21].

Stability analysis of averaging algorithms via quadratic Lyapunov functions is of interest due to its implications on convergence rate. In [1], the authors note that there exists no common quadratic Lyapunov function for the system including 10 or more number of agents but they give no analytical explanation. In [2], Olshevsky and Tsitsiklis analytically show that there exists no common quadratic Lyapunov function for the equal neighbour model in [3] for systems including 8 or more number of nodes even if associated undirected graphs are connected, which also ensures that consensus is guaranteed.

1.3. The Main Contributions of This Thesis

In this thesis, we study quadratic Lyapunov functions in the context of stability analysis of consensus algorithms. The main contributions of this thesis are listed below:

- We first generalise the results of Olshevsky and Tsitsiklis in [2] to directed graphs which have a spanning tree; more specifically we show that there exists no common quadratic Lyapunov function for the equal neighbour algorithm for such graphs for $n \geq 5$.
- Secondly, we propose a new method, named reduction to lower order, to investigate the existence of quadratic Lyapunov functions in the lower order space by computing singular values. Based on this method, we define a special class of linear systems for which there exists a special type of common quadratic Lyapunov

functions.

- The third contribution of this thesis is to demonstrate that our proposed approach provides a common quadratic Lyapunov function for the averaging based algorithms under certain conditions defined in [16]. Moreover, it provides an alternative proof for the results of [10].

1.4. Organization of the Thesis

The rest of the thesis is organized as follows:

Chapter 2: In Section 2.1, we give basic definitions related to averaging algorithms. Section 2.2 includes the graph theoretical representation of the consensus problem and several definitions related to graph theory. Section 2.3 discusses Lyapunov functions and quadratic Lyapunov functions. Lastly, we present several examples to clarify the concepts that is introduced throughout the chapter.

Chapter 3: In Section 3.1, we review the basic convergence results on consensus. Section 3.2 investigates several averaging based consensus schemes for which quadratic Lyapunov functions exist. Lastly, in Section 3.3, we analyse the existence of quadratic Lyapunov functions for the equal neighbor averaging model whose associated graphs are connected or have a spanning tree.

Chapter 4: In Section 4.1, we first propose a reduced order approach in order to check whether a quadratic function is a quadratic Lyapunov function for an agreement algorithm or not. Then we provide a special class of algorithms for which existence of a common quadratic Lyapunov function is guaranteed. We subsequently use this result to provide a common quadratic Lyapunov function for the averaging based algorithms under certain conditions defined in [16]. In addition, showing existence of a common quadratic Lyapunov function is an alternative proof for the results in [16].

Chapter 5: We state concluding remarks and further possible research directions.

2. AVERAGING BASED CONSENSUS ALGORITHM AND MATHEMATICAL PRELIMINARIES

In this chapter, we first present distributed averaging based consensus algorithms for multi-agent systems. Then, we describe special types of nearest neighbor algorithms. Later on, we present the graph theoretical representation of multi-agent systems and several definitions from graph theory. Finally, we introduce Lyapunov functions and quadratic Lyapunov functions.

2.1. Averaging Algorithm

Consider a group of n agents with $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T \in \mathbb{R}^n$, which is a vector of values on which agents are to reach consensus. Agents try to reach consensus as soon as possible by transmitting information via communication links. Achieving consensus for a set of connected agents is generally easy. However, in real time applications such as mobile robots and sensor networks, one can encounter the lack of connectivity. Therefore, we are looking for distributed algorithms that provide consensus.

Let each agent independently update its value in discrete time according to the averaging algorithm

$$x_i(t+1) = \sum_{j=1}^{n} a_{ij}(t)x_j(t), \;\; i = 1, 2, ..., n \tag{2.1}$$

where $a_{ij}(t)$ are non-negative weighting coefficients.

Equation 2.1 can also be written in matrix form as

$$x(t+1) = A(t)x(t) \tag{2.2}$$

where $x(t) = [x_1(t), x_2(t), ..., x_n(t)]^T$ and $A(t) = [a_{ij}(t)]_{n\times n}$. This formulation of consensus algorithms can be considered as an example of switched linear systems.

Let a network of agents be represented by a time varying graph $G(t) = (V, E(t))$ where $V = \{1, 2, ..., n\}$ is the set of vertices and $E(t)$ is the set of edges. Agent i can send messages to agent j at time t if and only if $(i, j) \in E(t)$. The set of neighbours of agent i at time t is defined as the set of nodes that can send messages to agent i at time t, i.e., $\mathcal{N}_i(t) = \{j|(j, i) \in E(t)\}$.

We introduce the following set of assumptions on the weighting coefficients, $a_{ij}(t)$, in order to study convergence properties of averaging algorithms.

Assumption 2.1. *(i) There exists some $\alpha > 0$ such that $a_{ii}(t) > \alpha$ for all $i = 1, 2, ..., n$ and $t \geq 0$.*

(ii) $a_{ij}(t) = 0$ if $(j, i) \notin E(t)$ and $a_{ij}(t) \in [\alpha, 1)$, if $j \in \mathcal{N}_i(t)$.

(iii) $\sum_{j=1}^{n} a_{ij}(t) = 1$ for all $i = 1, 2, ..., n$ and for all $t \geq 0$.

Assumption 2.1 implies that the associated linear system matrix, $A(t)$, is non-negative and its row-sums are equal to 1. Such matrices are referred to as *row-stochastic matrices*.

2.1.1. Three Heuristic Averaging Algorithms

One the most popular averaging algorithms is known as the equal neighbour algorithm. Consider the update equation Equation 2.1. In the equal neighbour algorithm, the weighting coefficients, $a_{ij}(t)$, are chosen as

$$a_{ij}(t) = \begin{cases} \frac{1}{d_i(t)+1}, & \text{if } j \in N_i(t), \\ 0, & \text{if } j \notin N_i(t), \end{cases} \tag{2.3}$$

where $d_i(t)$ is the number of the nodes in $\mathcal{N}_i(t)$. The equal neighbour algorithm is proposed for a group of n autonomous agents in [3].

Another popular averaging algorithm is known as the Metropolis model [27]. In this model, the weighting coefficients are assigned as follows:

$$\begin{aligned} a_{ij}(t) &= \begin{cases} \frac{1}{\max(d_i(t),d_j(t))+1}, & \text{if } j \in N_i(t), i \neq j, \\ 0, & \text{if } j \notin N_i(t), \end{cases} \\ a_{ii}(t) &= 1 - \textstyle\sum_j a_{ij}(t),\ i = 1, ..., n. \end{aligned} \tag{2.4}$$

The Metropolis model can be considered as a special case of the equal neighbour algorithms; it has the advantage of symmetry which can be both exploited in convergence analysis and convergence rate analysis. We will examine this model in more detail in Section 3.2.

In this thesis, we also propose a new heuristic nearest neighbor consensus algorithm, named *priority to social nodes* algorithm, with the hope that it converges faster. Consider a connected undirected graph. Let a node assign priorities to its neighbors based on the number of their neighbors, i.e., the coefficients in the proposed updating algorithm are chosen proportional to the number of neighbors of the nodes as follows:

$$\begin{aligned} a_{ij}(t) &= \begin{cases} \frac{d_j(t)}{\sum_{k \in N_i(t)} d_k(t)}, & \text{if } j \in N_i(t), \\ 0, & \text{if } j \notin N_i(t), \end{cases} \\ a_{ii}(t) &= 1 - \textstyle\sum_j a_{ij}(t),\ i = 1, ..., n. \end{aligned} \tag{2.5}$$

2.2. Graph Preliminaries

In a graph $G = (V, E)$, if there is a directed edge from vertex i to vertex j, in other words $(i, j) \in E$, vertex i is defined as a parent node and vertex j is defined as a child node. A directed path in a graph G is a list of edges $(i_1, i_2), (i_2, i_3), (i_3, i_4), ...$ in that graph.

Definition 2.1. *(Undirected Graph) For a graph $G = (V, E)$, if all edges are bidirectional, i.e., $(i, j) \in E$ for all $(j, i) \in E$, then G is said to be an undirected graph.*

Definition 2.2. *(Connected Graph) For an undirected graph $G = (V, E)$ if there exists a path from agent i to agent j for all $i, j \in V$, then G is said to be a connected graph.*

Definition 2.3. *(Strongly Connected Graph) For a directed graph $G = (V, E)$ if there exists a directed path from agent i to agent j for all $i, j \in V$, then G is said to be a strongly connected graph.*

Definition 2.4. *(Directed Tree) A directed tree is a directed graph, in which every node, except the root, has exactly one parent.*

Definition 2.5. *(Spanning Tree) A spanning tree of a directed graph is a directed tree which connects all the nodes of the graph.*

Consider a set of graphs $G = \{\mathcal{G}_1, \mathcal{G}_2, ..., \mathcal{G}_m\}$ with a common vertex set $V = \{1, 2, ..., n\}$. The union of this collection of graphs $\mathcal{G}$ is a graph with vertex set V and the edge set equalling the union of the edge sets of all of the graphs in the set.

2.3. Lyapunov Functions

In this thesis, we investigate whether a common quadratic Lyapunov function exists for the switched linear system Equation 2.2. Existence of a common quadratic Lyapunov function for switching system matrices implies that the system is asymptotically stable. Moreover, it can be utilized as an indicator of the convergence rate of the consensus algorithm in our case. Therefore, existence of common quadratic Lyapunov functions for consensus algorithms is of interest.

Definition 2.6. *(Common Lyapunov Function) Consider the switched linear system in Equation 2.2. Let $\mathcal{A}$ be a finite set of $n \times n$ stochastic system matrices and $\boldsymbol{e} \in \mathbb{R}^n$ be $[1\,1\,...\,1]^T$. A function $V : \mathbb{R}^n \to \mathbb{R}$ is said to be a common Lyapunov function if it satisfies the following two conditions:*

a) $V(x) = 0$ for all $x \in Span\{\boldsymbol{e}\}$ and $V(x) > 0$, otherwise.

b) $V(Ax) \leq V(x)$ for all $A \in \mathcal{A}$ and for all $x \in \mathbb{R}^n$.

In the class of Lyapunov functions, the quadratic ones are of interest because of their potential implications on convergence rate analysis.

Definition 2.7. *(Common Quadratic Lyapunov Function) Consider the switched linear system in Equation 2.2. Let $\mathcal{A}$ be a finite set of $n \times n$ stochastic system matrices. A function $Q : \mathbb{R}^n \to \mathbb{R}$ is said to be a common quadratic Lyapunov function if it is of the form $Q(x) = x^T M x$ and satisfies the following conditions:*

a) The matrix $M \in \mathbb{R}^{n \times n}$ is non-zero, symmetric and nonnegative definite,

b) We have $Q(Ax) \leq Q(x)$, for all $A \in \mathcal{A}$ and for all $x \in \mathbb{R}^n$,

c) Equation $Q(x) = 0$ is satisfied only for $x \in Span\{\boldsymbol{e}\}$.

2.4. Examples

Consider the network with time-invariant topology depicted in Figure 2.1. Note that the associated undirected graph is connected.

In this case, the averaging matrices defined in Section 2.1.1 are listed below.

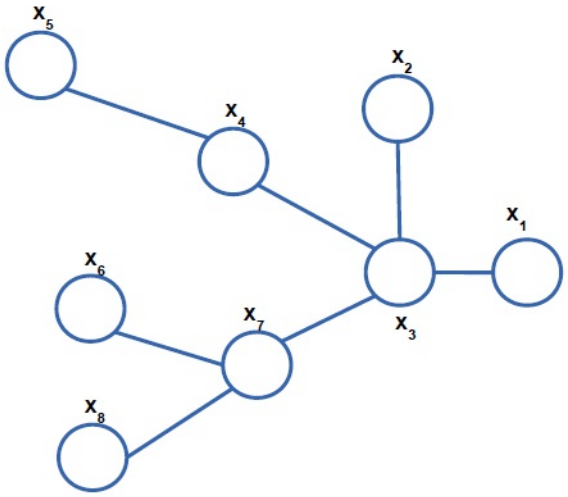

Figure 2.1. An example of connected network.

System matrix for the equal neighbor model Equation 2.3:

$$A_{eq} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \tag{2.6}$$

System matrix for the Metropolis model Equation 2.4:

$$A_{mm} = \begin{bmatrix} \frac{4}{5} & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{5} & \frac{7}{15} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{5} & 0 & 0 & \frac{1}{4} & \frac{3}{10} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{3}{4} \end{bmatrix} \tag{2.7}$$

System matrix for the priority to social nodes model Equation 2.5:

$$A_{psn} = \begin{bmatrix} \frac{2}{7} & 0 & \frac{5}{7} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{2}{7} & \frac{5}{7} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{8} & \frac{1}{8} & \frac{5}{16} & \frac{3}{16} & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{3}{10} & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{3}{5} & \frac{2}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & \frac{5}{13} & 0 & 0 & \frac{2}{13} & \frac{4}{13} & \frac{2}{13} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \end{bmatrix} \tag{2.8}$$

Now let $x(t) \in \mathbb{R}^8$ be a vector with the initial values $x_1(0) = 2, x_2(0) = 3, x_3(0) = 1.4, x_4(0) = 5/3, x_5(0) = 3.5, x_6(0) = 1, x_7(0) = 1.75, x_8(0) = 0.5$. Let each node update its values according to Equation 2.2.

We now present the simulation results for the equal neighbor model, where the averaging matrix is as in Equation 2.6; for the Metropolis model, where the averaging matrix is as in Equation 2.7; and for the priority to social nodes model, where the averaging matrix is as in Equation 2.8, respectively.

All of the algorithms achieve consensus as depicted in Figures $2.2 - 2.4$. Note also that agents converge to different values when using each algorithm. In this case, the consensus values for the equal neighbor model, the Metropolis model and the priority to social nodes model are 2.045, 2.101 and 1.946, respectively.

Another important observation from Figures $2.2 - 2.4$ is related to how fast each algorithm converges. We know that the convergence speed is related to the second largest eigenvalue modulus. Let $\lambda_2(A)$ denote the second largest eigenvalue of the matrix A. For the example matrices, we compute the second largest eigenvalues as follows:

$$\lambda_2(A_{eq}) \approx 0.866 \tag{2.9}$$

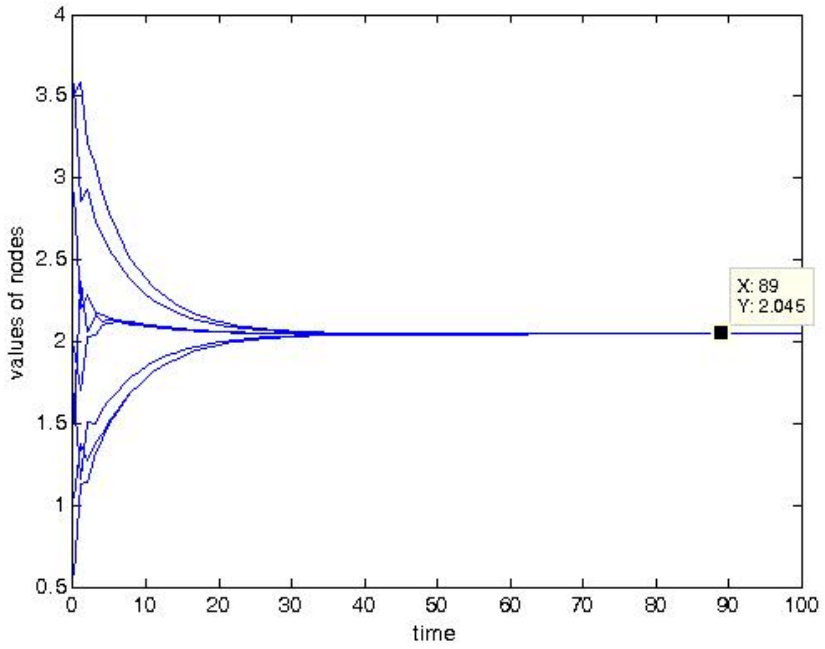

Figure 2.2. The Equal Neighbor Model.

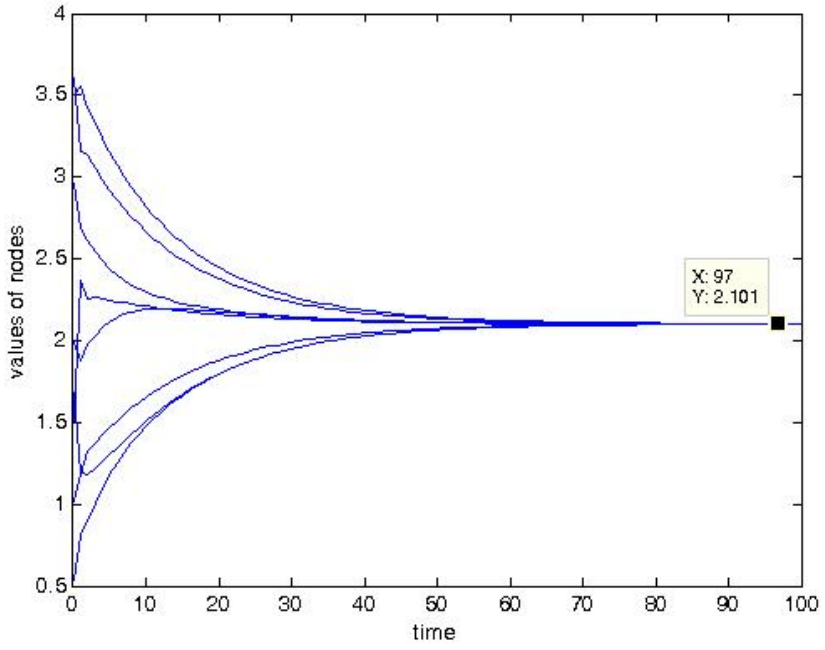

Figure 2.3. The Metropolis Model.

$$\lambda_2(A_{mm}) \approx 0.932 \tag{2.10}$$

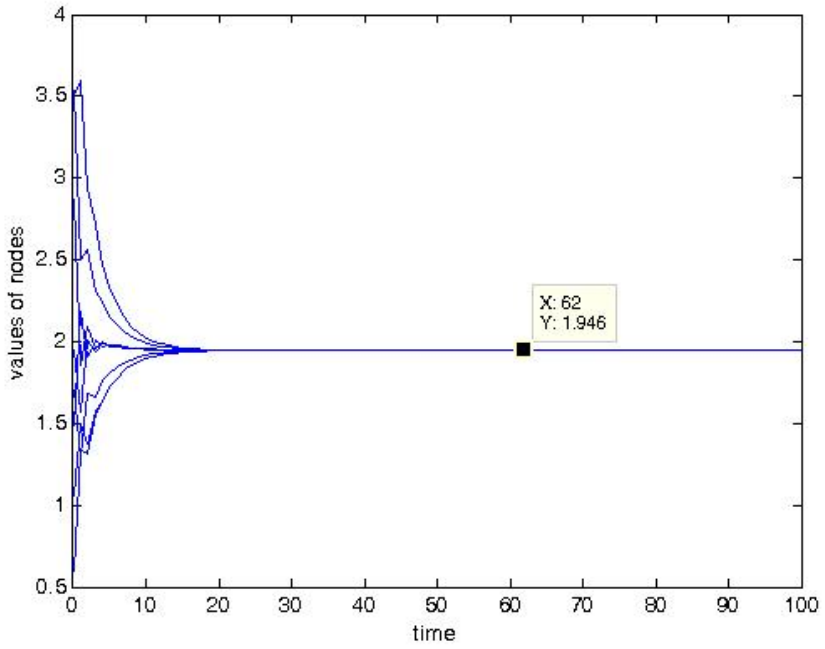

Figure 2.4. The Priority to Social Nodes Model.

$$\lambda_2(A_{psn}) \approx 0.736 \tag{2.11}$$

Note that, Figures $2.2 - 2.4$ present this result.

Now consider the switched system as

$$x(t+1) = A(t)x(t), \;\; A(t) \in \{A_{eq}, A_{psn}, A_{mm}\} \text{ for all } t \tag{2.12}$$

where the matrices, A_{eq}, A_{mm}, and A_{psn} are defined in Equation 2.6−Equation 2.8. Suppose that the system is switched with $A(t)$ being randomly selected from the set $\{A_{eq}, A_{psn}, A_{mm}\}$ with equal probabilities at each time t. Consensus can also be observed in this case as shown in Figure 2.5. The value to which the agents converge is determined by $A(t)$ and the initial values [16].

Now define the sample variance $V(x)$ as

$$V(x) = \sum_{i=1}^{n} (x_i - \bar{x}) \tag{2.13}$$

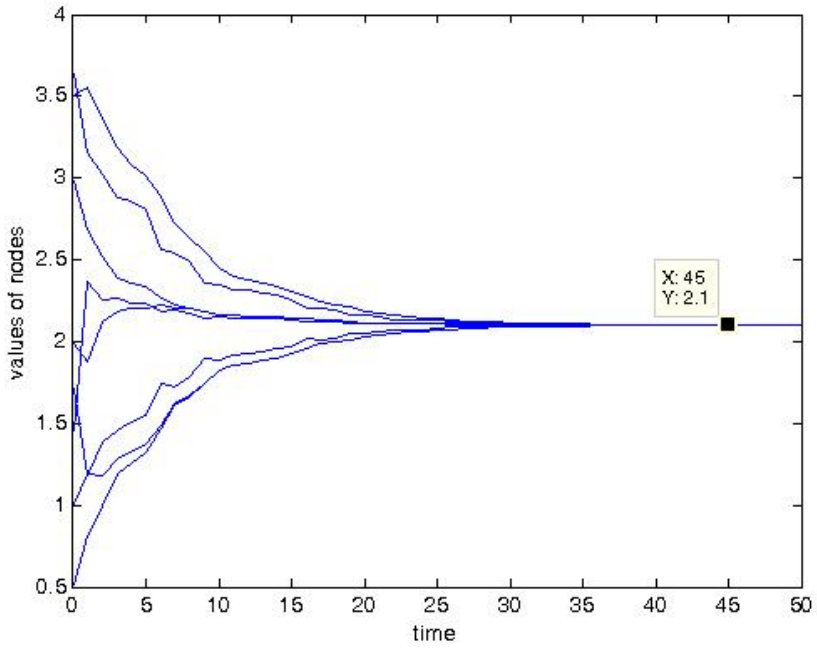

Figure 2.5. The Switched System.

where $\bar{x} = (1/n)\sum_{i=1}^{n} x_i$. Notice that variance is a quadratic function of x and thus, $V(x) = x^T M x$ for a suitable symmetric matrix $M \in \mathbb{R}^{n\times n}$. This makes sense because variance is a measure of the difference between the values of nodes. As shown in Figures $2.6 - 2.7$ the variance is monotonically decreasing. Whether variance is a quadratic Lyapunov function for any stochastic matrix has to be analytically investigated, and this is one of the main discussions in this thesis.

2.5. Summary of the Chapter

In this chapter, we have introduced the averaging based consensus algorithms and some related mathematical preliminaries. These definitions and descriptions will be used to analyse the convergence properties of averaging algorithms in the following chapters.

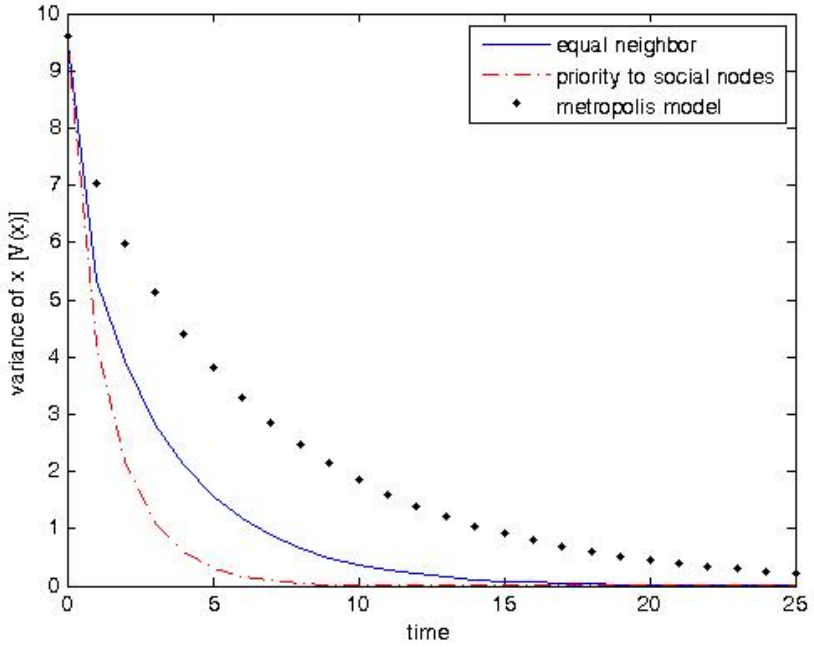

Figure 2.6. Variances are monotonically decreasing.

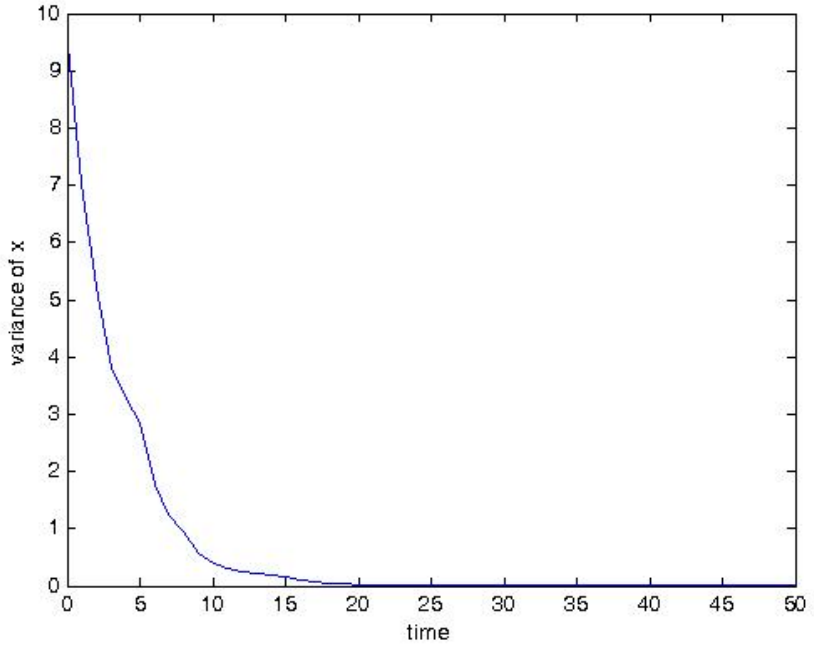

Figure 2.7. Variance In Switched System.

3. QUADRATIC LYAPUNOV FUNCTIONS IN STABILITY ANALYSIS OF CONSENSUS ALGORITHMS

In Chapter 2, we have introduced the averaging algorithms and related mathematical preliminaries which help us in their analysis. In this chapter, we first review the convergence results on consensus for different network topologies. Secondly, we demonstrate that there exists a common quadratic Lyapunov function for systems that are denoted by the Metropolis model and for systems that have only 2 or 3 nodes. Later on, we extend the results of [2] to directed graphs. We show that there exists no common quadratic Lyapunov function for such systems when $n \geq 5$.

3.1. Review of Convergence Results on Consensus

Consider a group of n agents with the initial values $x_1(0)$, $x_2(0)$, ..., $x_n(0)$. They communicate with each other at each time t via communication links determined by the directed graph $G(t) = (V, E(t))$ and each node updates its value according to Equation 2.1. In [1], the authors analysed the equal neighbour algorithm, proposed in [3], described in Section 2.1.1.

Lemma 3.1. *(Theorem 1 in [1]) Let $x(0) = [x_1(0), x_2(0), ..., x_n(0)]^T$ be fixed, $G(t)$ be a connected graph for all $t \in \{0, 1, ...\}$ and each agent update its value based on Equation 2.1. The weighting coefficients, a_{ij}, are as defined in Equation 2.3. Then*

$$\lim_{t\to\infty} x(t) = x_{ss}\boldsymbol{e} \tag{3.1}$$

where $\boldsymbol{e} = [1, 1, ..., 1]^T$ and x_{ss} is a number depending only on $x(0)$ and $G(t)$.

As expected, a more likely situation in real time applications is when $G(t)$ is not connected for all $t \in \{0, 1, ...\}$, but the union of $\{G(t), G(t+1), ..., G(t+\tau)\}$ is jointly connected for an arbitrary τ.

Lemma 3.2. *(Theorem 2 in [1]) Let $x(0) = [x_1(0), x_2(0), ..., x_n(0)]^T$ be fixed, for an*

arbitrary but finite τ, the collection of $\{G(t), G(t+1), ..., G(t+\tau)\}$ be a jointly connected graph for all $t \in \{0, 1, ...\}$ and each agent update its value based on Equation 2.1. The weighting coefficients, a_{ij}, are as defined in Equation 2.3. Then

$$\lim_{t\to\infty} x(t) = x_{ss}\boldsymbol{e} \tag{3.2}$$

where $\boldsymbol{e} = [1, 1, ..., 1]^T$ and x_{ss} is a number depending only on $x(0)$ and $G(t)$.

Lemma 3.1 is a special case of Lemma 3.2 and Lemma 3.2 is proved in [1] using the results of [20] which is on products of indecomposable, aperiodic, stochastic matrices. Even though the authors do not state in the theorem, the proof of the theorem covers that under Assumption 2.1 for a finite set of the averaging matrices and the same connectivity conditions in the theorem, consensus is still guaranteed.

The results in [1] are related to undirected graphs. For directed graphs, in [19], Olfati-Saber and Muray show that if the directed graphs are strongly connected and balanced and there exists a finite set of the weighting coefficients, a_{ij}, satisfying Assumption 2.1, consensus is guaranteed for the update equation Equation 2.1. Then, Moreau relaxed strong connectivity condition in [12]. Ren and Beard state a more relaxed necessary condition for consensus in [16].

Lemma 3.3. *(Theorem 3.10 in [16]) Let $x(0) = [x_1(0), x_2(0), ..., x_n(0)]^T$ be fixed, for an arbitrary but finite τ, union of the collection of $\{G(t), G(t+1), ..., G(t+\tau)\}$ have a spanning tree for all $t \in \{0, 1, ...\}$ and each agent update its value based on Equation 2.1. The set of averaging matrices is a finite set and satisfies Assumption 2.1. Then*

$$\lim_{t\to\infty} x(t) = x_{ss}\boldsymbol{e} \tag{3.3}$$

where $\boldsymbol{e} = [1, 1, ..., 1]^T$ and x_{ss} is a number depending only on $x(0)$ and $G(t)$. Moreover, consensus is achieved asymptotically, and if the union of the graphs after some finite time does not have a spanning tree, consensus cannot be achieved asymptotically.

In [16], Ren and Beard also note that the result in Lemma 3.3 is also valid for infinite set of averaging coefficients a_{ij}, where for all $a_{ij} \in [\alpha, 1)$ for an arbitrary α satisfying $0 < \alpha < 1$. The proof of this statement is based on [20].

3.2. Existence of Quadratic Lyapunov Functions

3.2.1. Metropolis Model

The Metropolis algorithm has the advantage of symmetry which can be exploited to show the existence of a common quadratic Lyapunov function.

Proposition 3.1. *([2]) Let $\mathcal{B}$ a set of $n \times n$ stochastic matrices. Suppose that there exists a positive vector π such that $\pi^T \mathbf{e} = 1$, and $\pi^T A = \pi^T$ for all $A \in \mathcal{B}$, there exists a quadratic Lyapunov function, $Q(x) = x^T M x$, for the switched system, $x(t+1) = A(t)x(t)$ where $A(t) \in \mathcal{B}$ for all t such that $M = H^T D H$, $H = I - \mathbf{e}\pi^T$ and D is a diagonal matrix, whose $i-th$ diagonal entry is equal to π_i.*

Remark 3.1. *Let $\mathcal{C}$ be a set of $n \times n$ stochastic matrices having connected graphs and coefficients defined as in Equation 2.4. Then, there exists a quadratic Lyapunov function, $Q(x) = x^T M x$, for the switched system $x(t+1) = A(t)x(t)$ where $A(t) \in \mathcal{B}$ for all t such that*

$$M = \begin{bmatrix} (n-1)/n & -1/n & \dots & -1/n \\ -1/n & (n-1)/n & \dots & -1/n \\ \vdots & \vdots & \ddots & \vdots \\ -1/n & -1/n & ... & (n-1)/n \end{bmatrix}. \tag{3.4}$$

Proof. Let $\mathcal{C}$ be a set of $n \times n$ stochastic matrices having connected undirected associated networks and coefficients as in Equation 2.4. Note that the Metropolis model matrices are symmetric, i.e., $A^T = A$ for all $A \in \mathcal{C}$. Recall that $A\mathbf{e} = \mathbf{e}$, then $A^T\mathbf{e} = \mathbf{e}$ and $\mathbf{e}^T A = \mathbf{e}^T$. From Lemma 3.1, we obtain $H = I - \frac{\mathbf{e}\mathbf{e}^T}{n}$ and $D = I$. Then, M can

be computed as

$$M = H^T I H = \begin{bmatrix} (n-1)/n & -1/n & \dots & -1/n \\ -1/n & (n-1)/n & \dots & -1/n \\ \vdots & \vdots & \ddots & \vdots \\ -1/n & -1/n & ... & (n-1)/n \end{bmatrix}. \tag{3.5}$$

□

3.2.2. Analytical Study: Equal Neighbor Model with at most Three Agents

Consider a network with $n \leq 3$. In this case, it can be analytically shown that there exist quadratic Lyapunov functions for the equal neighbour algorithm matrices for the connected undirected graphs which have only two or three nodes.

For $n = 2$, variance is a quadratic Lyapunov function for all row-stochastic system matrices, even when the algorithm is not the equal neighbour algorithm. In this case, the difference between the values of the two nodes is a decreasing function, unless they are equal to each other.

For connected graphs which have three nodes, there are only two possible equal neighbour algorithm matrices. One of them is all nodes connected and the other is nodes being connected pairwise. The other graphs are only permutations of these configurations. The possible equal neighbour matrices are as follows:

$$A_1 = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix} \tag{3.6}$$

$$A_2 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \end{bmatrix} \tag{3.7}$$

Obviously, agents reach consensus in the first step and the consensus value is the mean of initial values when the update matrix is A_1.

When the update matrix is A_2, to prove that variance is a quadratic Lyapunov function, let the state be

$$x(t) = [x_1(t)\ x_2(t)\ x_3(t)]^T. \tag{3.8}$$

Then, the variance of $x(t)$ is given by

$$V(x(t)) = ((2x_1 - x_2 - x_3)/3)^2 + ((2x_2 - x_1 - x_3)/3)^2 + ((2x_3 - x_1 - x_2)/3)^2. \tag{3.9}$$

Consider the state vector, $x(t+1)$, one step later;

$$x(t+1) = A_2 x(t) = \begin{bmatrix} (x_1 + x_2)/2 \\ (x_1 + x_2 + x_3)/3 \\ (x_2 + x_3)/2 \end{bmatrix} \tag{3.10}$$

The variance of $x(t+1)$ can be computed as

$$V(x(t+1)) = [(4x_1+x_2-5x_3)/18]^2+[(x_1+x_3-2x_2)/18]^2+[(4x_3+x_2-5x_1)/18]^2. \tag{3.11}$$

To prove that variance is a quadratic Lyapunov function for the averaging algorithm for networks including only 3 nodes, it is enough to show that

$$V(x(t+1)) - V(x(t)) \leq 0 \tag{3.12}$$

for all $x(t)$.

For simplicity, let

$$a = (x_1 - x_2)/18,\ b = (x_1 - x_3)/18,\ c = (x_2 - x_3)/18 = b - a. \tag{3.13}$$

Then, Equation 3.9 and Equation 3.11 can be rewritten as

$$V(x(t)) = (6a + 6b)^2 + (6b - 12a)^2 + (6a - 12b)^2 \tag{3.14}$$

$$V(x(t+1)) = (5b - a)^2 + (2a - b)^2 + (-a - 4b)^2 \tag{3.15}$$

to obtain

$$\begin{aligned} V(x(t+1)) - V(x(t)) &= -(210a^2 - 210ab + 174b^2) && (3.16)\\ &= -(\sqrt{105}a - \sqrt{105}b)^2 - 105a^2 - 69b^2. && (3.17) \end{aligned}$$

Note that for all a, b we have $V(x(t+1)) - V(x(t)) \leq 0$ and equality is satisfied only when $a = b = 0$, i.e., when

$$x_1 = x_2 = x_3 \tag{3.18}$$

Therefore, we can conclude that variance is a quadratic Lyapunov function for undirected and connected equal neighbour algorithm for the graphs which have three nodes.

3.2.3. Numerical Study: Equal Neighbor Model with $n \leq 7$

Above we analytically show that variance is a common quadratic Lyapunov function. We try to extend this result for larger n and are able to demonstrate numerically that a common quadratic Lyapunov exists for $n \leq 7$. However, the analytical calculations are too cumbersome to present in this thesis.

3.3. Nonexistence of Common Quadratic Lyapunov Functions

In [1], Jadbabaei *et al.* state that there exists no common quadratic Lyapunov function for the class of equal neighbor averaging algorithms proposed in [3] whose associated graphs have 10 or more number of vertices and are connected, i.e., no common quadratic Lyapunov function exists for this class even if consensus is guaranteed for the algorithm. However, they proposed no analytical example. In [2], Olshevsky and Tsitsiklis analytically show that there exists no common quadratic Lyapunov function for the class of equal neighbour averaging algorithms proposed in [3] whose associated graphs have 8 or more number of vertices and are connected.

In their proof, they first note that *variance* must be a common quadratic Lyapunov function for the class of all possible equal neighbour averaging algorithm matrices. Then, they construct a counter example for which variance is not a quadratic Lyapunov function.

Consider permutations of n elements as a bijective mapping $\xi : \{1, ..., n\} \rightarrow \{1, ..., n\}$. Let P_ξ be defined as corresponding permutation matrix such as $P_\xi x = x_\xi$.

Lemma 3.4. *(Lemma 6 and 7 in [2]) Let $\mathcal{A} \subset \mathbb{R}^{n \times n}$ be the set of stochastic matrices A satisfying Assumption 2.1 and all positive entries on any row of A are equal. Let $Q(x) = x^T M x$ be a quadratic Lyapunov function for associated switched linear system defined by Equation 2.2 where $A(t) \in \mathcal{A}$ for all t and $\mathcal{P}$ be a set of all permutation*

matrices. Define Z as

$$Z = \sum_{P \in \mathcal{P}} P^T M P \tag{3.19}$$

Then, Z is also a quadratic Lyapunov function for this switched linear system. Moreover,

$$x^T Z x = \alpha V(x), \ \ \textit{for all } x \in \mathbb{R}^n \tag{3.20}$$

Lemma 3.5. *(Theorem 5 in [2]) Let $\mathcal{A} \subset \mathbb{R}^{n \times n}$ be the set of stochastic matrices A satisfying Assumption 2.1, having connected associated graphs and all positive entries on any row of A are equal. There exists no common quadratic Lyapunov function for linear systems defined by matrices in set $\mathcal{A}$ for $n \geq 8$.*

In [2], the authors give the example network shown in Figure 3.3, with the initial conditions $x_1 = 5$, $x_2 = x_3 = x_4 = 2$, $x_5 = 0$, $x_6 = x_7 = -3$, $x_8 = 5$ and $x_9 = ... = x_n = 0$. Then, $V(x) = 80$. Let each agent update its value according to Equation 2.1, where the weighting coefficients, a_{ij}, are as in Equation 2.3. The new vector $y = Ax$, which is obtained at the first iteration, has values $y_1 = 11/5$, $y_2 = y_3 = y_4 = 7/2$, $y_5 = 0$, $y_6 = y_7 = -4$, $y_8 = -11/4$, and $y_9 = ... = y_n = 0$ and $V(y) \approx 80.68$. Thus, $V(y) > V(x)$ and $V(x)$ is not a quadratic Lyapunov function for the system $x(t+1) = Ax(t)$. Therefore, there exists no common quadratic Lyapunov function for $n \geq 8$.

The first contribution of this thesis is to extend the results of Lemma 3.5 to directed graphs.

Theorem 3.1. *Let $\mathcal{A} \subset \mathbb{R}^{n \times n}$ be a set of stochastic matrices A satisfying Assumption 2.1, whose associated graphs have a spanning tree and all positive entries on any row of A are equal. There exists no common quadratic Lyapunov function for linear systems defined by matrices in set $\mathcal{A}$ for $n \geq 5$.*

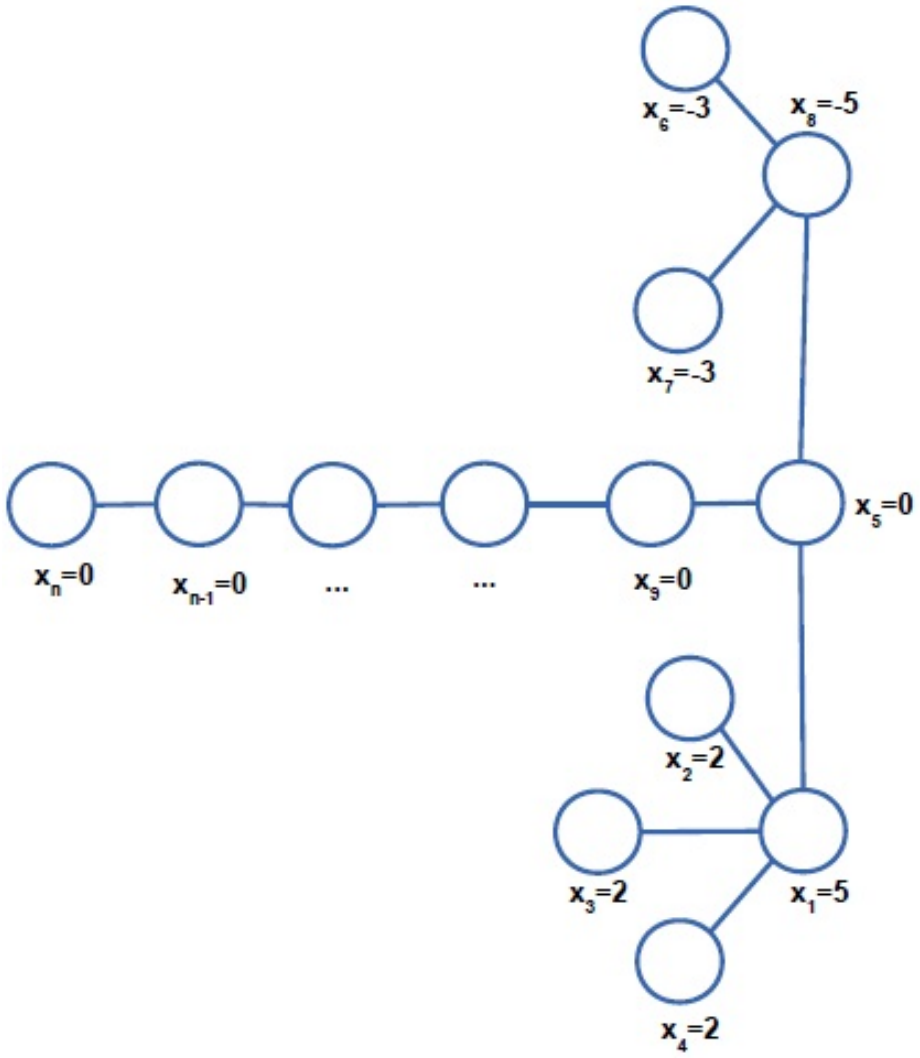

Figure 3.1. $V(x)$ is not a common Lyapunov function for $n \geq 8$ [2].

Proof. If there exists any common quadratic Lyapunov function for the linear systems defined by the matrices in the set $\mathcal{A}$, variance $V(x)$ must also be a common quadratic Lyapunov function according to Lemma 3.4, for which a counter example is developed.

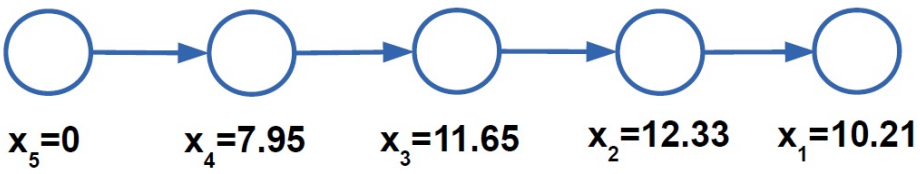

Figure 3.2. $V(x)$ is not a common Lyapunov function when $n \geq 5$.

Consider the vector of agents x with components $x_1 = 10.21$, $x_2 = 12.33$, $x_3 = 11.65$, $x_4 = 7.95$, $x_5 = 0$. Then, $V(x) = 500.21$. Let each agent update its value according to Equation 2.1, where the weighting coefficients a_{ij} are as in Equation 2.3 and associated graph has the form in Figure 3.3. The new vector $y = Ax$, which is obtained at the first iteration, has components $y_1 = 11.27$, $y_2 = 11.99$, $y_3 = 9.80$, $y_4 = 3.975$, $y_5 = 0$, and $V(y) \approx 541.47$. Thus, $V(y) > V(x)$ and $V(x)$ is not a quadratic Lyapunov function. Therefore, there exists no common quadratic Lyapunov function for $n \geq 5$. □

3.4. Summary of the Chapter

In this chapter, we have reviewed the related convergence results on consensus which basically state the necessary and sufficient condition for consensus as the associated graphs having a spanning tree frequently enough. Later on, we have shown that there exists no common quadratic Lyapunov function for the equal neighbor model in networks with five or more nodes having a spanning tree, i.e., where consensus is guaranteed. Hence, one can conclude that there exists no common quadratic Lyapunov function for averaging algorithms in networks with 5 or more nodes even if consensus is observed. In the following chapter, we will show that a common quadratic Lyapunov function can be generated for averaging algorithms in an alternative way using finite matrix products.

4. ANALYSIS IN REDUCED ORDER SPACE

The main contribution of this thesis is presented in this chapter. We first propose a method called reduction to lower order to examine the existence of a quadratic Lyapunov function for a consensus system. We further utilize this method to show that a special class of quadratic Lyapunov functions exists for a special type of switched linear systems that represent the averaging algorithms for the networks with varying topologies. Subsequently, we use this result to provide a common quadratic Lyapunov function for the averaging based algorithms under the conditions stated in [16]. Moreover, showing existence of a common quadratic Lyapunov function is an alternative proof for the results in [16].

4.1. Reduction to Lower Order

In this section, we propose an approach which can help us determine whether a quadratic function is a quadratic Lyapunov function for a switched linear system, or not. Switching linear system matrices (averaging matrices) can be analysed in reduced order space with this approach. It can also help us define several classes of averaging matrices for which the existence of a common quadratic Lyapunov function is guaranteed.

Firstly, let us recall the definition *singular values* of a matrix.

Definition 4.1. *The singular values $\sigma_1, ..., \sigma_r$ of an $m \times n$ matrix A are the positive square roots of the eigenvalues of $A^T A$.*

Note that the singular values of a matrix are always non-negative. The standard convention is to label the singular values in decreasing order, i.e., $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_r \geq 0$.

We now state the second main contribution of this thesis.

Theorem 4.1. *For any $n \times n$ averaging matrix A and full row rank matrix $P \in \mathbb{R}^{(n-1)\times n}$ such that $P\boldsymbol{e} = \boldsymbol{0}$, there exists a quadratic Lyapunov function $Q(x) = x^T M x$ such that $M = P^T P$ if and only if all singular values of A_r are less than 1 where*

$$A_r = PAP^+ \tag{4.1}$$

and P^+ is the pseudo inverse of P denoted as

$$P^+ = P^T(PP^T)^{-1}. \tag{4.2}$$

Proof. (Sufficiency) Suppose $\sigma_1(A_r) < 1$; this implies

$$x^{*T} A_r^T A_r x^* - x^{*T} x^* < 0 \text{ for all } x^* \in \mathbb{R}^{n-1}. \tag{4.3}$$

By using Equation 4.1 and Equation 4.2 in above, we obtain

$$(P^+x^*)^T A^T P^T PAP^+ x^* - (P^+x^*)^T P^T PP^+ x^* < 0. \tag{4.4}$$

$P\mathbf{e} = 0$, thus P^+ is full column rank and the column space of P^+ is orthogonal to $\mathbf{e}$. Therefore, the column space of P^+ and $\mathbf{e}$ form a basis for $\mathbb{R}^n$, i.e., any $x \in \mathbb{R}^n$ can be obtained by using an x^* as follows

$$x = P^+x^* + k\mathbf{e} \tag{4.5}$$

where $x^* = Px$ and $k \in \mathbb{R}$. Adding $k\mathbf{e}$ does not change the Inequality 4.4; only less than sign should be less than or equal to sign because of $k\mathbf{e}$ term (when $x^* = 0$ left hand side of the inequality should be equal to 0), i.e.,

$$(P^+x^* + k\mathbf{e})^T A^T P^T PA(P^+x^* + k\mathbf{e}) - (P^+x^* + k\mathbf{e})^T P^T P(P^+x^* + k\mathbf{e}) \leq 0 \tag{4.6}$$

is satisfied for all $x^* \in \mathbb{R}^{n-1}$ and for all $k \in \mathbb{R}$. Recall that the column space of P^+

and $\mathbf{e}$ form a basis for $\mathbb{R}^n$. Then, if Equation 4.6 is satisfied for all $x^* \in \mathbb{R}^{n-1}$ and for all $k \in \mathbb{R}$, the following equation should also be satisfied

$$x^T A^T P^T P A x - x^T P^T P x \leq 0 \text{ for all } x \in \mathbb{R}^n. \tag{4.7}$$

Let $M = P^T P$. Then $Q(x) = x^T M x$ qualifies as a quadratic Lyapunov function for the system.

(Necessity) Suppose $Q(x) = x^T M x$ is a quadratic Lyapunov function for the system with system matrix A where $M = P^T P$. Then we have,

$$x^T A^T M A x - x^T M x \leq 0 \text{ for all } x \in \mathbb{R}^n. \tag{4.8}$$

Substitute Equation 4.5 into the above inequality and use $A\mathbf{e} = \mathbf{e}$, $M\mathbf{e} = P^T P\mathbf{e} = \mathbf{0}$ to obtain

$$(P^+ x^*)^T A^T M A P^+ x^* - (P^+ x^*)^T M P^+ x^* < 0 \tag{4.9}$$

where the equality disappears due to $\mathcal{N}(M) = \{\mathbf{e}\}$ and the column space of P^+ being orthogonal to $\mathbf{e}$.

$$x^{*T} P^{+T} A^T M A P^+ x^* - x^{*T} P^{+T} M P^+ x^* < 0 \tag{4.10}$$

Substitute $P^T P$ for M, and utilize Equation 4.1 and Equation 4.2 in the above inequality to obtain

$$x^{*T} A_r^T A_r x^* - x^{*T} x^* < 0 \tag{4.11}$$

This ensures that the singular values of A_r are less than 1. □

Remark 4.1. *One of the P matrices that satisfies $P\mathbf{e} = 0$ is given by*

$$P = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{-2}{\sqrt{6}} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \\ \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \frac{1}{\sqrt{n(n-1)}} & \dots & \frac{-(n-1)}{\sqrt{n(n-1)}} \end{bmatrix}. \tag{4.12}$$

In this case,

$$M = P^T P = I_{n\times n} - \frac{1}{n} J_{n\times n} \tag{4.13}$$

where $J_{n\times n}$ is $n \times n$ ones matrix. In [2], $Q(x) = x^T M x$ is the variance of x which Olshevsky and Tsitsiklis use in order to verify that there exists no common quadratic Lyapunov function for the equal neighbor algorithm for networks including 8 or more number of nodes.

4.1.1. Examples

In this section, we provide two examples to illustrate the results of Lemma 3.5 and Theorem 3.1.

Example 4.1. *To illustrate the results of Lemma 3.5 for undirected graphs, consider the network in Figure 3.3 with the system matrix*

$$A = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}. \tag{4.14}$$

The associated reduced order matrix $A_r = PAP^+$ can be computed as

$$A_r = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4} & \frac{3}{4\sqrt{15}} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{4\sqrt{15}} & \frac{7}{20} & \frac{4}{5\sqrt{6}} & 0 & 0 \\ 0 & 0 & \frac{-2}{3\sqrt{10}} & \frac{7}{10\sqrt{6}} & \frac{4}{15} & \frac{5}{3\sqrt{35}} & \frac{5}{2\sqrt{105}} \\ 0 & 0 & \frac{-2}{3\sqrt{14}} & \frac{1}{2\sqrt{210}} & \frac{17}{6\sqrt{35}} & \frac{19}{42} & \frac{-1}{14\sqrt{3}} \\ 0 & 0 & \frac{-1}{42} & \frac{1}{4\sqrt{70}} & \frac{17}{2\sqrt{105}} & \frac{-1}{14\sqrt{3}} & \frac{13}{28} \end{bmatrix}, \tag{4.15}$$

whose maximum singular value is $\sigma_1(A) \approx 1.0123$. Since $\sigma_1(A) > 1$, the variance is not a quadratic Lyapunov function for the linear system $x(t+1) = Ax(t)$. According to Lemma 3.4, there exists no common quadratic Lyapunov function for the set of stochastic matrices $A \in \mathbb{R}^{8\times 8}$ satisfying Assumption 2.1, having connected associated graphs and all positive entries on any row of A are equal.

Example 4.2. *To illustrate the results of Theorem 3.1 for directed networks, consider the network in Figure 3.3 with the system matrix*

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.16}$$

In this case, the associated reduced order matrix $A_r = PAP^+$ can be computed as

$$A_r = \begin{bmatrix} \frac{1}{4} & \frac{3}{4\sqrt{3}} & 0 & 0 \\ -\frac{1}{4\sqrt{3}} & \frac{5}{12} & \frac{2}{3\sqrt{2}} & 0 \\ -\frac{1}{4\sqrt{6}} & -\frac{1}{12\sqrt{2}} & \frac{11}{24} & \frac{15}{8\sqrt{15}} \\ -\frac{1}{4\sqrt{10}} & \frac{1}{4\sqrt{30}} & -\frac{1}{8\sqrt{15}} & \frac{7}{8} \end{bmatrix}. \tag{4.17}$$

Since the maximum singular value of A_r is $\sigma_1(A) \approx 1.33$, the variance is not a quadratic

Lyapunov function for the linear system $x(t+1) = Ax(t)$. Then, according to Lemma 3.4, there exists no common quadratic Lyapunov function for the set of stochastic matrices $A \in \mathbb{R}^{5\times5}$ satisfying Assumption 2.1, whose associated graphs have a spanning tree and all positive entries on any row of A are equal.

4.2. Convergence Analysis and Existence of Quadratic Lyapunov Functions

Consider the switched linear system in Equation 2.2 and suppose that Assumption 2.1 is valid. One of the useful tools related to stochastic matrices is the coefficient of ergodicity. The coefficient of ergodicity for a stochastic matrix A, $\tau(A)$ is defined in [14] as:

$$\tau(A) = \frac{1}{2}\max_{i,j}\sum_{k}|a_{ik} - a_{jk}|. \tag{4.18}$$

The coefficient of ergodicity satisfies $0 \leq \tau(A) \leq 1$. Furthermore, if $\tau(A) < 1$, the matrix A is called a scrambling matrix [14].

Now we propose a special type of reduction matrix which will help us use the coefficient of ergodicity in defining a new class of switching linear system matrices (averaging matrices) for which existence of common quadratic Lyapunov function is guaranteed.

Consider the reduction matrices we have discussed in Section 4.1. There is a special one among these matrices, which we can use to relate the existence of quadratic Lyapunov function for a system matrix to the coefficient of ergodicity. Let $A = [a_{ij}] \in$

$\mathbb{R}^{n\times n}$ be the linear system matrix. Let the reduction matrix $P \in \mathbb{R}^{(n-1)\times n}$ be given as

$$P = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & -1 \\ 0 & 1 & \dots & 0 & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & -1 \\ 0 & 0 & \dots & 0 & 1 & -1 \end{bmatrix}. \tag{4.19}$$

The pseudo-inverse of P can be computed as

$$P^{+} = \begin{bmatrix} \frac{n-1}{n} & \frac{-1}{n} & \frac{-1}{n} & \dots & \frac{-1}{n} & \frac{-1}{n} \\ \frac{-1}{n} & \frac{n-1}{n} & \frac{-1}{n} & \dots & \frac{-1}{n} & \frac{-1}{n} \\ \frac{-1}{n} & \frac{-1}{n} & \frac{n-1}{n} & \dots & \frac{-1}{n} & \frac{-1}{n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-1}{n} & \frac{-1}{n} & \frac{-1}{n} & \dots & \frac{-1}{n} & \frac{(n-1)}{n} \\ \frac{-1}{n} & \frac{-1}{n} & \frac{-1}{n} & \dots & \frac{-1}{n} & \frac{-1}{n} \end{bmatrix}. \tag{4.20}$$

A simple bound of the largest singular value of matrix A, $\sigma_1(A)$, is defined in [14] as follows:

$$\sigma_1(A) \leq (\|A\|_1 \|A\|_\infty)^{1/2} \tag{4.21}$$

According to Lemma 4.1, if the maximum singular value of $A_r = PAP^{+}$ is less than 1, then $Q(x) = x^T Mx$ is a quadratic Lyapunov function for the discrete time

linear system $x(t+1) = Ax(t)$ where

$$M = P^T P = \begin{bmatrix} 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \\ -1 & -1 & \dots & -1 & n-1 \end{bmatrix}. \tag{4.22}$$

If $\sigma_1(A_r) < 1$ is satisfied, then $Q(x) = x^T Mx$ is a quadratic Lyapunov function for the discrete time linear system $x(t+1) = Ax(t)$. Using Inequality 4.21, if

$$(\|A_r\|_1 \|A_r\|_\infty)^{1/2} < 1, \tag{4.23}$$

then we can conclude that $Q(x) = x^T Mx$ is a quadratic Lyapunov function for the discrete time linear system $x(t+1) = Ax(t)$. The infinity norm of matrix A_r is defined in [15] as

$$\|A_r\|_\infty = \max_{1 \le i \le n-1} \sum_{j=1}^{n-1} |a_{r_{ij}}|. \tag{4.24}$$

In this case, the reduced order system matrix $A_r \in \mathbb{R}^{(n-1)\times(n-1)}$ can be computed as

$$A_r = PAP^+ = [a_{r,ij}] \tag{4.25}$$

where

$$a_{r,ij} = a_{ij} - a_{nj} \tag{4.26}$$

Then, the infinity norm of A_r can be defined as

$$\|A_r\|_\infty = \max_{1\leq i\leq n-1} \sum_{j=1}^{n-1} |a_{ij} - a_{nj}|. \tag{4.27}$$

An upper bound can be obtained for $\|A_r\|_\infty$ by using the coefficient of ergodicity defined in Equation 4.18 as

$$\|A_r\|_\infty = \max_{1\leq i\leq n-1} \sum_{j=1}^{n-1} |a_{ij} - a_{nj}| \leq 2\tau(A). \tag{4.28}$$

Based on the relations between matrix norms, an upper bound for $\|A_r\|_1$ is defined in [15] as

$$\|A_r\|_1 \leq n\|A_r\|_\infty \leq 2n\tau(A). \tag{4.29}$$

The above discussion allows us to state the following result.

Lemma 4.1. *If*

$$\tau(A) < \frac{1}{2\sqrt{n}} \tag{4.30}$$

then $Q(x) = x^T Mx$ is a quadratic Lyapunov function for the discrete time linear system $x(t+1) = Ax(t)$ where A is a row stochastic matrix that satisfies Assumption (2.1).

We conclude that a quadratic Lyapunov function can be determined via using Lemma 4.1.

Theorem 4.2. *Consider a finite set of $n\times n$ matrices $\mathcal{A} = \{A_1, A_2, ..., A_N\}$ which satisfies Assumption 2.1 and whose associated graphs have a spanning tree. Let π_k be the set of matrices which contains all products of matrices in $\mathcal{A}$ of length $k \geq 1$, e.g. $\pi_1 = \mathcal{A} = \{A_1, A_2, ..., A_N\}$, $\pi_2 = \mathcal{A}\times\pi_1 = \{A_1^2, A_1A_2..., A_1A_N, A_2A_1, A_2^2..., A_2A_N, ..., A_NA_1, A_NA_2,$*

$..., A_N^2\}$, $\pi_k = \mathcal{A} \times \pi_{k-1}$, $k \geq 1$. *There exists a common quadratic Lyapunov function for all matrix products in* π_N *for an arbitrary but finite* N *such that* $Q(x) = x^T Mx$ *where* M *is denoted as in Equation 4.22.*

Proof. In [10], the authors show that all matrices in π_{n-1} are scrambling, when $\mathcal{A}$ is a finite set of matrices that satisfies Assumption 2.1 and whose associated graphs have a spanning tree, in other words $\tau(W) < 1$ for all $W \in \pi_{n-1}$.

This implies that there exists a finite K such that for all possible choices of $W_1, W_2, ..., W_K \in \pi_{n-1}$,

$$\tau(W_1 W_2 ... W_K) \leq \tau(W_1)\tau(W_2)...\tau(W_K) < \frac{1}{2\sqrt{n}} \tag{4.31}$$

[14]. According to Lemma 4.1, $Q(x) = x^T Mx$ is a common quadratic Lyapunov function for all matrices $W \in \pi_N$ where $N = K(n-1)$ and M is denoted as in Equation 4.22. □

It is possible to determine an upper bound for N by using the worst case scenario analysis.

Theorem 4.3. *Consider a finite set of* $n \times n$ *matrices* $\mathcal{A} = \{A_1, A_2, ..., A_N\}$ *which satisfies Assumption 2.1 and whose associated graphs have a spanning tree. Let* π_{n-1} *be the set of matrices which contains all products of matrices in* $\mathcal{A}$ *of length* $n-1 \geq 1$. *Assume that* $W_{max} \in \pi_{n-1}$ *is the matrix with the maximum value of coefficient of ergodicity, i.e.,* $\tau(W_{max}) \geq \tau(W)$ *for all* $W \in \pi_{n-1}$. *Let* m *be the smallest integer which satisfies* $m \geq \log_{\tau(W_{max})} \frac{1}{2\sqrt{n}}$. *Then, there exists a common quadratic Lyapunov function for all* $W \in \pi_N$ *for* $N = m(n-1)$ *such that* $Q(x) = x^T Mx$ *where* M *is denoted as in Equation 4.22.*

Proof. From [14], we have

$$\tau(W_1W_2...W_K) \leq \tau(W_1)\tau(W_2)...\tau(W_K) \ . \tag{4.32}$$

Let

$$W_{max} = \arg \max_{W \in \pi_{n-1}} \tau(W). \tag{4.33}$$

to obtain

$$\tau(W_1W_2...W_K) \leq \tau^K(W_{max}). \tag{4.34}$$

Let m be the smallest integer satisfying $m \geq \log_{\tau(W_{max})} \frac{1}{2\sqrt{n}}$ and $N = m(n-1)$, then $\tau(W) < \frac{1}{2\sqrt{n}}$ for all $W \in \pi_N$. According to Lemma 4.1, $Q(x) = x^T Mx$ is a common quadratic Lyapunov function for the switching system matrix products $W \in \pi_N$. □

The results in Theorem 4.2 can be extended to the case where the network $G(t)$ does not have a spanning tree for all t but the union of the networks $G(t), G(t+1), ..., G(t+\tau)$ has a spanning tree for all t for a finite τ.

Example 4.3. *Consider a discrete time switched linear system* $x(t+1) = A(t)x(t)$ *where* $A(t) \in \{A_1, A_2\}$ *with*

$$A_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix} \tag{4.35}$$

$$A_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \tag{4.36}$$

First note that Assumption 2.1 is satisfied for this system and both matrices have connected associated graphs. Therefore, consensus is guaranteed [1]. However, an example of common quadratic Lyapunov functions cannot be found by either analytic or numeric methods based on convex optimization. Nevertheless, by the approach suggested in this thesis consensus can be proven by showing the existence of a quadratic Lyapunov function for augmented system.

Consider the set $\pi_4 = \{A_1^4,\ A_2A_1^3,\ A_1A_2A_1^2,\ A_1^2A_2A_1, A_1^3A_2,\ A_2^2A_1^2,\ A_2A_1A_2A_1,\ A_2 A_1^2A_2,\ A_1A_2^2A_1,\ A_1A_2A_1A_2,\ A_1^2A_2^2,\ A_1A_2^3,\ A_2A_1A_2^2,\ A_2^2A_1A_2,\ A_2^3A_1,\ A_2^4\}$. $\tau(W) < 1$ *for all* $W \in \pi_4$ *because the associated graphs are undirected and connected. We have* $\tau(W_{max}) = 0.97$ *and* $\log_{\tau(W_{max})} \frac{1}{2\sqrt{8}} = 57$. *Then, define the augmented system as* $x(t+k) = W(t)x(t)$ *where* $k = 4 \times 57$ *and* $W(t) \in \pi_k$. *It is obviously a loose bound. It can be shown that variance is a quadratic Lyapunov function for the augmented system for* $k = 2$. *However, this approach provides us an upper limit for all types of averaging algorithms.*

Theorem 4.4. *Consider a discrete time system Equation 2.2 switched with* $A(t)$ *being selected from a finite set of* $n \times n$ *averaging matrices* $\mathcal{A} = \{A_1, A_2, ..., A_N\}$ *which satisfies Assumption 2.1 and there exists an infinite sequence of uniformly bounded, nonoverlapping time intervals* $[t, t+\tau_i)$, $i = 1, 2, ...$, *starting at* $t = 0$, *such that the union of the associated graphs* $G(t) \cup G(t+1) \cup ... \cup G(t+\tau_i)$ *at each interval* $[t, t+\tau_i)$ *have a spanning tree. The system Equation 2.2 achieves consensus asymptotically.*

Proof. Note that we suppose uniformly bounded and nonoverlapping time intervals $[t, t+\tau_i)$. There exists a maximum time constant, τ_{max}, as follows:

$$\tau_{max} = \max_{i=1,2,3,...} \tau_i \tag{4.37}$$

Consider the discrete time system Equation 2.2 that is described in the theorem. Define an augmented system as follows:

$$x(t+\tau_{max}) = B(t)x(t) \tag{4.38}$$

where

$$B(t) = A(t)A(t+1)...A(t+\tau_{max}-1). \tag{4.39}$$

$\mathcal{B}$ is a finite set of stochastic matrices such that $B(t) \in \mathcal{B}$ for all t, because set $\mathcal{A}$ is a finite set and τ_{max} exists, i.e., time intervals are uniformly bounded. Moreover, every matrix $B \in \mathcal{B}$ satisfies Assumption 2.1 and has associated graphs that have a spanning tree.

Let π_k be the set of matrices which contains all products of matrices in $\mathcal{B}$ of length $k \geq 1$, e.g. $\pi_1 = \mathcal{B} = \{B_1, B_2, ..., B_L\}$, $\pi_2 = \mathcal{B} \times \pi_1 = \{B_1^2, B_1B_2, ..., B_1B_N, B_2B_1, B_2^2, ..., B_2B_N, ..., B_NB_1, B_NB_2, ..., B_N^2\}$, $\pi_k = \mathcal{B} \times \pi_{k-1}$, $k \geq 1$. In [10], the authors show that all matrices in π_{n-1} are scrambling, i.e. $\tau(W) < 1$ for all $W \in \pi_{n-1}$.

Based on Theorem 4.2, this implies that there exists a finite K such that for all possible choices of $W_1, W_2, ..., W_K \in \pi_{n-1}$,

$$\tau(W_1W_2...W_K) \leq \tau(W_1)\tau(W_2)...\tau(W_K) < \frac{1}{2\sqrt{n}} \tag{4.40}$$

[14]. According to Lemma 4.1, $Q(x) = x^TMx$ is a common quadratic Lyapunov function for all matrices $W \in \pi_N$ where $N = K(n-1)$ and M is denoted as in Equation

4.22. We conclude that the augmented system, Equation 4.38 is asymptotically stable, and so is the system that is described in the theorem. □

4.3. Summary of the Chapter

In this chapter, we have proposed a new method to investigate the existence of quadratic Lyapunov functions for averaging algorithms in reduced order space. We have also provided a common quadratic Lyapuov function for systems whose products of averaging matrices satisfy the conditions determined in [16]. Generating a common quadratic Lyapunov function is an alternative way of proving convergence of the algorithms that satisfies the conditions in [16].

5. CONCLUSIONS

In this thesis, we discuss averaging based consensus algorithms. We first state the consensus problem for multi-agent systems in Chapter 2 and provide the mathematical preliminaries necessary to examine convergence and convergence rate of these algorithms. In Chapter 3, we review the literature on convergence analysis of averaging algorithms and present necessary and sufficient conditions for convergence in directed and undirected graphs. Later, we discuss quadratic Lyapunov functions in the context of averaging algorithms. Quadratic Lyapunov functions are of interest because of their implications on convergence rate of averaging algorithms. The first contribution of the thesis is to generalise the results in [2] to directed graphs. We show that there exists no common quadratic Lyapunov function for the systems denoted by equal neighbor averaging model whose associated graphs are connected or having a spanning tree, i.e., consensus is guaranteed.

In the fourth and the main chapter, we first propose a reduced order approach in order to check whether a quadratic function is a quadratic Lyapunov function for an agreement algorithm or not. This method has been utilized to determine a new class of averaging algorithms for which existence of a common quadratic Lyapunov function is guaranteed. By using this class, we provide a common quadratic Lyapunov function for the system, denoted by matrix products, even if there exists no common quadratic Lyapunov functions for original matrices under the necessary and sufficient condition for consensus presented in [16]. Moreover, we determine an upper bound for the number of the products of matrices for the existence of a common quadratic Lyapunov function by using the coefficient of ergodicity and the result in Theorem 4.3.

We are aware of the fact that this bound is a loose one. However, note that this bound is for the general case, and not for special types of algorithms. Tighter bounds can be found for special types of algorithms by using the proposed strategy. In addition, by employing different types of reduction matrices the original upper bound can be made more stringent.

One open research subject in the context of this thesis is to investigate the convergence rate of the heuristic algorithm Equation 2.5 proposed in Section 2.1.1. Although simulations numerically show that this proposed algorithm tends to converge faster than the other two popular heuristic methods, e.g., the equal neighbor algorithm and the Metropolis model, the class of graphs for which this is true has to be justified analytically. In this context, results in recent Ph.D. theses [25] and [28] are relevant.

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